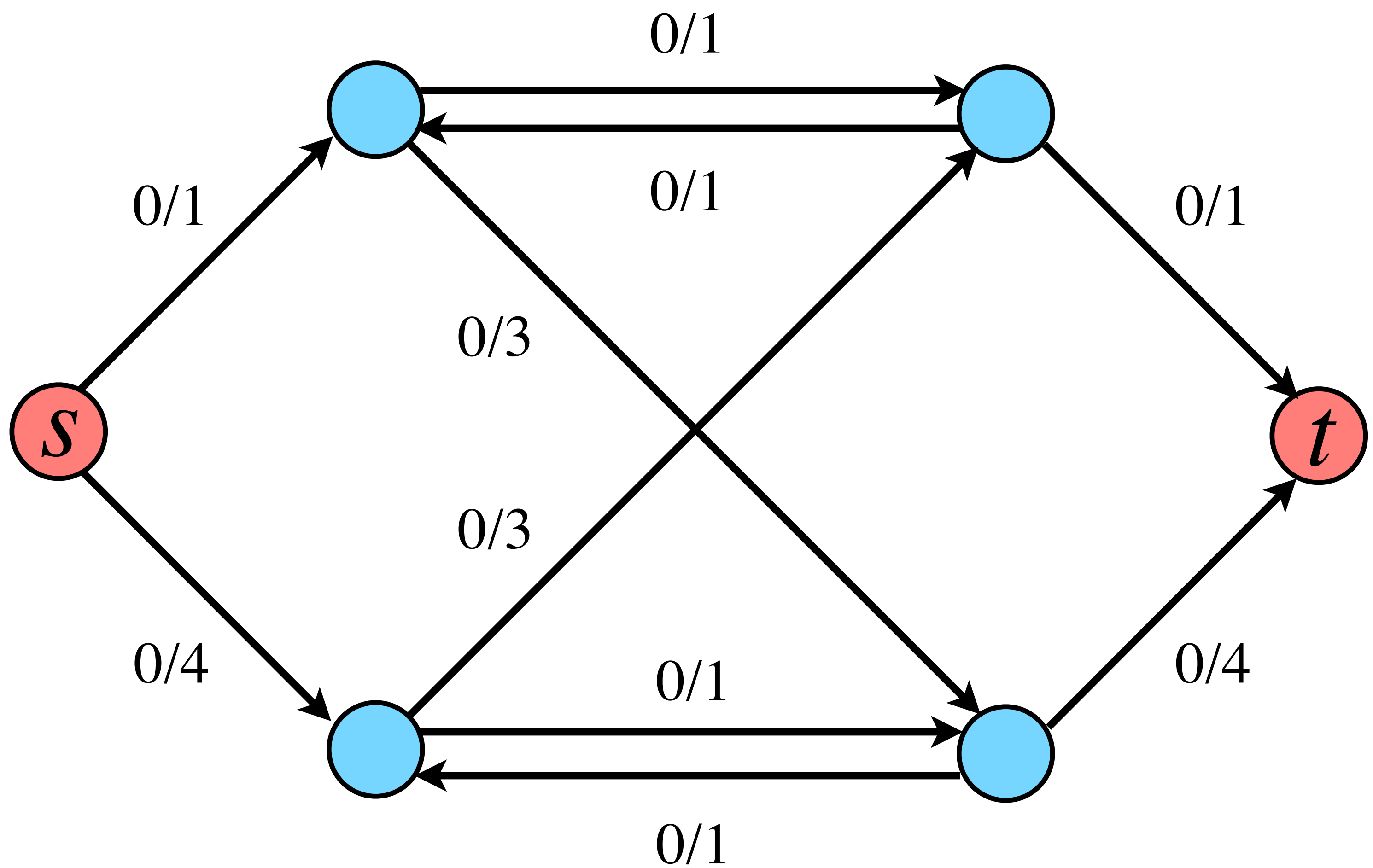
Advanced Algorithms

LP Duality

尹一通 Nanjing University, 2022 Fall

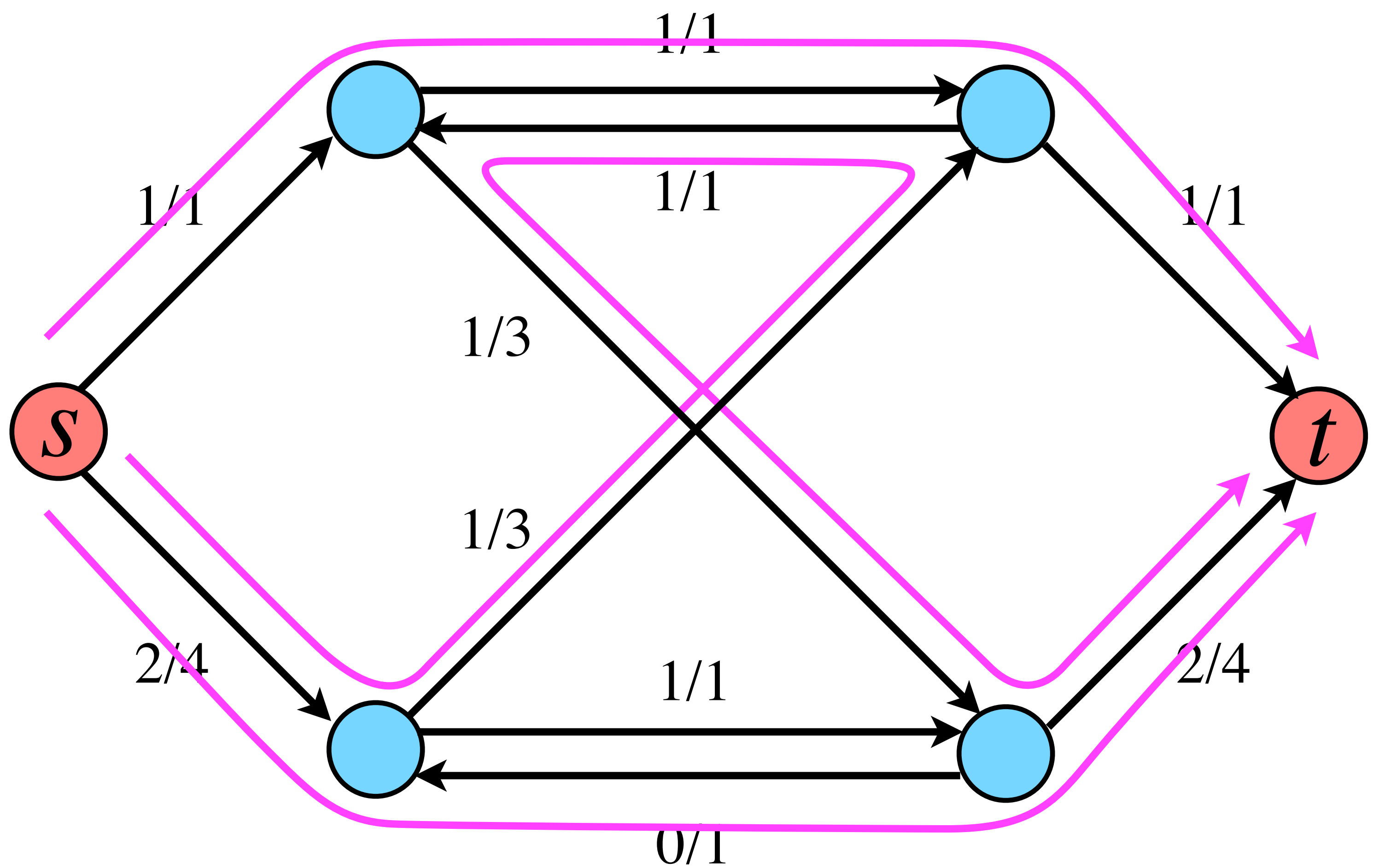
Flow Network

- Digraph: $D(V, E)$
- source: $s \in V$ sink: $t \in V$
- Capacity $c : E \to \mathbb{R}_{\geq 0}$



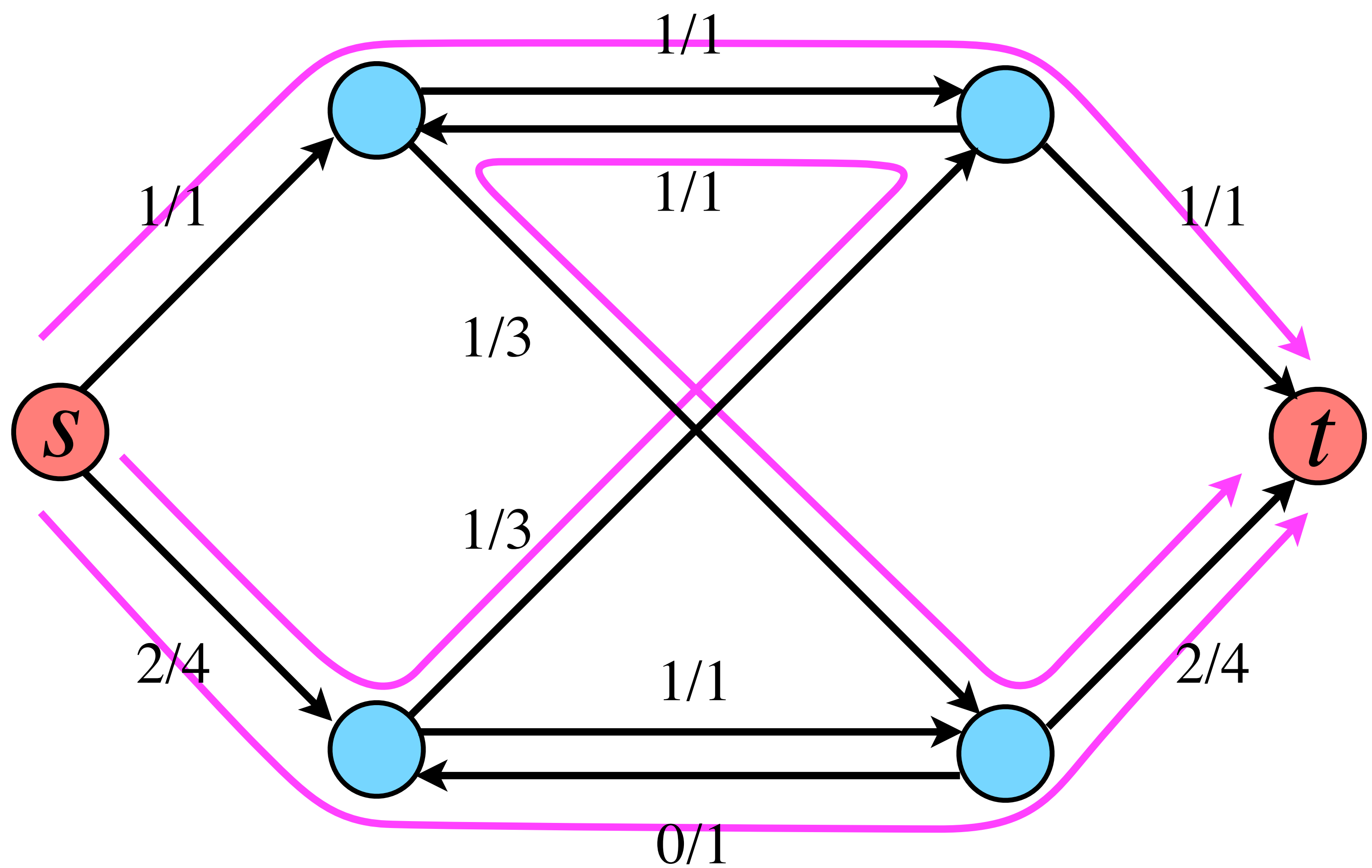
Network Flow

- Digraph: $D(V, E)$
- source: $s \in V$ sink: $t \in V$
- Capacity $c : E \to \mathbb{R}_{\geq 0}$
- Flow $f : E \to \mathbb{R}_{\geq 0}$



