Foundations of Data Science

Probability Space

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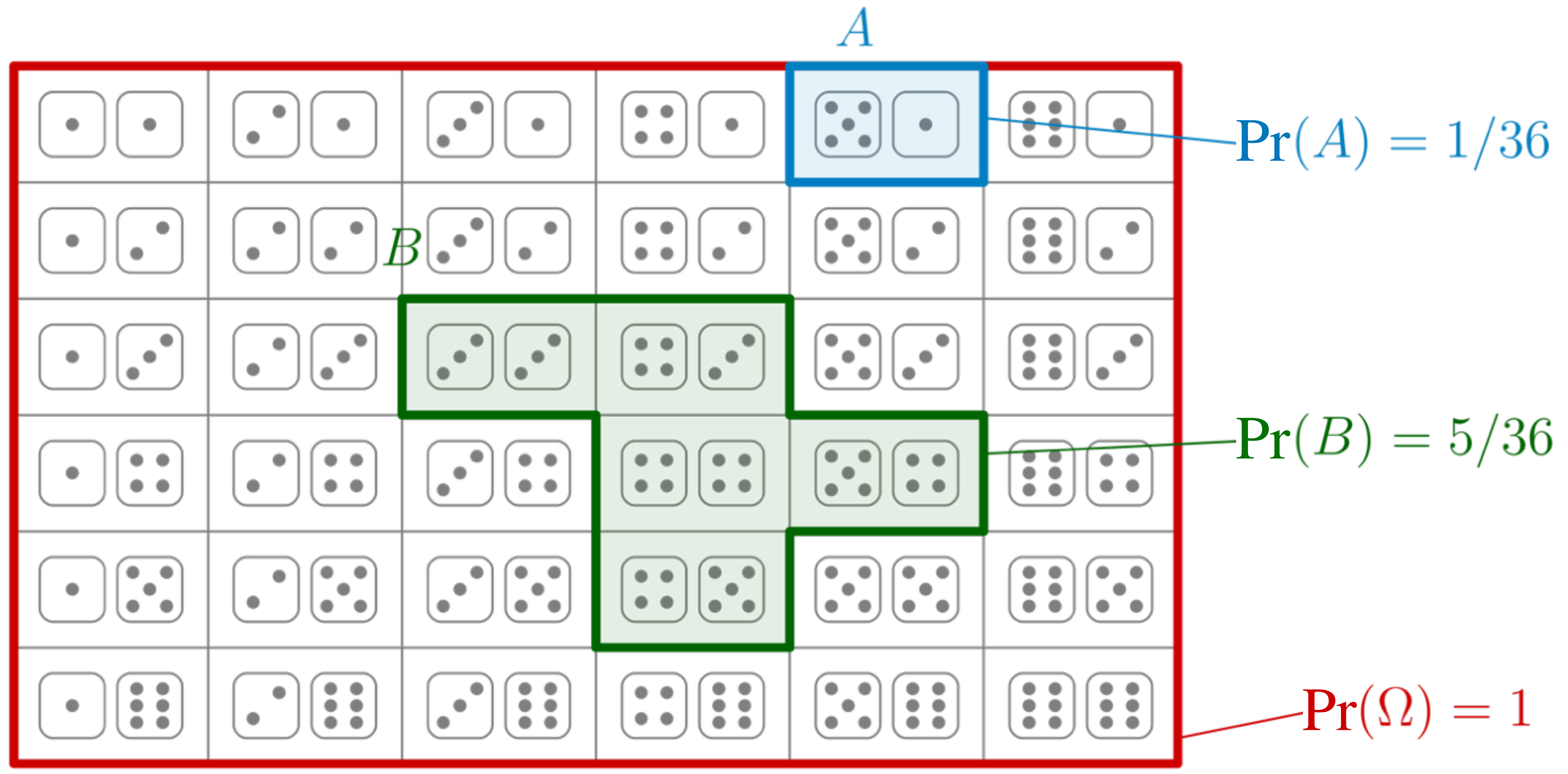
Probability Space

Sample Space (样本空间)

- Sample space Ω: set of all possible outcomes of an experiment (samples).
  - **Example:** all sides of a dice; all outcomes of a sequence of coin tosses; …
- Each $\omega \in \Omega$ is called a sample (样本) or elementary event (基本事件).
- An event (事件) is a subset $A \subseteq \Omega$ of the sample space.



Discrete Probability Space

$(\Omega, \Pr)$

- Sample space Ω: set of all possible outcomes of an experiment (**samples**).
  - **Example:** all sides of a dice; all outcomes of a sequence of coin tosses; …
- Each $\omega \in \Omega$ is called a **sample** (样本) or **elementary event** (基本事件).
- For **discrete** probability space (where Ω is *finite* or *countably infinite*):
  - **probability mass function** (*pmf*) $p : \Omega \to [0,1]$ satisfies $\sum_{\omega \in \Omega} p(\omega) = 1$
  - the probability of event $A \subseteq \Omega$ is given by $\Pr(A) = \sum_{\omega \in A} p(\omega)$

Sample Space and Events

- <u>Sample space</u> Ω: set of all possible outcomes of an experiment (<u>samples</u>).
  - **Example:** all sides of a dice; all outcomes of a sequence of coin tosses; …
- A family $\Sigma \subseteq 2^{\Omega}$ of subsets of Ω, called <u>events</u> (事件), satisfies:
  - $\emptyset$ and Ω are events (the ***<u>impossible event</u>*** and ***<u>certain event</u>***);
    “不可能事件” “必然事件”
  - if A is an event, then so is its complement $A^c = \Omega \backslash A$;
  - if (countably many) $A_1, A_2, \ldots$ are events, then so is $\bigcup_i A_i$ (and $\bigcap_i A_i$)

σ-Algebra (σ-代数)

- A family $\Sigma \subseteq 2^\Omega$ of subsets of Ω is called a **σ-algebra** or **σ-field**, if:
  - $\emptyset \in \Sigma$
  - $A \in \Sigma \implies A^c \in \Sigma$ (where $A^c = \Omega \setminus A$ denotes A's compliment in Ω)
  - $A_1, A_2, \ldots \in \Sigma \implies \bigcup_i A_i \in \Sigma$ (for countably many $A_1, A_2, \ldots \in \Sigma$)
- **Examples:**
  - $\Sigma = 2^\Omega$
  - $\Sigma = \{\emptyset, \Omega\}$
  - $\Sigma = \{\emptyset, A, A^c, \Omega\}$ for any $A \subseteq \Omega$

Sets as Events

Notation	Set interpretation	Event interpretation
$\omega \in \Omega$	Member of Ω	Elementary event
$A \subseteq \Omega$	Subset of Ω	Event A occurs
A^c	Complement of A	Event A does not occur
$A \cap B$	Intersection	Both A and B
$A \cup B$	Union	Either A or B or both
$A \backslash B$	Difference	A, but not B
$A \oplus B$	Symmetric difference	Either A or B, but not both
$\varnothing$	Empty set	Impossible event
Ω	Whole space	Certain event
$A \subseteq B$	Inclusion	A implies B
$A \cap B = \varnothing$	Set disjointness	A and B cannot both occur

Probability Space (概率空间)

$(\Omega, \Sigma, \Pr)$

Andrey Kolmogorov
Андре́й Колмого́ров
(1903-1987)

- Let $\Sigma \subseteq 2^{\Omega}$ be a **σ-algebra**.
- A **probability measure** (概率测度), also called **probability law** (概率律), is a function $\Pr : \Sigma \to [0,1]$ satisfying:
  - (*unitary/normalized*) $\Pr(\Omega) = 1$;
  - (*σ-additive*) for **disjoint (不相容)** $A_1, A_2, \ldots \in \Sigma$: $\Pr\left(\bigcup_i A_i\right) = \sum_i \Pr(A_i)$.
- The triple $(\Omega, \Sigma, \Pr)$ is called a **probability space**.

Classical Examples of Probability Space

- 古典概型 (classic probability): ***discrete uniform probability law***

  Finite sample space Ω, each outcome $\omega \in \Omega$ has equal probability.

