# Approximation Algorithms
# Greedy and Local Search

Advanced Algorithms

Nanjing University, Fall 2018

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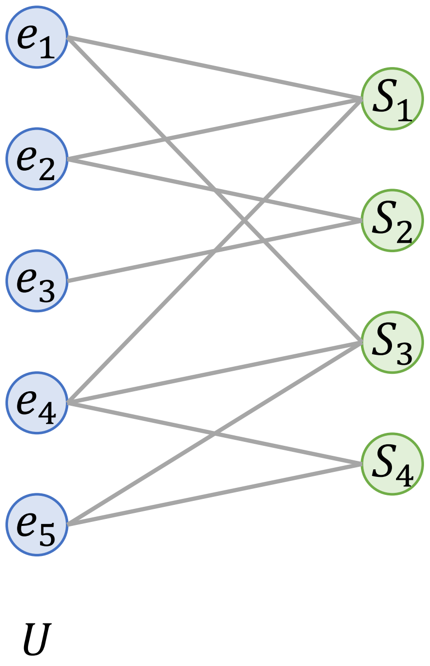

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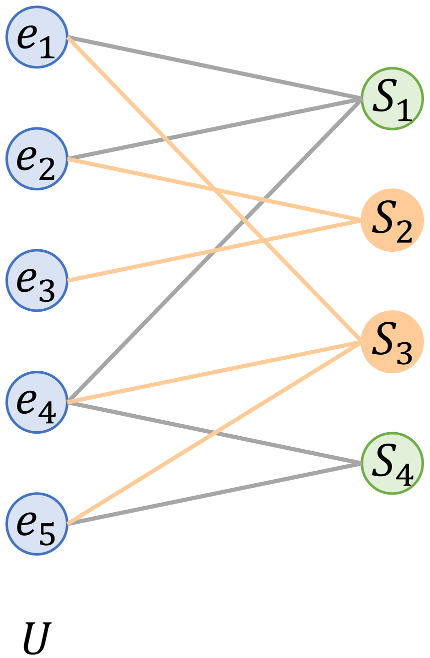

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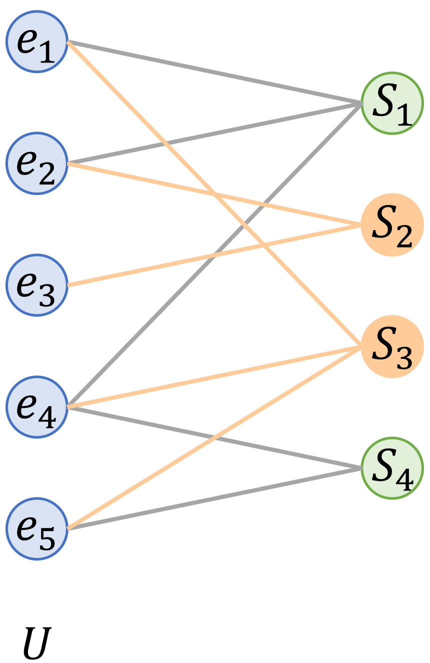


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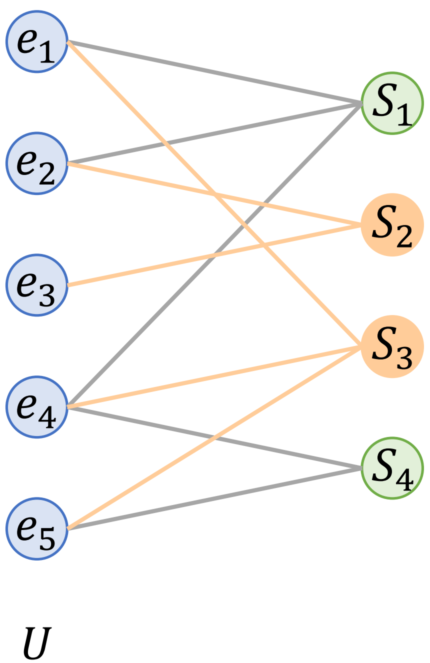


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- Can we find good enough solutions efficiently?

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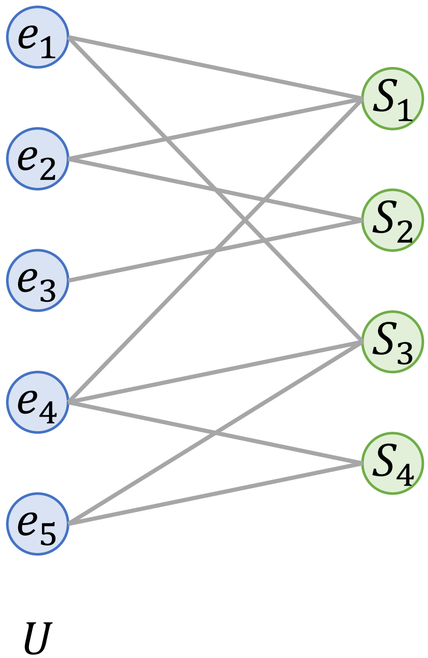


**<u>GreedyCover:</u>**

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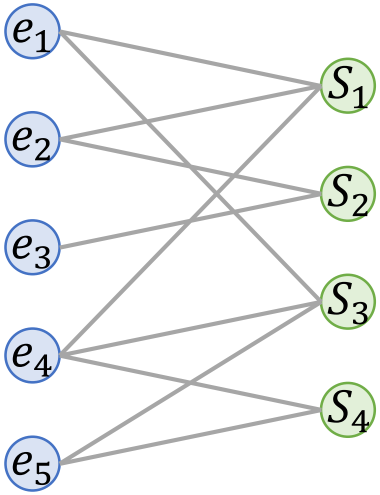


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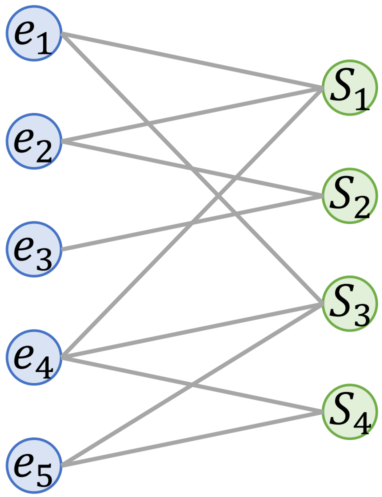


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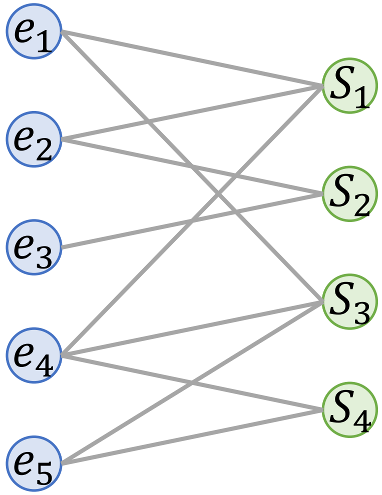


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For minimization problems, we want $\mathrm{SOL}(I)/\mathrm{OPT}(I) \leq \alpha$ where $\alpha \geq 1$

For maximization problems, we want $\mathrm{SOL}(I)/\mathrm{OPT}(I) \geq \alpha$ where $\alpha \leq 1$

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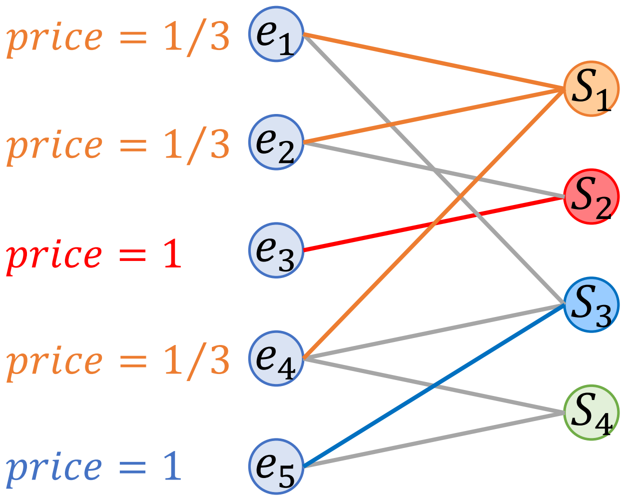


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$$|C| = \sum_{e \in U} price(e)$$

- Initially, there must exist some subset that covers its elements with price at most $\mathrm{OPT}(I)/n$.
- Therefore, price of elements in the first subset covered by **GreedyCover** is at most $\mathrm{OPT}(I)/n$.
- After $k$ elements in $t$ subsets are covered by **GreedyCover**, there must exist some subset such that the price of its uncovered elements is at most $\mathrm{OPT}(I_t)/(n-k) \leq \mathrm{OPT}(I)/(n-k)$.
- In general, **GreedyCover** pays at most $\mathrm{OPT}(I)/(n-k+1)$ to cover the $k^{\text{th}}$ chosen element.