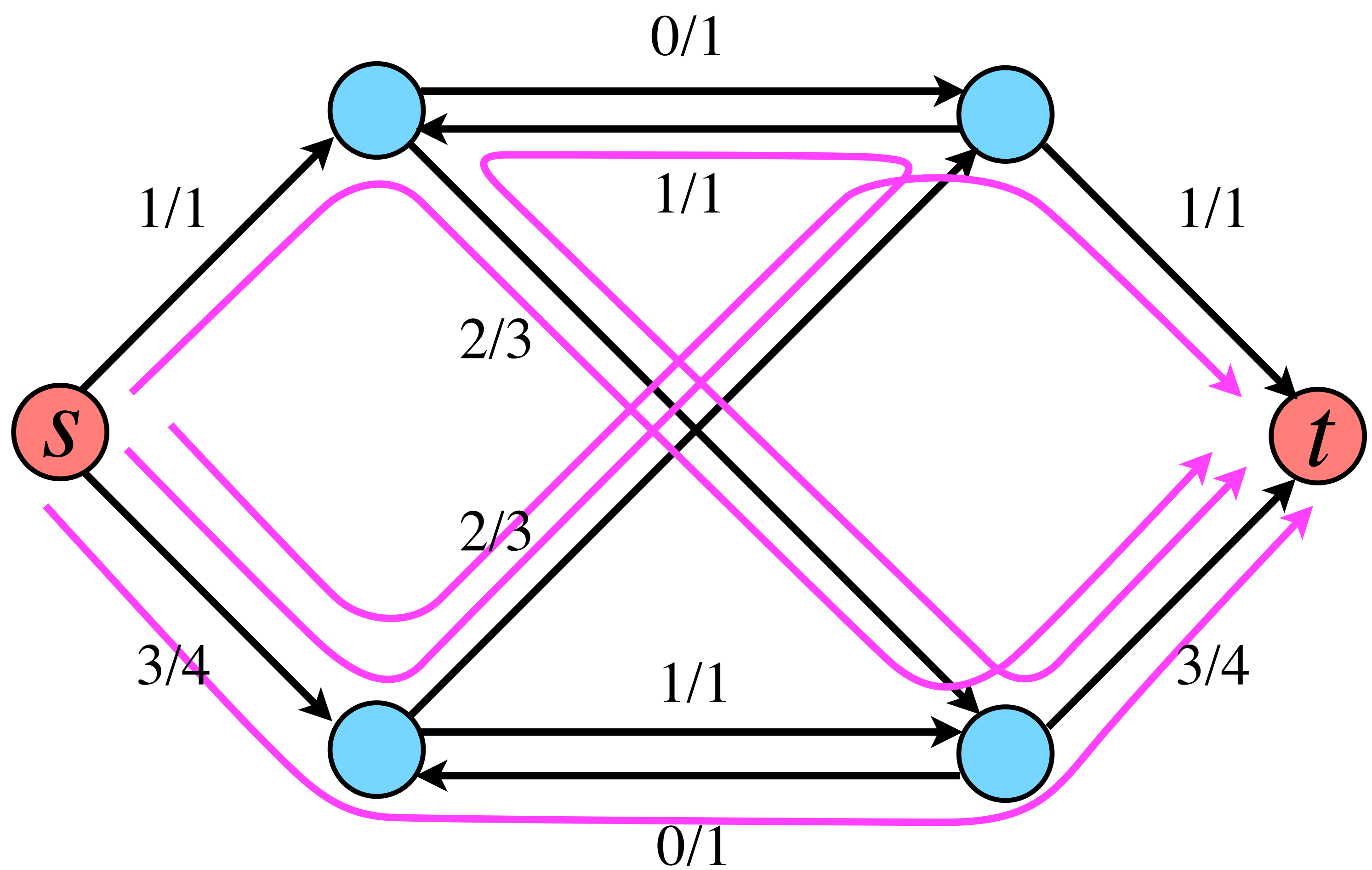
- Capacity: $\forall (u, v) \in E, \quad f_{uv} \leq c_{uv}$
- Conservation: $\forall u \in V \setminus \{s, t\}, \quad \sum_{(w,u) \in E} f_{wu} = \sum_{(u,v) \in E} f_{uv}$

Network Flow



- Capacity: $\forall (u, v) \in E, \quad f_{uv} \leq c_{uv}$
- Conservation: $\forall u \in V \backslash \{s, t\}, \quad \sum_{(w,u) \in E} f_{wu} = \sum_{(u,v) \in E} f_{uv}$
- Value of flow: $$\sum_{(s,u) \in E} f_{su} = \sum_{(v,t) \in E} f_{vt}$$

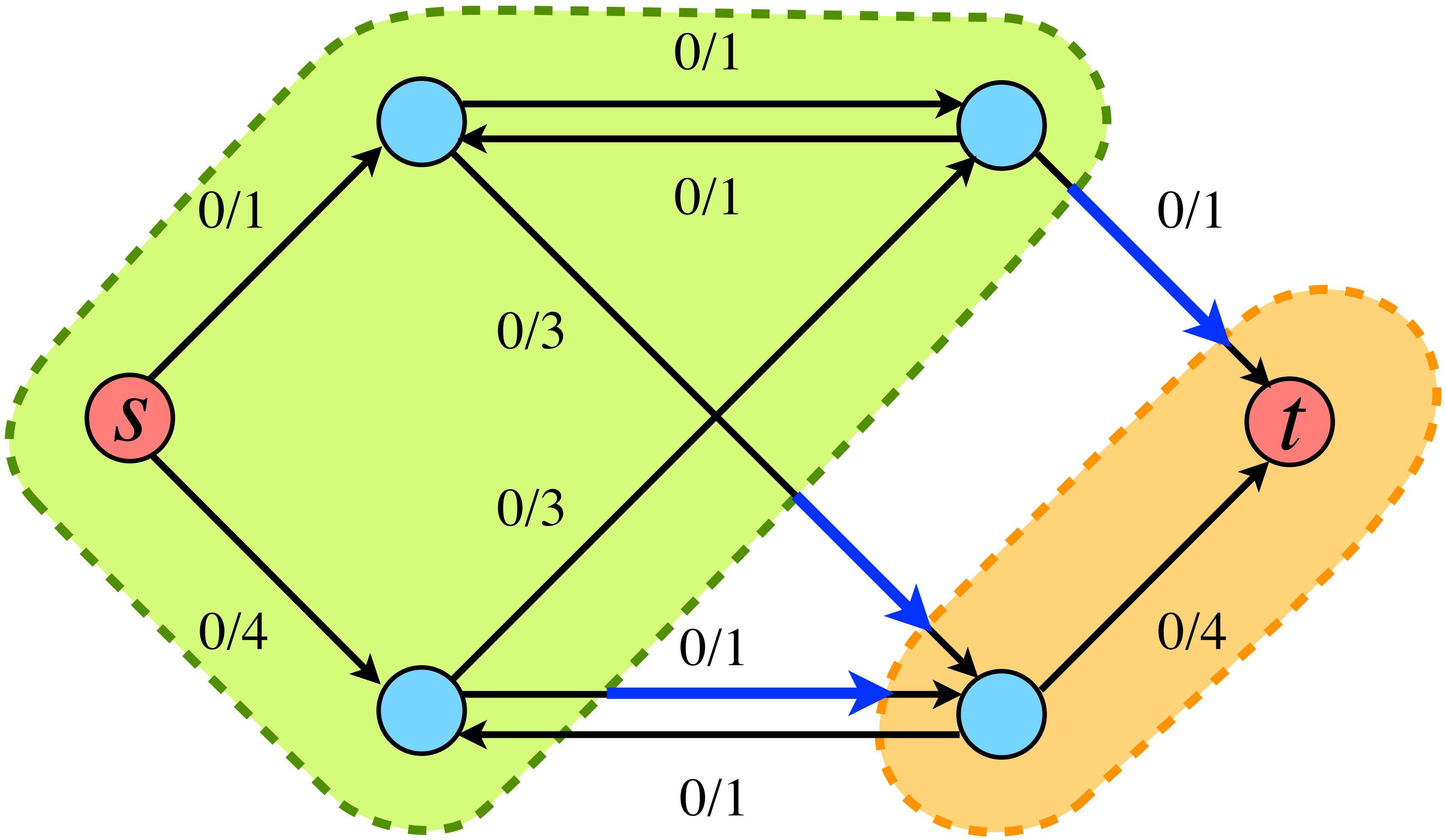
Maximum Flow



- **Capacity**: $\forall (u,v) \in E, \quad f_{uv} \leq c_{uv}$
- **Conservation**: $\forall u \in V \setminus \{s,t\}, \quad \sum_{(w,u)\in E} f_{wu} = \sum_{(u,v)\in E} f_{uv}$
- **Value of flow**: $$\sum_{(s,u)\in E} f_{su} = \sum_{(v,t)\in E} f_{vt}$$