  For every event $A \subseteq \Omega$: $\Pr(A) = \dfrac{|A|}{|\Omega|}$

- 几何概型 (geometric probability): continuous probability space such that

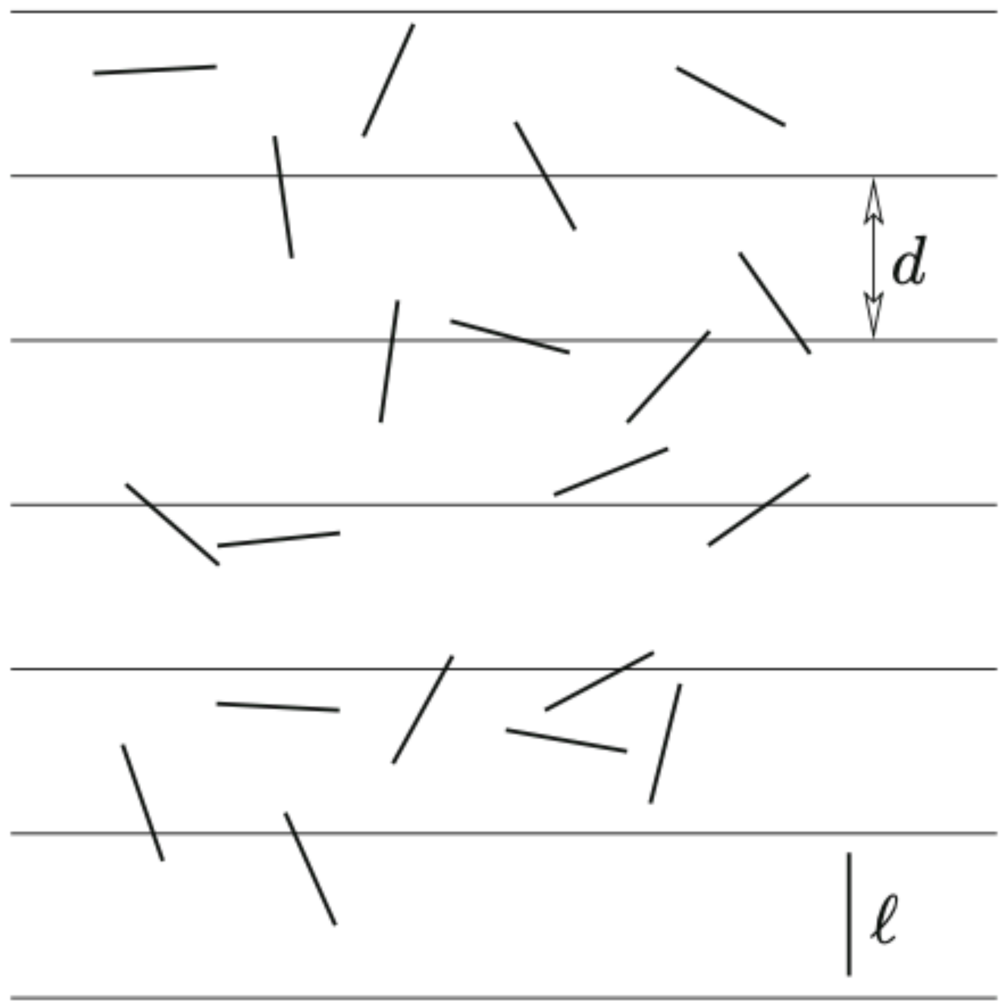
  For every event $A \in \Sigma$: $\Pr(A) = \dfrac{\mathrm{Vol}(A)}{\mathrm{Vol}(\Omega)}$

  - Bertrand's paradox
  - Buffon's needle problem

$\Pr \propto \angle$

Buffon's Needle Problem (蒲丰投针问题)

(Georges-Louis Leclerc de Buffon in 1733, and in 1777)



- Suppose that you drop a short needle of length ℓ on ruled paper, with distance d between parallel lines.
- What is the probability that the needle comes to lie in a position where it crosses one of the lines?
- For $\ell < d$, this probability is calculated as:

$$\Pr(A) = \frac{\text{Vol}(A)}{\text{Vol}(\Omega)} = \frac{2}{d\pi}\int_0^{\pi} \frac{\ell}{2}\sin(x)\,\mathrm{d}x = \frac{2\ell}{d\pi}$$

$x \in [0,\pi]$: angle between the needle and the parallel line below it

$y \in [0,d/2]$: distance from the center of the needle to the closest parallel line

- A ***Monte Carlo method*** for computing π

Event $A = \left\{ (x,y) \in [0,\pi] \times \left[0,\frac{d}{2}\right] \mid y \le \frac{\ell}{2}\sin(x) \right\}$

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Basic Properties of Probability

All followings can be deduced from the **axioms** of probability space:

- $\Pr(A^c) = 1 - \Pr(A)$
- $\Pr(\varnothing) = 0$ $\quad\quad$ $\Pr(A) > 0 \implies A \neq \varnothing$ (the probabilistic method)
- $\Pr(A \setminus B) = \Pr(A) - \Pr(A \cap B)$
- $A \subseteq B \implies \Pr(A) \leq \Pr(B)$
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
- ***Not even wrong***: "自然数是偶数的概率为1/2"
  (然而"[0,1]中均匀实数是有理数的概率为0"却是正确的)

Union Bound

- **Union bound** (Boole's inequality): for events $A_1, A_2, \ldots A_n \in \Sigma$

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \Pr(A_i)$$

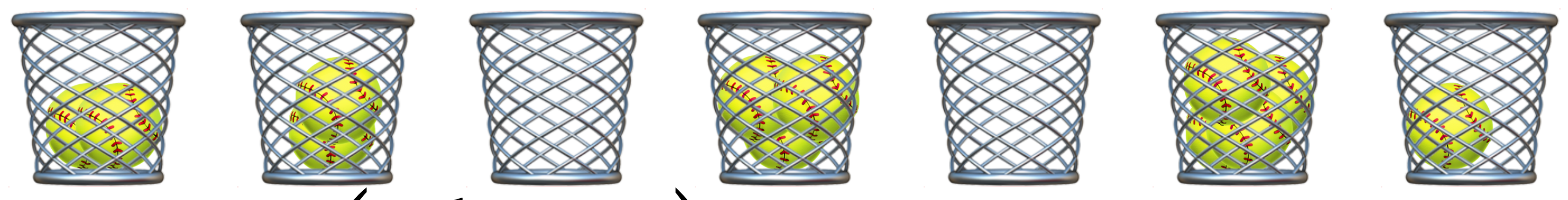
- **Example:** A system has n types of errors, each occurring with probability at most p

Let A_i be the event that type-i error occurs.

$$\Pr[\text{ no error occurs }] = \Pr\left(\bigcap_{i=1}^{n} A_i^c\right) = 1 - \Pr\left(\bigcup_{i=1}^{n} A_i\right) \ge 1 - np$$

Holds unconditionally.
(tight if all bad events are disjoint)

Balls into Bins



- Throwing n balls into n bins, every bin receives at most $O\left(\frac{\ln n}{\ln\ln n}\right)$ bins **_w.h.p._** (***with high probability***, with probability $1 - O(1/n)$)

- **Proof**: Define event A : some bin receives $\geq k$ balls (k to be fixed)

  and events A_i : bin-i receives $\geq k$ balls

  Then by union bound: $\Pr(A) = \Pr\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \Pr(A_i) \leq \frac{1}{n} \implies \Pr(A^c) \geq 1 - \frac{1}{n}$

  For each $S \in \binom{[n]}{k}$, define event $A_{i,S}$: bin-i receives the balls in S

  By union bound: $\Pr(A_i) = \Pr\left(\bigcup_{S\in\binom{[n]}{k}} A_{i,S}\right) \leq \sum_{S\in\binom{[n]}{k}} \Pr\left(A_{i,S}\right) = \binom{n}{k}\frac{1}{n^k} \leq \left(\frac{\mathrm{e}n}{k}\right)^k \frac{1}{n^k} \leq \left(\frac{\mathrm{e}}{k}\right)^k \leq \frac{1}{n^2}$

  Choose $k = 3\ln n/\ln\ln n$

Principles of Inclusion-Exclusion

- **Principle of inclusion-exclusion**: for events $A_1, A_2, \ldots A_n \in \Sigma$,

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i) - \sum_{i<j} \Pr(A_i \cap A_j) + \sum_{i<j<k} \Pr(A_i \cap A_j \cap A_k) - \cdots$$

$$= \sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ S \neq \varnothing}} (-1)^{|S|-1} \Pr\left(\bigcap_{i \in S} A_i\right)$$

- **Boole-Bonferroni Inequality**: for events $A_1, A_2, \ldots A_n \in \Sigma$, for any $k \geq 0$

$$\sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ 1 \leq |S| \leq 2k}} (-1)^{|S|-1} \Pr\left(\bigcap_{i \in S} A_i\right) \leq \Pr\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ 1 \leq |S| \leq 2k+1}} (-1)^{|S|-1} \Pr\left(\bigcap_{i \in S} A_i\right)$$

Derangement (错排)

(*le problème des rencontres, 1708*)

- The probability that a random permutation $\pi : [n] \xrightarrow[\text{onto}]{\text{1-1}} [n]$ has no fixed point (i.e. there is no $i \in [n]$ such that $\pi(i) = i$).

- Let A_i be the event that $\pi(i) = i$.

$$\Pr\left(\bigcap_{i\in S} A_i\right) = \frac{(n-|S|)!}{n!}$$

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} \sum_{S\in\binom{\{1,2,\ldots,n\}}{k}} (-1)^{k-1} \Pr\left(\bigcap_{i\in S} A_i\right) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{(n-k)!}{n!} = -\sum_{k=1}^{n} \frac{(-1)^k}{k!}$$

$$\Pr[\,\pi \text{ has no fixed point}\,] = \Pr\left(\bigcap_{i=1}^{n} A_i^c\right) = 1 - \Pr\left(\bigcup_{i=1}^{n} A_i\right) = 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \to \frac{1}{\mathrm{e}} \text{ as } n \to \infty$$

Continuity of Probability Measures*

- Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be an increasing sequence of events, and write A for their limit

$$A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i \, .$$

Then $\Pr(A) = \lim_{i \to \infty} \Pr(A_i)$.

