Approximation Algorithms
LP Relaxation and Rounding

Advanced Algorithms

Nanjing University, Fall 2018

Linear Programming (LP)

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- Revenue for A is 1 per kg and the revenue for B is 6 per kg.
- The demand for A is 200kg per day and the demand for B is 300kg per day.
- The factory can produce at most 400kg each day.
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matrix $A = \{a_{ij}\}_{m \times n}$
sets $M \subseteq [m]$ and $N \subseteq [n]$

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\begin{array}{lll}
\text{minimize} & \vec{c}^T \vec{x} & \\
\text{subject to} & \vec{a_i}\, \vec{x} \geq b_i & i \in M \\
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$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} -200 \\ -300 \\ -400 \end{bmatrix}$$

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$$x_j \text{ unconstrained} \iff \begin{cases} x_j^+ \geq 0 \\ x_j^- \geq 0 \\ x_j = x_j^+ - x_j^- \end{cases}$$

Convex Polytope

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Hyperplane:

Subspace of dimension $n - 1$

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Halfspace:

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(Convex) Polytope:

Bounded and nonempty intersection of finite number of halfspaces

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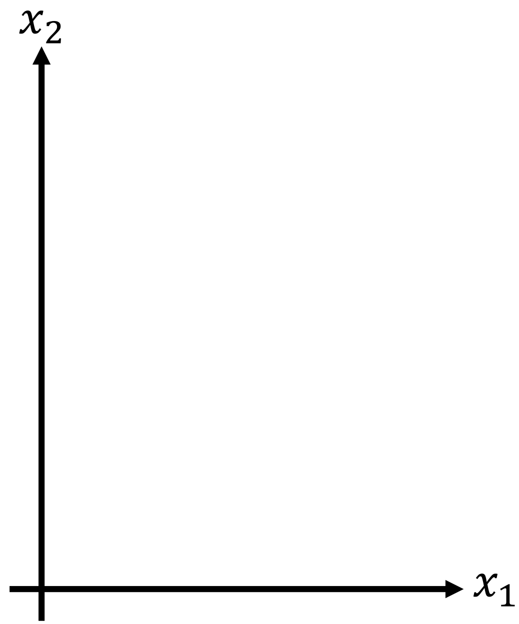

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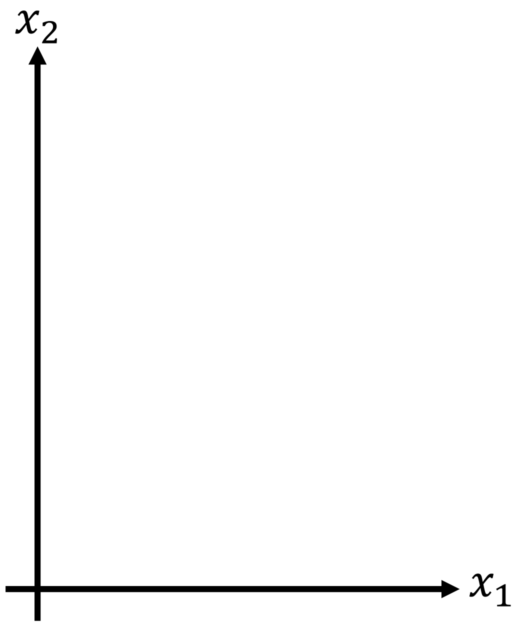

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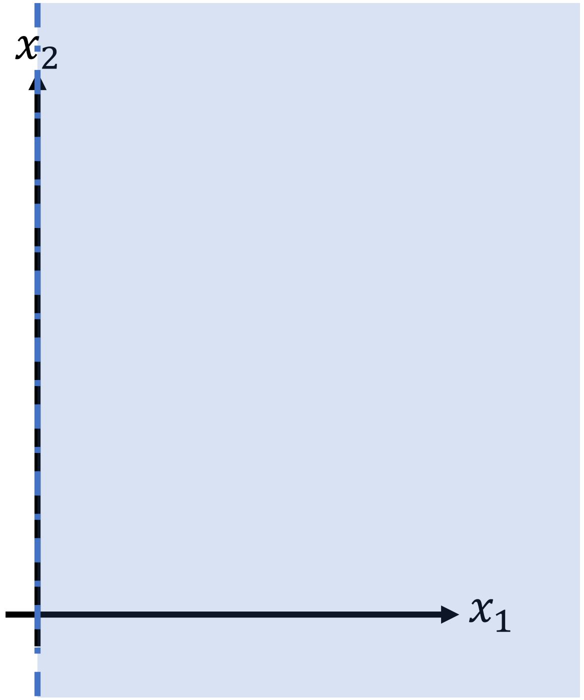

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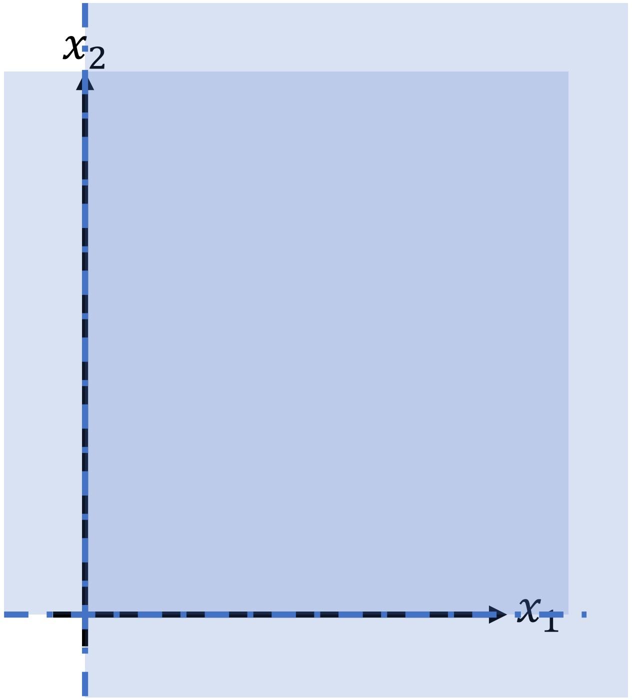

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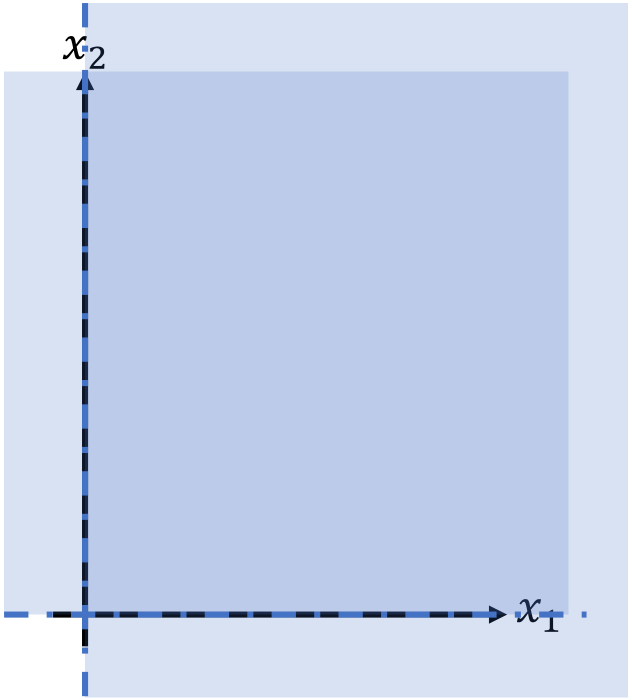

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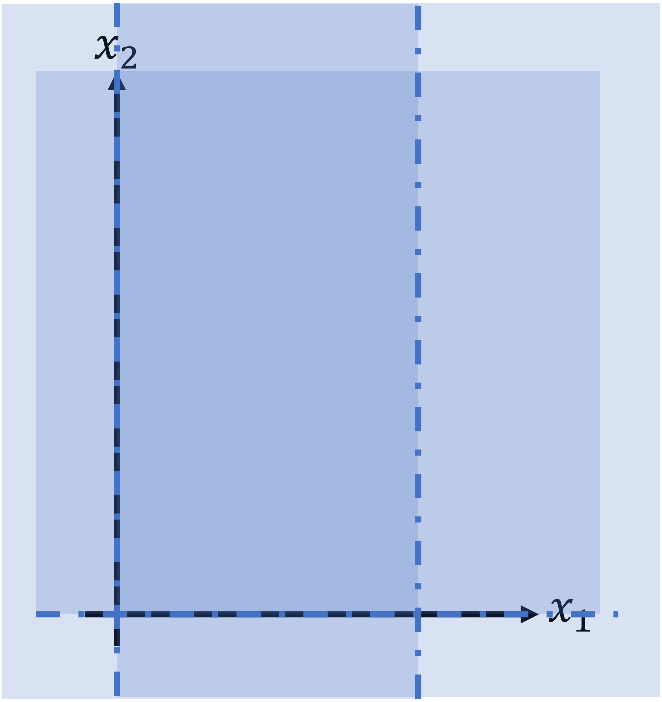