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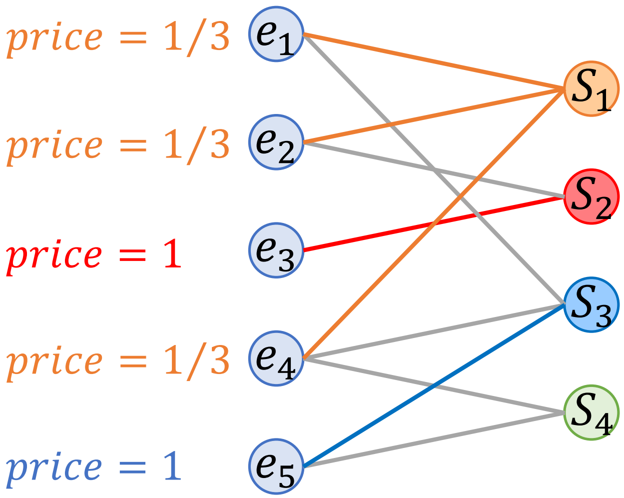


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Enumerate $e_k$ in the order in which they are covered by **GreedyCover**:

$$price(e_k) \leq \frac{\mathrm{OPT}(I)}{n - k + 1}$$

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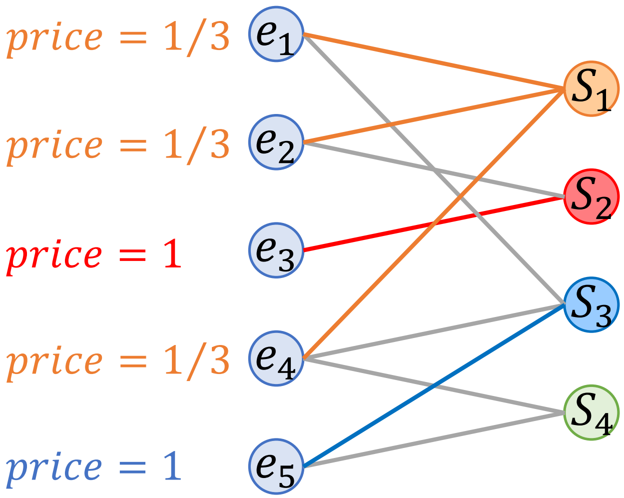


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$$|C| = \sum_{e \in U} price(e) \leq \sum_{k=1}^{n} \frac{\mathrm{OPT}(I)}{n-k+1} = H_n \cdot \mathrm{OPT}(I)$$

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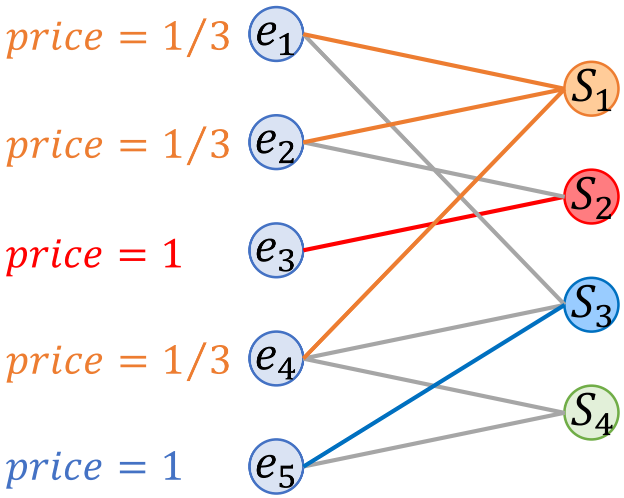


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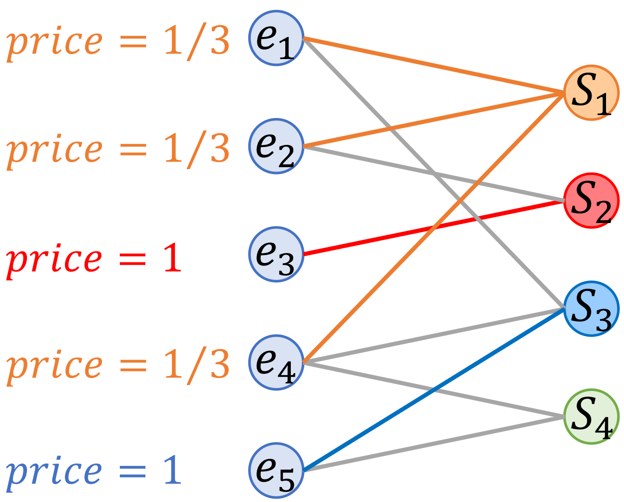


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- [Ras, Safra 1997] For some constant $c$, there is no poly-time $c \ln(n)$ approx. algorithm unless NP = P.
- [Dinur, Steuer 2014] There is no poly-time $\left(1 - o(1)\right) \ln(n)$ approx. algorithm unless NP = P.

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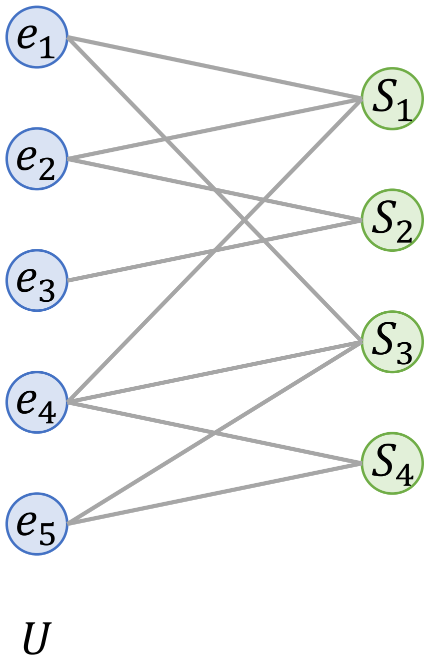


- This problem is NP-hard.
- We have $O(\ln n)$ approx. alg.

- *Frequency* of an element:
  # of subsets the element is in.
- Use $f_I$ to denote the frequency of the most frequent element in instance $I$.

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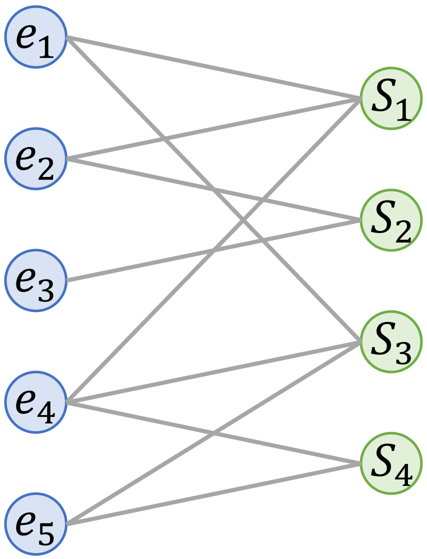

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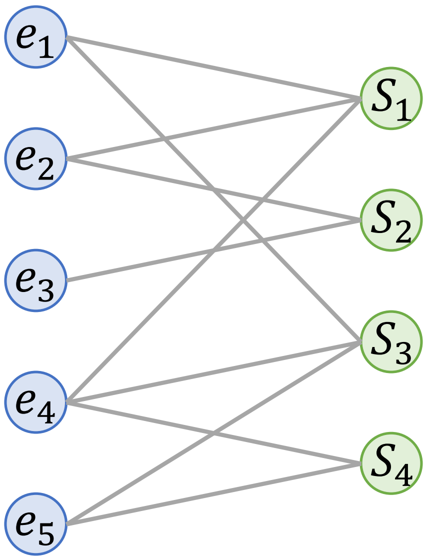


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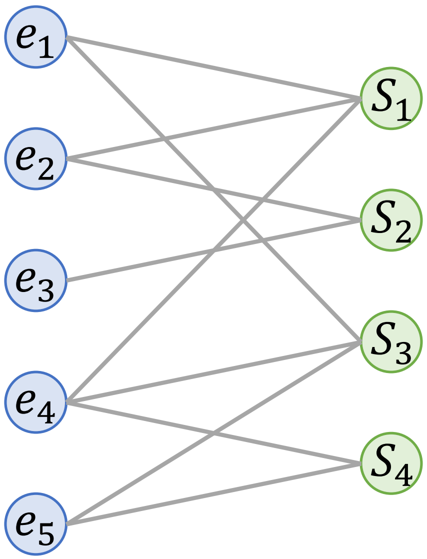


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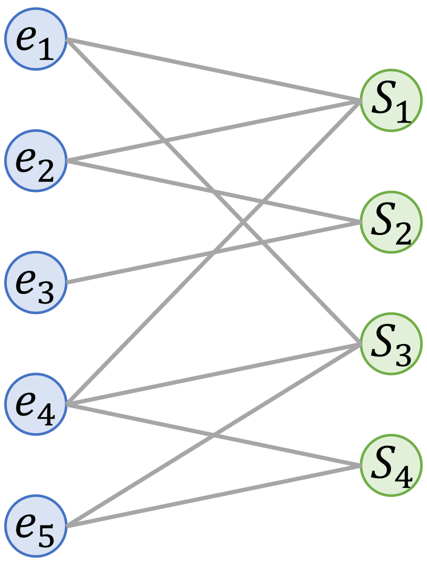


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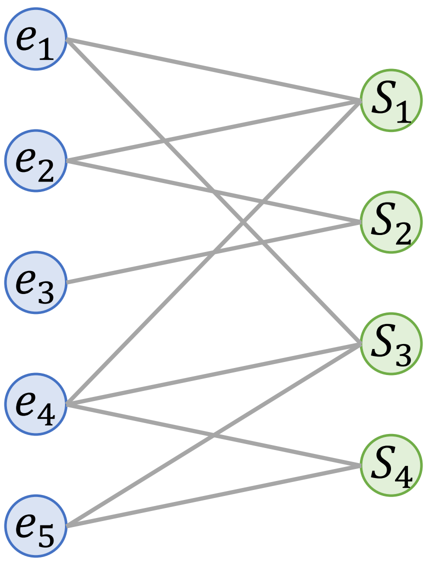


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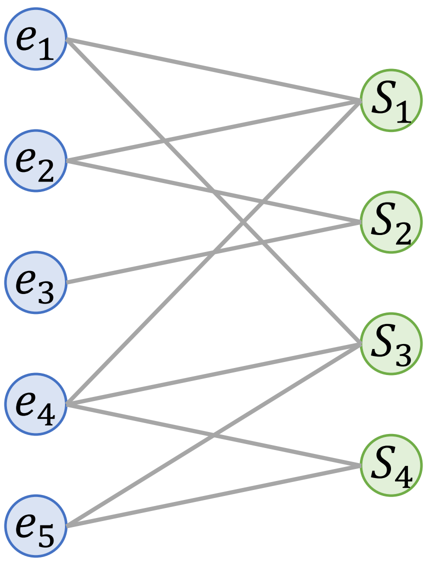