- **Proof**: Express A as a disjoint union $A = A_1 \uplus (A_2 \backslash A_1) \uplus (A_3 \backslash A_2) \uplus \cdots$. Then

$$\begin{aligned}
\Pr(A) &= \Pr(A_1) + \sum_{i=1}^{\infty} \Pr(A_{i+1} \backslash A_i) \\
&= \Pr(A_1) + \lim_{n \to \infty} \sum_{i=1}^{n-1} [\Pr(A_{i+1}) - \Pr(A_i)] \\
&= \lim_{n \to \infty} \Pr(A_n)
\end{aligned}$$

Continuity of Probability Measures*

- Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be an increasing sequence of events, and write A for their limit

$$A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i \,.$$

  Then $\Pr(A) = \lim_{i \to \infty} \Pr(A_i)$.

- Let $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$ be an decreasing sequence of events, and write B for their limit

$$B = \bigcap_{i=1}^{\infty} B_i = \lim_{i \to \infty} B_i \,.$$

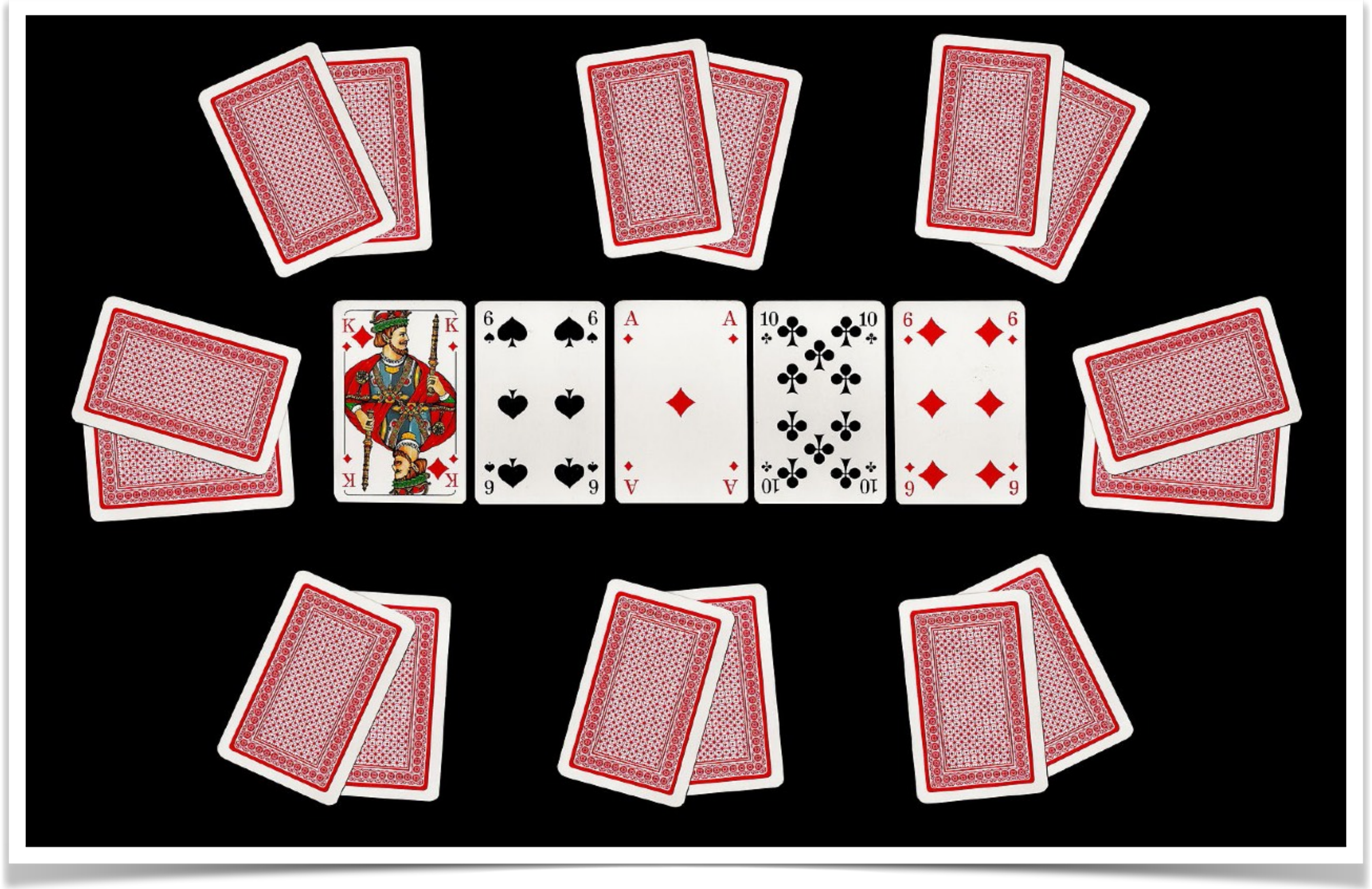
  Then $\Pr(B) = \lim_{i \to \infty} \Pr(B_i)$.

- **Proof**: Consider the complements $B_1^c \subseteq B_2^c \subseteq B_3^c \subseteq \ldots$ which is an increasing sequence.

Null and Almost Surely Events*

- An event $A \in \Sigma$ is called **null** if $\Pr(A) = 0$.
  - A null event is not necessarily the impossible event $\emptyset$.
- An event $A \in \Sigma$ occurs **almost surely** (**a.s.**) if $\Pr(A) = 1$.
  - An event that occurs a.s., is not necessarily the certain event Ω.
- A probability space is called **complete**, if all subsets of null events are events.
  - Without loss of generality: we only consider complete probability spaces
    (if we start with an incomplete one, we can complete it without changing the probabilities)

Conditional Probability



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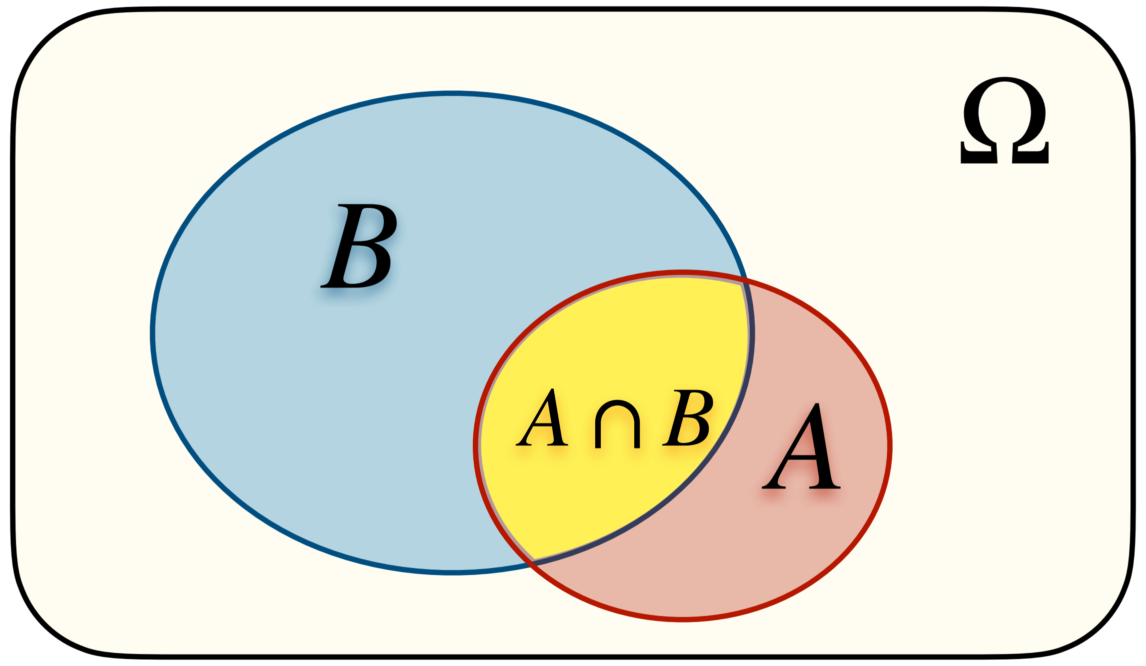


- Frequently, we need to make such statement:

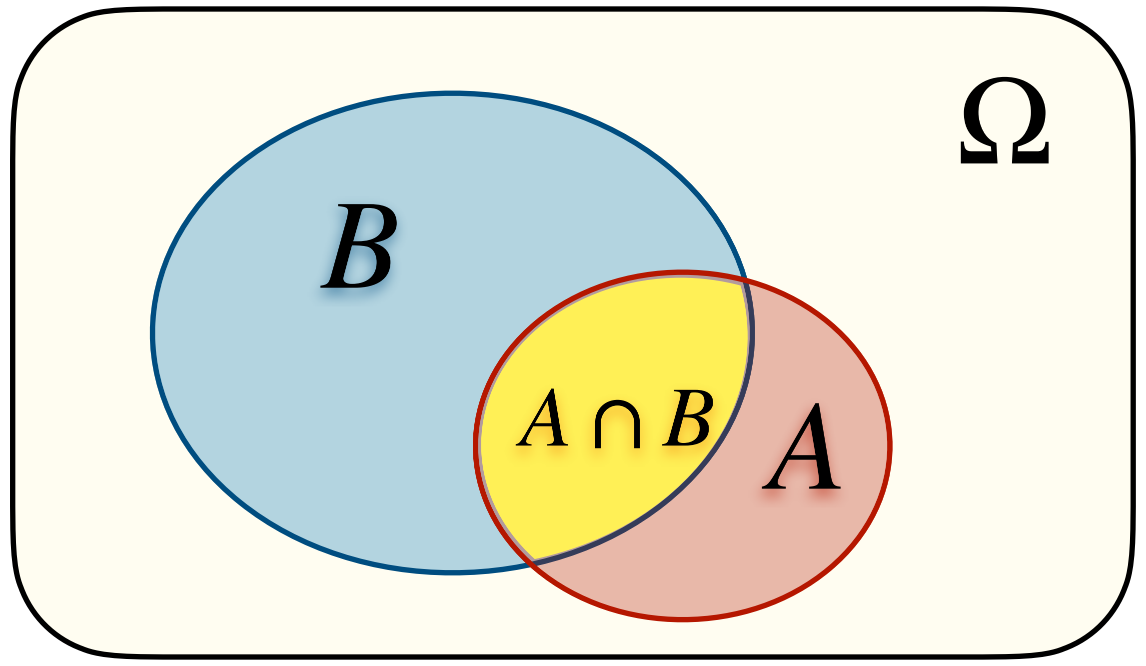
  "*The probability of A is p, given that B occurs.*"

