Advanced Algorithms

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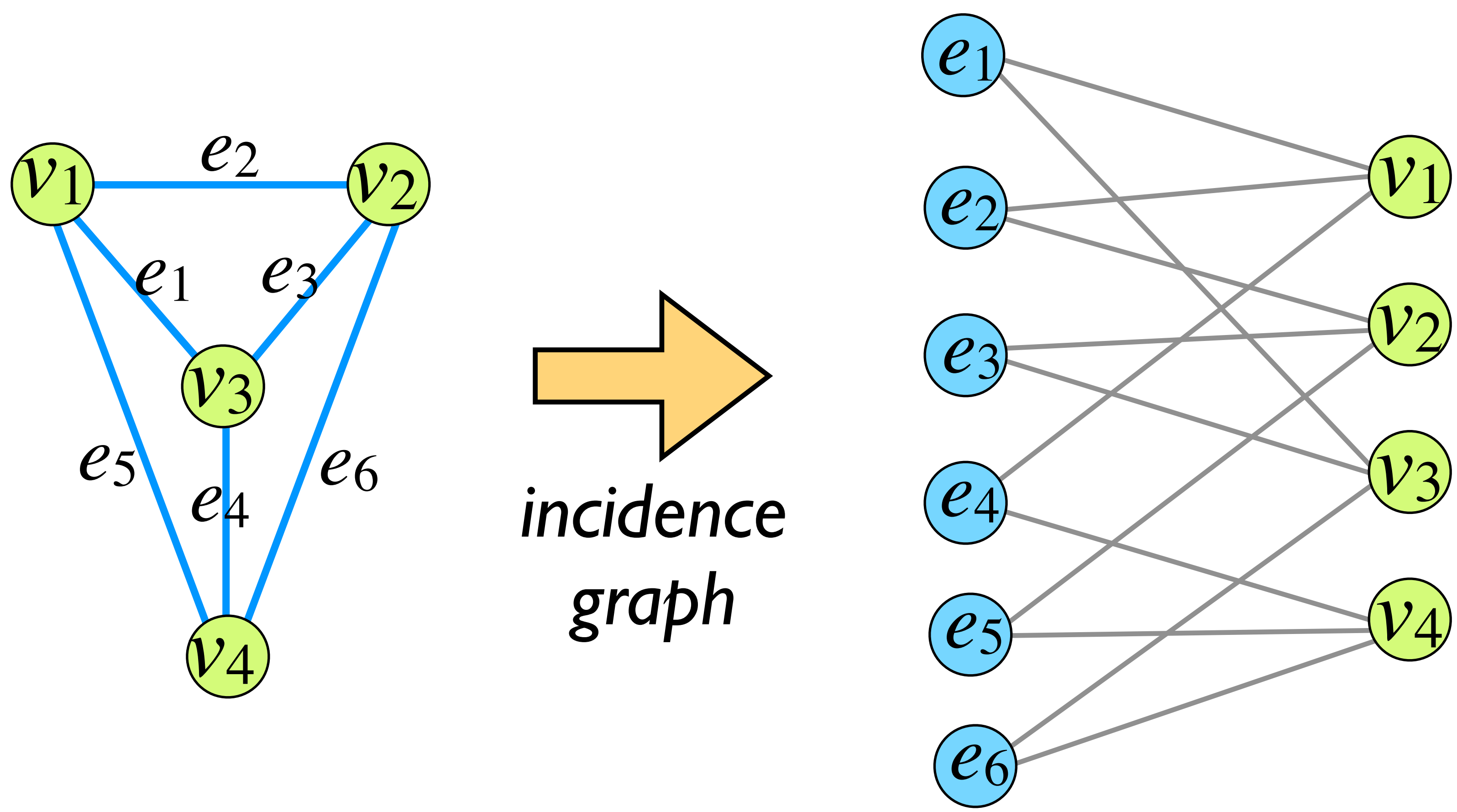
LP-based Algorithms

- LP rounding:
  - Relax the integer program to LP;
  - round the optimal LP solution to a *nearby* feasible integral solution.
- The primal-dual schema:
  - Find a pair of solutions to the primal and dual programs which are *close* to each other.

Vertex Cover

Instance: An undirected graph $G(V,E)$

Find the smallest $C \subseteq V$ that every edge has at least one endpoint in C.



instance of set cover
with *frequency* =2

Instance: An undirected graph $G(V,E)$

Find the smallest $C \subseteq V$ that every edge has at least one endpoint in C.

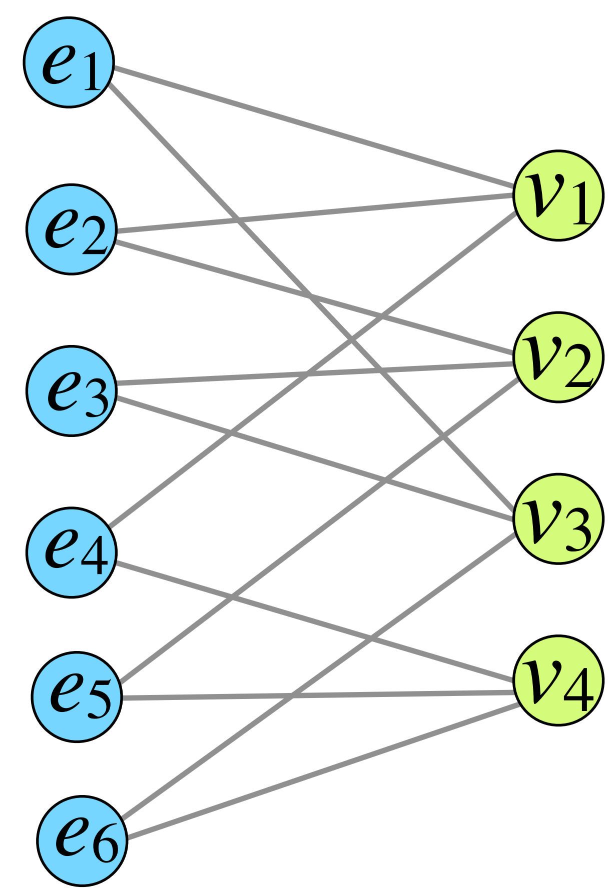
Find a *maximal matching* M;
return the set $C = \{v : uv \in M\}$ of *matched* vertices;

maximality $\Rightarrow$ C is vertex cover

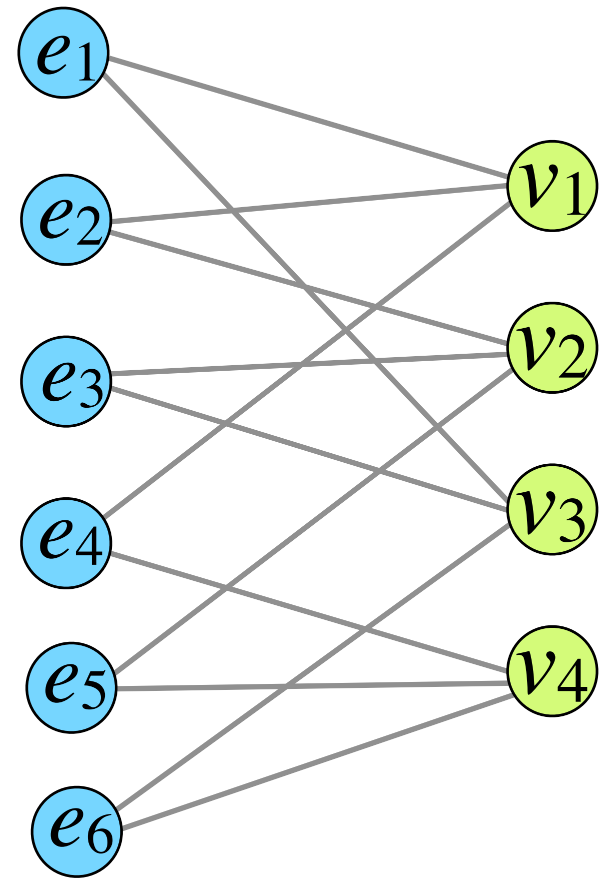
matching $\Rightarrow$ $|M| \le \mathrm{OPT}_{\mathrm{VC}}$