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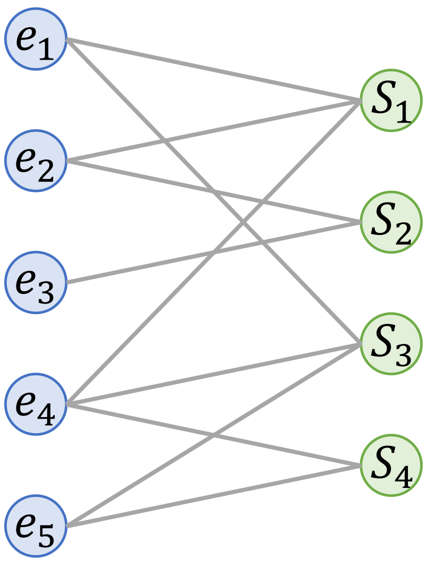


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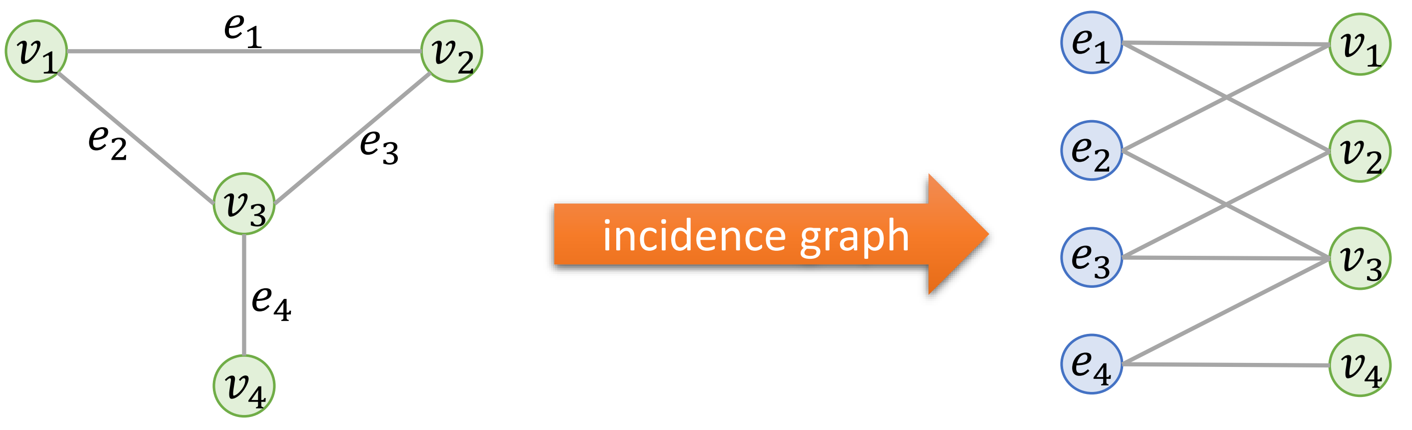

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**Instance:** An undirected simple graph $G = (V, E)$.
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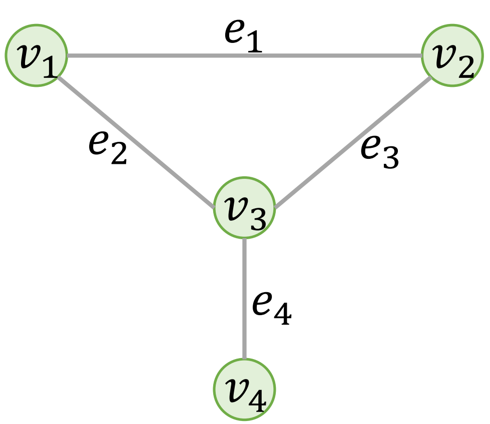


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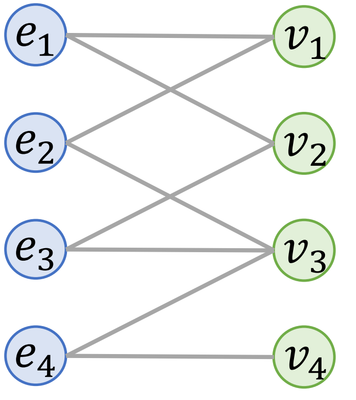


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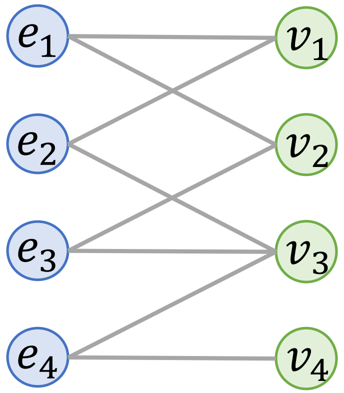

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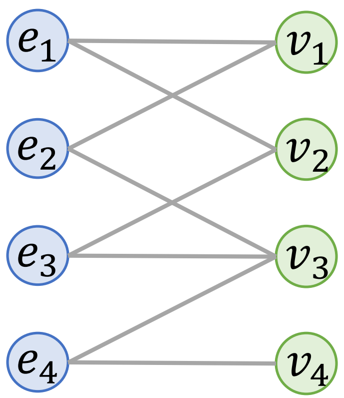


A 2-approximation algorithm for the vertex cover problem

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Find arbitrary maximal matching $M$ of the input graph.
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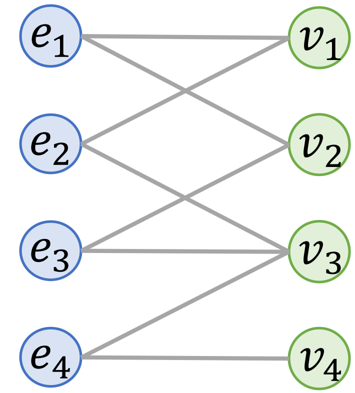


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- There is no poly-time <1.36-approx. alg. unless P = NP.
- Assuming the unique game conjecture, there is no poly-time (2-ε)-approx. alg.

# Scheduling

$m$ identical machines

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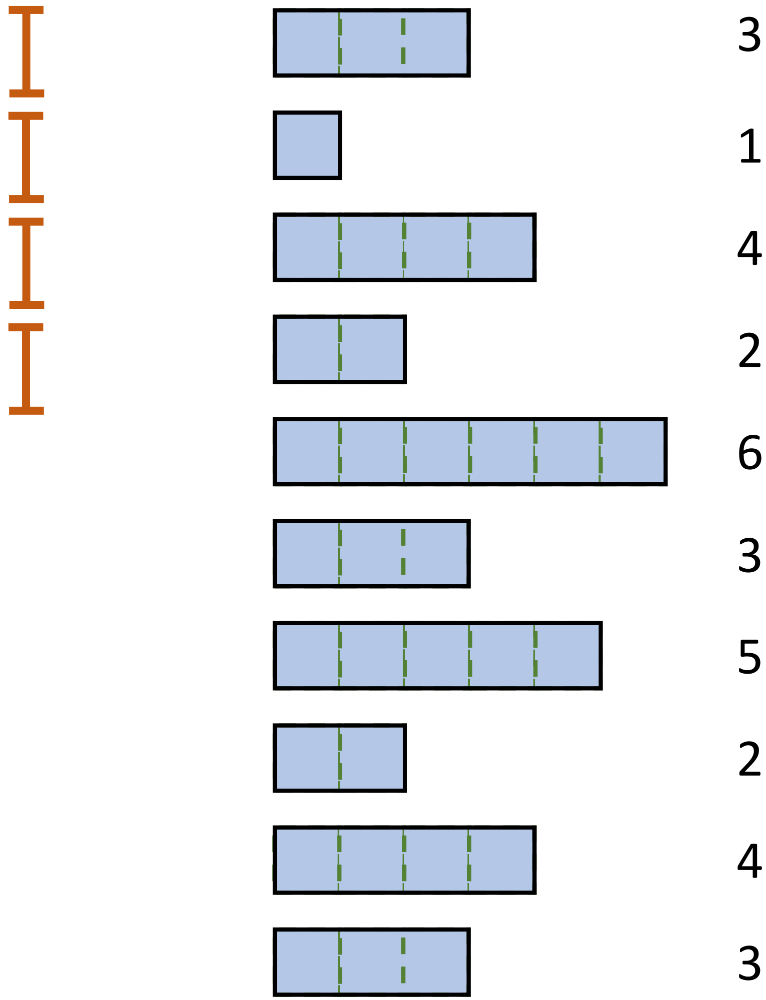
$n$ jobs

processing time $p_j$



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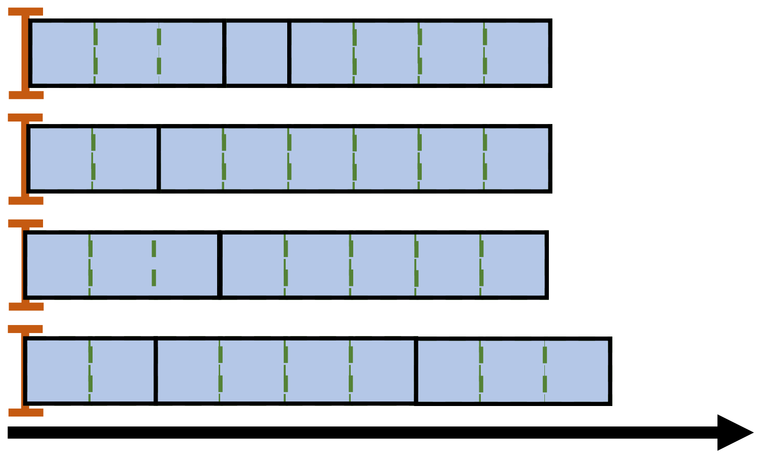
$m$ machines