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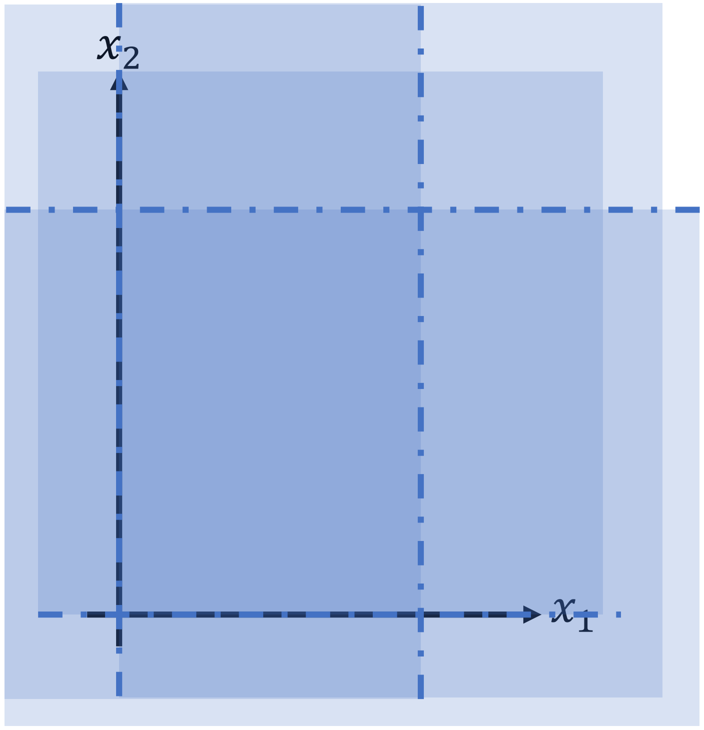

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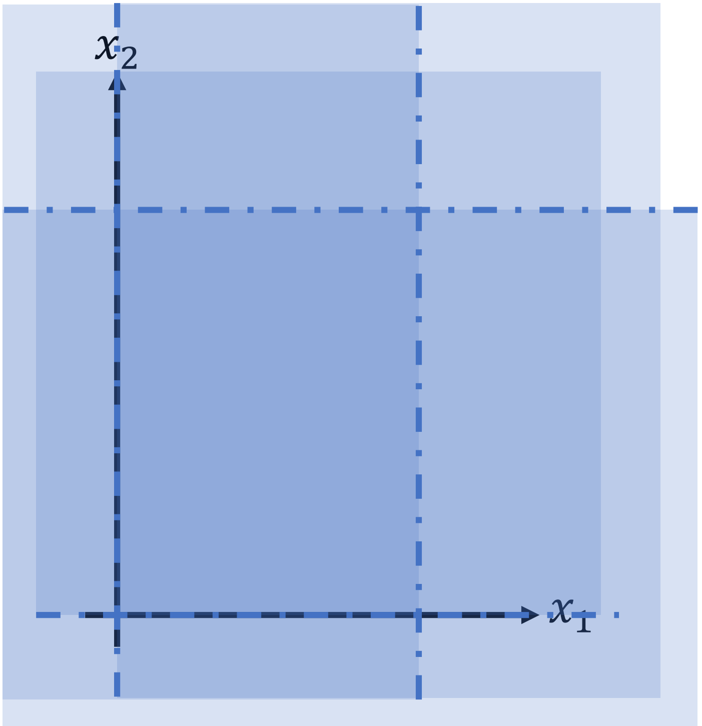

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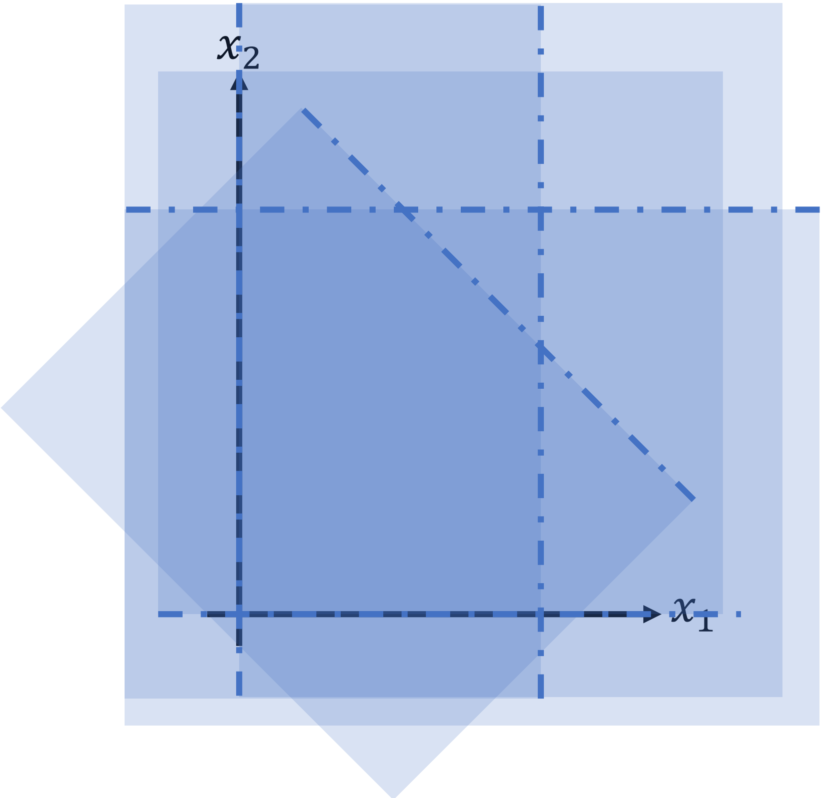

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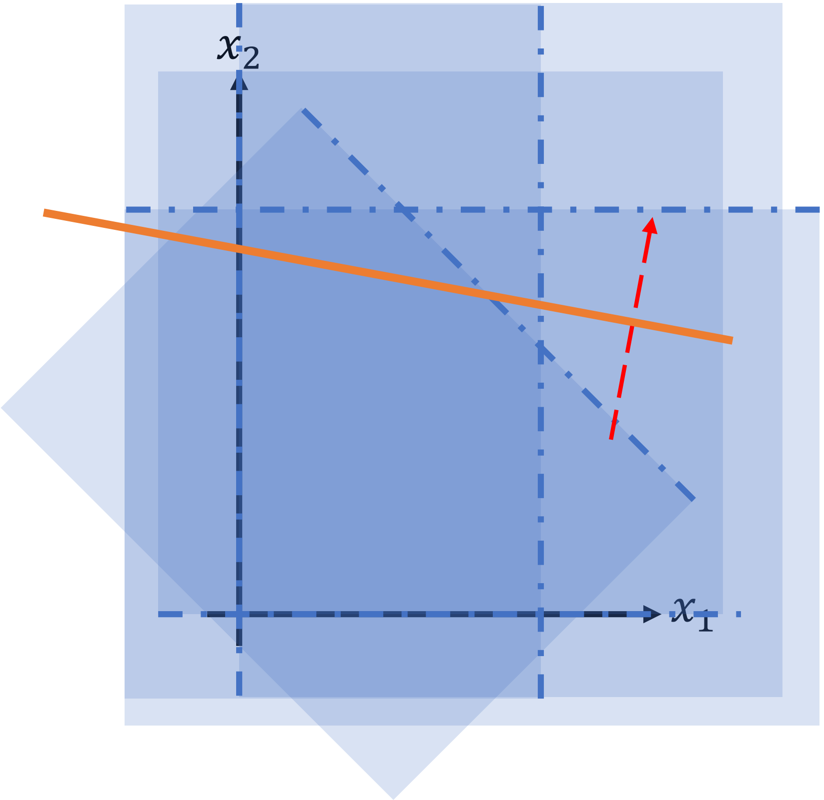

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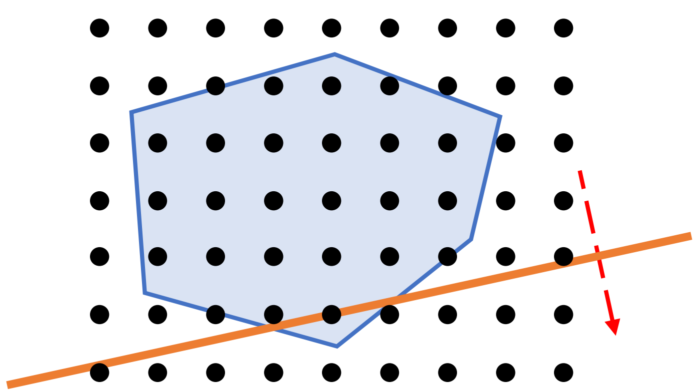

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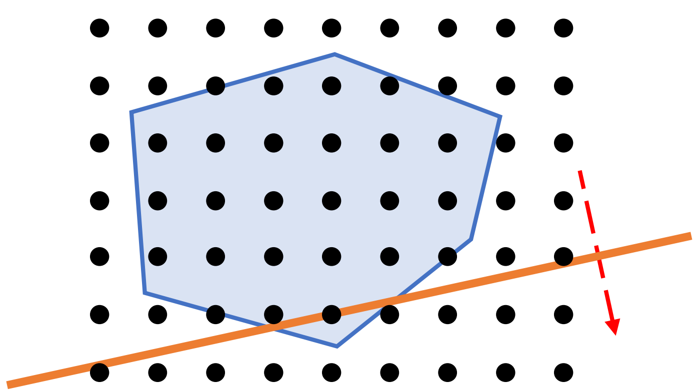
integer vectors $\vec{b}$ and $\vec{c}$

minimize $\vec{c}^T\vec{x}$

subject to $A\vec{x} \geq \vec{b}$

$\vec{x} \geq 0$

$\vec{x} \in \mathbb{Z}^n$



LP is polynomial time solvable,
ILP is NP-hard.