- For discrete uniform law: $p = \dfrac{|A \cap B|}{|B|} = \dfrac{|A \cap B|/|\Omega|}{|B|/|\Omega|} = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

- Let A be an event, and let B be an event that $\Pr(B) > 0$.
  The conditional probability that A **occurs given that** B **occurs** is defined to be

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Conditional Probability



- Let A be an event, and let B be an event that $\Pr(B) > 0$.
  The conditional probability that A **occurs given that** B **occurs** is defined to be

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

- $\Pr(\,\cdot \mid B)$ is a well-defined probability law:
  - sample space is B
  - $\Sigma^B = \{A \cap B \mid A \in \Sigma\}$ is a σ-algebra
  - the law $\Pr(\,\cdot \mid B)$ satisfies the probability axioms

Fair Coins out of a Biased One

(von Neumann’s Bernoulli factory)

- John von Neumann (1951): “Suppose you are given a coin for which the probability of **HEADS**, say p, is **unknown**. How can you use this coin to generate unbiased (fair) coin-flips.”

- **Protocol:** Repetitively flip the coin until a $\texttt{HT}$ or $\texttt{TH}$ is encountered, output $\texttt{H}$ if $\texttt{HT}$ is encountered, and output $\texttt{T}$ if otherwise.

- Consider any two consecutive coin flips:

$$\Pr(\texttt{HT} \mid \{\texttt{HT}, \texttt{TH}\}) = \Pr(\texttt{TH} \mid \{\texttt{HT}, \texttt{TH}\}) = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}$$

The Two Child Problem

(boy or girl paradox)

- Martin Gardner (1959): “Knowing that I have two children and at least one of them is girl, what is the probability that both children are girls?”

- Consider a uniform law $\Pr$ over $\Omega = \{\mathtt{BB}, \mathtt{BG}, \mathtt{GB}, \mathtt{GG}\}$

$$\Pr(\{\mathtt{GG}\} \mid \{\mathtt{BG}, \mathtt{GB}, \mathtt{GG}\}) = \frac{\Pr(\{\mathtt{GG}\})}{\Pr(\{\mathtt{BG}, \mathtt{GB}, \mathtt{GG}\})}$$

$$= \frac{1/4}{3/4} \quad = \frac{1}{3}$$

Laws for Conditional Probability

- **Chain rule**:

$$\Pr\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \Pr\left(A_i \mid \bigcap_{j<i} A_j\right)$$

- **Law of total probability**: For partition $B_1, B_2, \ldots, B_n$ of Ω,

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \cap B_i) = \sum_{i=1}^{n} \Pr(A \mid B_i) \Pr(B_i)$$

- **Bayes' law**: For partition $B_1, B_2, \ldots, B_n$ of Ω,

$$\Pr(B_i \mid A) = \frac{\Pr(B_i) \Pr(A \mid B_i)}{\Pr(A)} = \frac{\Pr(B_i) \Pr(A \mid B_i)}{\Pr(A \mid B_1) \Pr(B_1) + \cdots + \Pr(A \mid B_n) \Pr(B_n)}$$

Chain Rule
(General Product Rule / Law of Successive Conditioning)

- Assuming that all the involved conditions have positive probabilities, we have

$$\Pr\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \Pr\left(A_i \mid \bigcap_{j<i} A_j\right)$$

- **Proof:** Due to the telescopic product

$$\Pr\left(\bigcap_{i=1}^{n} A_i\right) = \frac{\Pr\left(\bigcap_{i=1}^{n} A_i\right)}{\Pr\left(\bigcap_{i=1}^{n-1} A_i\right)} \cdot \frac{\Pr\left(\bigcap_{i=1}^{n-1} A_i\right)}{\Pr\left(\bigcap_{i=1}^{n-2} A_i\right)} \cdots \frac{\Pr\left(A_1 \cap A_2\right)}{\Pr\left(A_1\right)} \cdot \Pr(A_1)$$

Birthday "Paradox"

"一个班级若想要100%地保证有两个人同一天过生日，需要班上有超过366人；
但若仅想让这件事发生的可能性超过99%，则班上有超过57人就足够了。"

- Consider uniform random mapping $f : [n] \to [m]$

$$\Pr[\, f \text{ is 1-1 }] = \frac{m!/(m-n)!}{m^n} = \prod_{i=1}^{n}\left(1 - \frac{i-1}{m}\right)$$

- <u>Balls-into-bins</u> model: throwing n balls into m bins one-by-one at random

$$\Pr[\text{every ball is thrown to an empty bin}] \quad = \epsilon \text{ for } n \approx \sqrt{2m\ln(1/\epsilon)}$$

$$= \prod_{i=1}^{n} \Pr[\text{ball } i \text{ is in thrown into an empty bin} \mid \text{every ball } j < i \text{ is in an empty bin}] = \prod_{i=1}^{n}\left(1 - \frac{i-1}{m}\right)$$

$$\approx \exp\left(-\sum_{i=1}^{n}\frac{i-1}{m}\right) \approx \exp\left(-\frac{n^2}{2m}\right)$$

Law of Total Probability

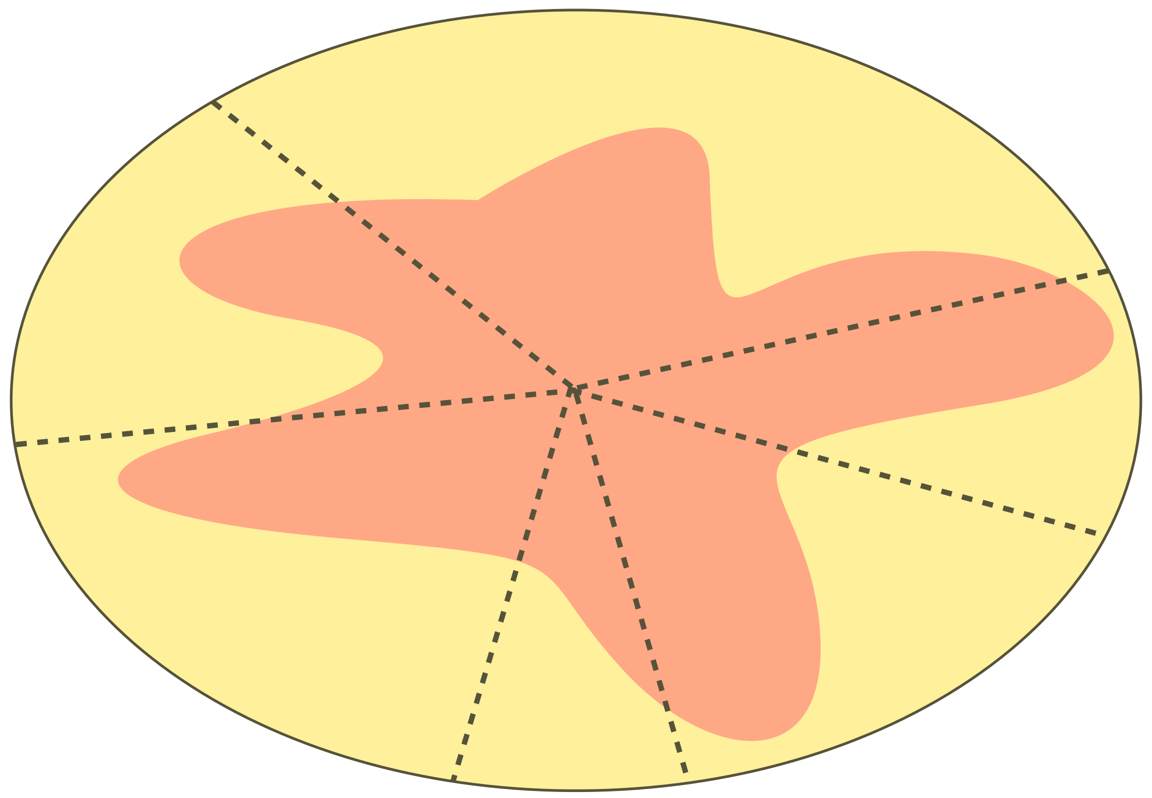
- Let events $B_1, B_2, \ldots, B_n$ be a partition of Ω such that $\Pr(B_i) > 0$ for all i. Then:

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \cap B_i) = \sum_{i=1}^{n} \Pr(A \mid B_i) \Pr(B_i)$$