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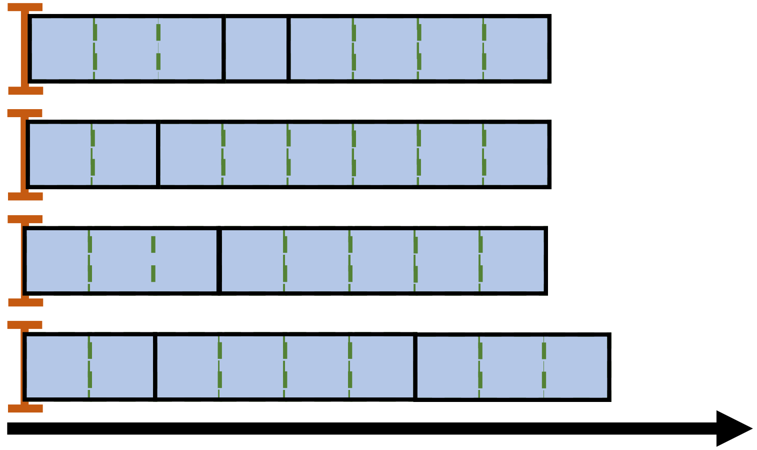

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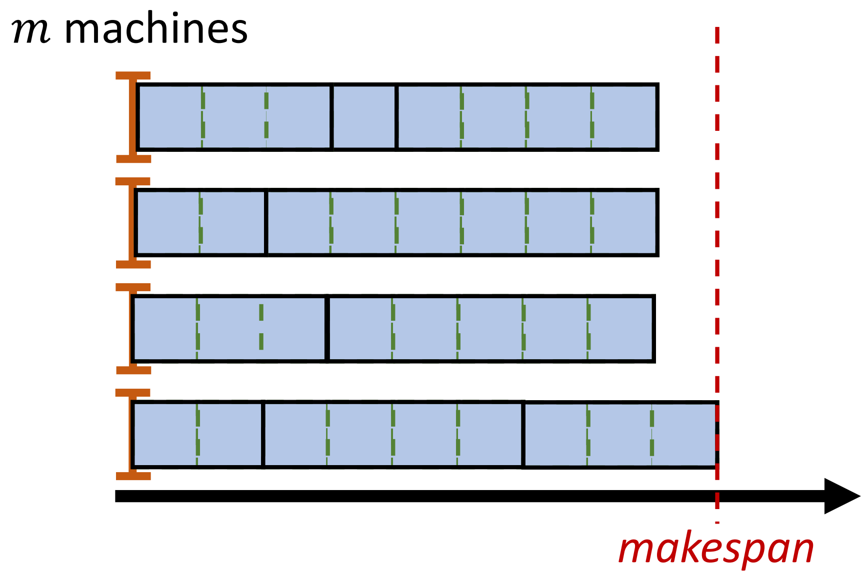


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**Makespan:**

$$C_{\max} = \max_i C_i$$

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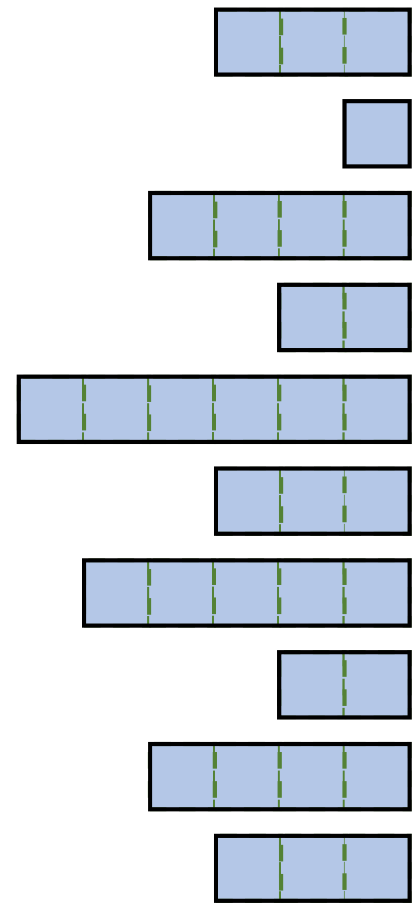
- The partition problem is one of Karp's 21 NP-complete problems.
- Thus the considered scheduling problem is NP-hard.

# Graham's **List** Algorithm for Scheduling

$m$ identical machines

$n$ jobs each with processing time $p_j$



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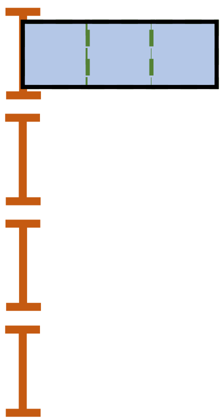

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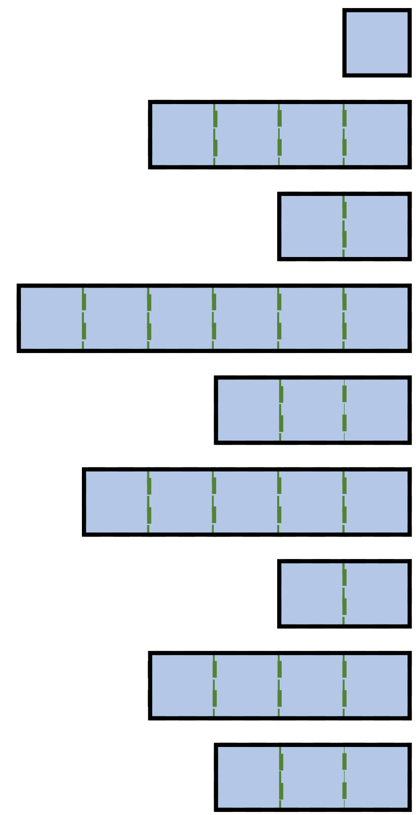

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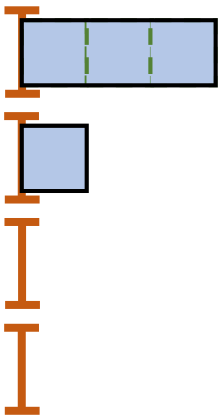

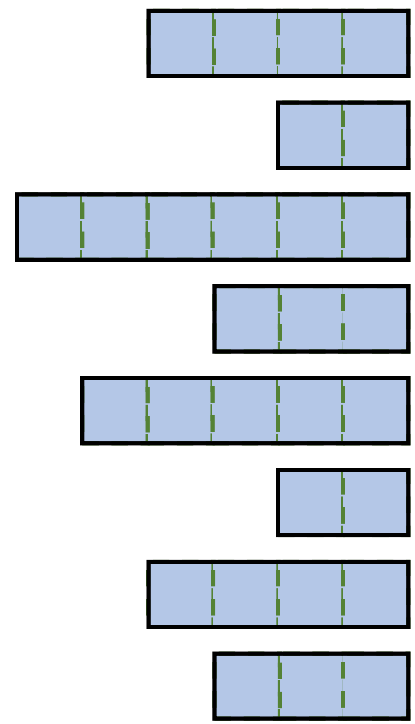

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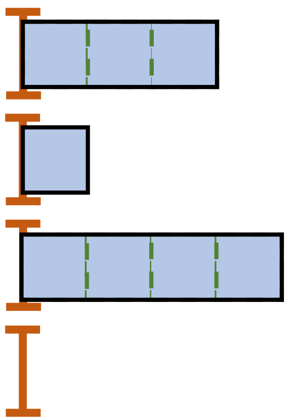

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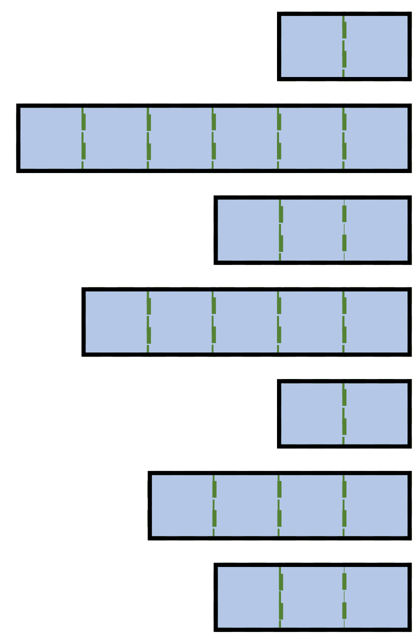

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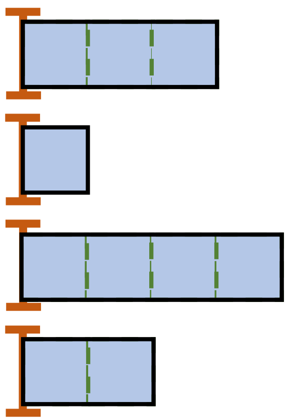

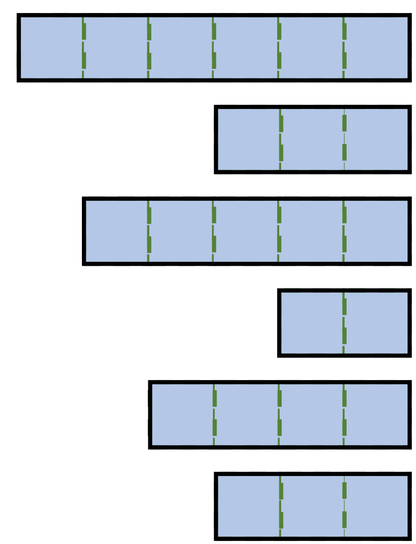

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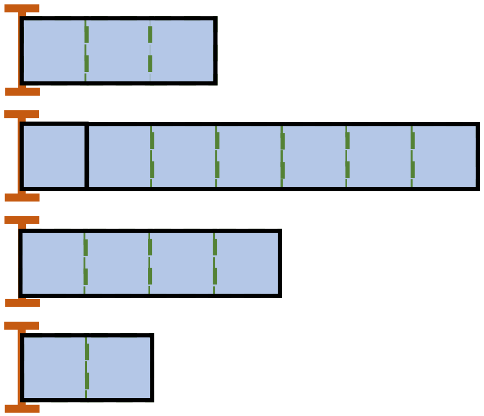