Vertex Cover

Instance: An undirected simple graph $G = (V, E)$.
Vertex Cover: Find smallest $C \subseteq V$ s.t. $\forall e \in E: e \cap C \neq \emptyset$.

Instance of set cover
with frequency = 2
for all elements.

Vertex Cover

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- This problem is NP-hard.
- $\ln n$-approx. alg. by greedy set cover.
- 2-approx. alg. by finding *maximal* matching.
- Assuming the *unique game conjecture*, there is no poly-time $(2 - \epsilon)$-approx. alg.

Vertex Cover as ILP

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LP Relaxation for Vertex Cover

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~~$x_v \in \{0,1\}, \quad v \in V$~~

$x_v \in [0,1], \quad v \in V$

LP is poly-time solvable,
so we can solve LP relaxation of vertex cover in poly-time.

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<u>VC as ILP:</u>

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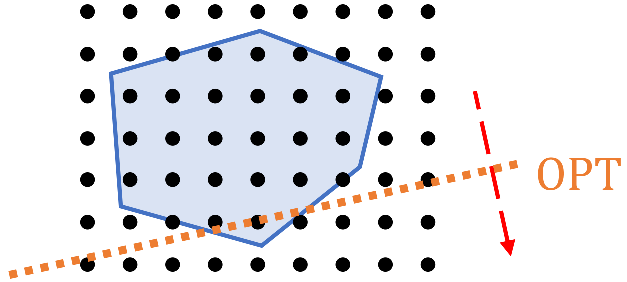

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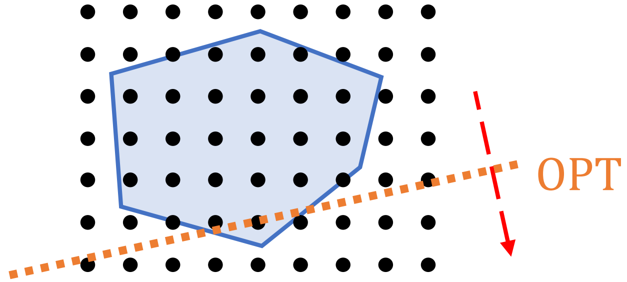

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fractional and optimal solution
$\overrightarrow{x^*} \in [0,1]^{|V|}$
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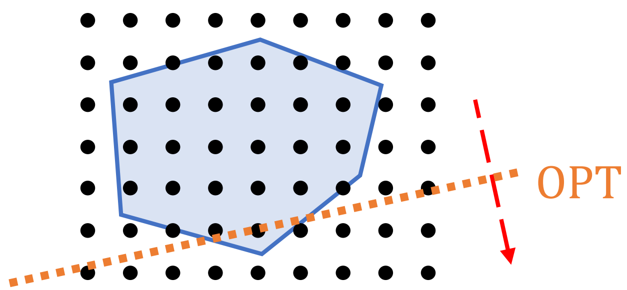

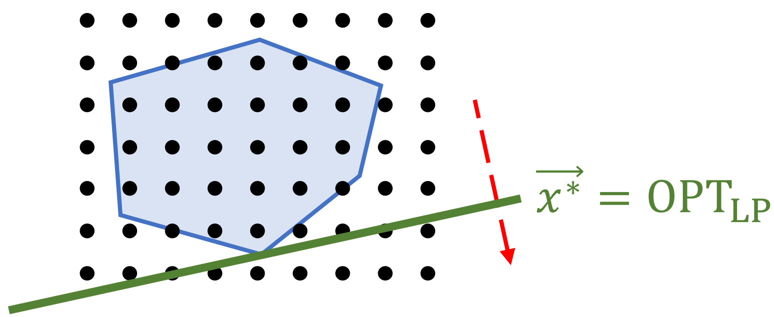

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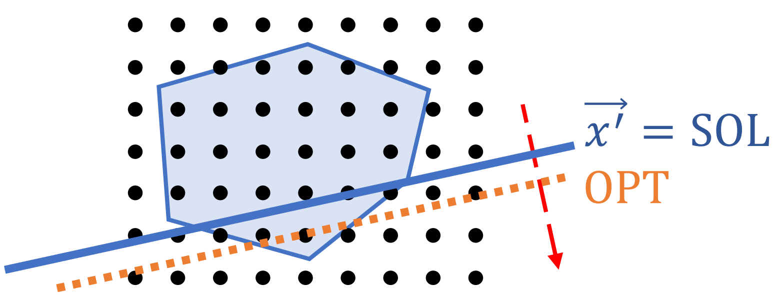

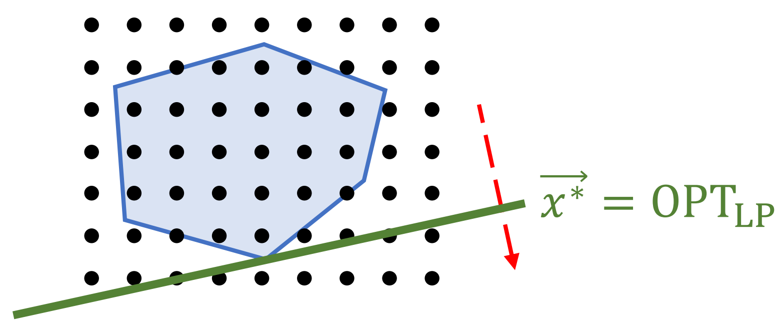

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rounding

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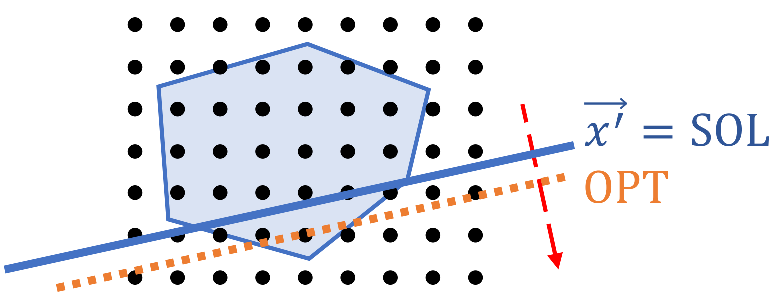

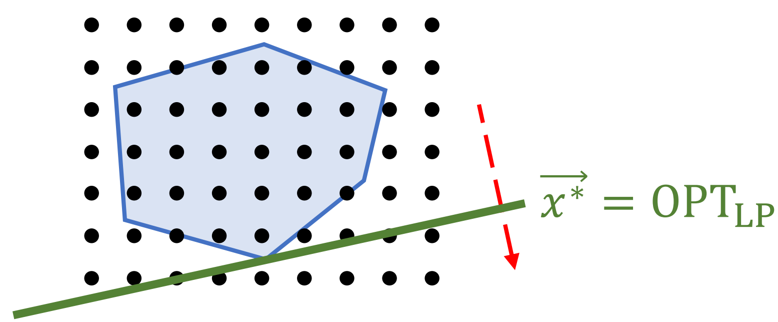

Relaxation & Rounding

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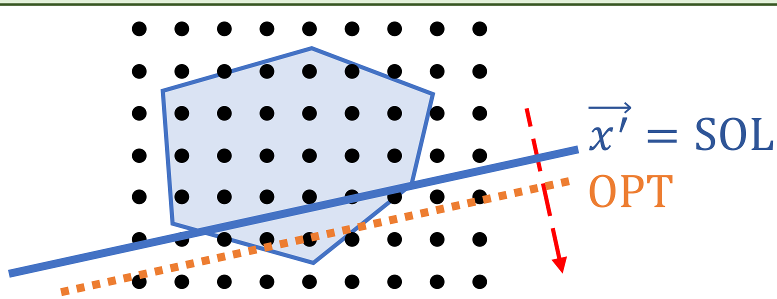

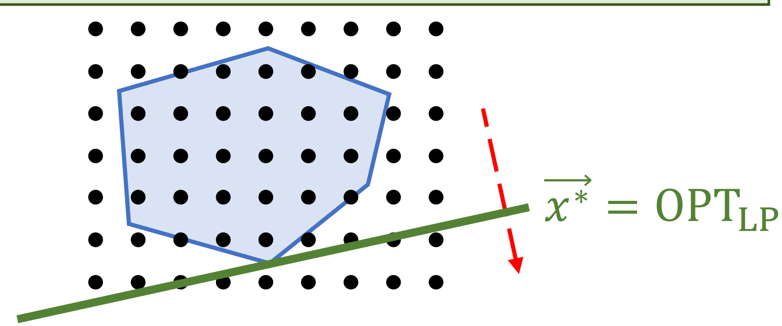

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Thus $\overrightarrow{x'}$ is an integral feasible solution!

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$$\text{OPT} \geq \text{OPT}_{\text{LP}} = \sum_{v \in V} x^*_v$$
$$\text{SOL} = \sum_{v \in V} x'_v \leq \sum_{v \in V} 2 \cdot x^*_v \leq 2 \cdot \text{OPT}$$

Relaxation & Rounding

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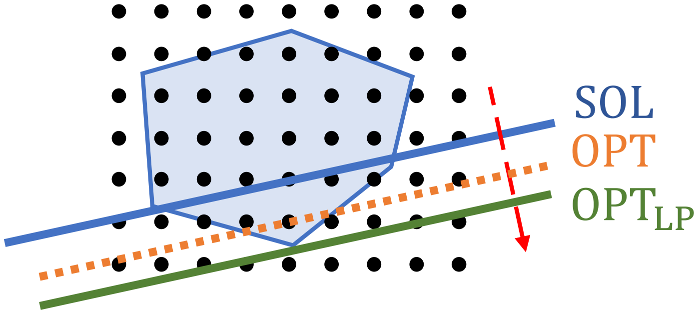
This is a poly-time 2-approx. alg. for VC.