(*weak duality*)

$|C| \le 2|M| \le 2\mathrm{OPT}$



Duality



vertex cover:	constraints	variables
	$\sum_{v\in e} x_v \geq 1$	$x_v \in \{0,1\}$
matching:	variables	constraints
	$y_e \in \{0,1\}$	$\sum_{e \ni v} y_e \leq 1$

Duality

Instance: graph $G(V,E)$

primal: minimize $\sum_{v \in V} x_v$

subject to $\sum_{v \in e} x_v \geq 1, \quad \forall e \in E$

$x_v \in \{0,1\}, \quad \forall v \in V$

vertex covers

dual: maximize $\sum_{e \in E} y_e$

subject to $\sum_{e \ni v} y_v \leq 1, \quad \forall v \in V$

$y_e \in \{0,1\}, \quad \forall e \in E$

matchings

Duality for LP-Relaxation

Instance: graph $G(V,E)$

primal: minimize $\sum_{v\in V} x_v$

subject to $\sum_{v\in e} x_v \geq 1, \quad \forall e \in E$

$x_v \geq 0, \quad \forall v \in V$

dual: maximize $\sum_{e\in E} y_e$

subject to $\sum_{e\ni v} y_v \leq 1, \quad \forall v \in V$

$y_e \geq 0, \quad \forall e \in E$

Estimate the Optima

$$
\begin{array}{lccccccccc}
\text{minimize} & \boxed{7x_1 & + & x_2 & + & 5x_3} & & \\
& & & \text{VI} & & & & \\
\text{subject to} & \boxed{x_1 & - & x_2 & + & 3x_3} & \geq & 10 \\
& & & + & & & & + \\
& \boxed{5x_1 & + & 2x_2 & - & x_3} & \geq & 6 \\
& & & & & & & = \\
& & x_1,\ x_2,\ x_3 \geq 0 & & & & & 16
\end{array}
$$

$16 \leq$ OPT $\leq$ any feasible solution

Estimate the Optima

$$
\begin{array}{llcc}
\text{minimize} & & \boxed{7x_1 \quad + \quad x_2 \quad + \quad 5x_3} & \\
 & & \text{VI} & \\
\text{subject to} & y_1 & \boxed{x_1 \quad - \quad x_2 \quad + \quad 3x_3} & \geq \quad 10\, y_1 \\
 & & + & + \\
 & y_2 & \boxed{5x_1 \quad + \quad 2x_2 \quad - \quad x_3} & \geq \quad 6\, y_2
\end{array}
$$

$$x_1,\ x_2,\ x_3 \geq 0$$

$$10y_1 + 6y_2 \leq \text{OPT}$$

for any

$$
\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1 \\
3y_1 & - & y_2 & \leq & 5
\end{array}
\qquad y_1,\ y_2 \geq 0
$$

Primal-Dual

Primal

$$\begin{array}{lrcrcrcl} \text{min} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \text{s.t.} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & \multicolumn{7}{l}{x_1,\ x_2,\ x_3 \geq 0} \end{array}$$

Dual

$$\begin{array}{lrcrcl} \text{max} & \multicolumn{5}{l}{10y_1 + 6y_2} \\ \text{s.t.} & y_1 & + & 5y_2 & \leq & 7 \\ & -y_1 & + & 2y_2 & \leq & 1 \\ & 3y_1 & - & y_2 & \leq & 5 \\ & \multicolumn{5}{l}{y_1,\ y_2 \geq 0} \end{array}$$

∀dual feasible ≤primal OPT

$\text{LP} \in \mathbf{NP} \cap \text{co}\mathbf{NP}$

Surviving Problem

  ······ 

price	c_1	c_2	·······	c_n	healthy
vitamin 1	a_{11}	a_{12}	·······	a_{1n}	$\geq b_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
vitamin m	a_{m1}	a_{m2}	·······	a_{mn}	$\geq b_m$
solution:	x_1	x_2	·······	x_n	

minimize the total price while keeping healthy

Surviving Problem

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

price	c_1	c_2	$\cdots\cdots$	c_n	healthy
vitamin 1	a_{11}	a_{12}	$\cdots\cdots$	a_{1n}	$\geq b_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
vitamin m	a_{m1}	a_{m2}	$\cdots\cdots$	a_{mn}	$\geq b_m$
solution:	x_1	x_2	$\cdots\cdots$	x_n	

minimize the total price while keeping healthy

LP Duality

Primal:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

$$\text{s.t.} \quad \boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

dual solution:	price					healthy
		c_1	c_2	$\cdots\cdots$	c_n	
y_1	vitamin 1	a_{11}	a_{12}	$\cdots\cdots$	a_{1n}	b_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
y_m	vitamin m	a_{m1}	a_{m2}	$\cdots\cdots$	a_{mn}	b_m

m types of vitamin pills, design a pricing system

competitive to n natural foods, max the total price

LP Duality

Primal:

$$\min \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \quad \geq \quad \max \ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

Dual:

Primal: $\min\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$

Dual: $\max\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ s.t. $\boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$, $\boldsymbol{y} \geq \boldsymbol{0}$

Monogamy: dual(dual(LP)) = LP

Weak Duality:

∀ feasible primal solution $\boldsymbol{x}$ and dual solution $\boldsymbol{y}$

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{y}^{\mathrm{T}}A\,\boldsymbol{x} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

LP Duality

Primal:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

$\geq$

Dual:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

$$\text{s.t.} \quad \boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

Weak Duality Theorem:

$\forall$ feasible primal solution $\boldsymbol{x}$ and dual solution $\boldsymbol{y}$

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

LP Duality

Primal:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$
$$\text{s.t.} \quad \boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

Strong Duality Theorem:

Primal LP has finite optimal solution $\boldsymbol{x}^*$
iff dual LP has finite optimal solution $\boldsymbol{y}^*$.

$$\boldsymbol{y}^{*\mathrm{T}}\boldsymbol{b} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*$$

Primal: $\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$

$\boldsymbol{x} \geq \mathbf{0}$

Dual: $\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$

s.t. $\boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$

$\boldsymbol{y} \geq \mathbf{0}$

$\forall$ feasible primal solution $\boldsymbol{x}$ and dual solution $\boldsymbol{y}$