Cut

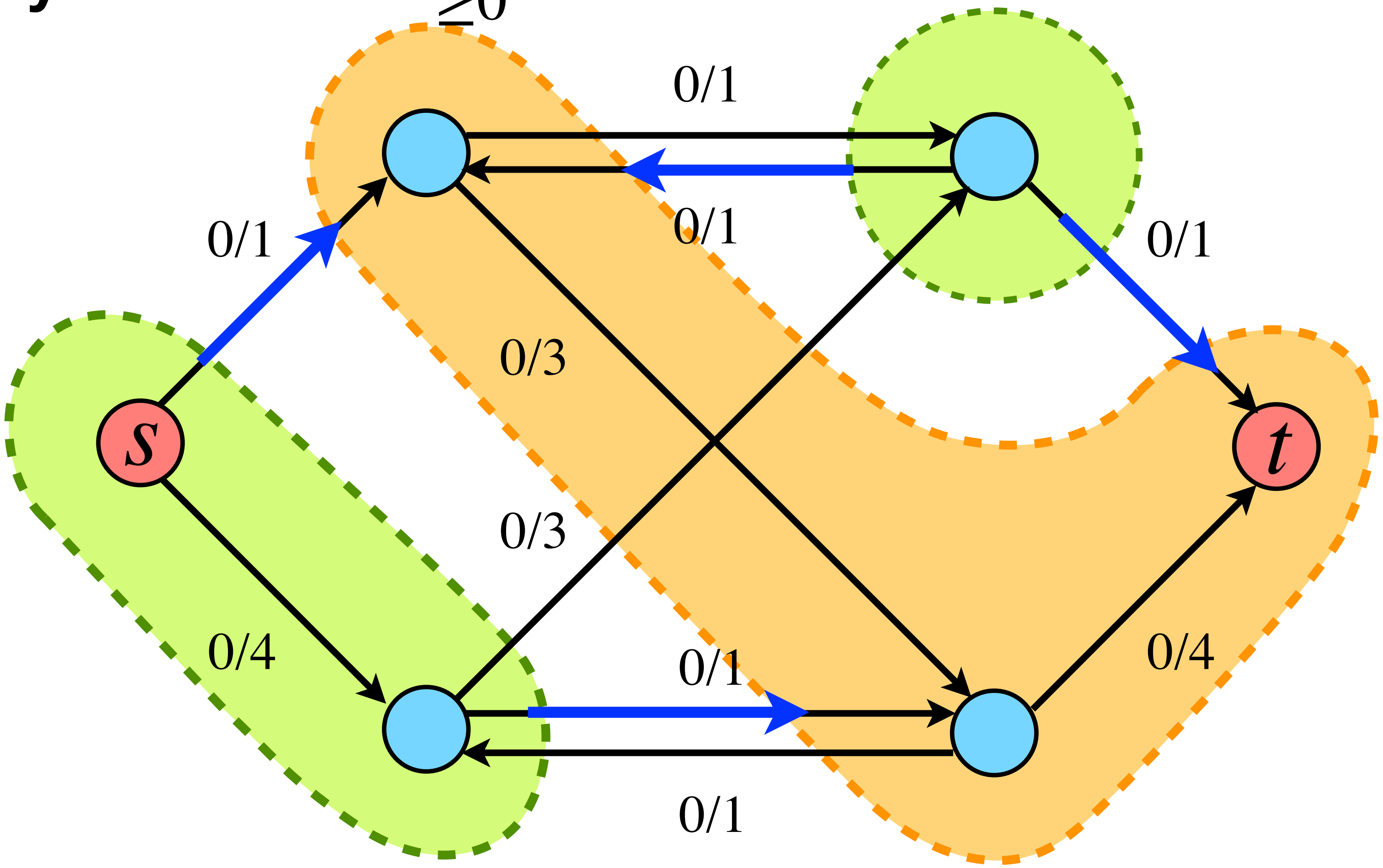
- Digraph: $D(V, E)$
- source: $s \in V$ sink: $t \in V$
- Capacity $c : E \to \mathbb{R}_{\geq 0}$
- Cut $S \subset V$, $s \in S, t \notin S$



- Value of cut: $$\sum_{u \in S, v \notin S, (u,v) \in E} c_{uv}$$

Minimum Cut

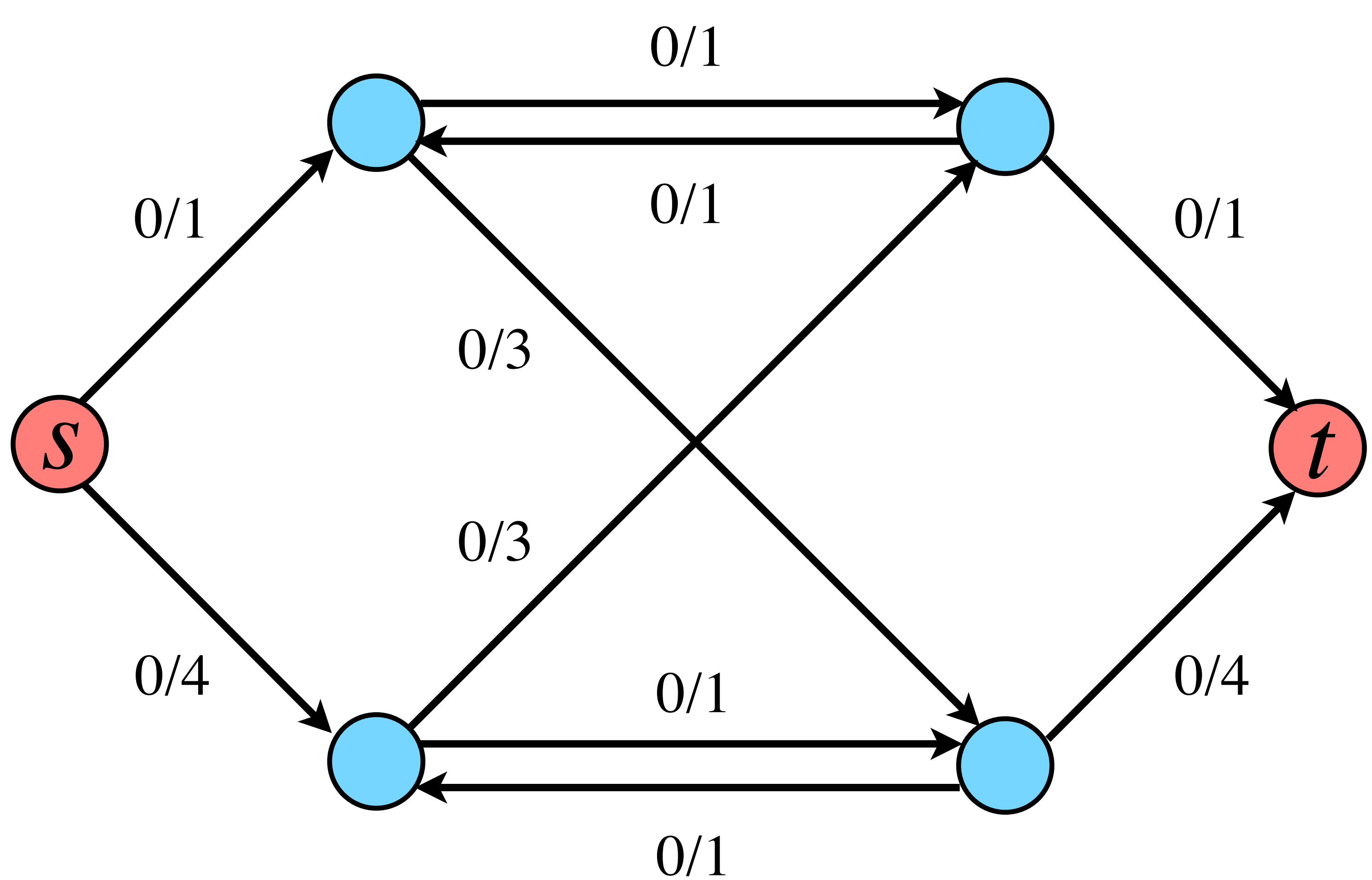
- Digraph: $D(V, E)$
- source: $s \in V$ sink: $t \in V$
- Capacity $c : E \to \mathbb{R}_{\geq 0}$
- Cut $S \subset V, s \in S, t \notin S$



- Value of cut: $$\sum_{u \in S, v \notin S, (u,v) \in E} c_{uv}$$

Fundamental Theorem of Flow

- Flow network: $D(V, E)$, $s, t \in V$, and $c : E \to \mathbb{R}_{\geq 0}$
  - Max-flow = min-cut
  - With integral capacity $c : E \to \mathbb{Z}_{\geq 0}$, the maximum flow is achieved by an integer flow $f : E \to \mathbb{Z}_{\geq 0}$.



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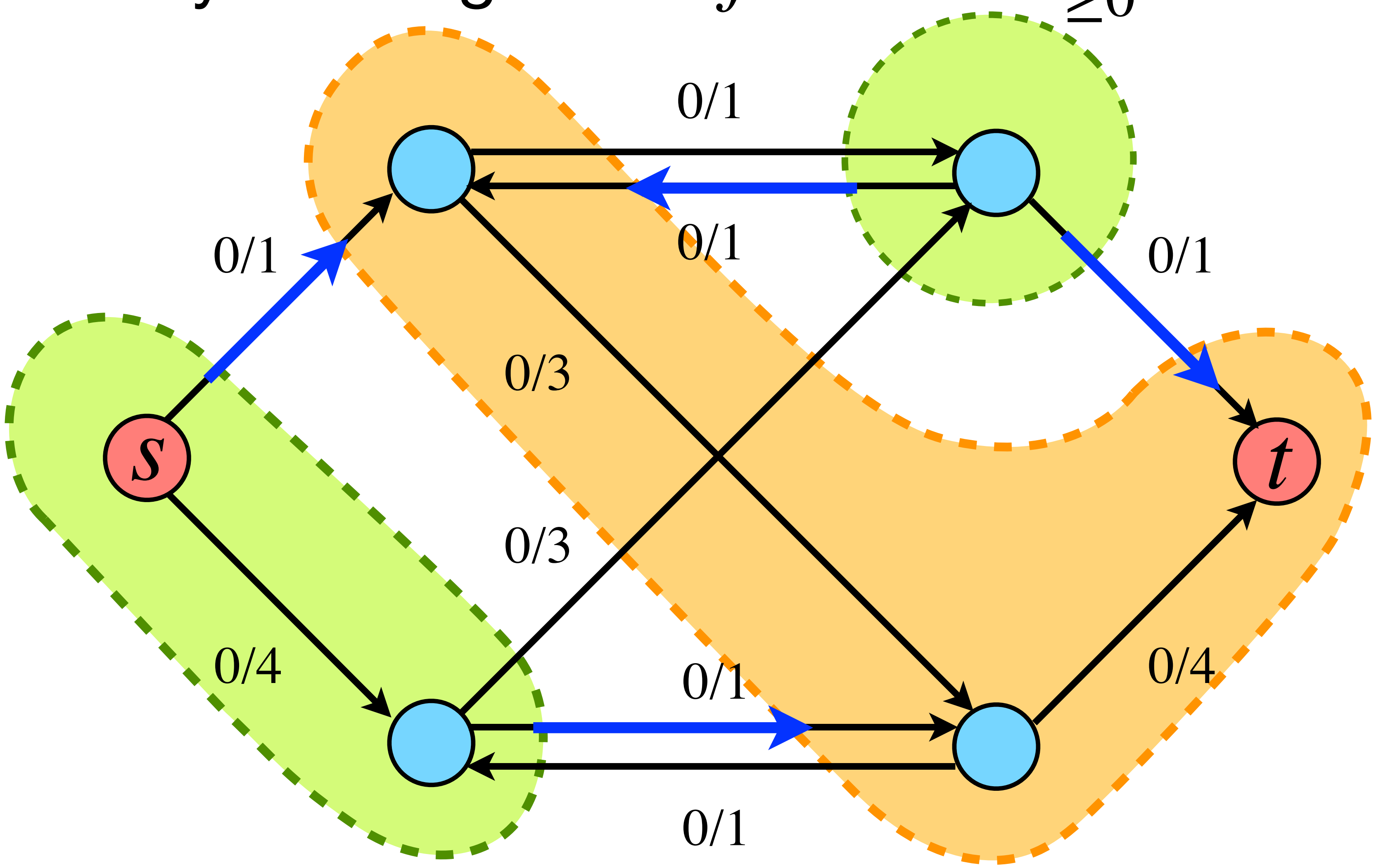