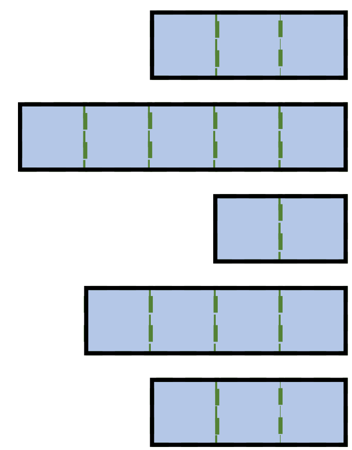

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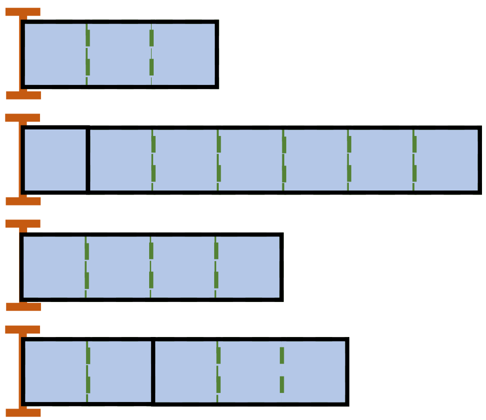

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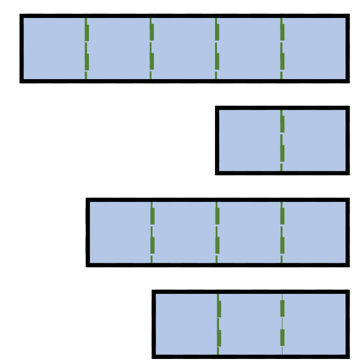

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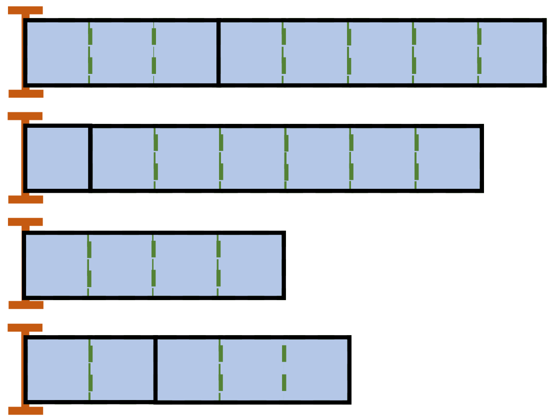

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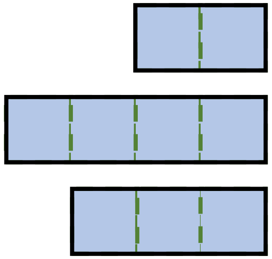

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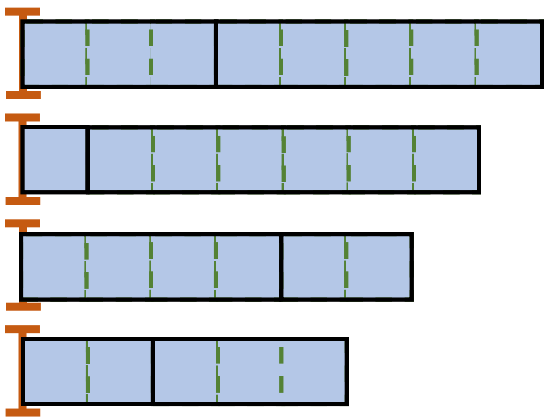

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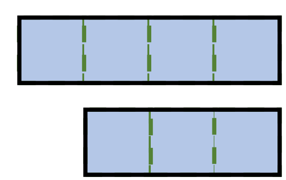

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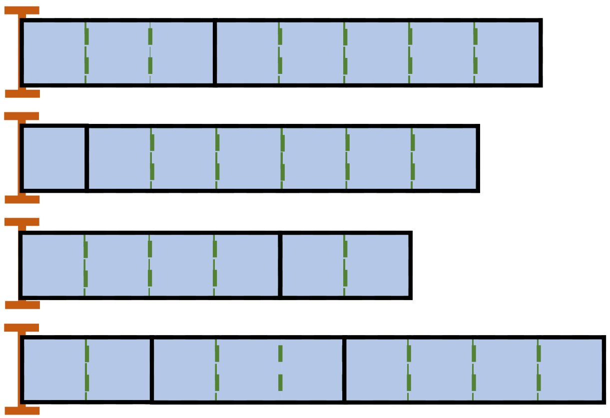

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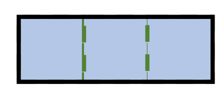

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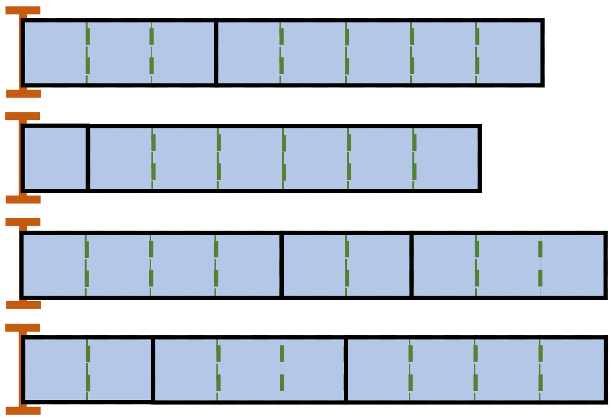

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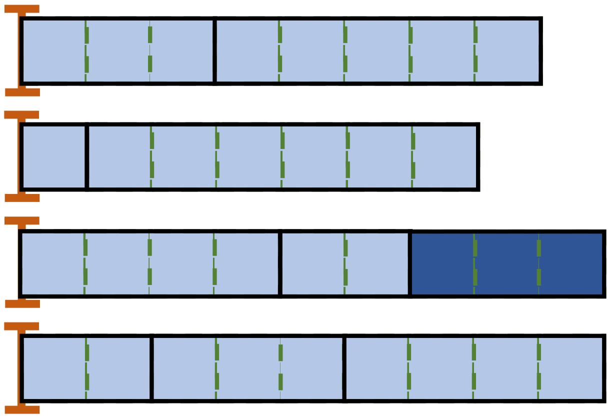

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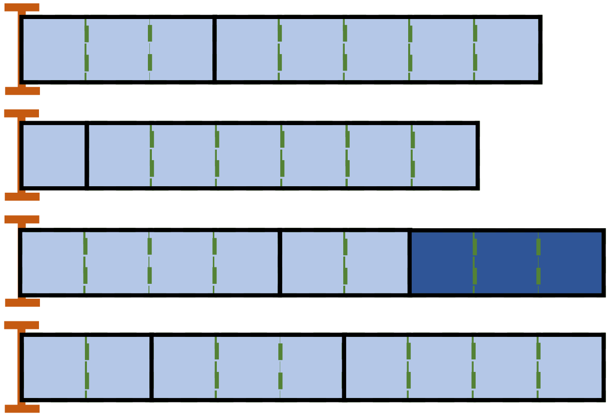

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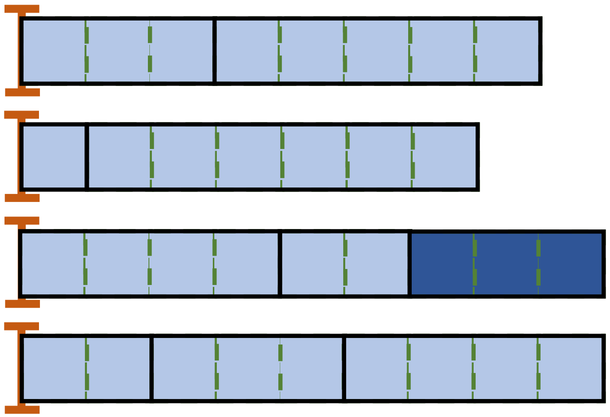

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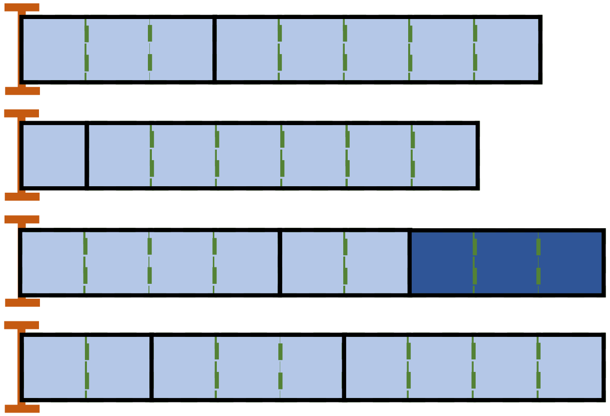

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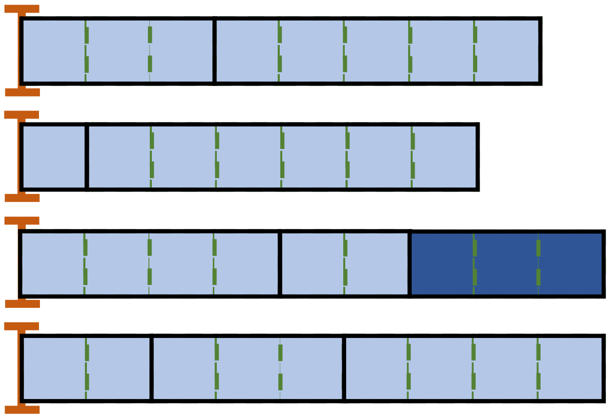

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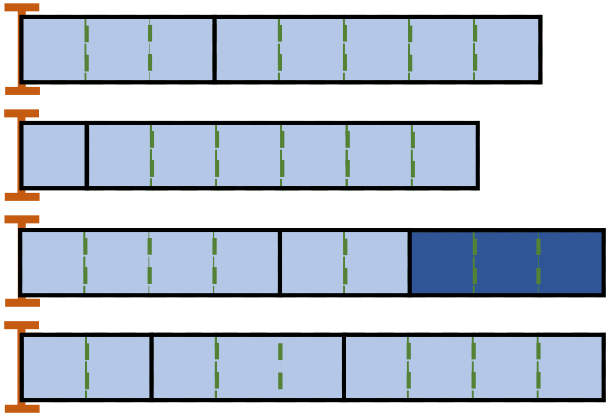