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{y}^{\mathrm{T}}A\,\boldsymbol{x} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

Strong Duality Theorem $\Rightarrow$ $\boldsymbol{x}$ and $\boldsymbol{y}$ are both optimal iff

$$\boxed{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{y}^{\mathrm{T}}A\,\boldsymbol{x} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}}$$

$\Updownarrow$

$$\sum_{i=1}^{m} b_i y_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right) y_i$$

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}y_i\right) x_j$$

$\Leftrightarrow$

$\forall i$: either $A_{i\cdot}\, x = b_i$ or $y_i = 0$

$\forall j$: either $y^{\mathrm{T}}A_{\cdot j} = c_j$ or $x_j = 0$

Complementary Slackness

Primal: $\min \; \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$

Dual: $\max \; \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ s.t. $\boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$, $\boldsymbol{y} \geq \boldsymbol{0}$

Complementary Slackness Conditions:

∀ feasible primal solution $\boldsymbol{x}$ and dual solution $\boldsymbol{y}$

$\boldsymbol{x}$ and $\boldsymbol{y}$ are both optimal iff

$\forall i$: either $A_{i\cdot}\, x = b_i$ or $y_i = 0$

$\forall j$: either $y^{\mathrm{T}} A_{\cdot j} = c_j$ or $x_j = 0$

Relaxed Complementary Slackness

Primal: min $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$
s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$
$\boldsymbol{x} \geq \mathbf{0}$

Dual: max $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$
s.t. $\boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$
$\boldsymbol{y} \geq \mathbf{0}$

$\forall$ feasible primal solution $\boldsymbol{x}$ and dual solution $\boldsymbol{y}$

for $\alpha, \beta \geq 1$:

$\forall i$: either $A_{i\cdot}\, x \leq \alpha\, b_i$ or $y_i = 0$

$\forall j$: either $y^{\mathrm{T}}A_{\cdot j} \geq c_j/\beta$ or $x_j = 0$

$\Rightarrow$ $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \alpha\beta\, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq \alpha\beta\, \mathrm{OPT}_{\mathrm{LP}}$

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\beta \sum_{i=1}^{m} a_{ij} y_i \right) x_j = \beta \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_i \right) y_j \leq \alpha\beta \sum_{i=1}^{m} b_i y_i$$

Primal-Dual Schema

Primal IP: $\min \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \in \mathbb{Z}_{\geq 0}$

Dual LP-relax: $\max \ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ s.t. $\boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$, $\boldsymbol{y} \geq \mathbf{0}$

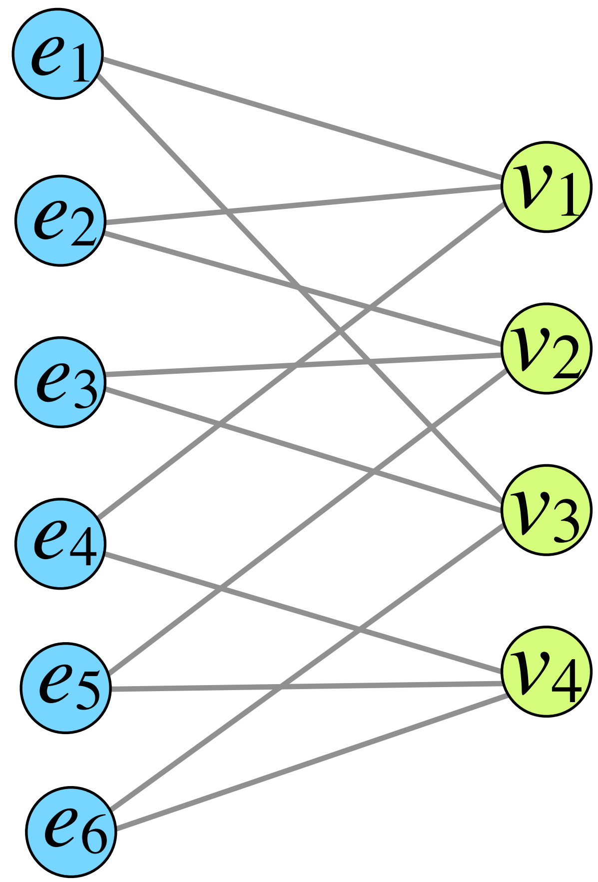
Find a primal *integral* solution $\boldsymbol{x}$ and a dual solution $\boldsymbol{y}$

for $\alpha, \beta \geq 1$:

$\forall i$: either $A_{i\cdot}\, x \leq \alpha\, b_i$ or $y_i = 0$

$\forall j$: either $y^{\mathrm{T}}A_{\cdot j} \geq c_j/\beta$ or $x_j = 0$

$\Rightarrow$ $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \alpha\beta\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq \alpha\beta\ \mathrm{OPT}_{\mathrm{LP}} \leq \alpha\beta\ \mathrm{OPT}_{\mathrm{IP}}$



primal:

$$\min \sum_{v \in V} x_v$$
$$\text{s.t.} \sum_{v \in e} x_v \geq 1, \quad \forall e \in E$$
$$x_v \in \{0,1\}, \quad \forall v \in V$$

dual-relax:

$$\min \sum_{e \in E} y_e$$
$$\text{s.t.} \sum_{e \ni v} y_e \leq 1, \quad \forall v \in V$$
$$y_e \geq 0, \quad \forall e \in E$$

vertex cover:

constraints $\sum_{v \in e} x_v \geq 1$

variables $x_v \in \{0,1\}$

matching:

variables $y_e \in \{0,1\}$

constraints $\sum_{e \ni v} y_e \leq 1$

feasible $(\boldsymbol{x}, \boldsymbol{y})$ such that:

$$\forall e: \; y_e > 0 \implies \sum_{v \in e} x_v \leq \alpha$$
$$\forall v: \; x_v = 1 \implies \sum_{e \ni v} y_e = 1$$

primal:

$$\min \sum_{v\in V} x_v$$
$$\text{s.t.} \sum_{v\in e} x_v \ge 1, \quad \forall e\in E$$
$$x_v \in \{0,1\}, \quad \forall v\in V$$

dual-relax:

$$\min \sum_{e\in E} y_e$$
$$\text{s.t.} \sum_{e\ni v} y_e \le 1, \quad \forall v\in V$$
$$y_e \ge 0, \quad \forall e\in E$$

event: "v is *tight* (*saturated*)" $\Leftrightarrow$ $\sum_{e\ni v} y_e = 1$

Initially $\boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{y} = \boldsymbol{0}$;

while $E \neq \emptyset$

pick an $e \in E$ and raise y_e ~~until some v goes tight~~ to 1;

set $x_v = 1$ for those ~~tight v~~ $v\in e$ and delete all $e \ni v$ from E;

every deleted e is incident to a v that $x_v = 1$
all edges are eventually deleted
$\Rightarrow$ $\forall e \in E$: $\sum_{v\in e} x_v \ge 1$
$\boldsymbol{x}$ is feasible

relaxed complementary slackness:

$\forall e$: either $\sum_{v\in e} x_v \le 2$ or $y_e = 0$
$\forall v$: either $\sum_{e\ni v} y_e = 1$ or $x_v = 0$