- **Proof**: $A \cap B_1, A \cap B_2, \ldots, A \cap B_n$ are disjoint and $A = \bigcup_{i=1}^{n} (A \cap B_i)$

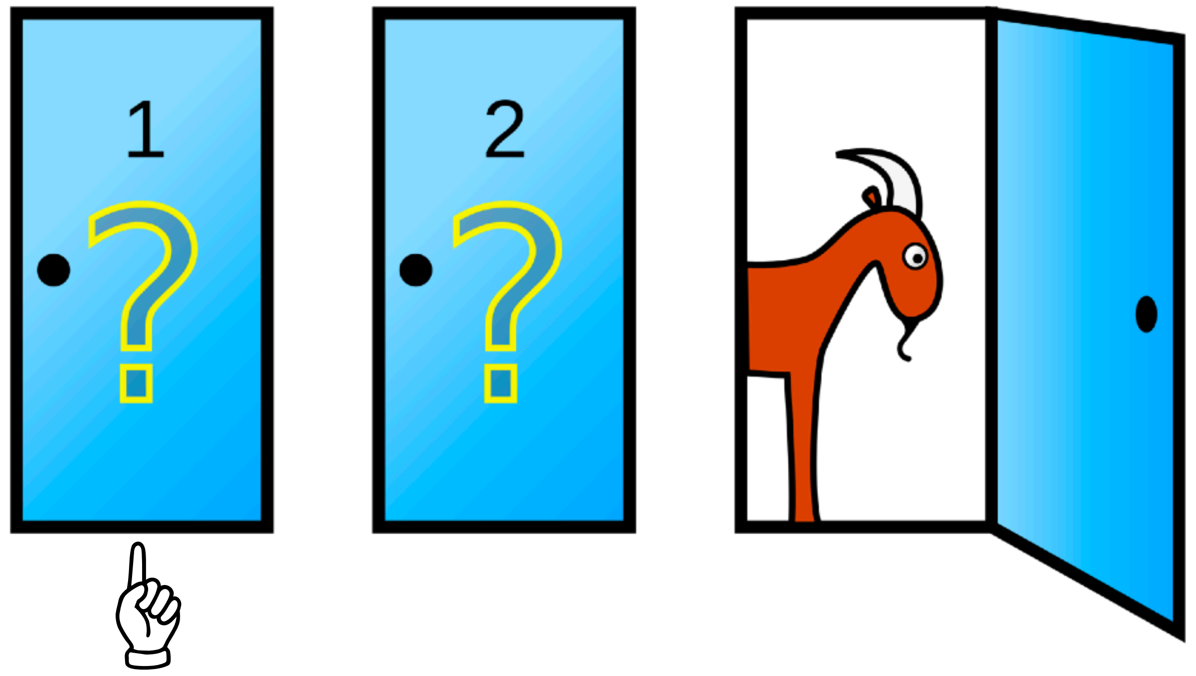
$$\implies \Pr(A) = \sum_{i=1}^{n} \Pr\left(A \cap B_i\right)$$

Moreover: $\Pr\left(A \cap B_i\right) = \Pr(A \mid B_i) \Pr(B_i)$.



Monty Hall Problem

(three doors problem)



- Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats.
- You pick a door, say No.1, and the host, who knows what's behind the doors, opens another door, say No.3, which has a goat. He then says to you, "Do you want to pick door No.2?" Is it to your advantage to switch your choice?

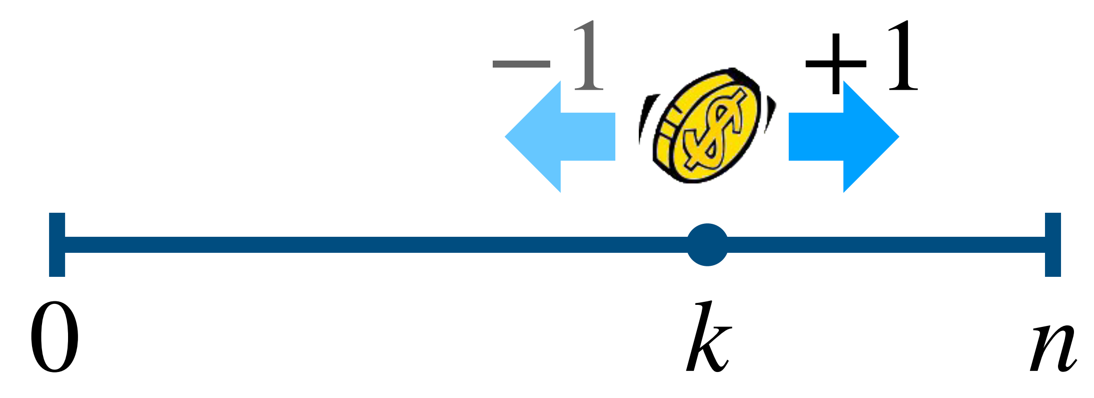
- Define event A : you win at last

  event B : you pick the car at first

$$\Pr(A) = \begin{cases} \Pr(B) = 1/3 & \text{if not switching} \\ \Pr(A \mid B)\Pr(B) + \Pr(A \mid B^c)\Pr(B^c) & \text{if switching} \\ = 0 + 1 \cdot 2/3 = 2/3 & \end{cases}$$

Gambler's Ruin

(Symmetric Random Walk in One-Dimension)



- A gambler plays a fair gambling game: At each step, he flips a fair coin, earns 1 point if it's HEADs, and loses 1 point if otherwise. He starts with k points, and will keep playing until either his points reaches 0 (lose) or $n > k$ (win).

- Define events A: the gambler loses; and B: the 1st coin flip returns HEADs

- Let $\Pr_k$ be the law that the gambler starts with k points.

$$\Pr_k(A) = \frac{1}{2}\Pr_k(A \mid B) + \frac{1}{2}\Pr_k(A \mid B^c) \ = \frac{1}{2}\Pr_{k+1}(A) + \frac{1}{2}\Pr_{k-1}(A)$$

$$\Pr_k(A) = \begin{cases} \frac{1}{2}(\Pr_{k+1}(A) + \Pr_{k-1}(A)) = 1 - \frac{k}{n} & \text{if } 0 < k < n \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k = n \end{cases}$$

Bayes' Law
(Bayes' Theorem)

- For events A, B that $\Pr(A), \Pr(B) > 0$, we have

$$\Pr(B \mid A) = \frac{\Pr(B)\Pr(A \mid B)}{\Pr(A)}$$

- Let events $B_1, B_2, \ldots, B_n$ be a partition of Ω such that $\Pr(B_i) > 0$ for all i.
  If event A has $\Pr(A) > 0$, then

$$\Pr(B_i \mid A) = \frac{\Pr(B_i)\Pr(A \mid B_i)}{\Pr(A)} = \frac{\Pr(B_i)\Pr(A \mid B_i)}{\Pr(A \mid B_1)\Pr(B_1) + \cdots + \Pr(A \mid B_n)\Pr(B_n)}$$

Dominating False Positives

- A rare disease occurs with probability 0.001.
- 5% testing error:
  - A person with the disease tested $\begin{cases} + & 95\,\% \\ - & 5\,\% \end{cases}$; a person without the disease tested $\begin{cases} + & 5\,\% \\ - & 95\,\% \end{cases}$
- **If a person is tested "+", what is the probability that he/she is ill?**

$$\Pr(\texttt{ill} \mid +) = \frac{\Pr(\texttt{ill})\Pr(+ \mid \texttt{ill})}{\Pr(+)} = \frac{\Pr(\texttt{ill})\Pr(+ \mid \texttt{ill})}{\Pr(+ \mid \texttt{ill})\Pr(\texttt{ill}) + \Pr(+ \mid \neg\texttt{ill})\Pr(\neg\texttt{ill})}$$

$$= \frac{0.001 \times 95\,\%}{95\% \times 0.001 + 5\% \times 0.999} \quad \approx 1.87\,\%$$

Simpson's Paradox

	Women		Men	
	Drug I	**Drug II**	**Drug I**	**Drug II**
Success	200	10	19	1000
Fail	1800	190	1	1000