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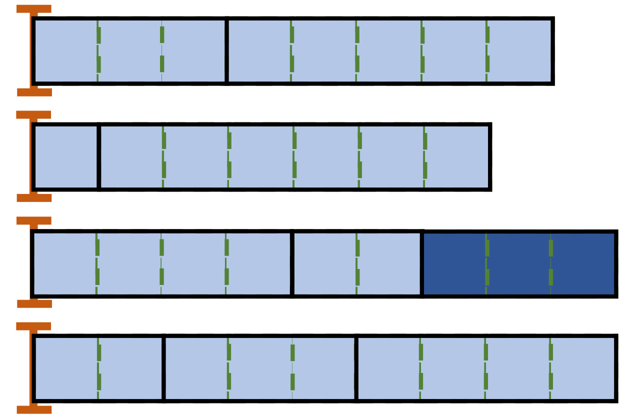

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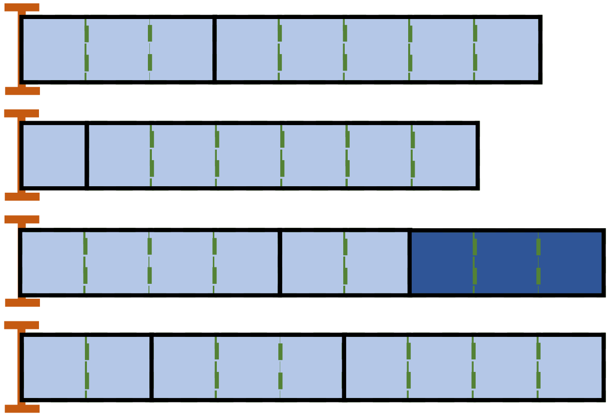

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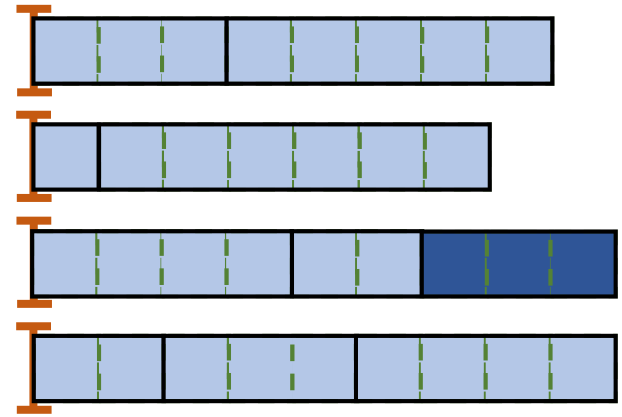

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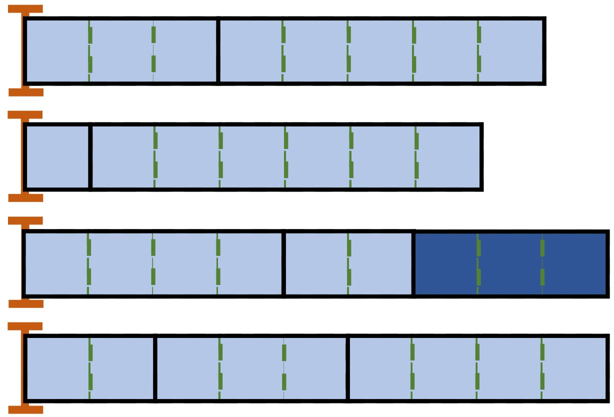

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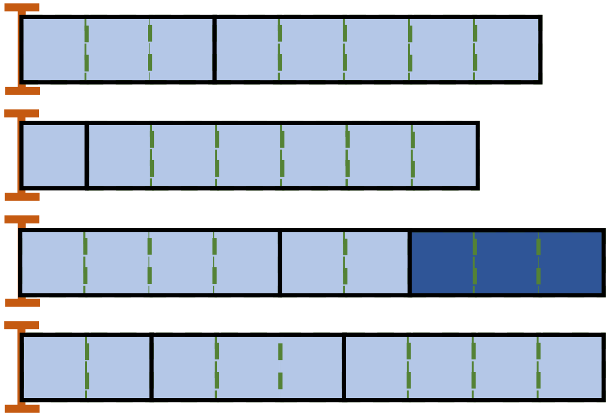

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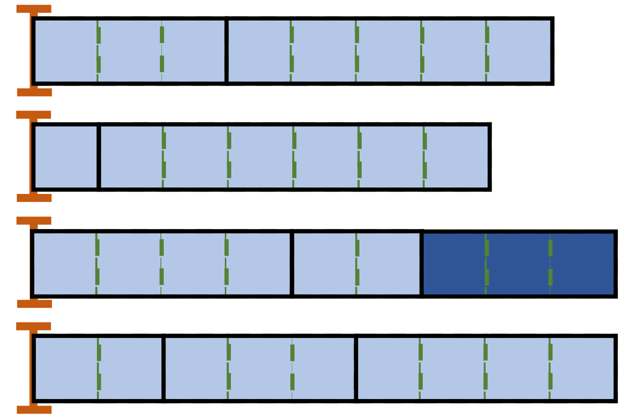

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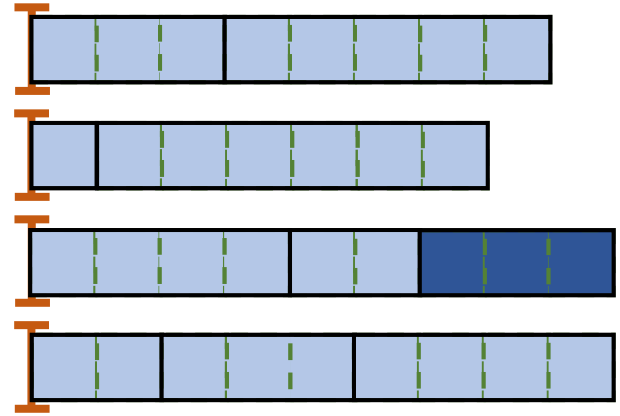

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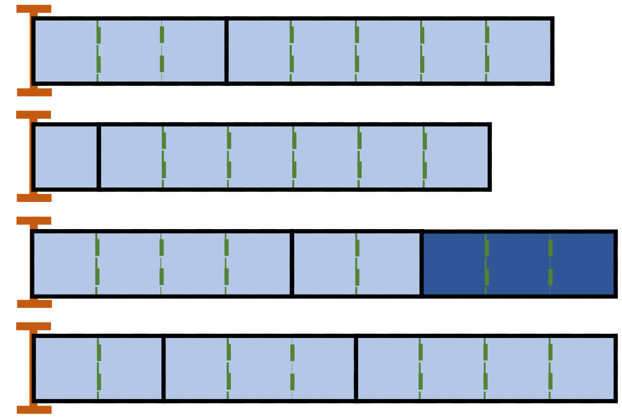

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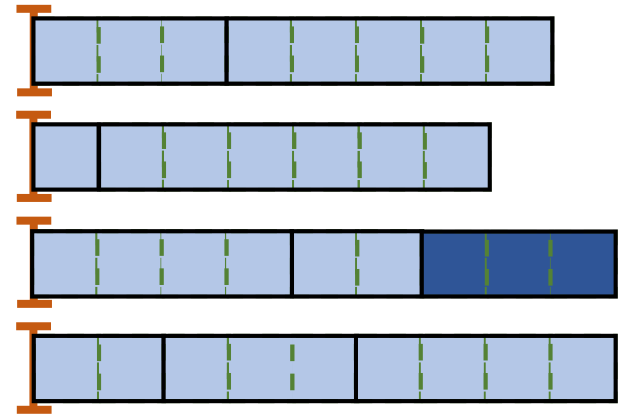

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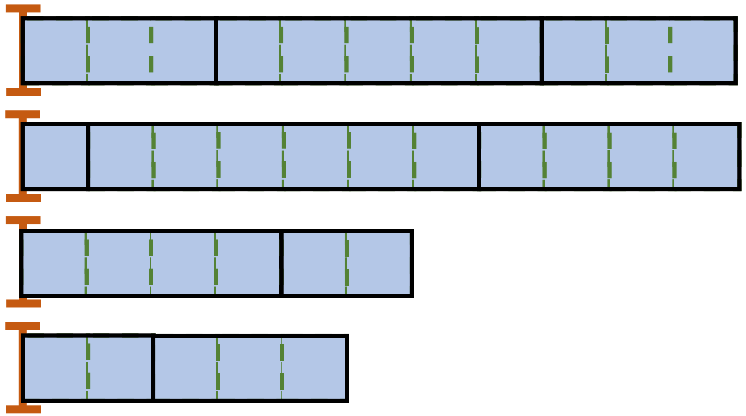
This bound is tight in the worst case. [Almost tight example: $m^2$ unit jobs followed by a length $m$ job. **List** generates makespan of $2m$ while $\mathrm{OPT} = m + 1$.]