$\Rightarrow$ $$\sum_{v\in V} x_v \le 2\cdot OPT$$

Initially $\boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{y} = \boldsymbol{0}$;

while $E \neq \emptyset$

pick an $e \in E$ and raise y_e to 1 ~~until some v goes tight~~;

set $x_v = 1$ for those ~~tight v~~ $v \in e$ and delete all $e \ni v$ from E;

Find a *maximal matching*;

return the set of *matched* vertices;

the returned set is a vertex cover

$$SOL \leq 2\ OPT$$

The Primal-Dual Schema

- Write down an LP-relaxation and its dual.

$$\begin{array}{ll} \min & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}_{\geq 0} \end{array} \qquad \begin{array}{ll} \max & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}} \\ & \boldsymbol{y} \geq \boldsymbol{0} \end{array}$$

- Start with a primal *infeasible* solution $\boldsymbol{x}$ and a dual *feasible* solution $\boldsymbol{y}$ (usually $\boldsymbol{x}=\boldsymbol{0}, \boldsymbol{y}=\boldsymbol{0}$).
- Raise $\boldsymbol{x}$ and $\boldsymbol{y}$ until $\boldsymbol{x}$ is feasible:
  - raise $\boldsymbol{y}$ until some dual constraints gets *tight* $\boldsymbol{y}^{\mathrm{T}}A_{\cdot j} = c_j$;
  - raise x_j (integrally) corresponding to the *tight* dual constraints.
- Show the *complementary slackness conditions*:

$$\boxed{\begin{array}{l} \forall i\text{: either } A_{i\cdot}\, x \leq \alpha\, b_i \text{ or } y_i = 0 \\ \forall j\text{: either } y^{\mathrm{T}}A_{\cdot j} \geq c_j/\beta \text{ or } x_j = 0 \end{array}} \Longrightarrow \begin{array}{l} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \alpha\beta\, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \\ \qquad \leq \alpha\beta\, \mathrm{OPT} \end{array}$$

Integrality Gap

LP relaxation of
vertex cover : given $G(V,E)$,

minimize $\sum_{v \in V} x_v$

subject to $\sum_{v \in e} x_v \geq 1, \quad e \in E$

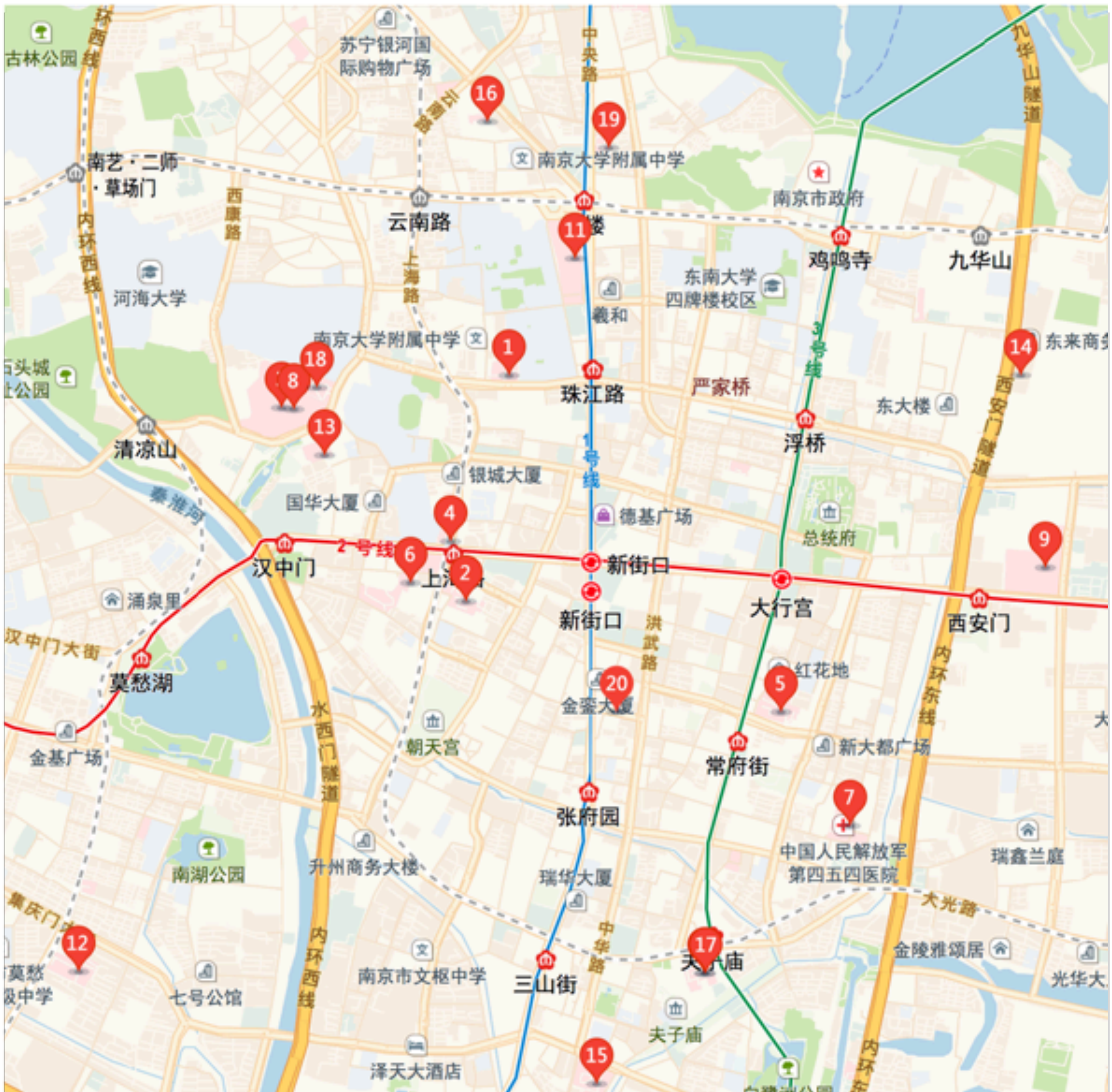
$\cancel{x_v \in \{0, 1\}}, \quad v \in V$