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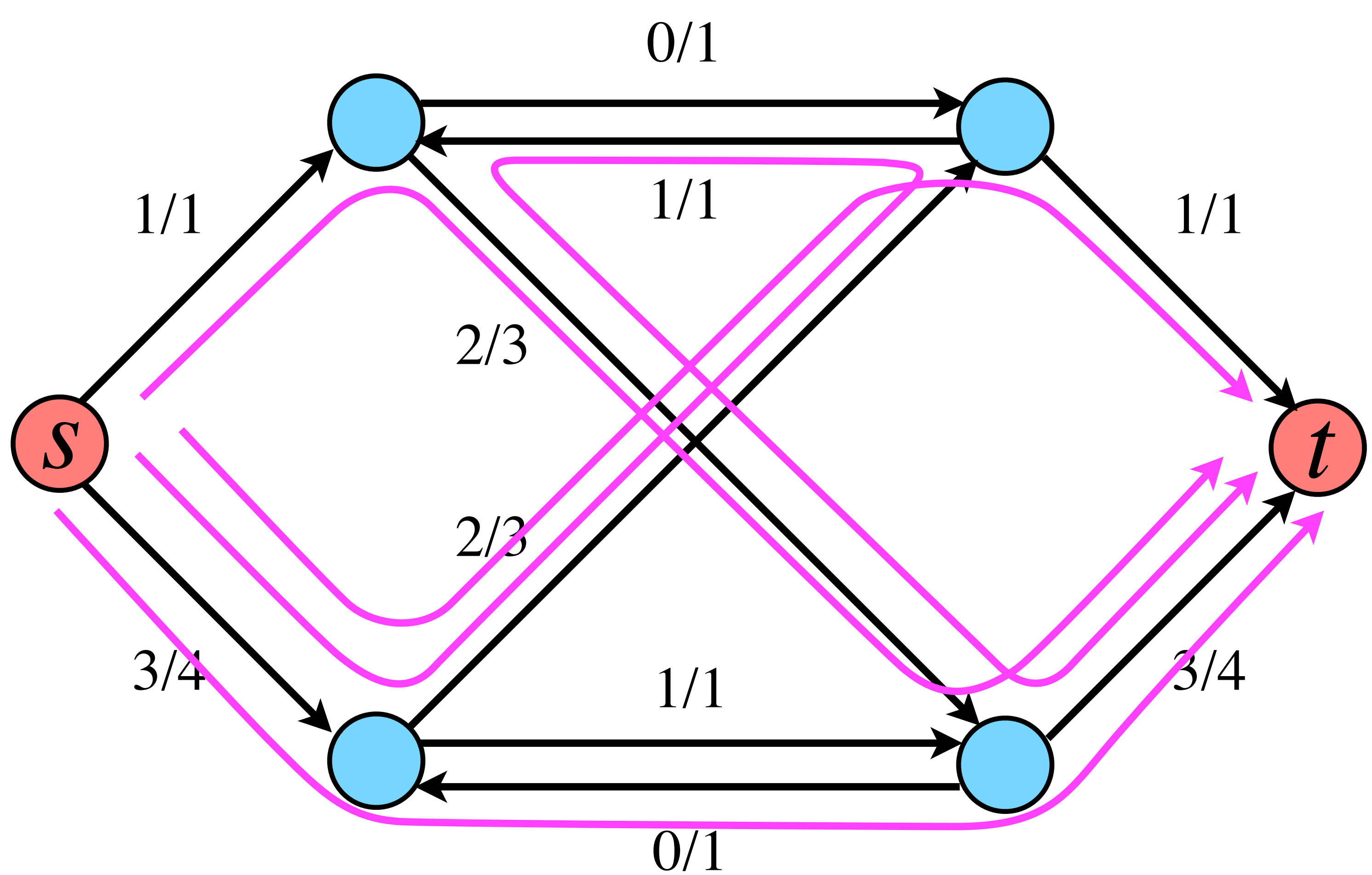

Fundamental Theorem of Flow

- Flow network: $D(V, E)$, $s, t \in V$, and $c : E \to \mathbb{R}_{\geq 0}$
  - Max-flow = min-cut
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- An *elementary* proof by **augmenting path.**
- An *advanced* proof by **LP duality** and **integrality**.

Estimate the Optima

$$
\begin{array}{llcl}
\text{minimize} & 7x_1 + x_2 + 5x_3 & & \\
 & \quad\quad\quad \text{VI} & & \\
\text{subject to} & x_1 - x_2 + 3x_3 & \geq & 10 \\
 & \quad\quad\quad + & & + \\
 & 5x_1 + 2x_2 - x_3 & \geq & 6 \\
 & & & \text{II} \\
 & x_1,\, x_2,\, x_3 \geq 0 & & 16
\end{array}
$$

16 ≤ OPT ≤ any feasible solution

Estimate the Optima

minimize $\quad 7x_1 + x_2 + 5x_3$

$\geq$

subject to
$$y_1 \quad x_1 - x_2 + 3x_3 \geq 10\, y_1$$
$$+ \qquad\qquad +$$
$$y_2 \quad 5x_1 + 2x_2 - x_3 \geq 6\, y_2$$

$$x_1, x_2, x_3 \geq 0$$

$$10y_1 + 6y_2 \leq \text{OPT}$$

for any
$$\begin{aligned} y_1 + 5y_2 &\leq 7 \\ -y_1 + 2y_2 &\leq 1 \\ 3y_1 - y_2 &\leq 5 \end{aligned} \qquad y_1, y_2 \geq 0$$

Primal-Dual

Primal:

$$\begin{array}{llllllll} \text{min} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \text{s.t.} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & x_1, x_2, x_3 \geq 0 & & & & \end{array}$$

Dual:

$$\begin{array}{llllllll} \text{max} & 10y_1 + 6y_2 & & & & \\ \text{s.t.} & y_1 & + & 5y_2 & \leq & 7 \\ & -y_1 & + & 2y_2 & \leq & 1 \\ & 3y_1 & - & y_2 & \leq & 5 \\ & & y_1, y_2 \geq 0 & & & \end{array}$$

∀dual feasible

≤ primal OPT

$\text{LP} \in \mathbf{NP} \cap \text{co}\mathbf{NP}$

Diet Problem

  · · · · 

calories	c_1	c_2	· · · · · ·	c_n	healthy
vitamin 1	a_{11}	a_{12}	· · · · · ·	a_{1n}	$\geq b_1$
⋮	⋮	⋮		⋮	⋮
vitamin m	a_{m1}	a_{m2}	· · · · · ·	a_{mn}	$\geq b_m$
solution:	x_1	x_2	· · · · · ·	x_n	

minimize the calories while keeping healthy

Surviving Problem

  · · · · 

price	c_1	c_2	· · · · ·	c_n	healthy
vitamin 1	a_{11}	a_{12}	· · · · ·	a_{1n}	$\geq b_1$
⋮	⋮	⋮		⋮	⋮
vitamin m	a_{m1}	a_{m2}	· · · · ·	a_{mn}	$\geq b_m$

solution: x_1 x_2 · · · · · x_n

minimize the total price while keeping healthy

Surviving Problem

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

price	c_1	c_2	$\cdots\cdots$	c_n	healthy
vitamin 1	a_{11}	a_{12}	$\cdots\cdots$	a_{1n}	$\geq b_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
vitamin m	a_{m1}	a_{m2}	$\cdots\cdots$	a_{mn}	$\geq b_m$
solution:	x_1	x_2	$\cdots\cdots$	x_n	

minimize the total price while keeping healthy

LP Duality

Primal:

$$\min \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual:

$$\max \ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

$$\text{s.t.} \quad A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

dual solution:	price					healthy
		c_1	c_2	$\cdots\cdots$	c_n	
y_1	vitamin 1	a_{11}	a_{12}	$\cdots\cdots$	a_{1n}	$\geq b_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
y_m	vitamin m	a_{m1}	a_{m2}	$\cdots\cdots$	a_{mn}	$\geq b_m$

m types of vitamin pills, competitive to n natural foods,

design a pricing system max the total price

LP Duality

Primal:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

$\geq$

Dual:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$
$$\text{s.t.} \quad A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

- **Monogamy**: dual(dual(LP)) = LP
- **Weak duality**:
  - ∀ feasible primal solution $\boldsymbol{x}$ and ∀ feasible dual solution $\boldsymbol{y}$

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{x} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

LP Duality

Primal:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

$$\text{s.t.} \quad A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

Weak Duality Theorem:

$\forall$ feasible primal solution $\boldsymbol{x}$ and $\forall$ feasible dual solution $\boldsymbol{y}$:

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

LP Duality

Primal:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

$$\text{s.t.} \quad A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

Strong Duality Theorem:

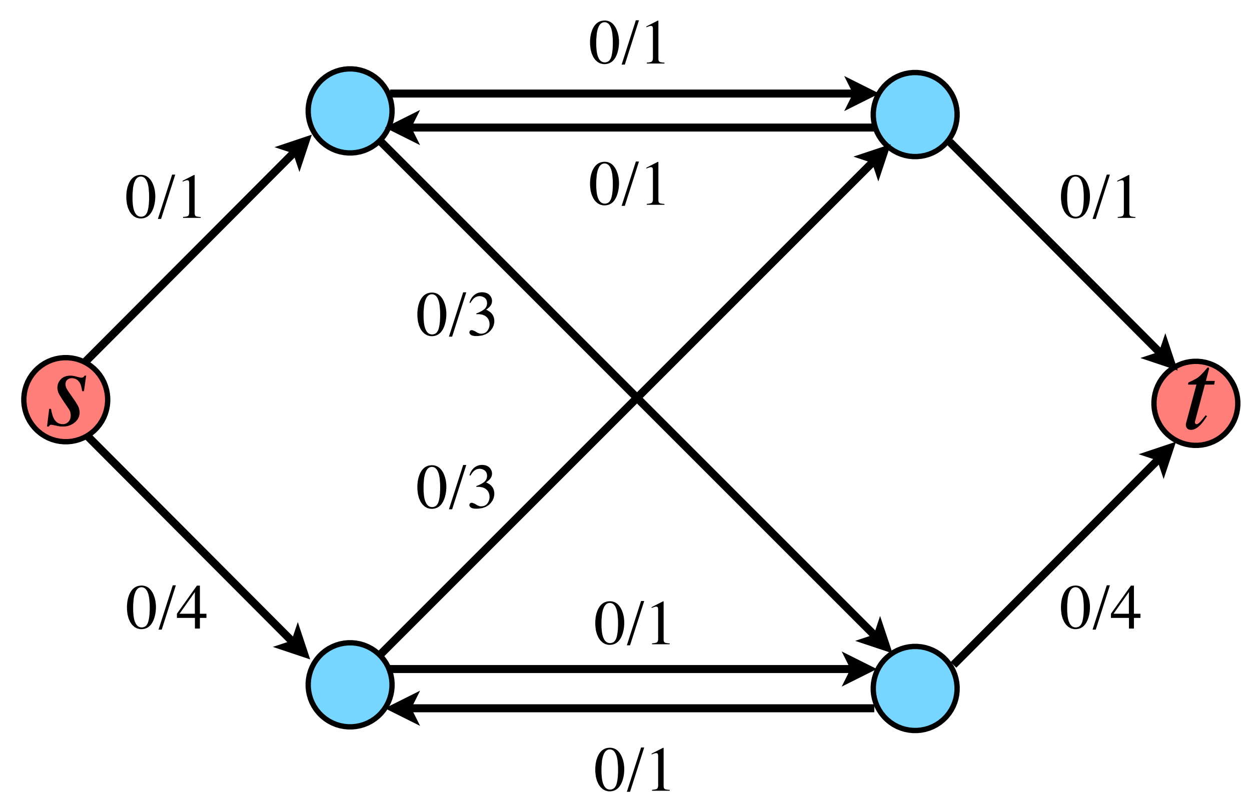
Primal LP has an optimal solution $\boldsymbol{x}^*$

$\iff$ Dual LP has an optimal solution $\boldsymbol{y}^*$

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^* = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*$$

Maximum Flow

- Digraph: $D(V, E)$
- Capacity $c : E \to \mathbb{R}_{\geq 0}$
- source: $s \in V$ sink: $t \in V$



$$\max \quad \sum_{u:(s,u)\in E} f_{su}$$

$$\text{s.t.} \quad f_{uv} \leq c_{uv} \qquad \forall (u, v) \in E$$

$$\sum_{w:(w,u)\in E} f_{wu} - \sum_{v:(u,v)\in E} f_{uv} = 0 \qquad \forall u \in V \setminus \{s, t\}$$

$$f_{uv} \geq 0 \qquad \forall (u, v) \in E$$

Maximum Flow

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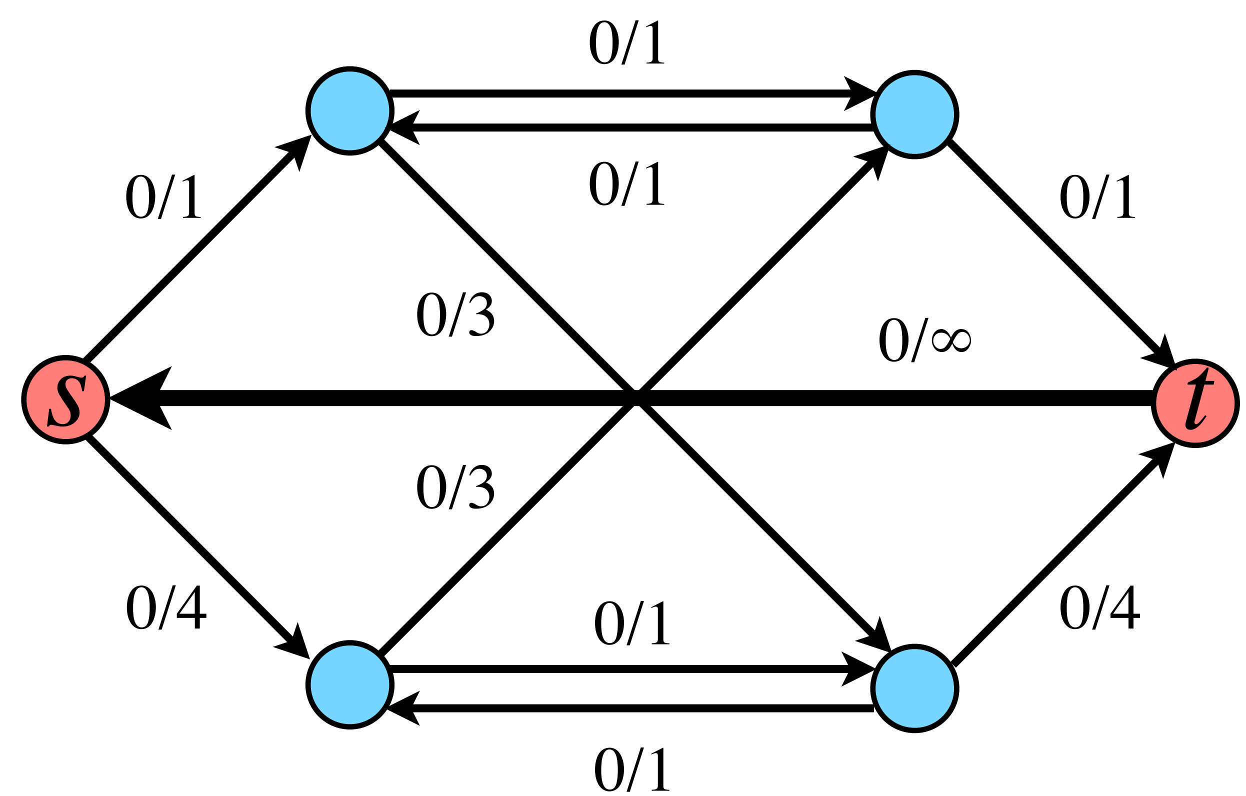
- source: $s \in V$ sink: $t \in V$



$$\max \quad f_{ts}$$

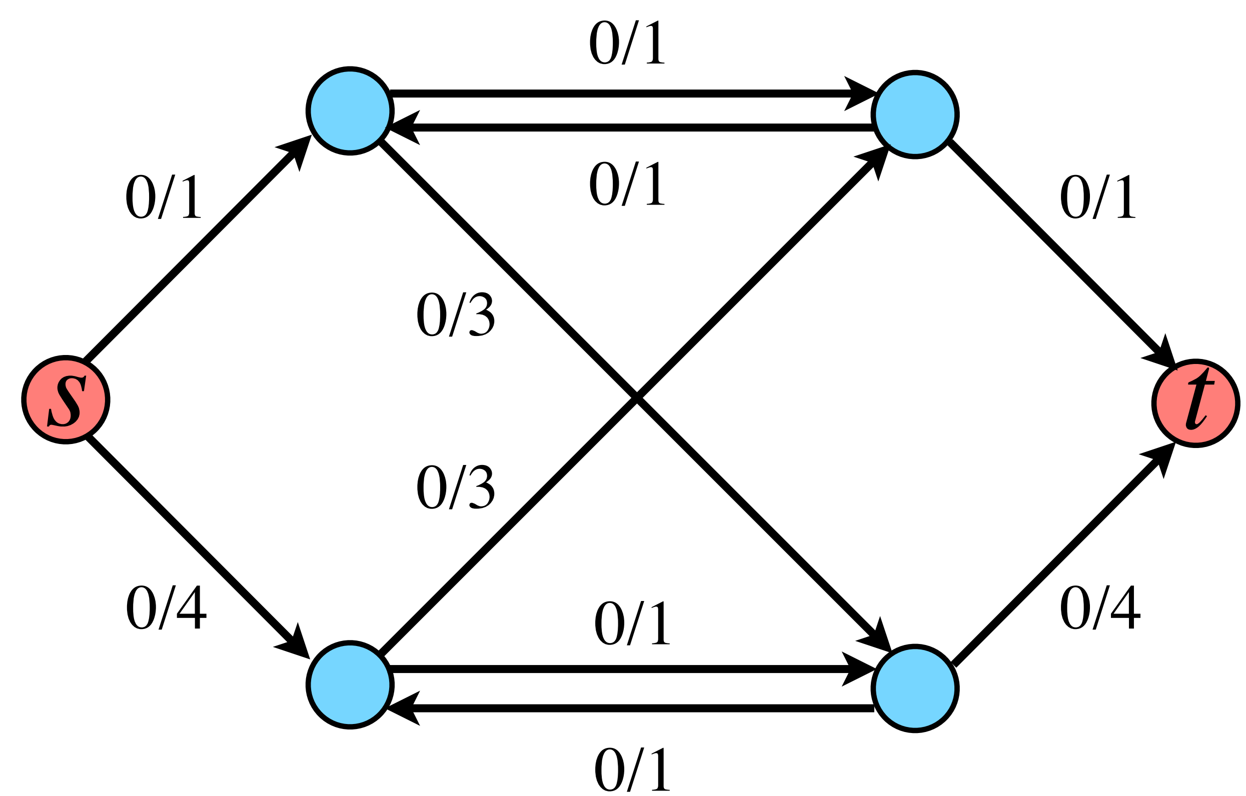
$$\textcolor{blue}{d_{uv}} \quad \text{s.t.} \quad f_{uv} \leq c_{uv} \qquad \forall (u,v) \in E$$

$$\textcolor{blue}{p_u} \qquad \sum_{w:(w,u)\in \textcolor{red}{E'}} f_{wu} - \sum_{v:(u,v)\in \textcolor{red}{E'}} f_{uv} \;\textcolor{red}{\leq}\; 0 \qquad \forall u \in V$$

$$f_{uv} \geq 0 \qquad \forall (u,v) \in \textcolor{red}{E'} = E \cup \{(t,s)\}$$