# **Local Search** for Scheduling

Start with an arbitrary solution:



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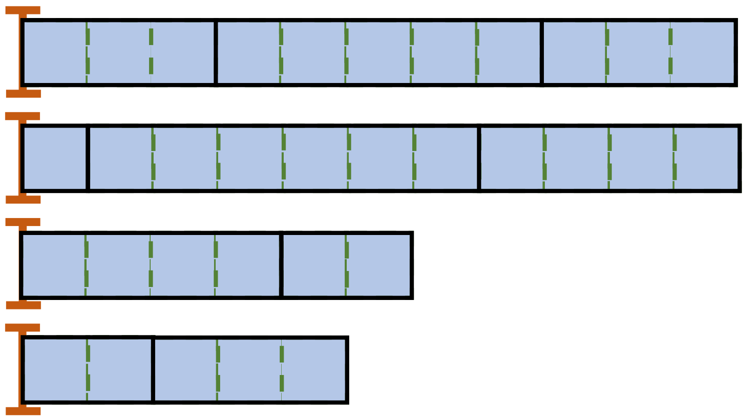

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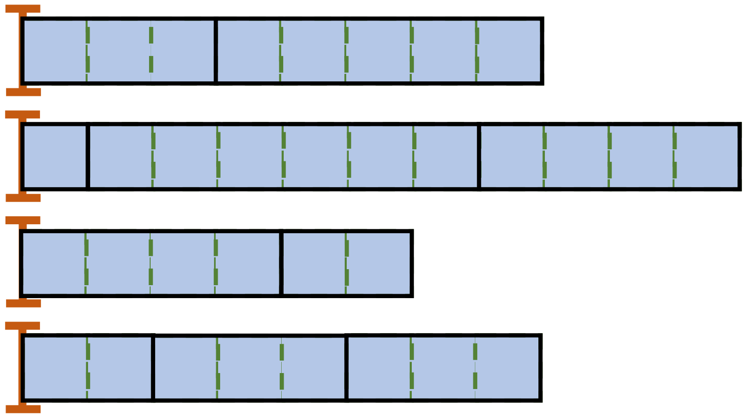

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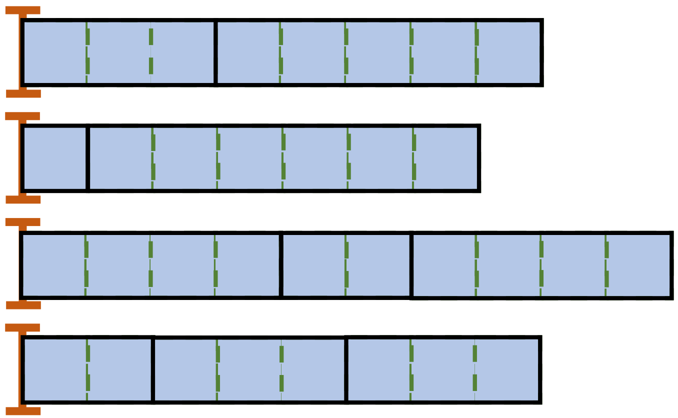

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The approximation ratio of this algorithm? $(2 - 1/m)$

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**List** (Graham 1966)**:**

For each job $j = 1,2, \cdots, n$ do:

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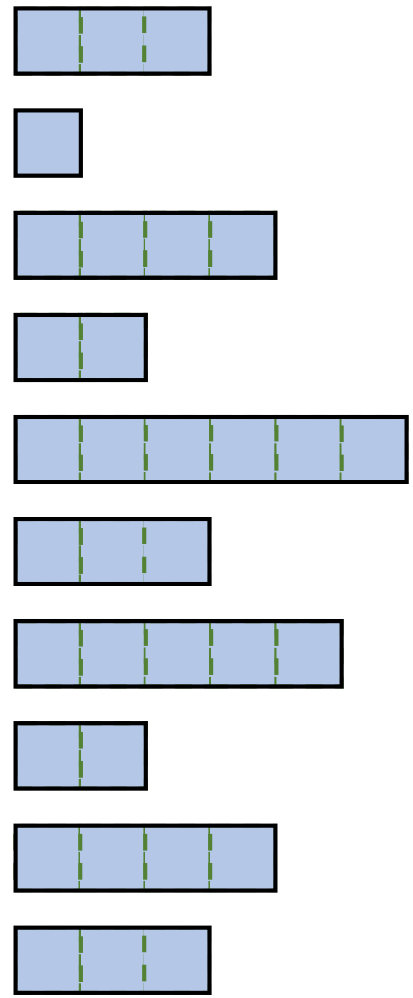
$m$ identical machines

$n$ jobs



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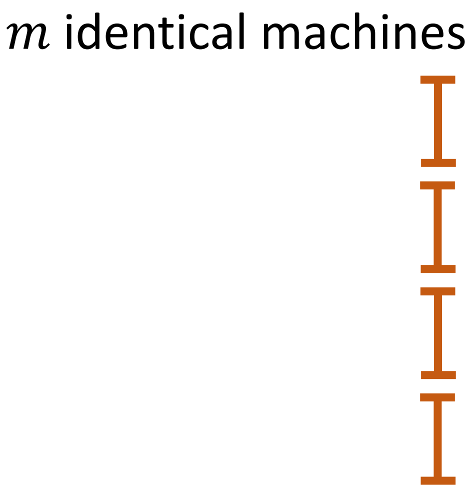


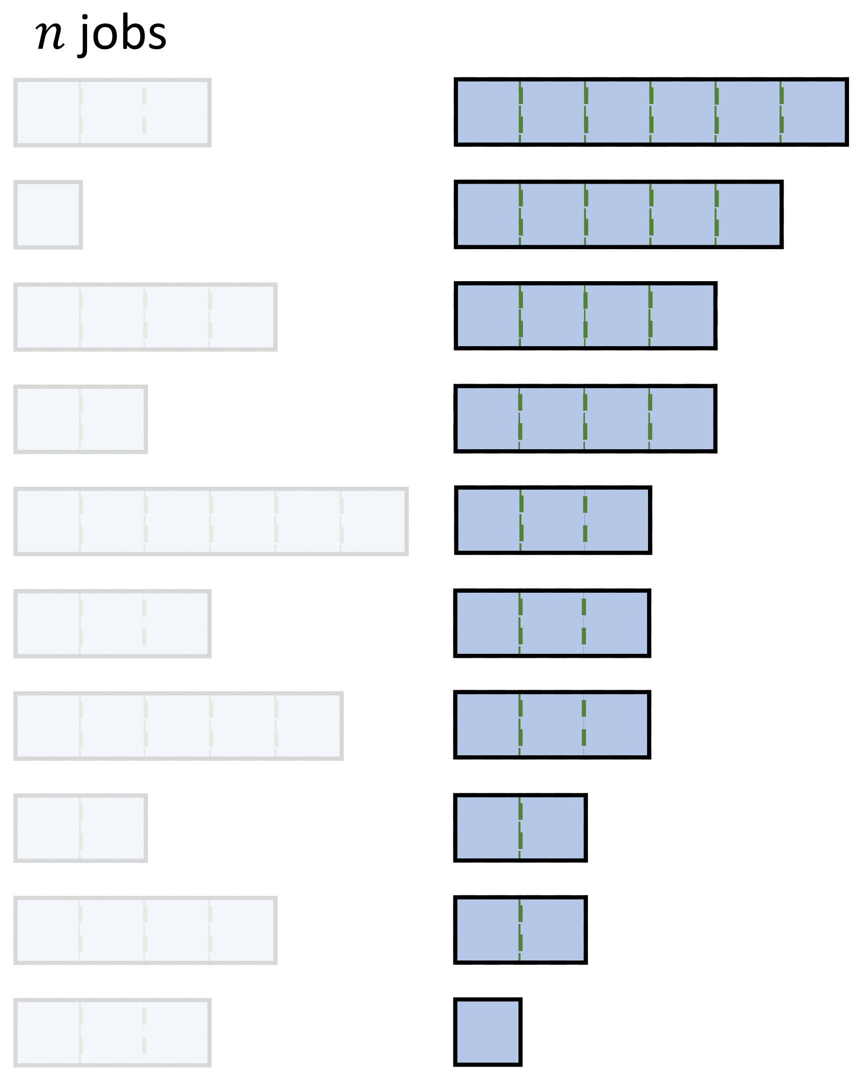


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# Longest Processing Time (LPT)

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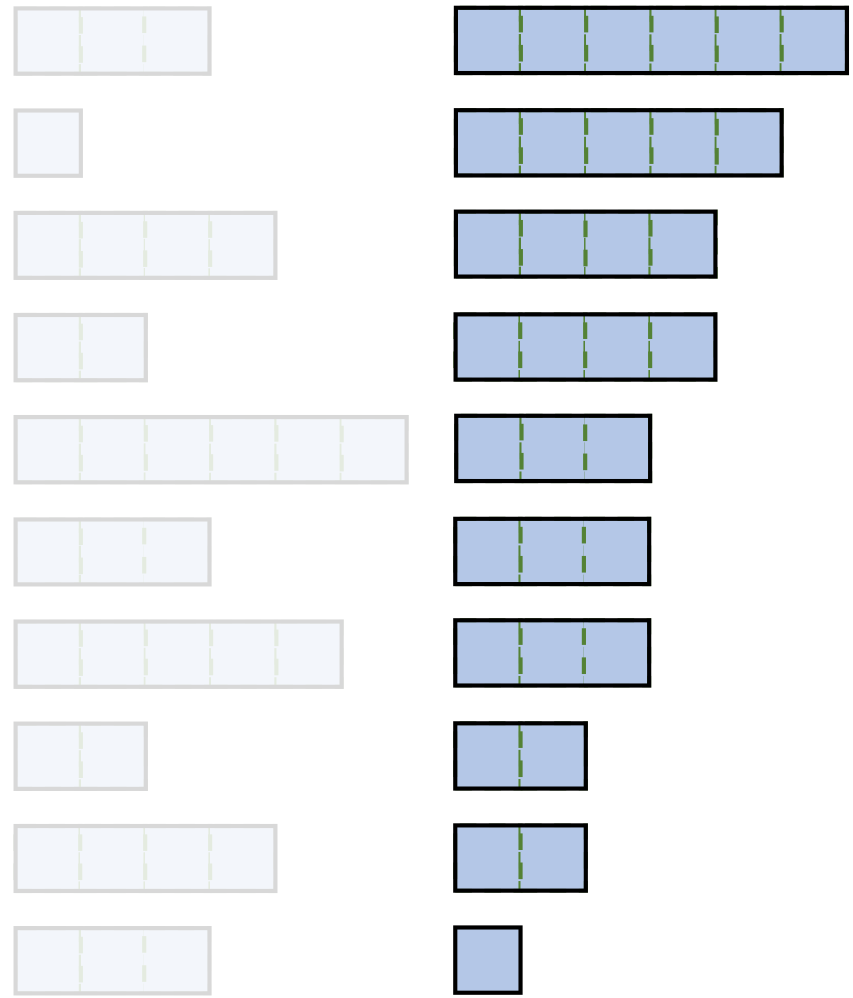