$x_v \in [0, 1],$

Integrality gap = $\sup_I \frac{\mathrm{OPT}(I)}{\mathrm{OPT}_{\mathrm{LP}}(I)}$

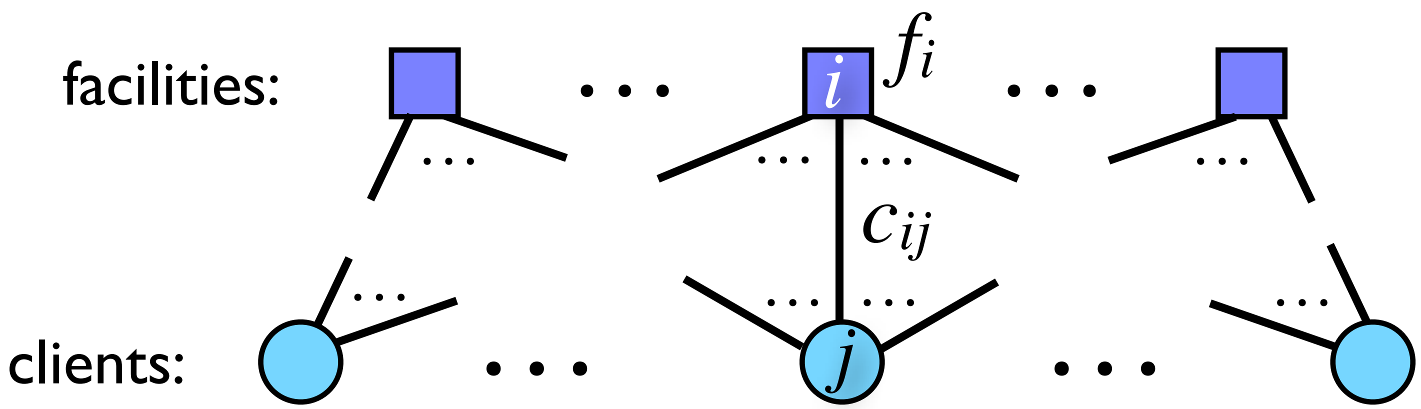
For the LP relaxation of vertex cover: integrality gap = 2

Facility Location



hospitals
in Nanjing

Facility Location

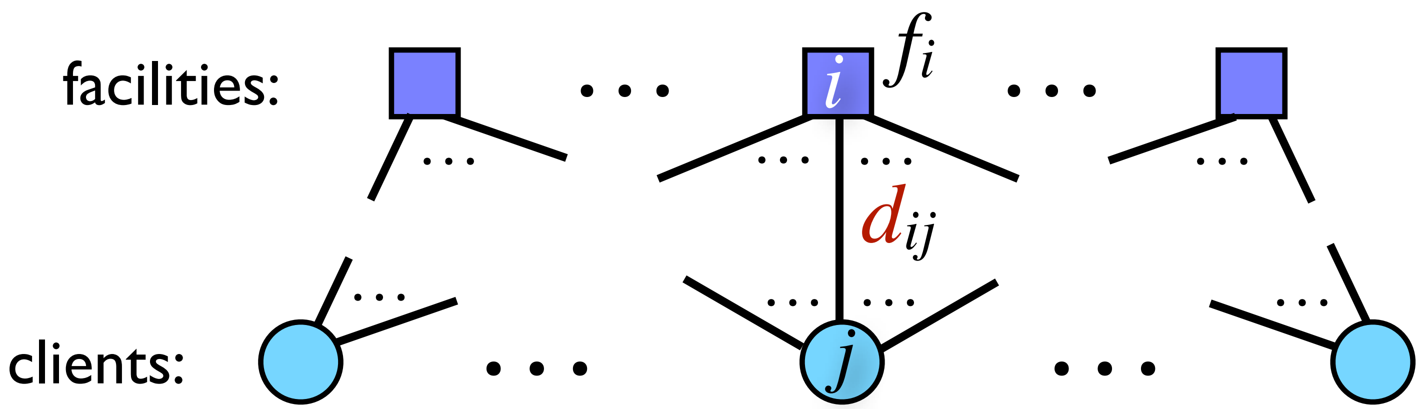


Instance: set F of facilities; set C of clients;
facility opening costs f: $F \rightarrow [0, \infty)$;
connection costs c: $F \times C \rightarrow [0, \infty)$;

Find a subset $I \subseteq F$ of *opening* facilities and a way ϕ: $C \rightarrow I$ of *connecting* all clients to them such that the total cost $\sum_{j \in C} c_{\phi(j),j} + \sum_{i \in I} f_i$ is minimized.

- uncapacitated facility location;
- **NP**-hard; AP(*Approximation Preserving*)-reduction from Set Cover;
- [Dinur, Steuer 2014] no poly-time $(1\text{-}o(1))\ln n$-approx. algorithm unless **NP** = **P**.

Metric Facility Location

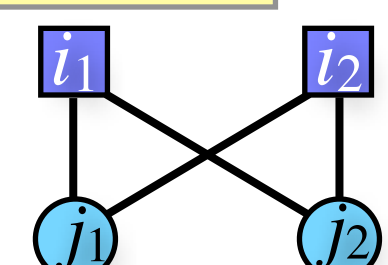


Instance: set F of facilities; set C of clients;
facility opening costs f: $F \rightarrow [0, \infty)$;
connection metric d: $F \times C \rightarrow [0, \infty)$;

Find a subset $I \subseteq F$ of *opening* facilities and a way ϕ: $C \rightarrow I$ of *connecting* all clients to them such that the total cost $\sum_{j \in C} d_{\phi(j),j} + \sum_{i \in I} f_i$ is minimized.

triangle inequality: $\forall i_1, i_2 \in F, \forall j_1, j_2 \in C$

$$d_{i_1 j_1} + d_{i_2 j_1} + d_{i_2 j_2} \geq d_{i_1 j_2}$$

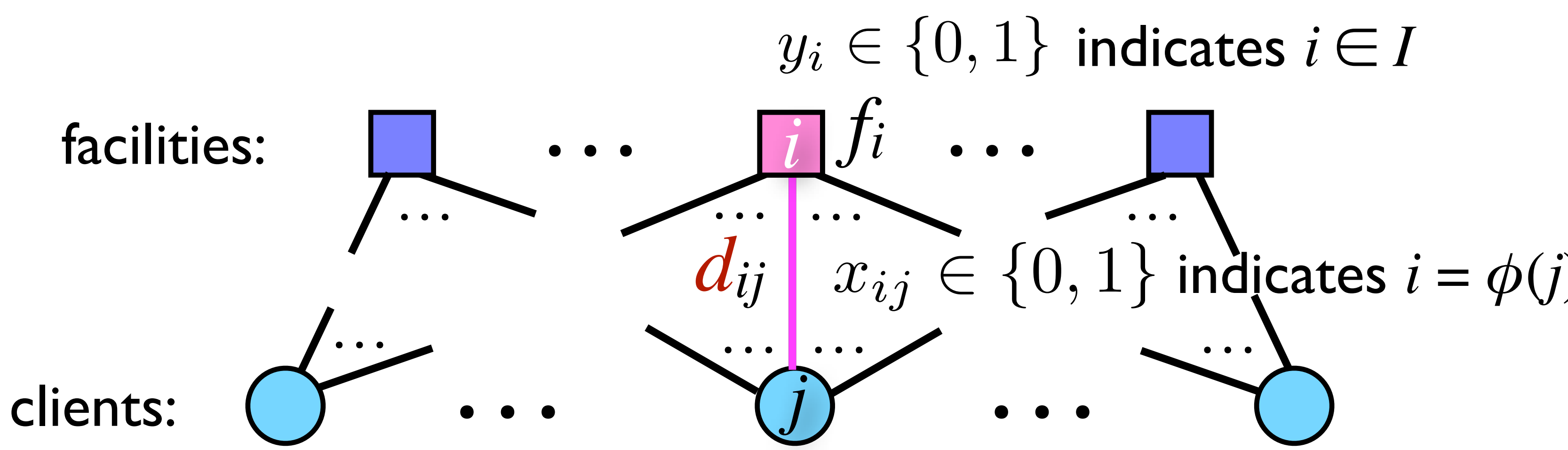


Instance: set F of facilities; set C of clients;

facility opening costs f: $F \rightarrow [0, \infty)$;

connection metric d: $F \times C \rightarrow [0, \infty)$;