Dual LP

- Digraph: $D(V, E)$
- Capacity $c : E \to \mathbb{R}_{\geq 0}$
- source: $s \in V$ sink: $t \in V$



$$\min \quad \sum_{(u,v)\in E} c_{uv} d_{uv}$$

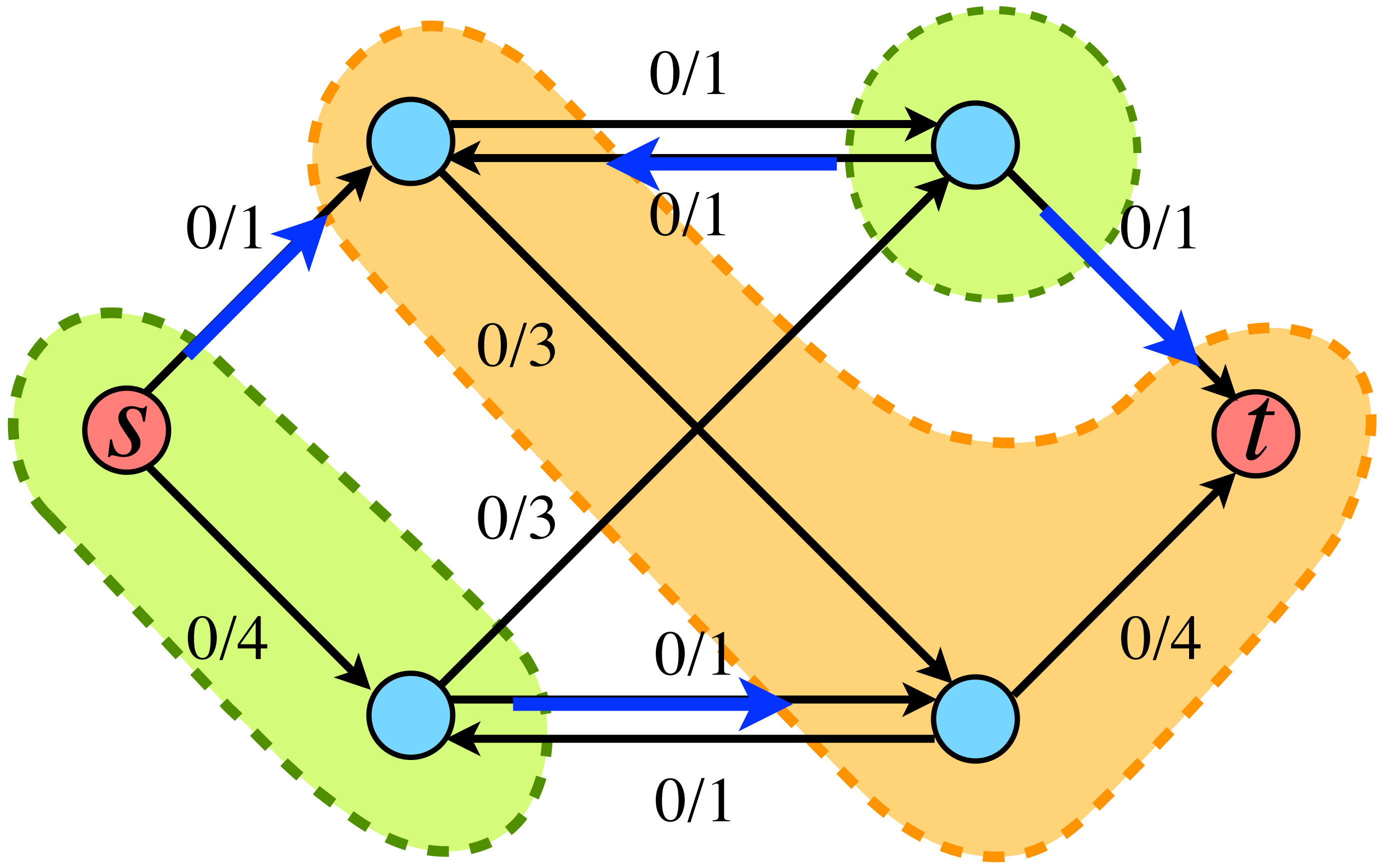
$$\text{s.t.} \quad d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \in E$$

$$p_s - p_t \geq 1$$

$$d_{uv} \geq 0 \qquad p_u \geq 0 \qquad \forall (u,v) \in E \qquad \forall u \in V$$

Minimum Cut

- Digraph: $D(V,E)$
- Capacity $c : E \to \mathbb{R}_{\geq 0}$
- source: $s \in V$ sink: $t \in V$



$$\min \sum_{(u,v)\in E} c_{uv} d_{uv}$$

s.t.

$$d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \in E$$

$$p_s - p_t \geq 1$$

$$d_{uv}, p_u \in \{0,1\} \qquad \forall (u,v) \in E \qquad \forall u \in V$$

Primal-Dual Schema

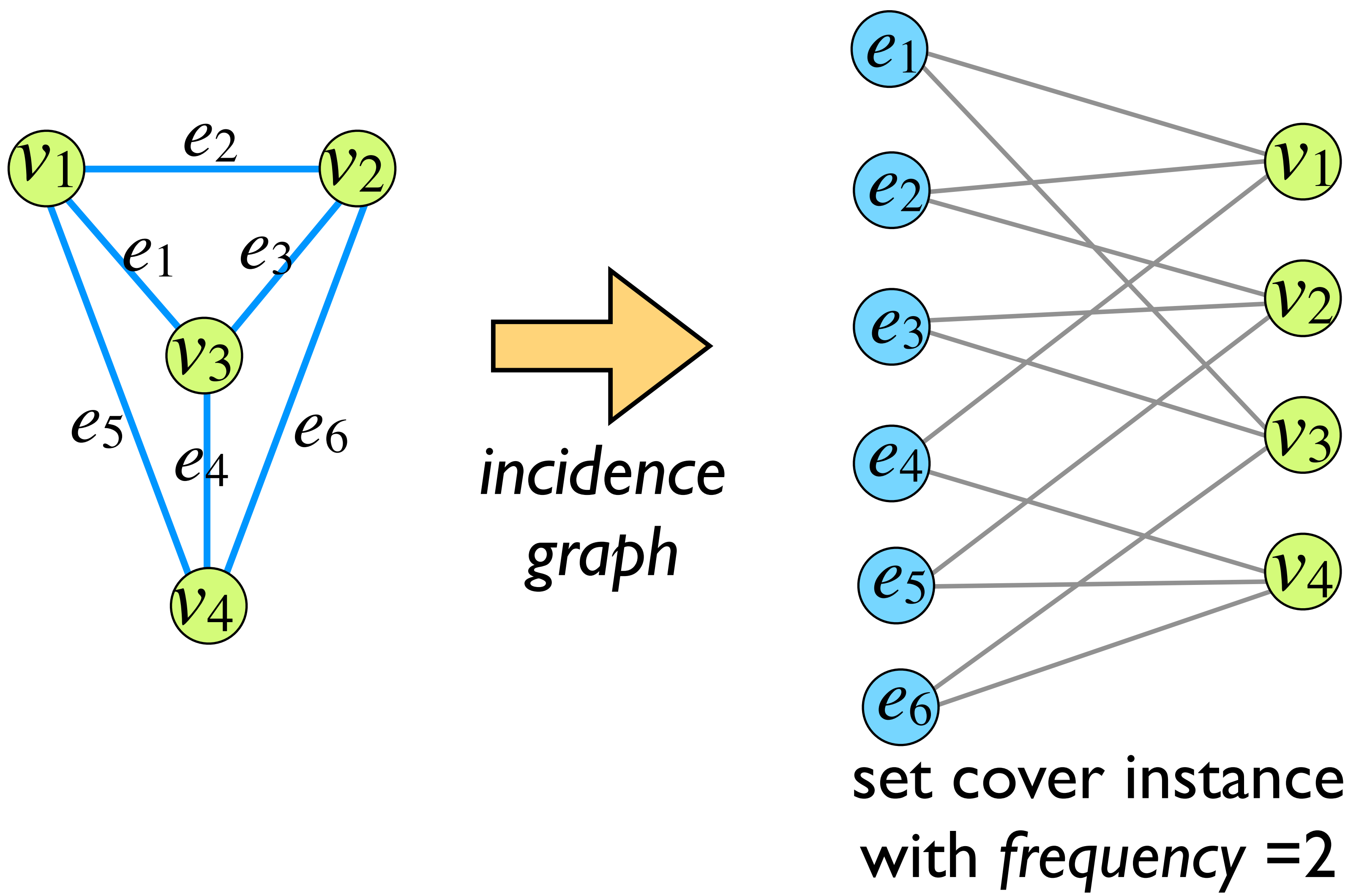
LP-based Algorithms

- **LP relaxation and rounding:**
  - Relax the **Integer Linear Program** to an **LP**.
  - Round the optimal LP solution to a ***feasible integral*** solution.
- Primal-dual schema:
  - Find a pair of feasible solutions to the primal and dual programs which are close to each other.

close to dual feasible solution $\Longrightarrow$ closer to OPT

Vertex Cover

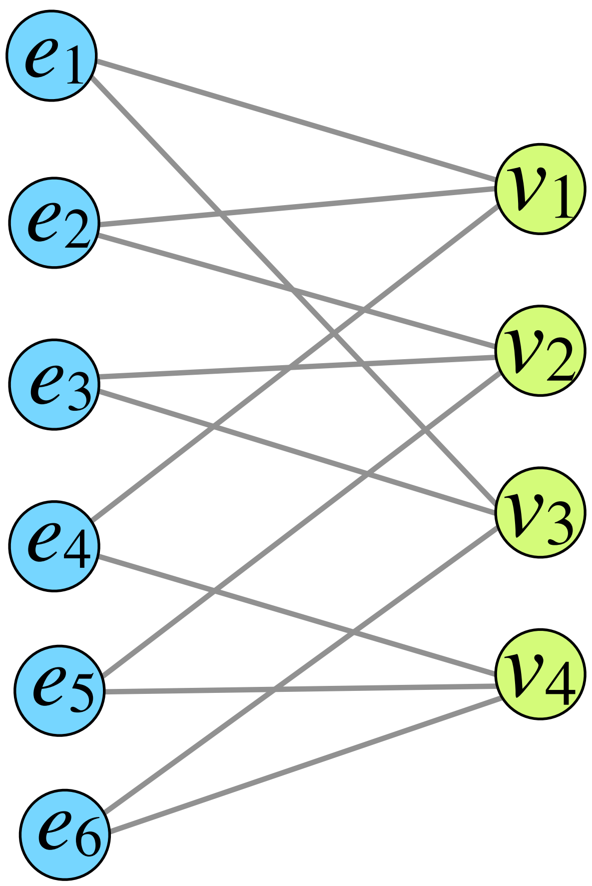
Instance: An undirected graph $G(V, E)$.
Find the smallest $C \subseteq V$ that intersects all edges.



Vertex Cover

Instance: An undirected graph $G(V, E)$.
Find the smallest $C \subseteq V$ that intersects all edges.