- Results of clinical trials for 2 drugs:
- Which drug is more effective?
  - **Drug-II is better**: overall success rate 219/2020 (I) $<$ 1010/2200 (II)
  - **Drug-I is better**: for women 1/10 (I) $>$ 1/20 (II), for men 19/20 (I) $>$ 1/2 (II)
- **In *Probability***: It's possible that for events A, B and partition $C_1, \ldots, C_n$ of Ω
  - in case for each C_i, the occurrence of B has positive influence on A:

$$\Pr(A \mid B \cap C_i) > \Pr(A \mid B^c \cap C_i) \text{ for all } i$$

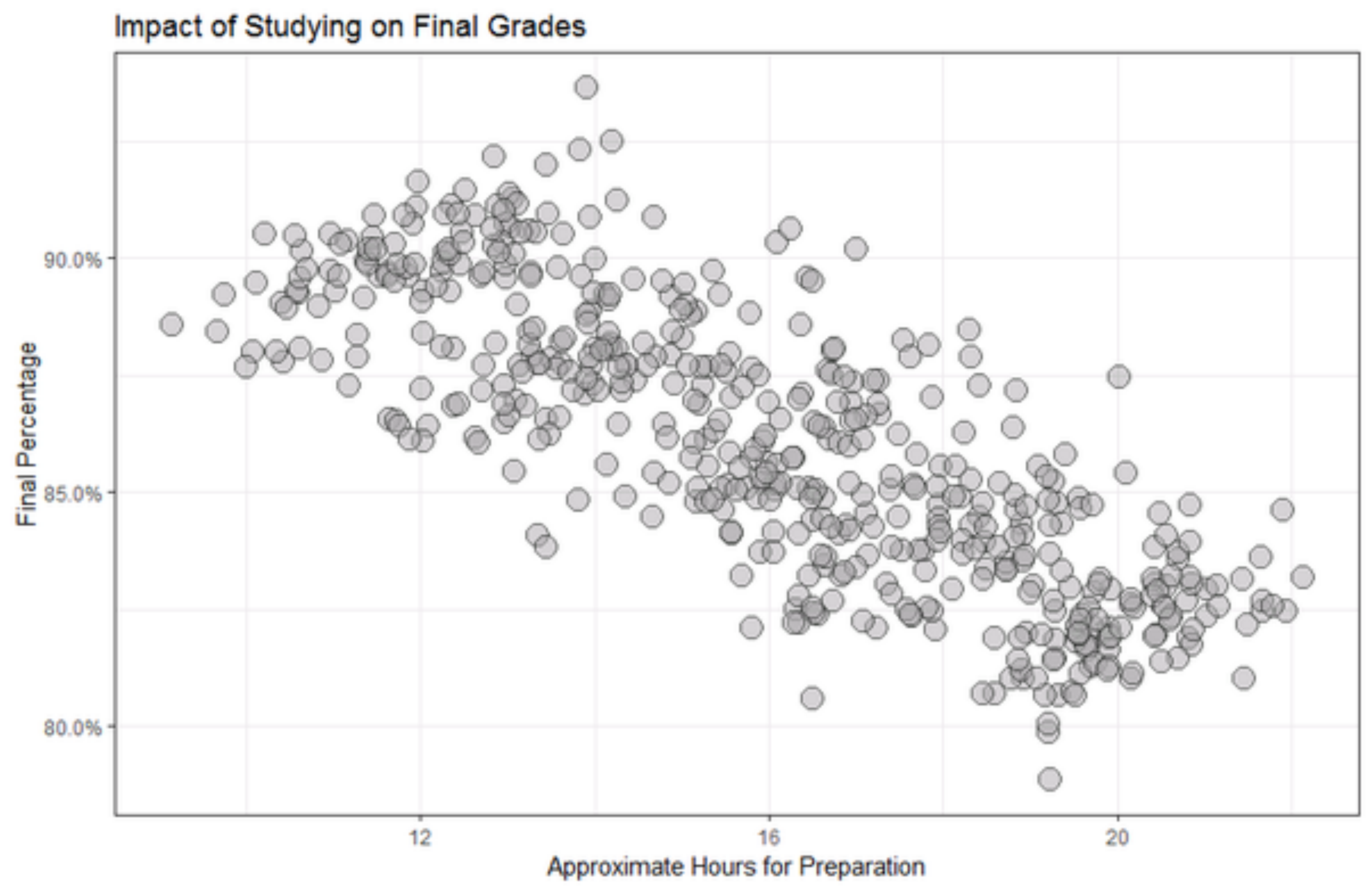
  - but overall, the occurrence of B has negative influence on A:

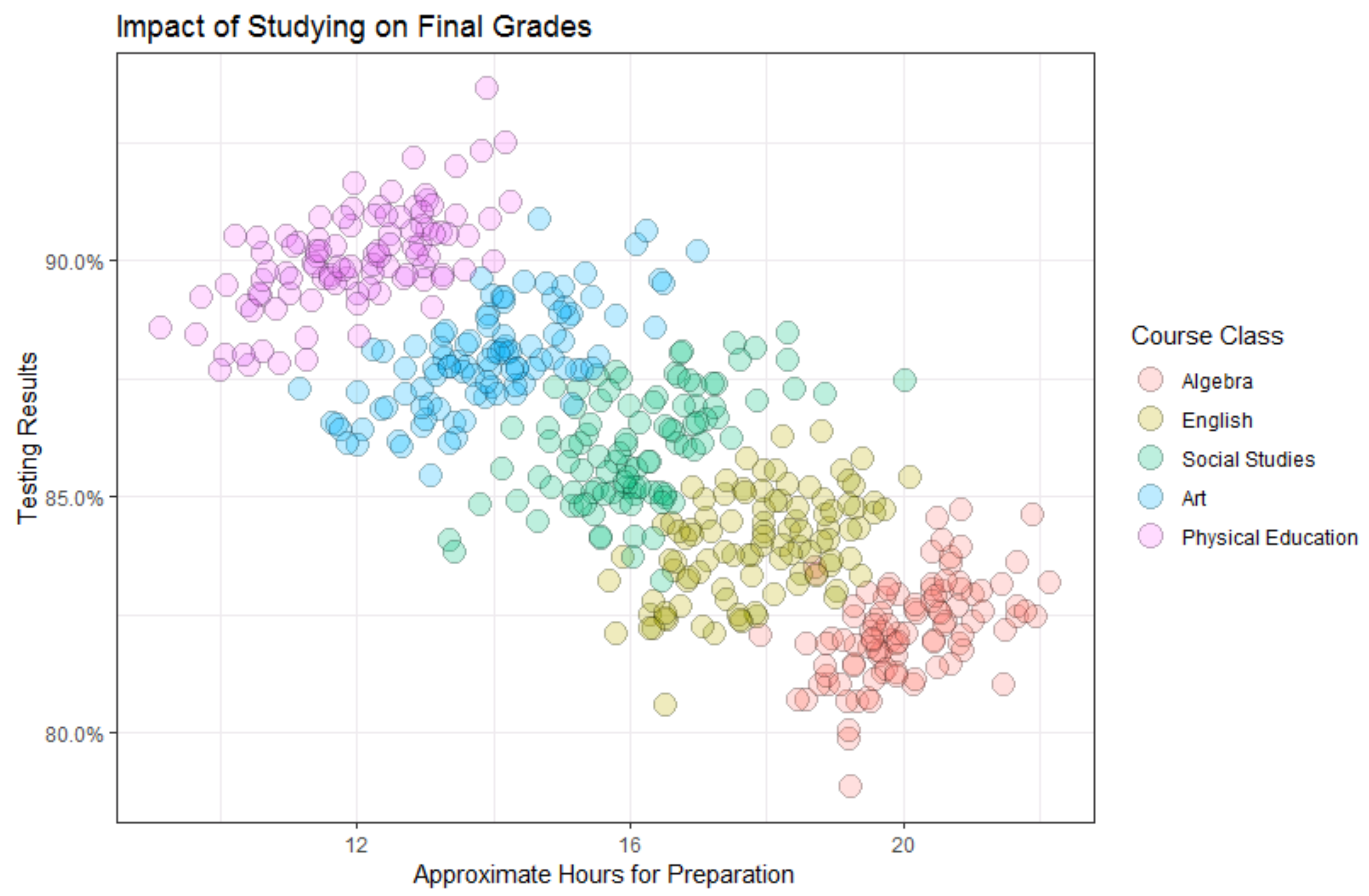
$$\Pr(A \mid B) < \Pr(A \mid B^c)$$

Simpson’s Paradox

(Edward H. Simpson in 1951; Karl Pearson in 1899; Udny Yule in 1903)

- **Example**: Correlation between hours for studying and grades.
  - Overall, it appears that lengths of studying have negative impact on grades. (*The longer the students study, the worse their grades are!*)
  - But truly the they are positively correlated in every course.





Independence

Independence of *Two* Events

- The occurrence of some event B changes the probability of another event A, from $\Pr(A)$ to $\Pr(A \mid B)$.

- If the occurrence of B has no influence on that of A, i.e. $\Pr(A \mid B) = \Pr(A)$, then A is said to be <u>**independent**</u> of B.

- The two events A and B are called <u>**independent**</u> if

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

- **Propositions**: if $\Pr(B) > 0$: $\Pr(A \mid B) = \Pr(A) \iff \Pr(A \cap B) = \Pr(A)\Pr(B)$

$$\Pr(A \cap B) = \Pr(A)\Pr(B) \iff \Pr(A \cap B^c) = \Pr(A)\Pr(B^c)$$

Independence of *Several* Events

- A family $\{A_i \mid i \in I\}$ of events is called **<u>(mutually) independent</u>** if for all finite subsets $J \subseteq I$

$$\Pr\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \Pr(A_i)$$

- An event A is called **<u>(mutually) independent</u>** of a family $\{B_i \mid i \in I\}$ of events if for all disjoint finite subsets $J^+, J^- \subseteq I$

$$\Pr(A) = \Pr\left(A \mid \bigcap_{i \in J^+} B_i \cap \bigcap_{i \in J^-} B_i^c\right)$$

Product Probability Space

- Probability space constructed from a sequence of *independent experiments*.
- Consider *discrete* probability spaces $(\Omega_1, p_1), (\Omega_2, p_2), \ldots, (\Omega_n, p_n)$.
- The product probability space (Ω, p) is constructed as:
  - sample space $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$
  - $\forall \omega = (\omega_1, \ldots, \omega_n) \in \Omega$: *pmf* $p(\omega) = p_1(\omega_1) \cdots p_n(\omega_n)$
- For general probability spaces $(\Omega_1, \Sigma_1, \Pr_1), \ldots, (\Omega_n, \Sigma_n, \Pr_n)$, the product probability space $(\Omega, \Sigma, \Pr)$ can be constructed similarly, where Σ is the unique smallest σ-algebra that contains $\Sigma_1 \times \cdots \times \Sigma_n$, and the law $\Pr$ is a natural extension onto such Σ from the product probabilities:

$$\forall A = (A_1, \ldots, A_n) \in \Sigma_1 \times \cdots \times \Sigma_n, \Pr(A) = \Pr(A_1) \cdots \Pr(A_n)$$

Dependency Structure

- The followings are all possible:
  - $A_1, A_2, \ldots, A_n$are mutually independent and $B_1, B_2, \ldots, B_n$ are mutually independent, but A_i and B_i are not independent for every $1 \leq i \leq n$.
  - For every $1 \leq i \leq n$, A_i and B_i are independent, but for every $1 \leq i < j \leq n$, neither A_i and A_j, nor B_i and B_j, are independent.
  - For an arbitrary undirected graph $G(V, E)$ on vertices $V = \{A_1, \ldots, A_n\}$, each A_i is mutually independent of all A_j's that are not adjacent to A_i in G.