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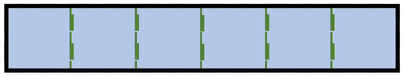
- We have shown **LPT** has approximation ratio (at most) $3/2$**.**
- By a more careful analysis, it can be shown **LPT** is actually a $4/3$ approximation algorithm.
- The problem of "minimum makespan on identical machines" has a **PTAS** (**P**olynomial **T**ime **A**pproximation **S**cheme).
  $\forall \epsilon > 0$, $\exists$poly-time $(1+\epsilon)$-approx. alg. for the problem

# Online Scheduling

$m$ identical machines

Jobs arrive (revealed) one-by-one



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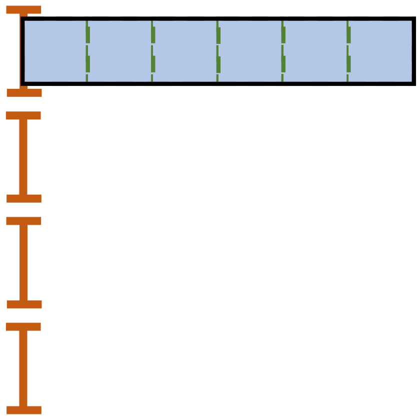

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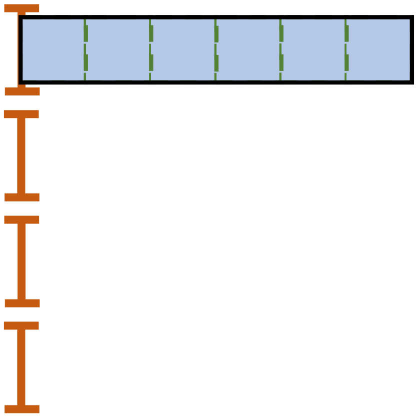

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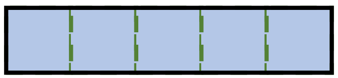

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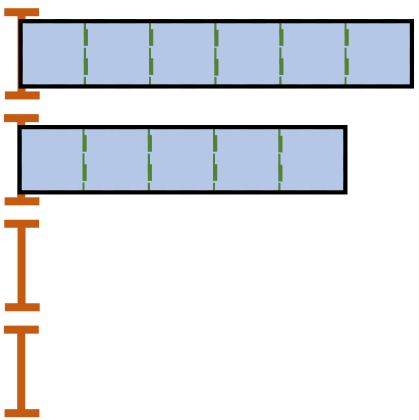

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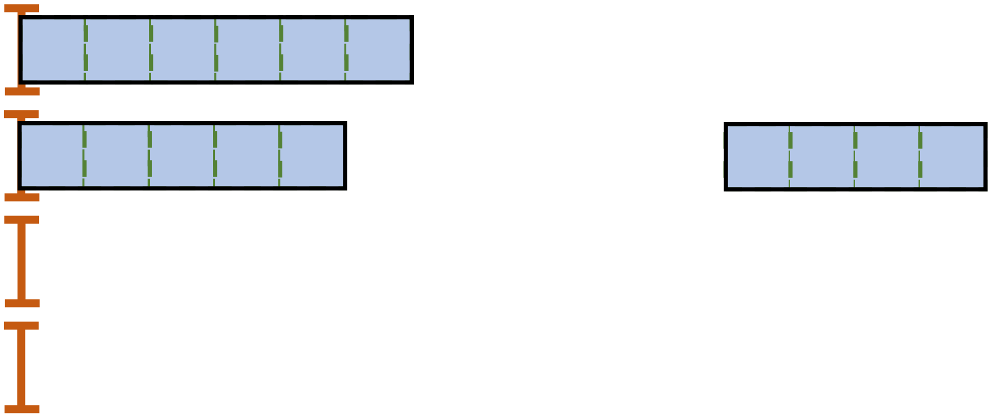

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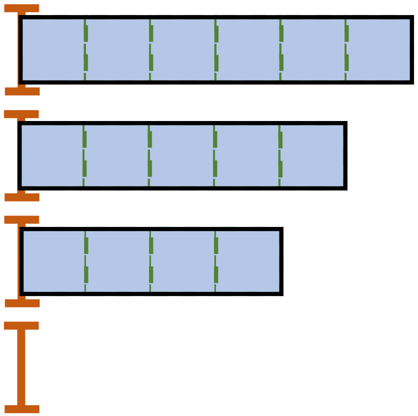

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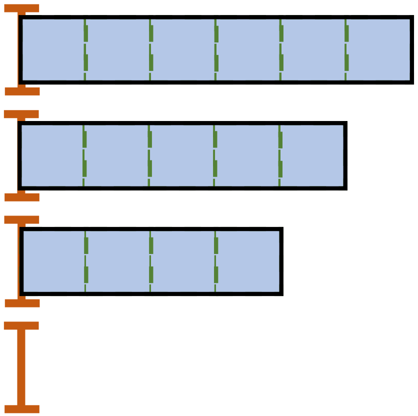

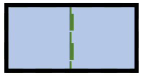

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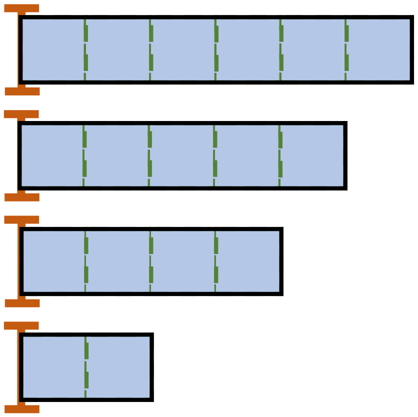

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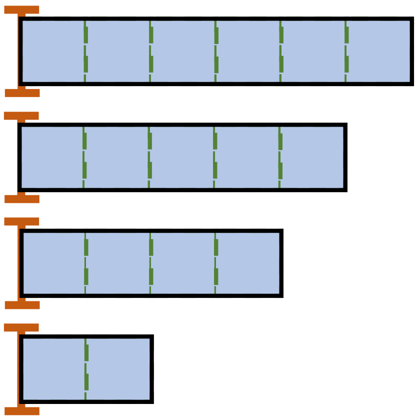

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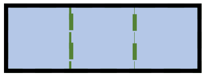

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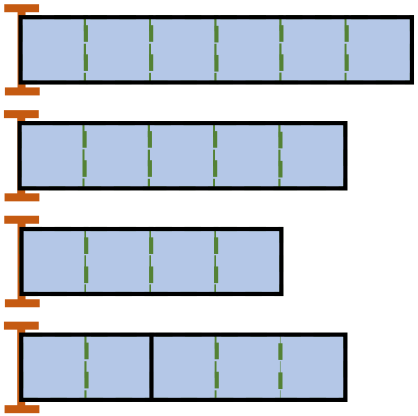

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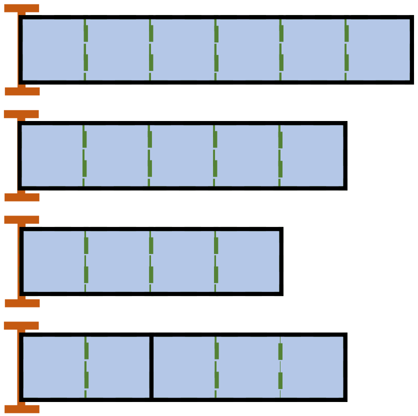

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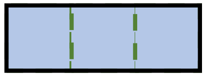

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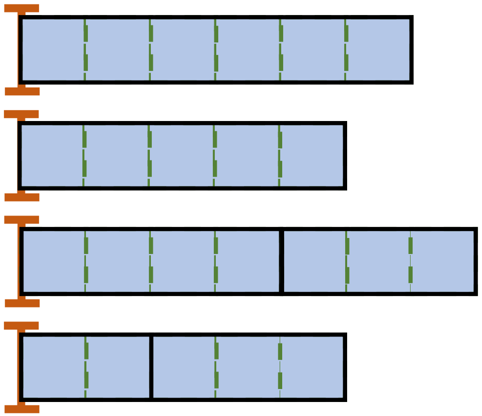

Jobs arrive (revealed) one-by-one

Schedule decision must be made *once* a job arrives, *without* seeing jobs in the future.

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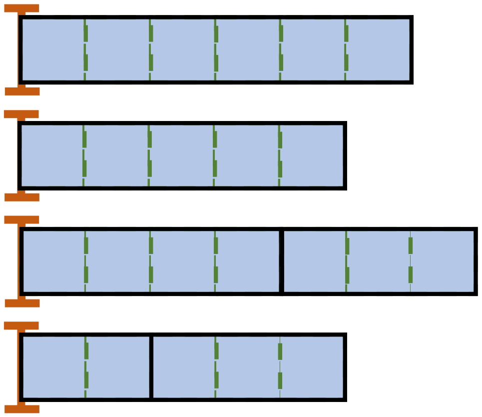

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**List** (Graham 1966)**:**

For each job $j = 1,2,\cdots,n$ do:

Assign job $j$ to a currently least loaded machine.

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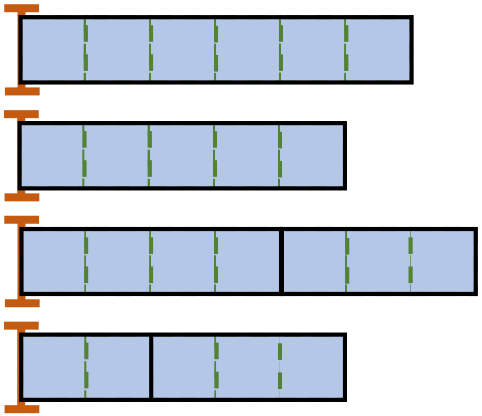

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**LPT** is not an online alg. for scheduling.

# Competitive Analysis

The competitive ratio of an **online algorithm** $\mathcal{A}$ is $\alpha$ if:

For every possible input sequence $I$ of the considered problem:

$$\frac{\text{solution value returned by online alg. } \mathcal{A} \text{ on } I}{\text{solution value returned by optimal offline alg. on } I} \leq \alpha$$

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**List** is a 2-competitive online algorithm