LP Relaxation & Rounding

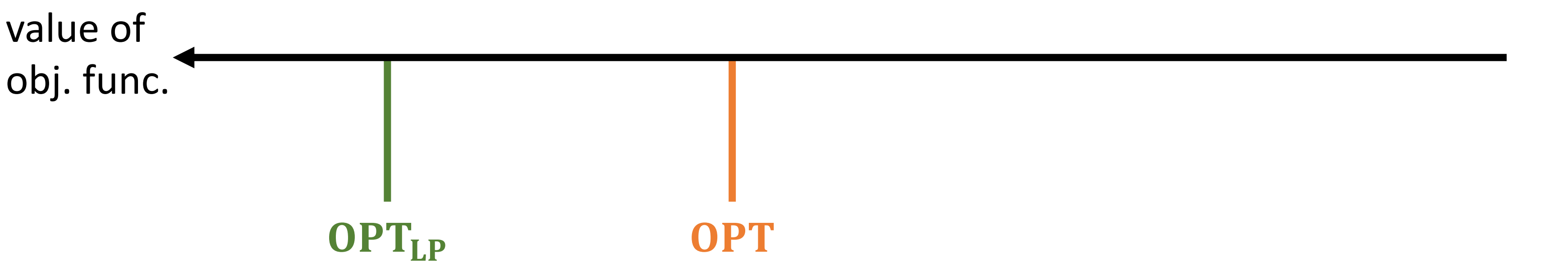
- Model the problem as an ILP.
  (Finding **OPT** of this ILP will be NP-hard.)
- Relaxation: relax the ILP to an LP.
- Find *optimal fraction* solution of the LP, call it $\mathbf{OPT_{LP}}$.
  (Can be done in poly-time via ellipsoid, interior-point, etc.)
- Rounding: round $\mathbf{OPT_{LP}}$ to a *feasible integral* solution **SOL**.
  (This is a tricky step: *how* to do rounding?)
- Show **SOL** is not far from **OPT**.
  (Notice $\mathbf{OPT_{LP}}$ provides a natural lower bound for **OPT**.)
  (Thus usually compare **SOL** with $\mathbf{OPT_{LP}}$.)



Integrality Gap

Consider a problem that can be modeled as a (minimization) ILP

Integrality Gap



Consider a problem that can be modeled as a (minimization) ILP

$\mathbf{OPT}(I)$: optimal value (of ILP) on instance I

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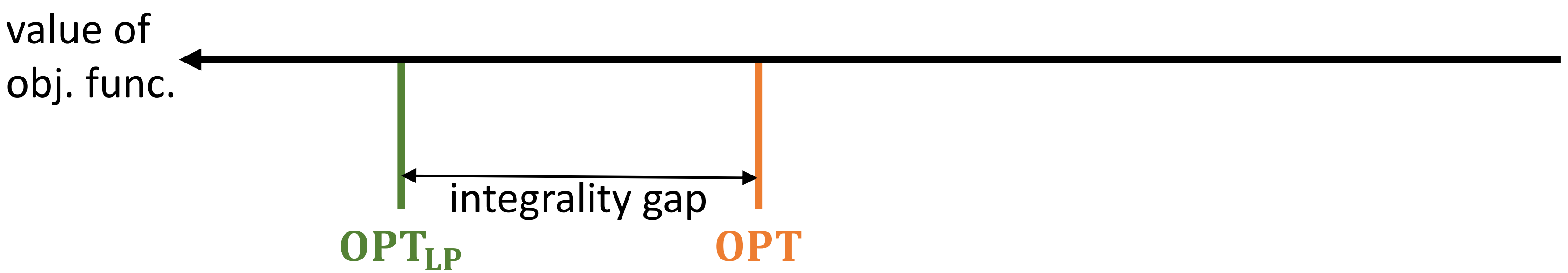

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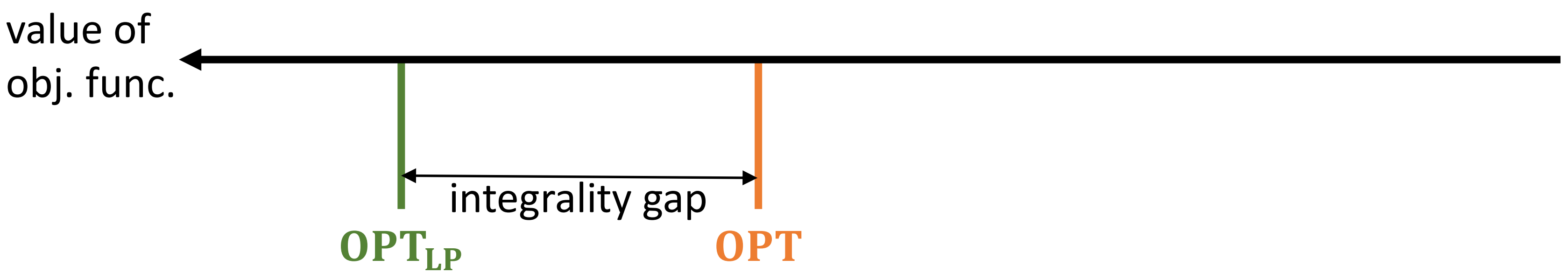

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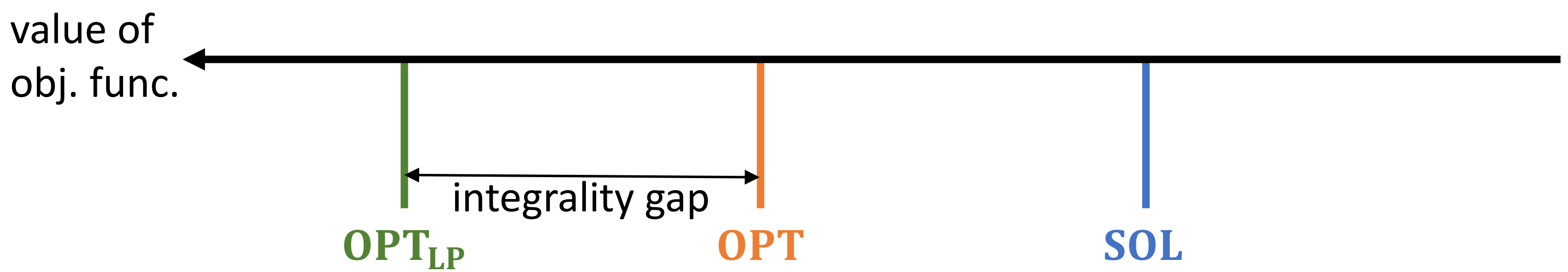

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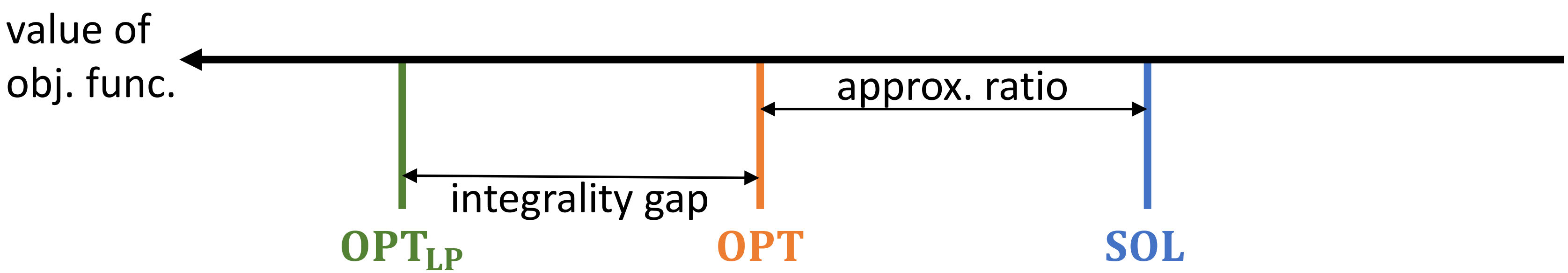

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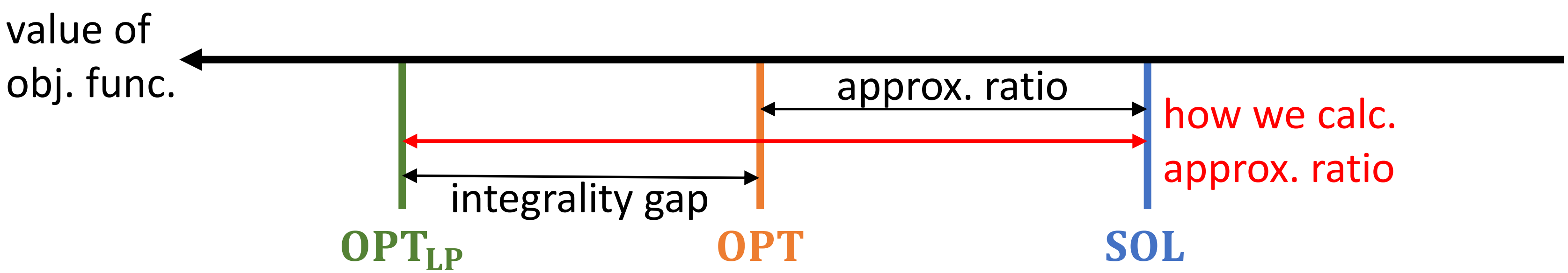