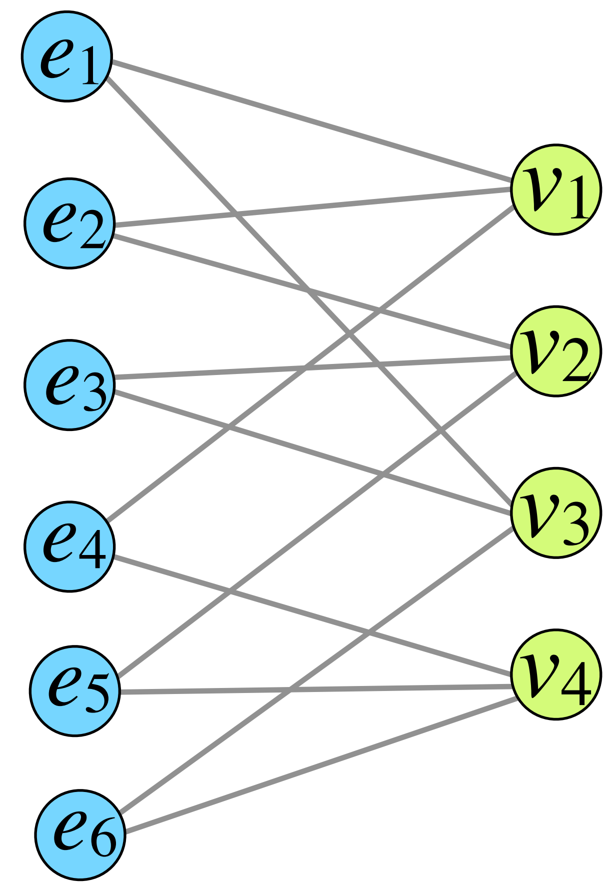
Find a ***maximal matching*** $M \subseteq E$;
return $C = \{v \mid \{u, v\} \in M\}$;



- Matching $\implies |M| \le OPT_{VC}$
  (weak duality)
- Maximality $\implies C$ is a vertex cover

$$|C| \le 2|M| \le 2OPT_{VC}$$

Duality



vertex cover:	constraints $\sum_{v \in e} x_v \geq 1$	variables $x_v \in \{0,1\}$
matching:	variables $y_e \in \{0,1\}$	constraints $\sum_{e \ni v} y_e \leq 1$

Duality

Instance: graph $G(V,E)$

primal: minimize $\sum_{v \in V} x_v$
(vertex cover)

subject to $\sum_{v \in e} x_v \geq 1, \quad \forall e \in E$

$x_v \in \{0,1\}, \quad \forall v \in V$

dual: maximize $\sum_{e \in E} y_e$
(matching)

subject to $\sum_{e \ni v} y_e \leq 1, \quad \forall v \in V$

$y_e \in \{0,1\}, \quad \forall e \in E$

Duality for LP-Relaxations

Instance: graph $G(V,E)$

primal: minimize $\sum_{v\in V} x_v$

subject to $\sum_{v\in e} x_v \geq 1, \quad \forall e \in E$

$x_v \geq 0, \quad \forall v \in V$

dual: maximize $\sum_{e\in E} y_e$

subject to $\sum_{e\ni v} y_e \leq 1, \quad \forall v \in V$

$y_e \geq 0, \quad \forall e \in E$

LP Duality

Primal:

$$\min \; \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual:

$$\max \; \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

$$\text{s.t.} \quad A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

Strong Duality Theorem:

Primal LP has an optimal solution $\boldsymbol{x}^*$

$\Longleftrightarrow$ Dual LP has an optimal solution $\boldsymbol{y}^*$

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}^* = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*$$

Complementary Slackness

Primal: min $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$

Dual: max $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$

s.t. $A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$

$\boldsymbol{y} \geq \boldsymbol{0}$

Theorem (Complementary Slackness Condition):

For feasible primal solution $\boldsymbol{x}$ and feasible dual solution $\boldsymbol{y}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ are both optimal iff:

- $\boldsymbol{y}^{\mathrm{T}}(A\boldsymbol{x} - \boldsymbol{b}) = 0;$ $\quad \forall i : y_i > 0 \implies A_{i\cdot}\boldsymbol{x} = b_i$
- $\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{c} - A^{T}\boldsymbol{y}) = 0;$ $\quad \forall j : x_j > 0 \implies A_{\cdot j}^{\mathrm{T}}\boldsymbol{y} = c_j$

Complementary Slackness

Primal: $\min \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$
s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$
$\boldsymbol{x} \geq \boldsymbol{0}$

Dual: $\max \ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$
s.t. $A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$
$\boldsymbol{y} \geq \boldsymbol{0}$

$\forall$ feasible primal solution $\boldsymbol{x}$ and feasible dual solution $\boldsymbol{y}$:

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{x} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$\boldsymbol{x}$ and $\boldsymbol{y}$ are both optimal $\iff$ $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

$\Updownarrow$

$$\forall i : y_i > 0 \implies A_{i\cdot}\boldsymbol{x} = b_i$$
$$\forall j : x_j > 0 \implies A_{\cdot j}^{\mathrm{T}}\boldsymbol{y} = c_j$$

$\iff$

- $\boldsymbol{y}^{\mathrm{T}}(A\boldsymbol{x} - \boldsymbol{b}) = 0;$
- $\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{c} - A^{T}\boldsymbol{y}) = 0;$

Complementary Slackness

Primal: $\min \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$

Dual: $\max \ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$

s.t. $A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$

$\boldsymbol{y} \geq \boldsymbol{0}$

Theorem (Complementary Slackness Condition):

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Complementary Slackness

Primal: $\min \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$

Dual: $\max \ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$

s.t. $A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$

$\boldsymbol{y} \geq \boldsymbol{0}$

Theorem:

$\forall$ feasible primal solution $\boldsymbol{x}$ and feasible dual solution $\boldsymbol{y}$,

if for $\alpha, \beta \geq 1$:

$$\forall i : y_i > 0 \implies A_{i\cdot}\boldsymbol{x} \leq \alpha b_i$$

$$\forall j : x_j > 0 \implies A_{\cdot j}^{\mathrm{T}}\boldsymbol{y} \geq c_j/\beta$$

$$\implies \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \alpha\beta\, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq \alpha\beta\, OPT_{LP}$$

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n}\left(\beta \sum_{i=1}^{m} a_{ij} y_i\right) x_j = \beta \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i \leq \alpha\beta \sum_{i=1}^{m} b_i y_i$$

Primal-Dual Schema

Primal IP:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$
$$\boldsymbol{x} \in \mathbb{Z}^n_{\geq 0}$$

Dual LP-Relax:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$
$$\text{s.t.} \quad A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

Primal-Dual Schema:

Find a pair $(\boldsymbol{x}, \boldsymbol{y})$ of feasible primal *integral* solution $\boldsymbol{x}$ and feasible dual solution $\boldsymbol{y}$ such that for some $\alpha, \beta \geq 1$:

$$\forall i : y_i > 0 \implies A_{i\cdot}\boldsymbol{x} \leq \alpha b_i$$
$$\forall j : x_j > 0 \implies A_{\cdot j}^{\mathrm{T}}\boldsymbol{y} \geq c_j/\beta$$

$$\implies \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \alpha\beta\, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq \alpha\beta\, OPT_{LP} \leq \alpha\beta\, OPT_{IP}$$

Primal-Dual Schema

Primal IP: min $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$
s.t. $A\boldsymbol{x} \geq \boldsymbol{b}$
$\boldsymbol{x} \in \mathbb{Z}^n_{\geq 0}$

Dual LP-Relax: max $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$
s.t. $A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$
$\boldsymbol{y} \geq \boldsymbol{0}$

Primal-Dual Schema:

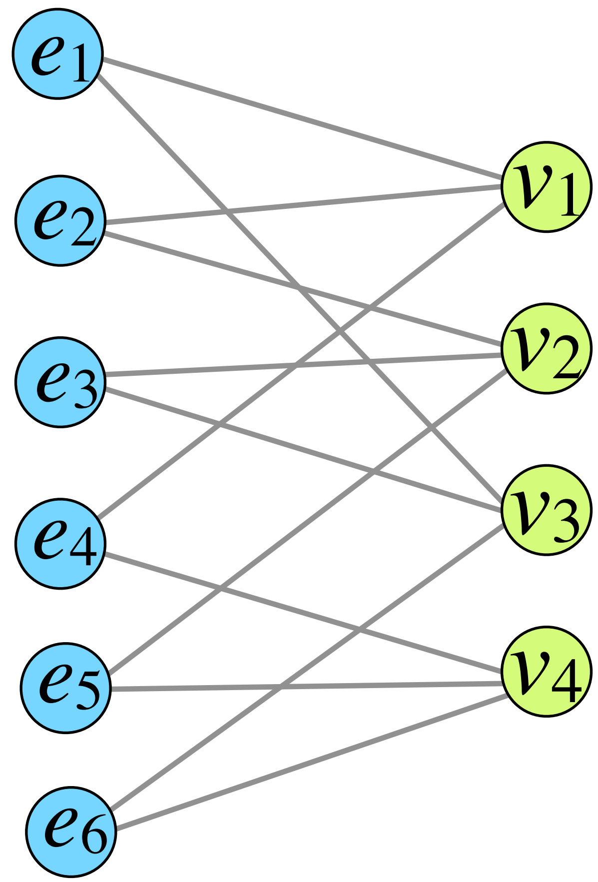
Raise a pair $(\boldsymbol{x}, \boldsymbol{y})$ of infeasible primal *integral* solution $\boldsymbol{x}$ and feasible dual solution $\boldsymbol{y}$ continuously, satisfying:

$$\forall i : y_i > 0 \implies A_{i\cdot}\boldsymbol{x} \leq \alpha b_i$$
$$\forall j : x_j > 0 \implies A_{\cdot j}^{\mathrm{T}}\boldsymbol{y} = c_j$$

for some $\alpha \geq 1$

until $\boldsymbol{x}$ becomes feasible.

$$\implies \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \alpha\, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq \alpha\, OPT_{LP} \leq \alpha\, OPT_{IP}$$



primal:

$$\min \sum_{v \in V} x_v$$

$$\text{s.t.} \quad \sum_{v \in e} x_v \geq 1, \quad \forall e \in E$$

$$x_v \in \{0,1\}, \quad \forall v \in V$$

dual-relax:

$$\max \sum_{e \in E} y_e$$

$$\text{s.t.} \quad \sum_{e \ni v} y_e \leq 1, \quad \forall v \in V$$

$$y_e \geq 0, \quad \forall e \in E$$

vertex cover:

constraints	variables
$\sum_{v\in e} x_v \geq 1$	$x_v \in \{0,1\}$

matching:

variables	constraints
$y_e \in \{0,1\}$	$\sum_{e\ni v} y_e \leq 1$

Find feasible $(\boldsymbol{x}, \boldsymbol{y})$ such that:

$$\forall e : y_e > 0 \implies \sum_{v \in e} x_v \leq \alpha$$

$$\forall v : x_v > 0 \implies \sum_{e \ni v} y_e = 1$$

primal:

$$\min \sum_{v \in V} x_v$$
$$\text{s.t.} \sum_{v \in e} x_v \geq 1, \quad \forall e \in E$$
$$x_v \in \{0,1\}, \quad \forall v \in V$$

dual-relax:

$$\max \sum_{e \in E} y_e$$
$$\text{s.t.} \sum_{e \ni v} y_e \leq 1, \quad \forall v \in V$$
$$y_e \geq 0, \quad \forall e \in E$$

initially $\boldsymbol{x} = \mathbf{0}, \boldsymbol{y} = \mathbf{0}$;

while $E \neq \emptyset$: (constraints currently violated by $\boldsymbol{x}$)

  pick an $e \in E$ and raise y_e ~~until $\sum_{e \ni v} y_e = 1$ for some $v \in V$;~~ to 1

  set $x_v = 1$ for all such $v \in e$ and remove all $e' \ni v$ from E;