Find ϕ: $C \rightarrow I \subseteq F$ to minimize $\displaystyle\sum_{j \in C} d_{\phi(j),j} + \sum_{i \in I} f_i$



LP-relaxation:

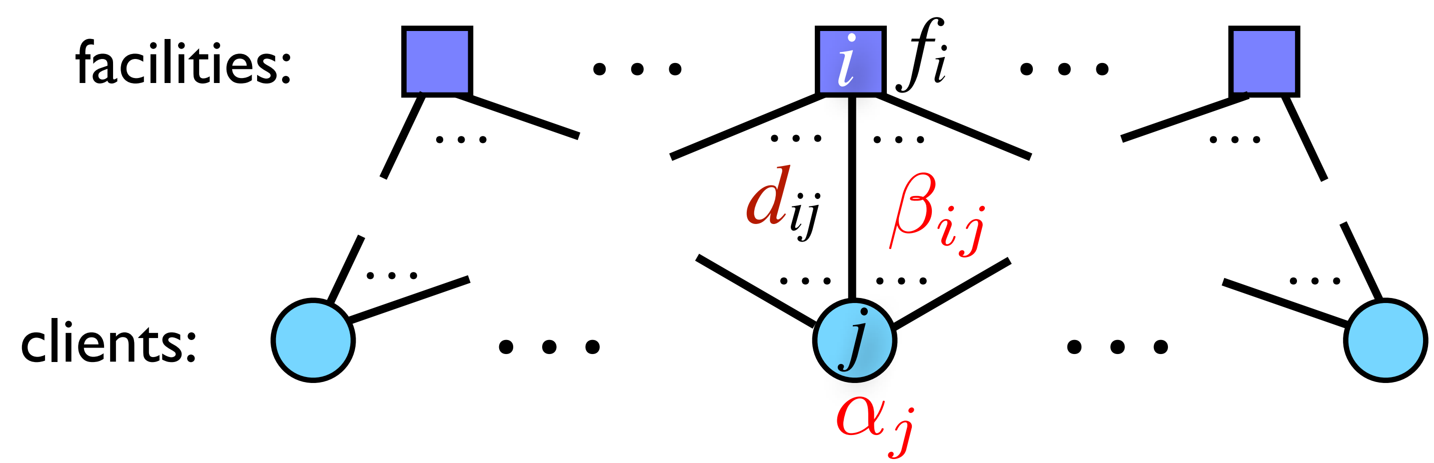
$$\min \sum_{i \in F, j \in C} d_{ij} x_{ij} + \sum_{i \in F} f_i y_i$$

s.t.

$$y_i \geq x_{ij}, \qquad \forall i \in F, j \in C$$

$$\sum_{i \in F} x_{ij} \geq 1, \qquad \forall j \in C$$

$$x_{ij}, y_i \geq 0, \quad \cancel{x_{ij}, y_i \in \{0,1\},} \qquad \forall i \in F, j \in C$$



Primal:

$$\min \sum_{i\in F, j\in C} d_{ij}x_{ij} + \sum_{i\in F} f_i y_i$$

$$\text{s.t.} \quad y_i - x_{ij} \geq 0, \qquad \forall i \in F, j \in C$$

$$\sum_{i\in F} x_{ij} \geq 1, \qquad \forall j \in C$$

$$x_{ij}, y_i \in \{0,1\}, \quad \forall i \in F, j \in C$$

Dual-relax:

$$\max \sum_{j\in C} \alpha_j$$

$$\text{s.t.} \quad \alpha_j - \beta_{ij} \leq d_{ij}, \quad \forall i \in F, j \in C$$

$$\sum_{j\in C} \beta_{ij} \leq f_i, \qquad \forall i \in F$$

$$\alpha_j, \beta_{ij} \geq 0, \qquad \forall i \in F, j \in C$$

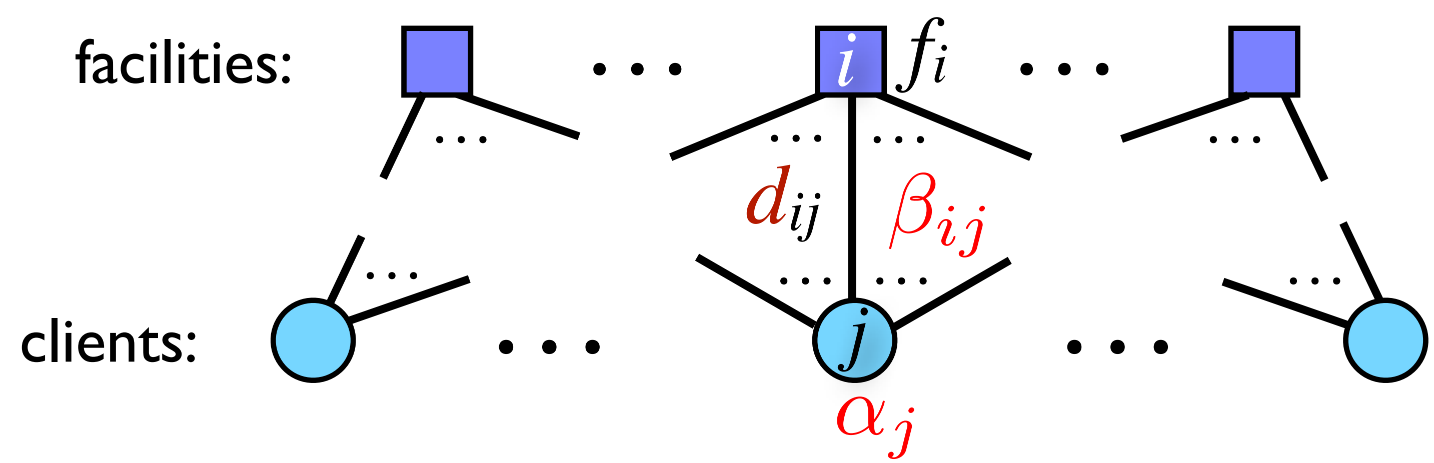
α_j: amount of value paid by client j to all facilities

$\beta_{ij} \geq \alpha_j - d_{ij}$: payment to facility i by client j (after deduction)

complimentary slackness conditions:
(if *ideally* held)

$x_{ij} = 1 \Rightarrow \alpha_j - \beta_{ij} = d_{ij}$; $\qquad \alpha_j > 0 \Rightarrow \sum_{i\in F} x_{ij} = 1$;