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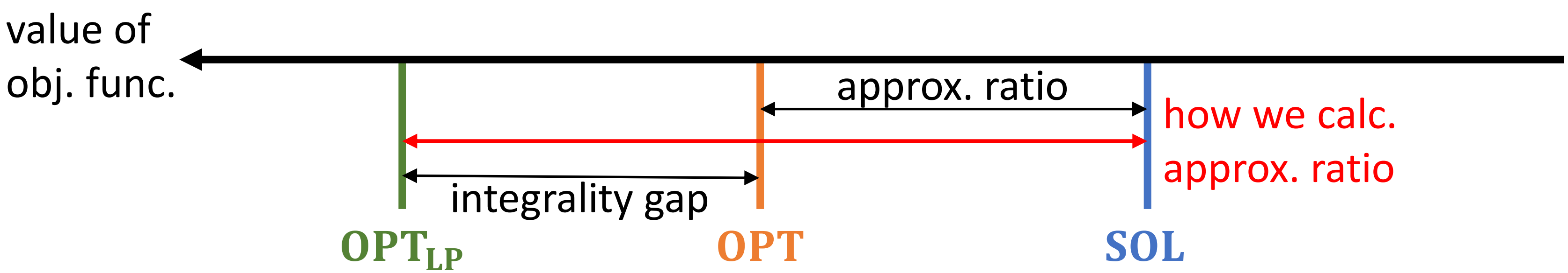

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Using LP relaxation & rounding, can approx. ratio beat integrality gap?

MAX-SAT

Instance: A set of clauses $C_1, C_2, \cdots, C_m$.
MAX-SAT: Find assignment $\vec{x} \in \{\text{true}, \text{false}\}^n$ that maximize number of *satisfied* clauses.

- Boolean variables: $x_1, x_2, \cdots, x_n$
- literal: x_i or $\overline{x_i}$
- clause: $\vee$ of literals

$$C_1 = (x_1 \vee \overline{x_2} \vee \overline{x_3})$$

$$C_2 = (x_1 \vee x_4)$$

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$$C_4 = (x_3)$$

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MAX-SAT is NP-hard, even MAX-E2SAT is NP-hard!
(Recall 2SAT is in P, and 3SAT is NP-hard.)

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$$\Pr[\text{clause } C \text{ is satisfied}] = 1 - 2^{-k} \geq \frac{1}{2}$$

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Does this imply a (1/2)-approx. alg. for MAX-SAT?

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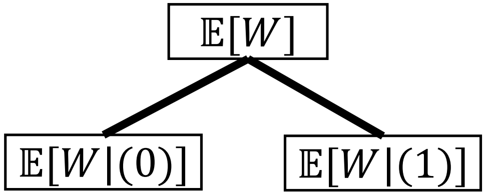

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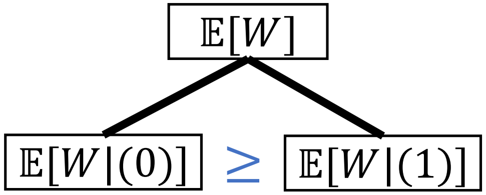

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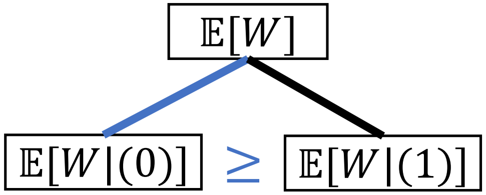

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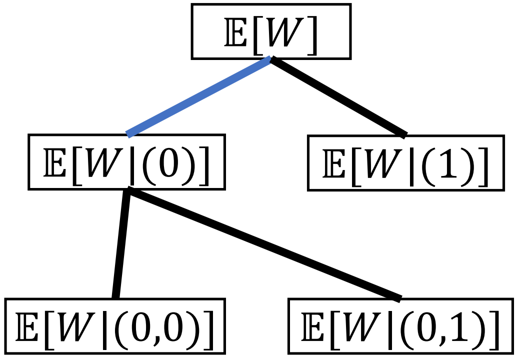

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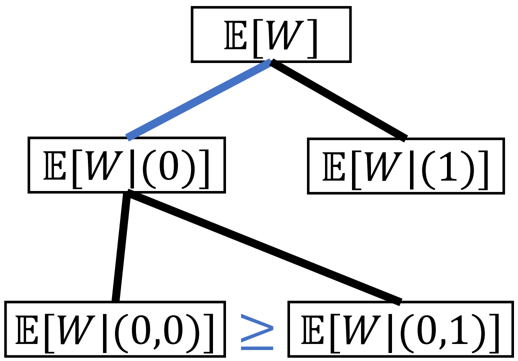

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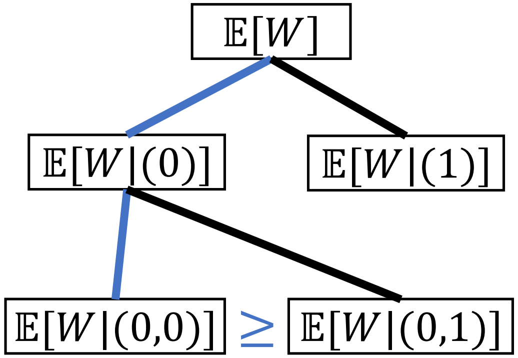

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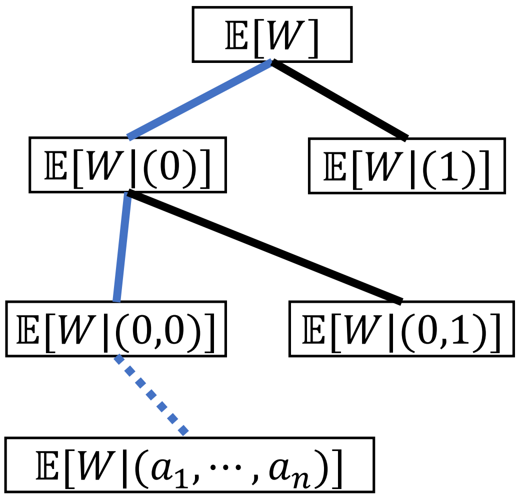

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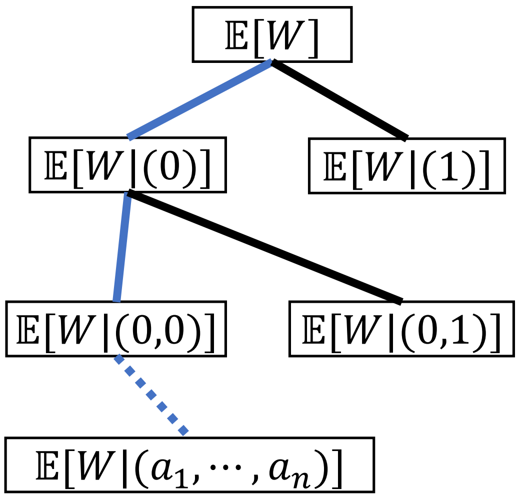

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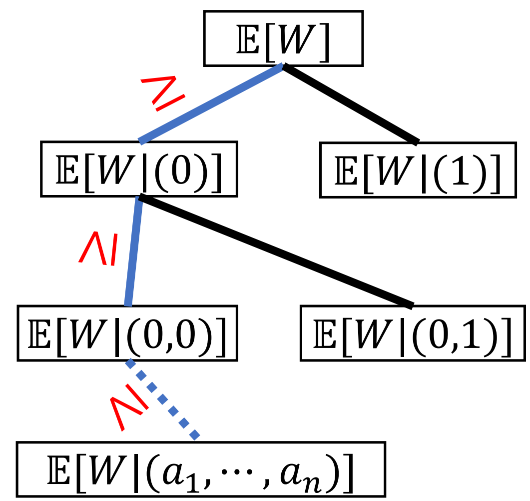

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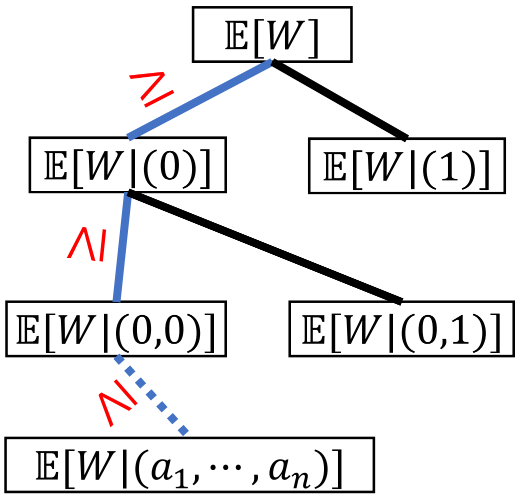

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For any partial assignment, $\mathbb{E}[W | (a_1, \cdots, a_i)]$ can be computed in poly-time.