Limited Independence

- A family $\{A_i \mid i \in I\}$ of events is called ***pairwise* independent** if for all distinct $i, j \in I$

$$\Pr(A_i \cap A_j) = \Pr(A_i)\Pr(A_j)$$

- Mutually independent events must be pairwise independent.
- Pairwise independent events are not necessarily mutually independent.
- **Example**: parities (XOR's) of random bits

A: coin-1 is $\mathtt{H}$; B: coin-2 is $\mathtt{H}$; C: coin-3 is $\mathtt{H}$;

D: coin-1 $\neq$ coin-2; E: coin-2 $\neq$ coin-3; F: coin-3 $\neq$ coin-1;

G: # of $\mathtt{H}$ in coins-1,2,3 is odd;

Triply Independent but not pairwise

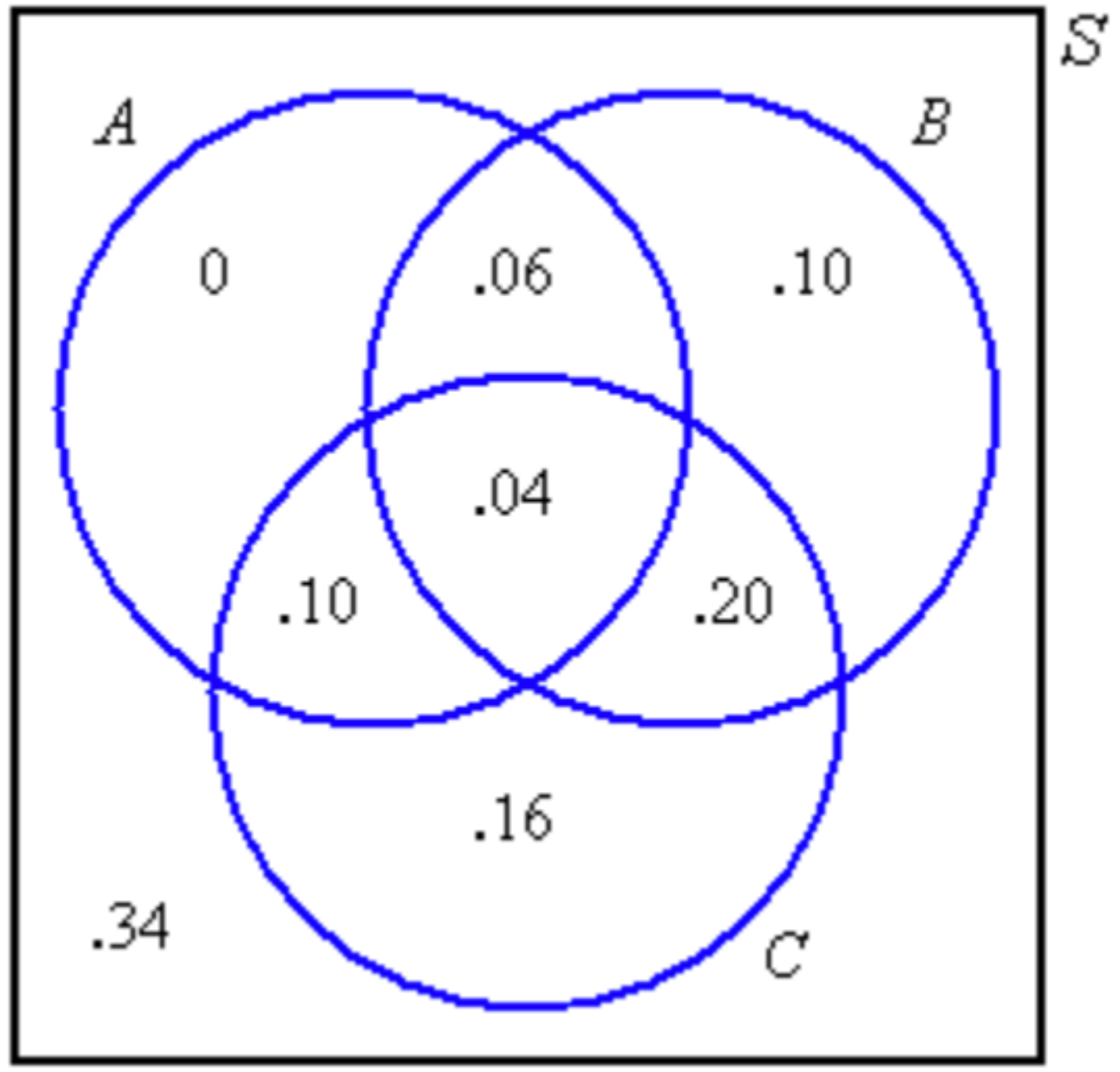


FIGURE 1

- $\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C)$ but no pairwise independence
- Example and figure is from George, Glyn, "Testing for the independence of three events," Mathematical Gazette 88, November 2004, 568

Error Reduction (one-sided case)

- Decision problem $f: \{0,1\}^* \to \{0,1\}$.
- Monte Carlo randomized algorithm $\mathscr{A}$ with ***one-sided*** error:
  - $\forall x \in \{0,1\}^*$: $f(x) = 1 \Longrightarrow \mathscr{A}(x) = 1$
  - $\forall x \in \{0,1\}^*$: $f(x) = 0 \Longrightarrow \Pr[\mathscr{A}(x) = 0] \geq p$
- $\mathscr{A}^n$: independently run $\mathscr{A}$ for n times, return $\wedge$ of the n outputs

$$f(x) = 0 \Longrightarrow \Pr[\mathscr{A}^n(x) = 1] \leq (1 - p)^n$$

- The one-sided error is reduced to ϵ by repeating $n \approx \frac{1}{p} \ln \frac{1}{\epsilon}$ times.

Binomial Probability

- Consider n *independent* tosses of a coin, in which each coin toss returns HEADs independently with probability p.

- We say that we have a sequence of **Bernoulli trials** (伯努利实验), in which each trial succeeds with probability p.

- **Binomial probability**: $p(k) = \Pr(k \text{ successes out of } n \text{ trials})$

$$= \sum_{S \in \binom{[n]}{k}} \Pr(\forall i \in S : i\text{th trial succeeds}) \Pr(\forall i \in [n] \setminus S : i\text{th trial fails})$$

$$= \sum_{S \in \binom{[n]}{k}} p^{|S|}(1-p)^{n-|S|} = \binom{n}{k} p^k (1-p)^{n-k}$$

$p(k)$ is a well-defined *pmf* on $\Omega = \{0, 1, \ldots, n\}$

$$\sum_{k=0}^{n} p(k) = 1$$ (binomial Thm.)

Controlling a Fair Voting

- In a society of n isolated (independent) and neutral (uniform) people, how many people are there enough to manipulate the result of a majority vote with 95% certainty.

- Consider n independent coin tosses of a fair coin.

$$\Pr[\,|\#\mathrm{HEADs} - \#\mathrm{TAILs}\,| \geq t] = \Pr[\#\mathrm{HEADs} \leq \tfrac{n}{2} - \tfrac{t}{2}] + \Pr[\#\mathrm{HEADs} \geq \tfrac{n}{2} + \tfrac{t}{2}]$$

$$= \sum_{k \leq (n-t)/2} \binom{n}{k} 2^{-n} + \sum_{k \geq (n+t)/2} \binom{n}{k} 2^{-n}$$

$$= 2^{1-n} \sum_{k \leq (n-t)/2} \binom{n}{k}$$

(entropy bound on the volume of a Hamming ball)

$$\leq 2^{1-n+nH\left(\frac{1}{2} - \frac{t}{2n}\right)}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$

$$H(x) \approx 1 - \frac{2}{\ln 2}\left(x - \frac{1}{2}\right)^2 + O\left(\left(x - \frac{1}{2}\right)^3\right)$$

$$\approx 2\exp\left(-\frac{t^2}{n}\right)$$

≤ 0.05 when $t \geq 2\sqrt{n}$

Error Reduction (two-sided case)