$y_i = 1 \Rightarrow \sum_{j\in C} \beta_{ij} = f_i$; $\qquad \beta_{ij} > 0 \Rightarrow y_i = x_{ij}$;



$$
\begin{aligned}
\textbf{min} \quad & \sum_{i\in F, j\in C} d_{ij}x_{ij} + \sum_{i\in F} f_i y_i \\
\textbf{s.t.} \quad & y_i - x_{ij} \ge 0, && \forall i \in F, j \in C \\
& \sum_{i\in F} x_{ij} \ge 1, && \forall j \in C \\
& x_{ij}, y_i \in \{0,1\}, && \forall i \in F, j \in C
\end{aligned}
$$

$$
\begin{aligned}
\textbf{max} \quad & \sum_{j\in C} \alpha_j \\
\textbf{s.t.} \quad & \alpha_j - \beta_{ij} \le d_{ij}, && \forall i \in F, j \in C \\
& \sum_{j\in C} \beta_{ij} \le f_i, && \forall i \in F \\
& \alpha_j, \beta_{ij} \ge 0, && \forall i \in F, j \in C
\end{aligned}
$$

Initially $\boldsymbol{\alpha} = \boldsymbol{0}$, $\boldsymbol{\beta} = \boldsymbol{0}$, no facility is open, no client is served;

raise α_j for all client j *simultaneously* at a *uniform continuous* rate:

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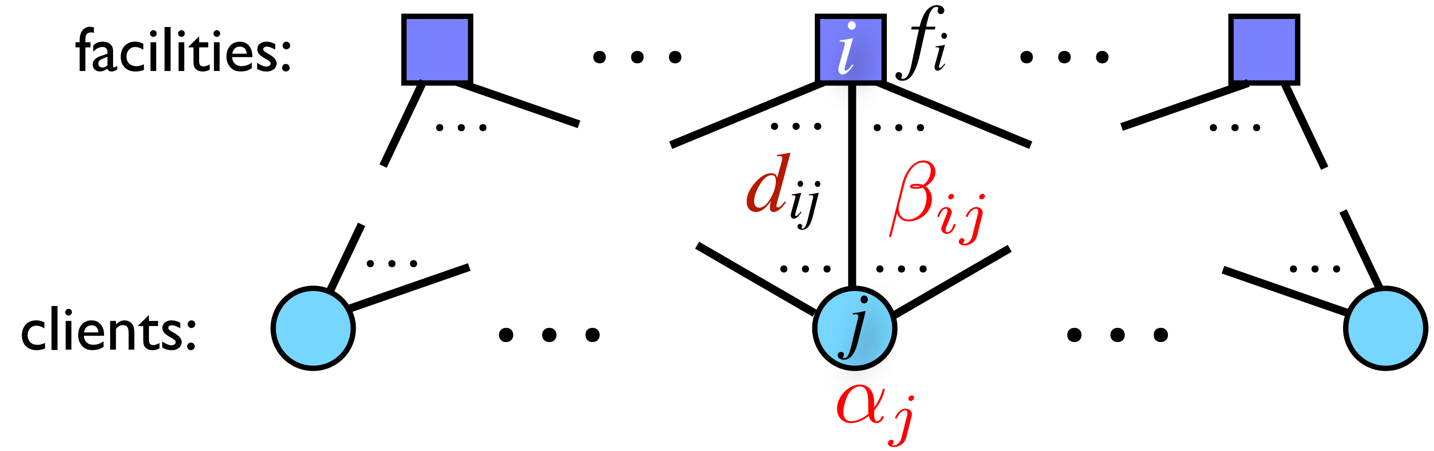

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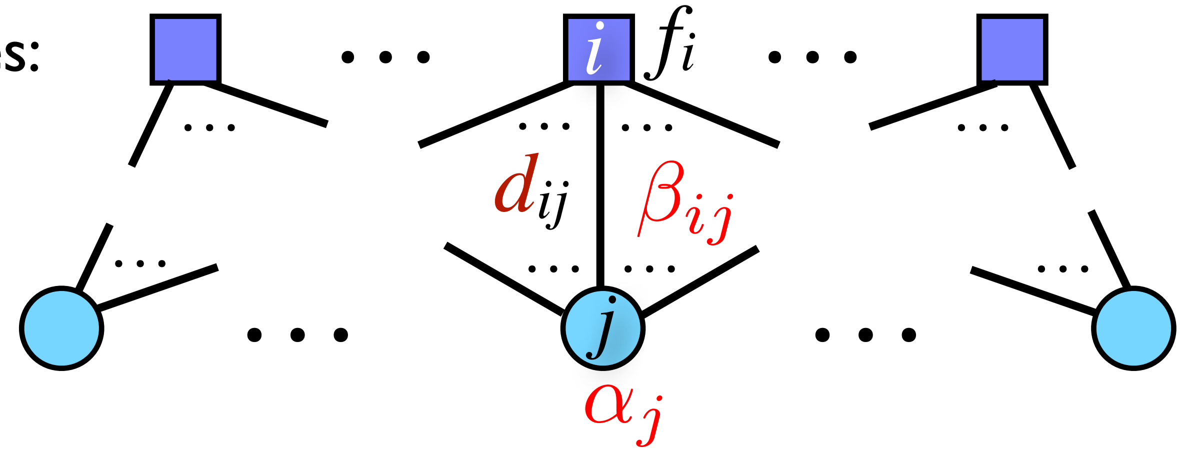
- The events that occur at the same time are processed in arbitrary order.
- Fully paid facilities are tentatively open: $\sum_{j\in C} \beta_{ij} = f_i$
- Fully paid edges to tentatively opening facilities are connected: $\alpha_j - \beta_{ij} = d_{ij}$
- Eventually all clients connect to tentatively opening facilities.

A client may connect to more than one facilities!



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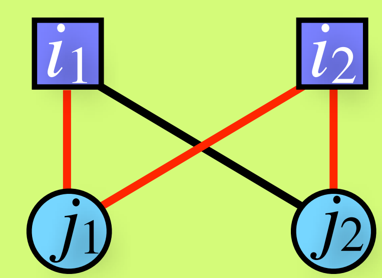

Phase II:

construct graph $G(V,E)$ where V={tentatively open facilities}

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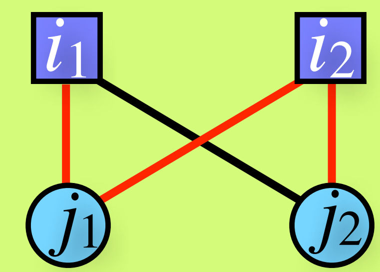