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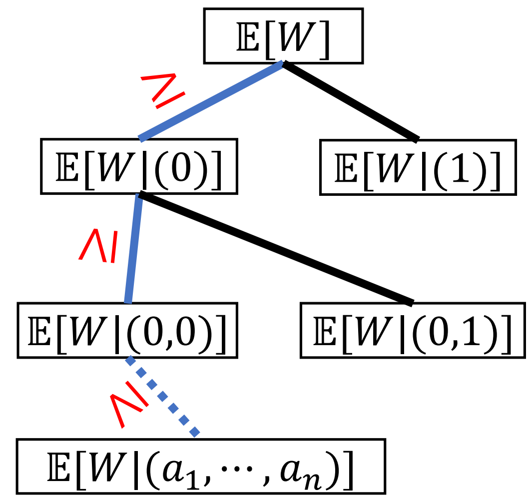
Poly-time deterministic (1/2)-approx. algorithm!



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MAX-SAT as ILP

Instance: A set of clauses $C_1, C_2, \cdots, C_m$.
MAX-SAT: Find assignment $\vec{x} \in \{\text{true}, \text{false}\}^n$ that maximize number of satisfied clauses.

Boolean variables: $x_1, x_2, \cdots, x_n \in \{0,1\}$
$x_i = 0$ means $x_i = \text{false}$
$x_i = 1$ means $x_i = \text{true}$

clause: $C_j = (l_1 \vee l_2 \vee \cdots \vee l_k)$ where $l_r \in \{x_i, \overline{x_i} \mid 1 \leq i \leq n\}$

C_j is satisfied

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S_j^+: set of i such that x_i is in C_j
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S_j^-: set of i such that $\overline{x_i}$ is in C_j

C_j is satisfied $\Longleftrightarrow$ $$\sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq 1$$

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MAX-SAT as ILP:

maximize $\sum_{j=1}^{m} y_j$

subject to

$$\sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j, \quad 1 \leq j \leq m$$

$$x_i \in \{0,1\}, \quad 1 \leq i \leq n$$

$$y_j \in \{0,1\}, \quad 1 \leq j \leq m$$

LP Relaxation

Instance: A set of clauses $C_1, C_2, \cdots, C_m$.
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Boolean variables: $x_1, x_2, \cdots, x_n \in \{0,1\}$

clause: $C_1, C_2, \cdots, C_m$

S_j^+: set of i such that x_i is in C_j

S_j^-: set of i such that $\overline{x_i}$ is in C_j

<u>LP relaxation of MAX-SAT-ILP:</u>

maximize $\sum_{j=1}^{m} y_j$

subject to $\sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j, \quad 1 \leq j \leq m$

~~$x_i \in \{0,1\}$~~, $x_i \in [0,1]$, $\quad 1 \leq i \leq n$

~~$y_j \in \{0,1\}$~~, $y_j \in [0,1]$, $\quad 1 \leq j \leq m$

Randomized Rounding

MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

s.t. $\sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \quad 1 \leq j \leq m$

$x_i \in \{0,1\} \quad 1 \leq i \leq n$

$y_j \in \{0,1\} \quad 1 \leq j \leq m$

LP relaxation of MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

s.t. $\sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \quad 1 \leq j \leq m$

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Randomized Rounding

MAX-SAT-ILP:

$$\max \sum_{j=1}^{m} y_j$$

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Optimal *integral* solution **OPT**

Randomized Rounding

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Optimal *integral* solution $\mathbf{OPT}$

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Optimal *fractional* solution $\mathbf{OPT_{LP}}$:

$\overrightarrow{x^*} \in [0,1]^n$, $\overrightarrow{y^*} \in [0,1]^m$

Randomized Rounding

MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

$$\text{s.t.} \quad \sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \qquad 1 \leq j \leq m$$

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Optimal *integral* solution $\mathbf{OPT}$

Clearly $\mathbf{OPT} \leq \mathbf{OPT_{LP}}$

Optimal *fractional* solution $\mathbf{OPT_{LP}}$:

$\overrightarrow{x^*} \in [0,1]^n$, $\overrightarrow{y^*} \in [0,1]^m$

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Randomly generate *integral* solution $\mathbf{SOL}$ from $\mathbf{OPT_{LP}}$:

$\overrightarrow{x'} \in \{0,1\}^n$, specifically, $x_i' = \begin{cases} 1 \text{ with probability } x_i^* \\ 0 \text{ with probability } 1 - x_i^* \end{cases}$

Randomized Rounding

MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

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Is $\mathbf{SOL}$ feasible?

$\overrightarrow{x'} \in \{0,1\}^n$, specifically, $x_i' = \begin{cases} 1 \text{ with probability } x_i^* \\ 0 \text{ with probability } 1 - x_i^* \end{cases}$

Randomized Rounding

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How good is $\mathbf{SOL}$?

Randomized Rounding

MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

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How good is $\mathbf{SOL}$?

$$\Pr[C_j \text{ satisfied in } \mathbf{SOL}] = 1 - \left(\prod_{i \in S_j^+} (1 - x_i^*)\right)\left(\prod_{i \in S_j^-} x_i^*\right)$$

Randomized Rounding

MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

$$\text{s.t.} \quad \sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \qquad 1 \leq j \leq m$$
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$$\Pr[C_j \text{ satisfied in } \mathbf{SOL}] = 1 - \left(\prod_{i \in S_j^+} (1 - x_i^*)\right)\left(\prod_{i \in S_j^-} x_i^*\right) \geq 1 - \left(1 - y_j^*/k\right)^k$$

Randomized Rounding

MAX-SAT-ILP:

$$\max \sum_{j=1}^{m} y_j$$

$$\text{s.t.} \quad \sum_{i\in S_j^+} x_i + \sum_{i\in S_j^-} (1-x_i) \geq y_j \qquad 1 \leq j \leq m$$

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$$\max \sum_{j=1}^{m} y_j$$

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$$x_i \in [0,1] \qquad 1 \leq i \leq n$$

$$y_j \in [0,1] \qquad 1 \leq j \leq m$$

Optimal *integral* solution $\mathbf{OPT}$

Optimal *fractional* solution $\mathbf{OPT_{LP}}$:

$^* \in [0,1]^m$

Clearly $\mathbf{OPT} \leq \mathbf{OPT}$

Randomly generate

arithmetic-geometric mean inequality:

for $a_1, a_2, \cdots, a_k \in \mathbb{R}^{\geq 0}$,

$(a_1 + \cdots + a_k)/k \geq (a_1 \times \cdots \times a_k)^{1/k}$

$\overrightarrow{x'} \in \{0,1\}^n$, specifically, $x_i' = \begin{cases} 1 \text{ with probability } x_i^* \\ 0 \text{ with probability } 1 - x_i^* \end{cases}$

How good is $\mathbf{SOL}$?

$$\Pr[C_j \text{ satisfied in } \mathbf{SOL}] = 1 - \left(\prod_{i\in S_j^+} (1-x_i^*)\right)\left(\prod_{i\in S_j^-} x_i^*\right) \geq 1 - \left(1 - y_j^*/k\right)^k$$

Randomized Rounding

MAX-SAT-ILP:

$$\max \sum_{j=1}^{m} y_j$$

$$\text{s.t.} \quad \sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \qquad 1 \leq j \leq m$$

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$$y_j \in \{0,1\} \qquad 1 \leq j \leq m$$

LP relaxation of MAX-SAT-ILP:

$$\max \sum_{j=1}^{m} y_j$$

$$\text{s.t.} \quad \sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \qquad 1 \leq j \leq m$$

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Optimal *integral* solution $\mathbf{OPT}$

Clearly $\mathbf{OPT} \leq \mathbf{OPT_{LP}}$

Optimal *fractional* solution $\mathbf{OPT_{LP}}$:

$\overrightarrow{x^*} \in [0,1]^n$, $\overrightarrow{y^*} \in [0,1]^m$

Randomly generate *integral* solution $\mathbf{SOL}$ from $\mathbf{OPT_{LP}}$:

$\overrightarrow{x'} \in \{0,1\}^n$, specifically, $x_i' = \begin{cases} 1 \text{ with probability } x_i^* \\ 0 \text{ with probability } 1 - x_i^* \end{cases}$

How good is $\mathbf{SOL}$?

$$\Pr[C_j \text{ satisfied in } \mathbf{SOL}] = 1 - \left(\prod_{i \in S_j^+} (1 - x_i^*)\right)\left(\prod_{i \in S_j^-} x_i^*\right) \geq 1 - \left(1 - y_j^*/k\right)^k$$

$$\geq [1 - (1 - 1/k)^k] \cdot y_j^*$$

Randomized Rounding

MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

s.t. $\sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \quad 1 \leq j \leq m$

$x_i \in \{0,1\} \quad 1 \leq i \leq n$

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LP relaxation of MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

s.t. $\sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \quad 1 \leq j \leq m$

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Optimal *integral* solution $\mathbf{OPT}$

Clearly $\mathbf{OPT} \leq \mathbf{OPT_{LP}}$

Optimal *fractional* solution $\mathbf{OPT_{LP}}$:

$\overrightarrow{x^*} \in [0,1]^n$, $\overrightarrow{y^*} \in [0,1]^m$

Randomly generate *integral* solution $\mathbf{SOL}$ from $\mathbf{OPT_{LP}}$:

$\overrightarrow{x'} \in \{0,1\}^n$, specifically, $x_i' = \begin{cases} 1 \text{ with probability } x_i^* \\ 0 \text{ with probability } 1 - x_i^* \end{cases}$

How good is $\mathbf{SOL}$?

$$\Pr[C_j \text{ satisfied in } \mathbf{SOL}] = 1 - \left(\prod_{i \in S_j^+} (1 - x_i^*)\right)\left(\prod_{i \in S_j^-} x_i^*\right) \geq 1 - \left(1 - y_j^*/k\right)^k$$

$$\geq [1 - (1 - 1/k)^k] \cdot y_j^*$$

$f(y_j^*) = 1 - \left(1 - y_j^*/k\right)^k$ is concave when $y_j^* \in [0,1]$

Randomized Rounding

MAX-SAT-ILP:

$$\max \sum_{j=1}^{m} y_j$$

$$\text{s.t.} \quad \sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \qquad 1 \leq j \leq m$$

$$x_i \in \{0,1\} \qquad 1 \leq i \leq n$$

$$y_j \in \{0,1\} \qquad 1 \leq j \leq m$$

LP relaxation of MAX-SAT-ILP:

$$\max \sum_{j=1}^{m} y_j$$

$$\text{s.t.} \quad \sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \qquad 1 \leq j \leq m$$

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Optimal *integral* solution $\mathbf{OPT}$

Clearly $\mathbf{OPT} \leq \mathbf{OPT_{LP}}$

Optimal *fractional* solution $\mathbf{OPT_{LP}}$:

$\overrightarrow{x^*} \in [0,1]^n$, $\overrightarrow{y^*} \in [0,1]^m$

Randomly generate *integral* solution $\mathbf{SOL}$ from $\mathbf{OPT_{LP}}$:

$\overrightarrow{x'} \in \{0,1\}^n$, specifically, $x'_i = \begin{cases} 1 \text{ with probability } x_i^* \\ 0 \text{ with probability } 1 - x_i^* \end{cases}$

How good is $\mathbf{SOL}$?

$$\Pr[C_j \text{ satisfied in } \mathbf{SOL}] = 1 - \left(\prod_{i \in S_j^+} (1 - x_i^*)\right)\left(\prod_{i \in S_j^-} x_i^*\right) \geq 1 - \left(1 - y_j^*/k\right)^k$$

$$\geq [1 - (1 - 1/k)^k] \cdot y_j^*$$

$$\geq (1 - 1/e) \cdot y_j^*$$

Randomized Rounding

MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

$$\text{s.t.} \quad \sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \qquad 1 \leq j \leq m$$
$$x_i \in \{0,1\} \qquad 1 \leq i \leq n$$
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LP relaxation of MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

$$\text{s.t.} \quad \sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \qquad 1 \leq j \leq m$$
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Optimal *integral* solution $\mathbf{OPT}$

Clearly $\mathbf{OPT} \leq \mathbf{OPT_{LP}}$

Optimal *fractional* solution $\mathbf{OPT_{LP}}$:

$\overrightarrow{x^*} \in [0,1]^n$, $\overrightarrow{y^*} \in [0,1]^m$

Randomly generate *integral* solution $\mathbf{SOL}$ from $\mathbf{OPT_{LP}}$:

$\overrightarrow{x'} \in \{0,1\}^n$, specifically, $x_i' = \begin{cases} 1 \text{ with probability } x_i^* \\ 0 \text{ with probability } 1 - x_i^* \end{cases}$

How good is $\mathbf{SOL}$?

$$\Pr[C_j \text{ satisfied in } \mathbf{SOL}] \geq (1 - 1/e) \cdot y_j^*$$

Randomized Rounding

MAX-SAT-ILP:

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max $\sum_{j=1}^{m} y_j$

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Optimal *integral* solution $\mathbf{OPT}$

Clearly $\mathbf{OPT} \leq \mathbf{OPT_{LP}}$

Optimal *fractional* solution $\mathbf{OPT_{LP}}$:

$\overrightarrow{x^*} \in [0,1]^n$, $\overrightarrow{y^*} \in [0,1]^m$

Randomly generate *integral* solution $\mathbf{SOL}$ from $\mathbf{OPT_{LP}}$:

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How good is $\mathbf{SOL}$?

$$\Pr[C_j \text{ satisfied in } \mathbf{SOL}] \geq (1 - 1/e) \cdot y_j^*$$

$$\mathbb{E}[\#\text{ clauses satisfied in } \mathbf{SOL}] \geq \sum_{j=1}^{m} \left(1 - \frac{1}{e}\right) \cdot y_j^*$$

$$= \left(1 - \frac{1}{e}\right) \cdot \mathbf{OPT_{LP}} \geq \left(1 - \frac{1}{e}\right) \cdot \mathbf{OPT}$$

Randomized Rounding

MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

s.t. $\sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \quad 1 \leq j \leq m$

$x_i \in \{0,1\} \quad 1 \leq i \leq n$

$y_j \in \{0,1\} \quad 1 \leq j \leq m$

LP relaxation of MAX-SAT-ILP:

max $\sum_{j=1}^{m} y_j$

s.t. $\sum_{i \in S_j^+} x_i + \sum_{i \in S_j^-} (1 - x_i) \geq y_j \quad 1 \leq j \leq m$

$x_i \in [0,1] \quad 1 \leq i \leq n$

$y_j \in [0,1] \quad 1 \leq j \leq m$

Optimal *integral* solution $\mathbf{OPT}$

Clearly $\mathbf{OPT} \leq \mathbf{OPT_{LP}}$

Optimal *fractional* solution $\mathbf{OPT_{LP}}$:

$\overrightarrow{x^*} \in [0,1]^n$, $\overrightarrow{y^*} \in [0,1]^m$

Randomly generate *integral* solution $\mathbf{SOL}$ from $\mathbf{OPT_{LP}}$:

$\overrightarrow{x'} \in \{0,1\}^n$, specifically, $x_i' = \begin{cases} 1 \text{ with probability } x_i^* \\ 0 \text{ with probability } 1 - x_i^* \end{cases}$

$\Pr[C_j \text{ satisfied in } \mathbf{SOL}] \geq (1 - 1/e) \cdot y_j^*$

$\mathbb{E}[\# \text{ clauses satisfied in } \mathbf{SOL}] \geq \sum_{j=1}^{m} \left(1 - \frac{1}{e}\right) \cdot y_j^*$

$= \left(1 - \frac{1}{e}\right) \cdot \mathbf{OPT_{LP}} \geq \left(1 - \frac{1}{e}\right) \cdot \mathbf{OPT}$

How good is $\mathbf{SOL}$?

Derandomize to get deterministic (1-1/e)-approx. alg.

Putting two algorithms together

Instance: A set of clauses $C_1, C_2, \cdots, C_m$.
MAX-SAT: Find $\vec{x} \in \{\text{true}, \text{false}\}^n$ that maximize # of satisfied clauses.

Assign each variable with true or false *uniformly* and *independently* at random.

Model problem as an ILP.
Obtain LP relaxation of the ILP.
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Suppose $C_j = (l_1 \vee l_2 \vee \cdots \vee l_k)$

- Random assignment:
  - 1/2-approximation
  - $\Pr[C_j \text{ satisfied}] = 1 - 2^{-k}$

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- LP relaxation and randomized rounding:
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- Random assignment:
  - 1/2-approximation
  - $\Pr[C_j \text{ satisfied}] = 1 - 2^{-k}$ Good when k is large.
- LP relaxation and randomized rounding:
  - (1-1/e)-approximation
  - $\Pr[C_j \text{ satisfied}] = [1 - (1 - 1/k)^k] \cdot y_j^*$ Good when k is small.