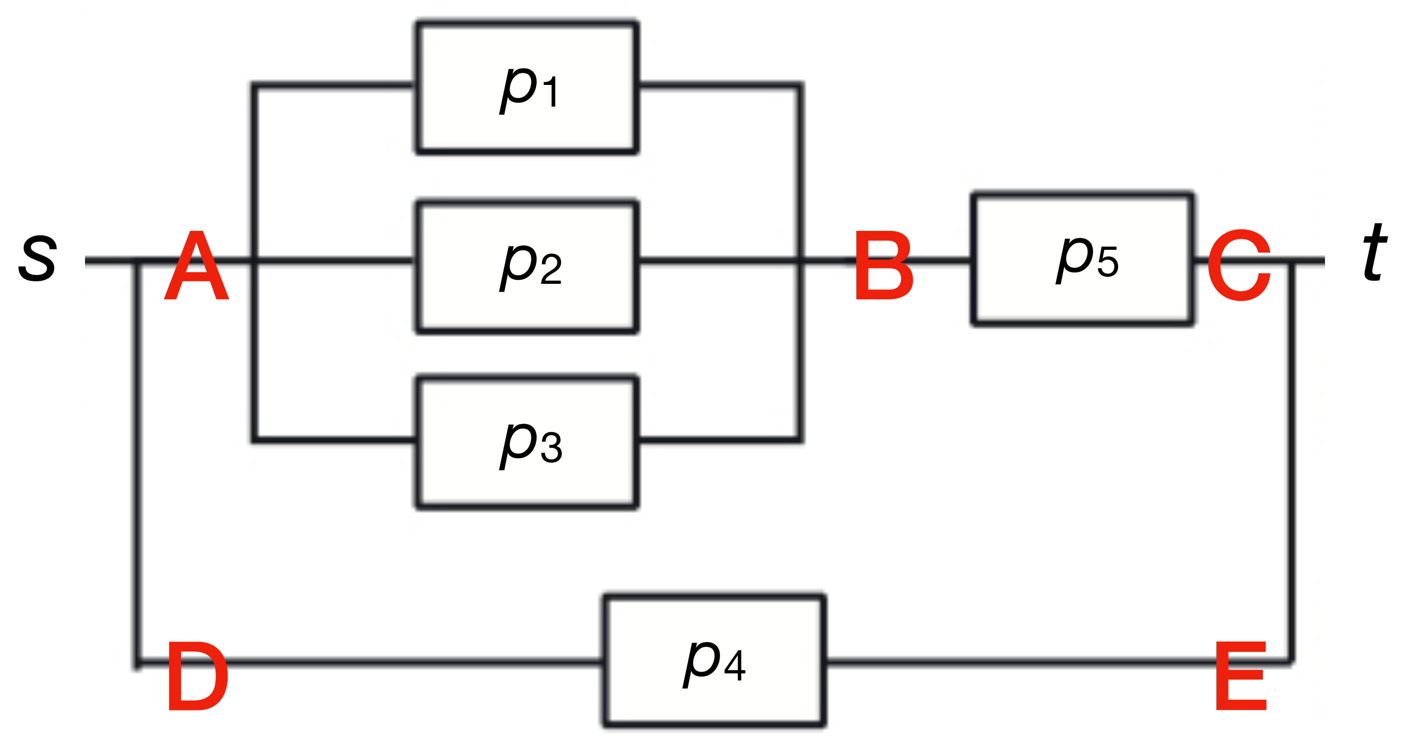
- Decision problem $f: \{0,1\}^* \to \{0,1\}$.
- Monte Carlo randomized algorithm $\mathscr{A}$ with ***two-sided*** error:
  - $\forall x \in \{0,1\}^*$: $\Pr[\mathscr{A}(x) = f(x)] \geq \frac{1}{2} + p$
- $\mathscr{A}^n$: independently run $\mathscr{A}$ for n times, return majority of the n outputs

$$\Pr[\mathscr{A}^n(x) \neq f(x)] \leq \sum_{k<\frac{n}{2}} \binom{n}{k} \left(\frac{1}{2}+p\right)^k \left(\frac{1}{2}-p\right)^{n-k} \leq \exp(-p^2 n)$$

$$\leq \epsilon \text{ when } n \geq \frac{1}{p^2} \ln \frac{1}{\epsilon}$$

- How to calculate this? (concentration inequalities)

Network Reliability



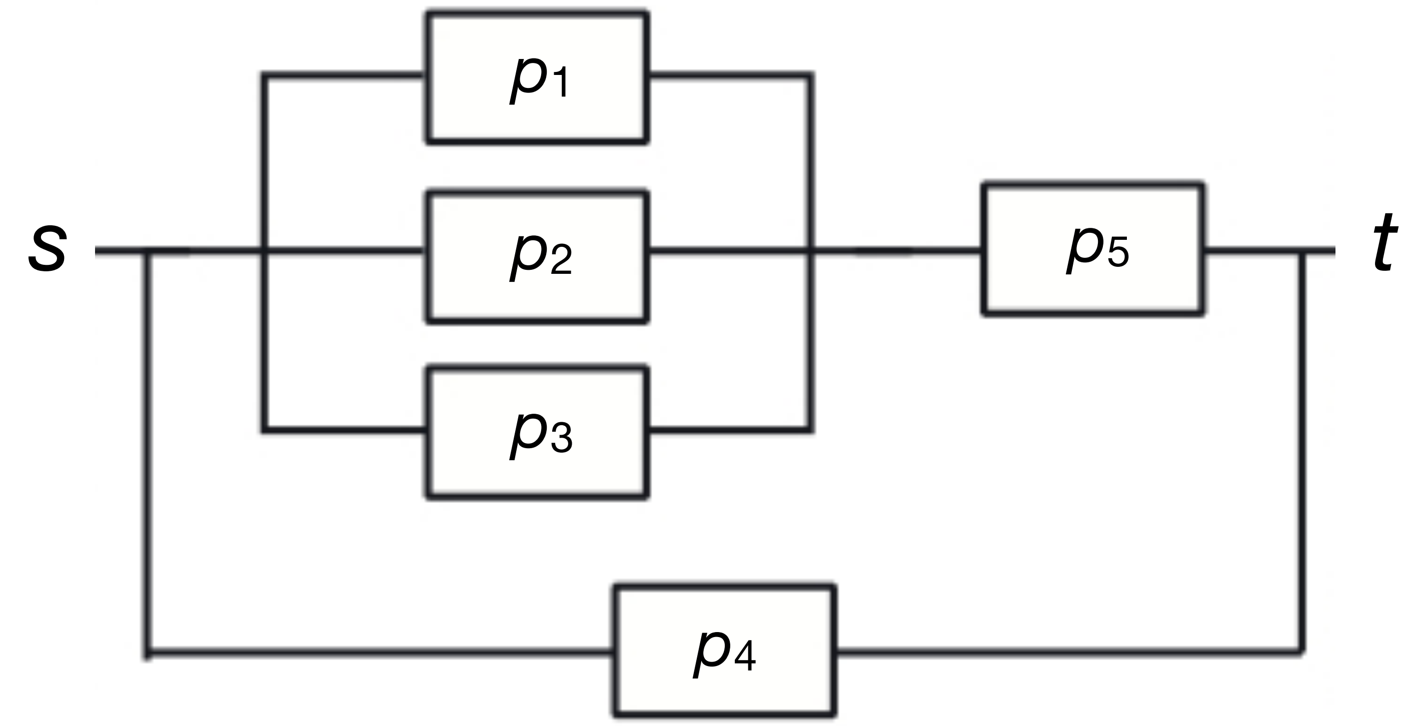
- A serial-parallel (串并联) network connects s to t.
- Suppose that each edge $e = uv$ connects uv independently with probability p_e.
- <u>s-t reliability</u> $P_{st} \triangleq \Pr[\, s \text{ and } t \text{ are connected}\,]$

$$= 1 - (1 - P_{AC})(1 - P_{DE}) \;= 1 - (1 - P_{AC})(1 - p_4)$$

$$P_{AC} = P_{AB}P_{BC} = P_{AB}p_5$$

$$P_{AB} = 1 - (1 - p_1)(1 - p_2)(1 - p_3)$$

Network Reliability



- A ~~serial-parallel (串并联)~~ network connects s to t.
- Suppose that each edge $e = uv$ connects uv independently with probability p_e.
- <u>s-t **reliability**</u> $P_{st} \triangleq \Pr[\, s$ and t are connected $]$
- <u>(*all-terminal*) **network reliability**</u>: $\triangleq \Pr[$ the resulting network is connected $]$
- For general networks:
  - s-t reliability is **#P-complete** (Leslie Valiant, 1979)
  - all-terminal network reliability is **#P-complete** (Mark Jerrum, 1981)

Conditional independence

- Two events A and B are <u>conditionally independent</u> given C if $\Pr(C) > 0$ and

$$\Pr(A \cap B \mid C) = \Pr(A \mid C) \Pr(B \mid C)$$

- If $\Pr(B \cap C) > 0$: $\Pr(A \cap B \mid C) = \Pr(A \mid C) \Pr(B \mid C) \iff \Pr(A \mid B \cap C) = \Pr(A \mid C)$

- Example: any two events are independent but not conditionally independent given the third event

A: coin-1 is $\mathtt{H}$; B: coin-2 is $\mathtt{H}$; C: coin-1 $\neq$ coin-2;

- Example: A and B are are not independent, but they are conditionally independent given C

A: X is tall; B: X knows a lot of math; C: X is 19 years old;

Suppose that X is a random person