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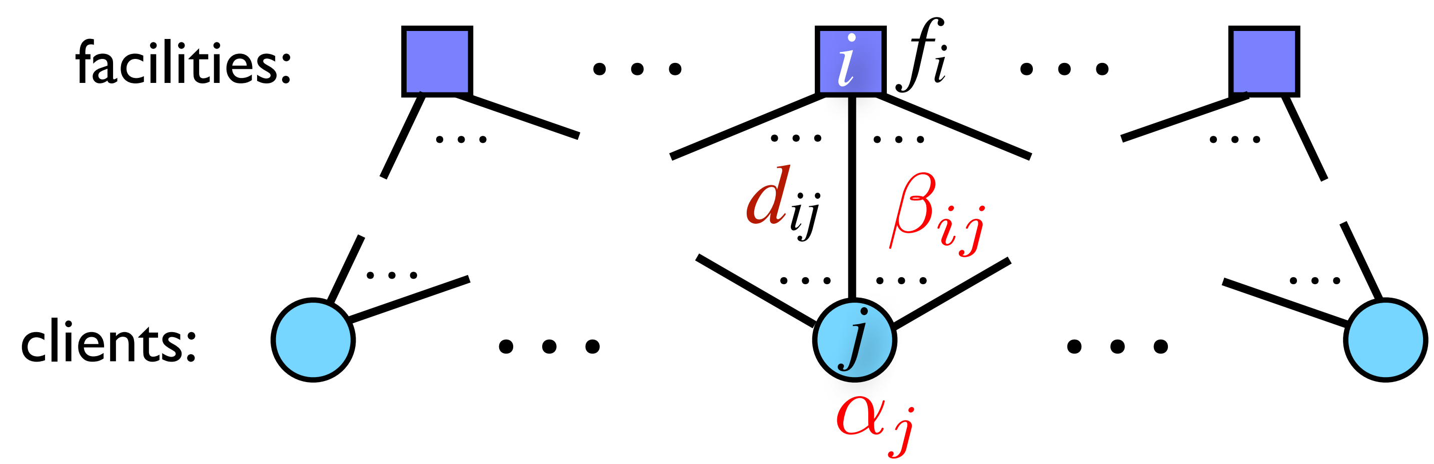
I is independent set $\Rightarrow$ Every client is connected to exact one open facilities.

(feasible)



Primal:

$$\begin{aligned}
\min \quad & \sum_{i\in F, j\in C} d_{ij}x_{ij} + \sum_{i\in F} f_i y_i \\
\text{s.t.} \quad & y_i - x_{ij} \ge 0, \quad \forall i\in F, j\in C \\
& \sum_{i\in F} x_{ij} \ge 1, \quad \forall j \in C \\
& x_{ij}, y_i \in \{0,1\}, \quad \forall i \in F, j\in C
\end{aligned}$$

Dual-relax:

$$\begin{aligned}
\max \quad & \sum_{j\in C} \alpha_j \\
\text{s.t.} \quad & \alpha_j - \beta_{ij} \le d_{ij}, \quad \forall i\in F, j\in C \\
& \sum_{j\in C} \beta_{ij} \le f_i, \quad \forall i \in F \\
& \alpha_j, \beta_{ij} \ge 0, \quad \forall i \in F, j\in C
\end{aligned}$$

α_j: amount of value paid by client j to all facilities

$\beta_{ij} \ge \alpha_j - d_{ij}$: payment to facility i by client j (after deduction)

complimentary slackness conditions: (if *ideally* held)

$x_{ij} = 1 \Rightarrow \alpha_j - \beta_{ij} = d_{ij}$; $\quad \alpha_j > 0 \Rightarrow \sum_{i\in F} x_{ij} = 1$;

$y_i = 1 \Rightarrow \sum_{j\in C} \beta_{ij} = f_i$; $\quad \beta_{ij} > 0 \Rightarrow y_i = x_{ij}$;

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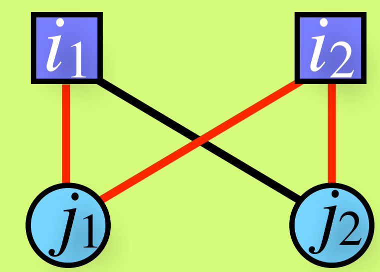

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$$SOL = \boxed{\sum_{i\in I} f_i + \sum_{j\text{:directly connected}} d_{\phi(j)j}} + \boxed{\sum_{j\text{:indirectly connected}} d_{\phi(j)j}} \leq 3\sum_{j\in C}\alpha_j \leq 3\ OPT$$

$$\leq \sum_{j\text{:directly connected}} \alpha_j$$

triangle inequality + maximality of I $\leq 3 \sum_{j\text{:indirectly connected}} \alpha_j$

$$\phi(j) = \begin{cases} i \text{ that } \beta_{ij} = \alpha_j - d_{ij} & \text{if } j \text{ is directly connected} \\ \text{nearest facility in } I & \text{if } j \text{ is indirectly connected} \end{cases}$$

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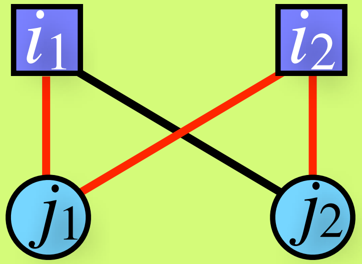

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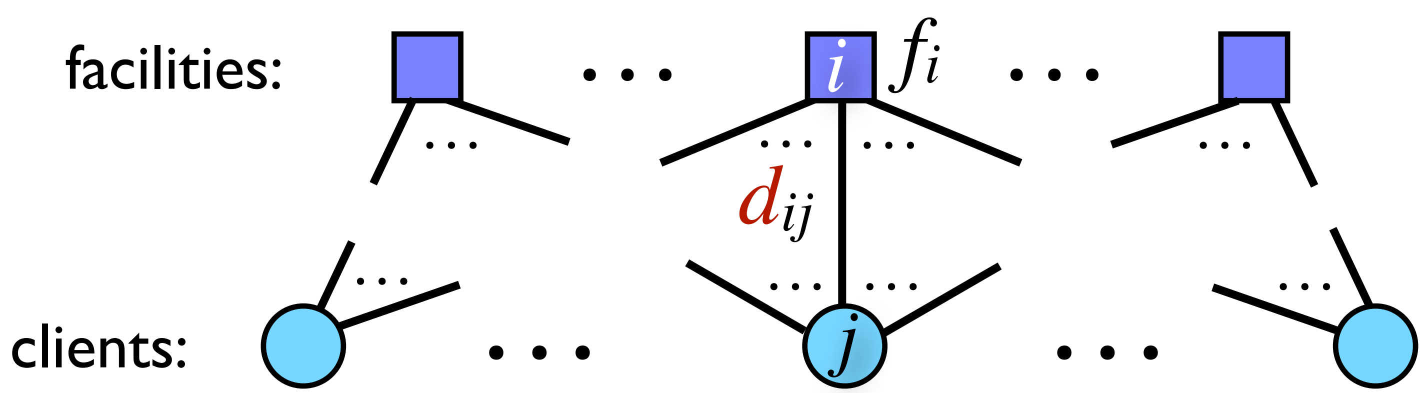
$$SOL \leq 3\ OPT$$

can be implemented discretely: in $O(m \log m)$ time, m=$|F||C|$

- sort all edges $(i,j) \in F\times C$ by non-decreasing d_{ij}
- dynamically maintain the time of next event by heap

Instance: set F of facilities; set C of clients;
facility opening costs f: $F \rightarrow [0, \infty)$;
connection metric d: $F \times C \rightarrow [0, \infty)$;

Find ϕ: $C \rightarrow I \subseteq F$ to minimize $\displaystyle\sum_{j \in C} d_{\phi(j),j} + \sum_{i \in I} f_i$



$$\min \sum_{i \in F, j \in C} d_{ij} x_{ij} + \sum_{i \in F} f_i y_i$$

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$$\sum_{i \in F} x_{ij} \geq 1, \qquad \forall j \in C$$

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- Integrality gap = 3
- no poly-time <1.463-approx. algorithm unless **NP**=**P**
- [Li 2011] 1.488-approx. algorithm