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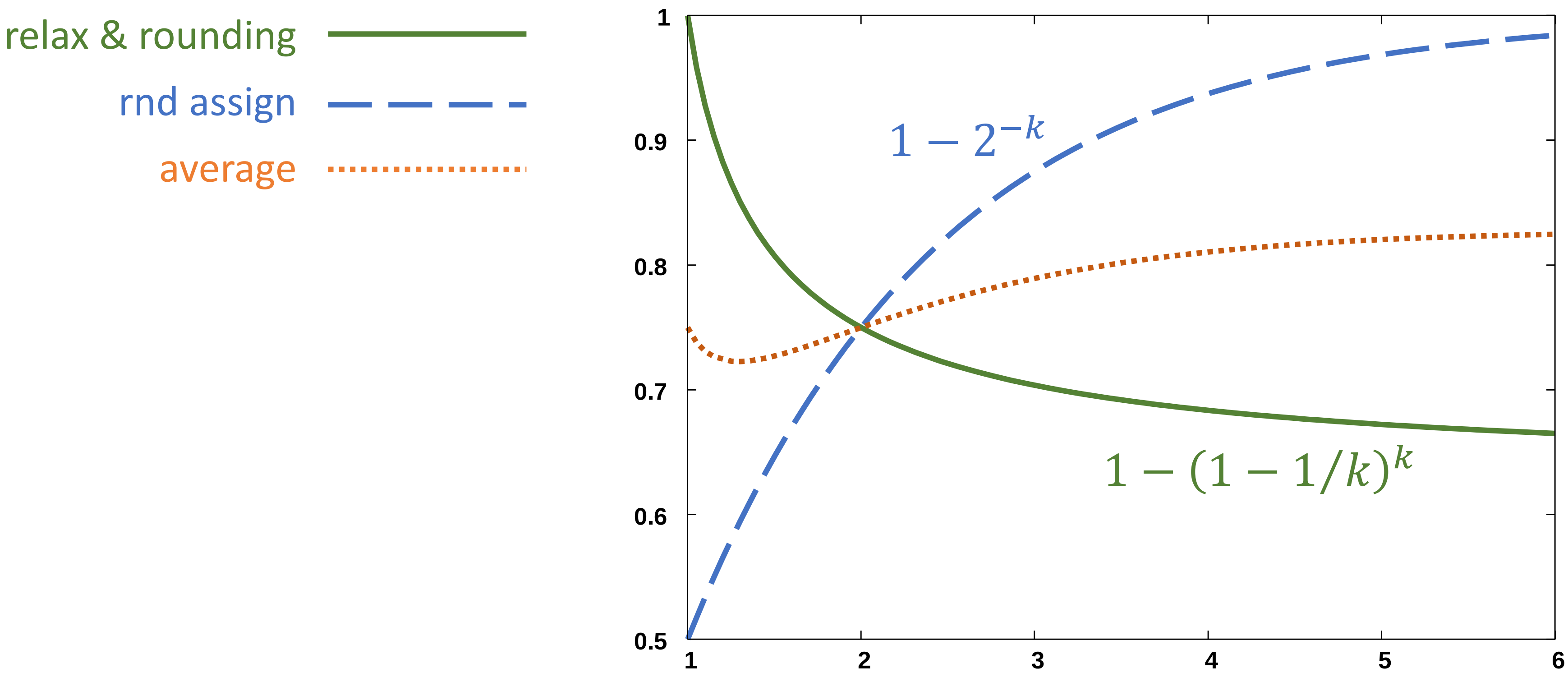

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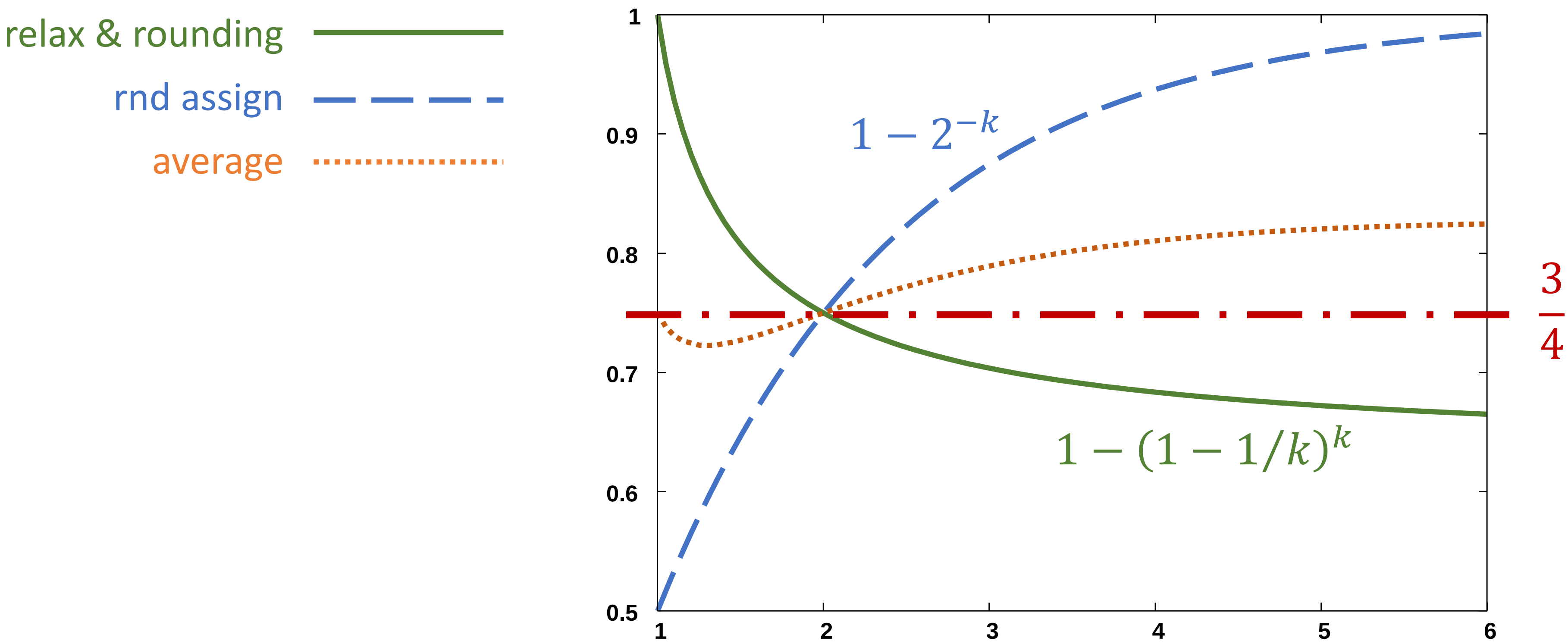

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Assume the solution of the LP relaxation & rounding alg. satisfies m_2 clauses.

$$\mathbb{E}[\#\text{ of satisfied clauses}] = \mathbb{E}[\max\{m_1, m_2\}] \geq \mathbb{E}\left[\frac{m_1 + m_2}{2}\right]$$

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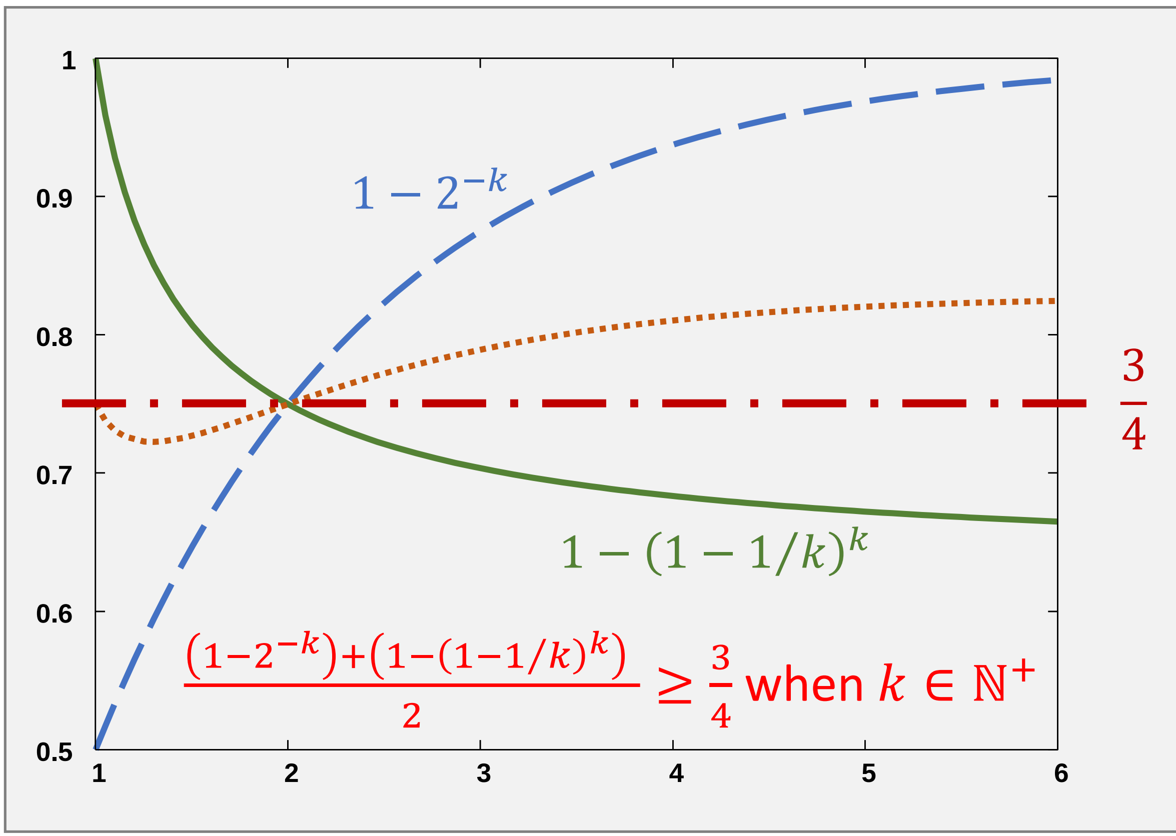

$_1$ clauses.

lg. satisfies m_2 clauses.

pt. sol. of relaxed-ILP

$$\frac{\quad + m_2}{2}\Big]$$

$$\left(1 - 1/k_j\right)^{k_j}\Big)\Big] \cdot \frac{y_j^*}{2}$$

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$$\geq \sum_{j=1}^{m} \frac{3}{4} \cdot y_j^* = \frac{3}{4} \cdot \text{OPT}_{\text{LP}} \geq \frac{3}{4} \cdot \text{OPT}$$

MAX-SAT

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- A rnd. alg. that satisfies at least $(3/4) \cdot \mathrm{OPT}$ clauses *in expectation*.
- Can derandomize above alg. via the method of conditional expectation.
- The integrality gap of the LP relaxation for MAX-SAT is $3/4$.
- MAX-3SAT has a $(7/8)$-approx. alg. by semidefinite programming, and cannot have better approx. alg. in poly-time unless P=NP.
- How about MAX-E3SAT?