Complementary slackness:

$$\forall e : y_e > 0 \implies \sum_{v \in e} x_v \leq \alpha = 2$$
$$\forall v : x_v > 0 \implies \sum_{e \ni v} y_e = 1$$

$$\implies \sum_{v \in V} x_v \leq 2 \sum_{e \in E} y_e \leq 2\, OPT$$

primal:

$$\min \sum_{v \in V} x_v$$

$$\text{s.t.} \sum_{v \in e} x_v \geq 1, \quad \forall e \in E$$

$$x_v \in \{0,1\}, \quad \forall v \in V$$

dual-relax:

$$\max \sum_{e \in E} y_e$$

$$\text{s.t.} \sum_{e \ni v} y_e \leq 1, \quad \forall v \in V$$

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set $x_v = 1$ for all such $v \in e$ and remove all $e' \ni v$ from E;

(constraints satisfied by current $\boldsymbol{x}$)

$$\forall e : y_e > 0 \implies \sum_{v \in e} x_v \leq \alpha = 2$$

$$\forall v : x_v > 0 \implies \sum_{e \ni v} y_e = 1$$

Complementary slackness:

$$\implies \sum_{v \in V} x_v \leq 2 \sum_{e \in E} y_e \leq 2\, OPT$$

primal:

$$\min \sum_{v\in V} x_v$$
$$\text{s.t.} \quad \sum_{v\in e} x_v \ge 1, \quad \forall e \in E$$
$$x_v \in \{0,1\}, \quad \forall v \in V$$

dual-relax:

$$\max \sum_{e\in E} y_e$$
$$\text{s.t.} \quad \sum_{e\ni v} y_e \le 1, \quad \forall v \in V$$
$$y_e \ge 0, \quad \forall e \in E$$

initially $\boldsymbol{x} = \mathbf{0}$, $\boldsymbol{y} = \mathbf{0}$;

while $E \neq \emptyset$: (constraints currently violated by $\boldsymbol{x}$)

pick an $e \in E$ and raise y_e ~~until $\sum_{e\ni v} y_e = 1$ for some $v \in V$;~~ to 1

set $x_v = 1$ for all such $v \in e$ and remove all $e' \ni v$ from E;

Find a *maximal matching* $M \subseteq E$;

return $C = \{v \mid \{u, v\} \in M\}$;

Primal-Dual Schema

- **Modeling**: Express the optimization problem as an **Integer Linear Program (ILP)** and write its dual relaxed program.

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}_{\geq 0} \end{aligned} \qquad \begin{aligned} \max \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}} \\ & \boldsymbol{y} \geq \boldsymbol{0} \end{aligned}$$

- **Initialization**: Start from a primal infeasible solution $\boldsymbol{x}$ and a dual feasible solution $\boldsymbol{y}$ (usually $\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{y} = \boldsymbol{0}$).

- Raise $\boldsymbol{x}$ and $\boldsymbol{y}$ until $\boldsymbol{x}$ becomes feasible:
  - raise $\boldsymbol{y}$ continuously until dual constraints getting tight $A_{\cdot j}^{\mathrm{T}}\boldsymbol{y} = c_j$;
  - raise corresponding x_j integrally so that $x_j > 0 \implies A_{\cdot j}^{\mathrm{T}}\boldsymbol{y} = c_j$.

- Verify **complementary slackness condition**:

$$\boxed{y_i > 0 \implies A_{i\cdot}\boldsymbol{x} \leq \alpha\, b_i} \implies \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \alpha\, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq \alpha\, OPT$$

Integrality Gap

- minimum vertex cover of $G(V, E)$:

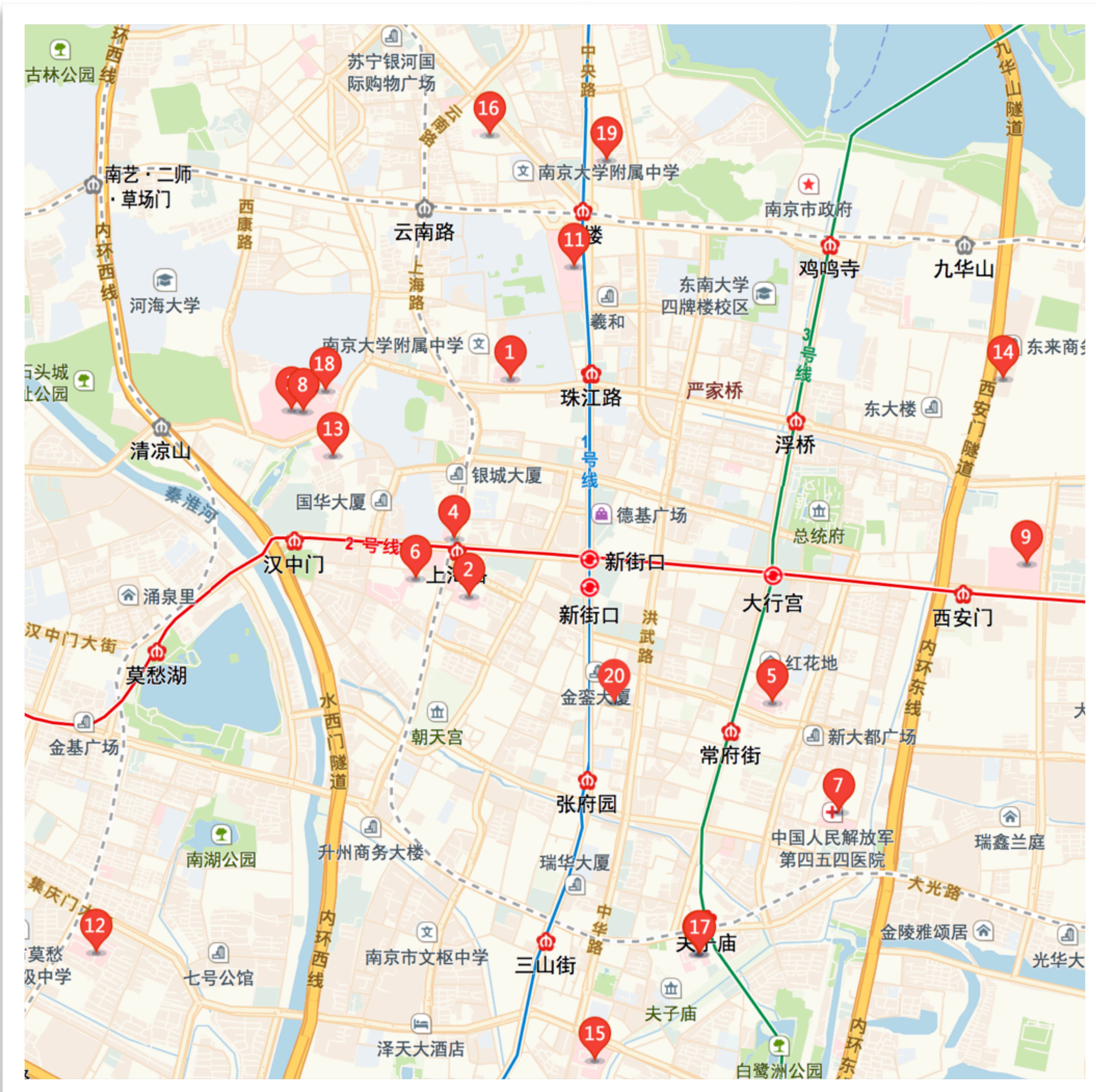
$$
\begin{aligned}
\text{minimize} \quad & \sum_{v \in V} x_v \\
\text{subject to} \quad & \sum_{v \in e} x_v \geq 1, && e \in E \\
& x_v \in \{0,1\}, && v \in V
\end{aligned}
$$

$$\text{integrality gap} = \sup_G \frac{\mathrm{OPT}(G)}{\mathrm{OPT}_{\mathrm{LP}}(G)}$$

- For LP relaxation of **vertex cover:** integrality gap $= 2$

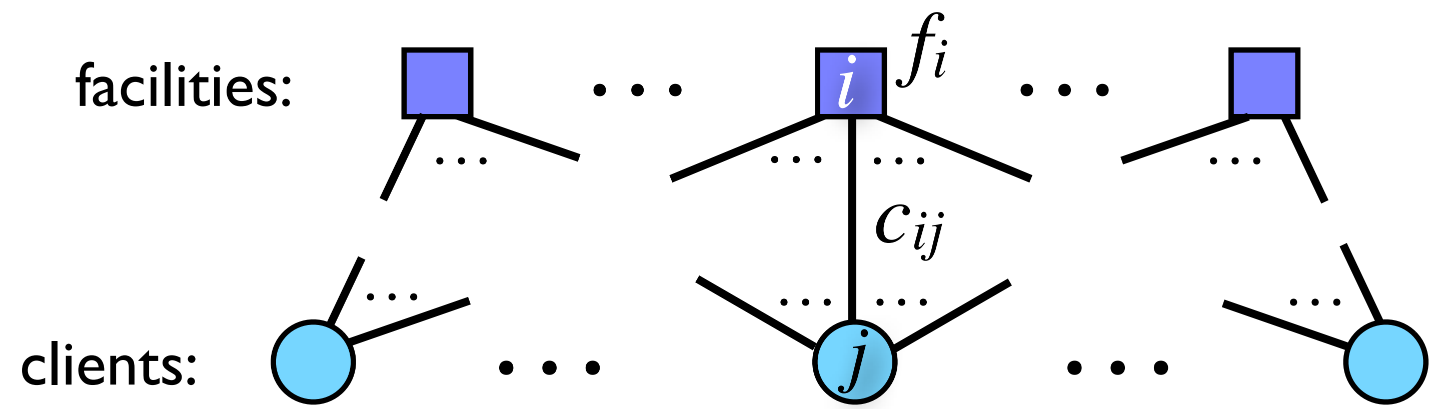
Facility Location

Facility Location



hospitals
in Nanjing

Facility Location

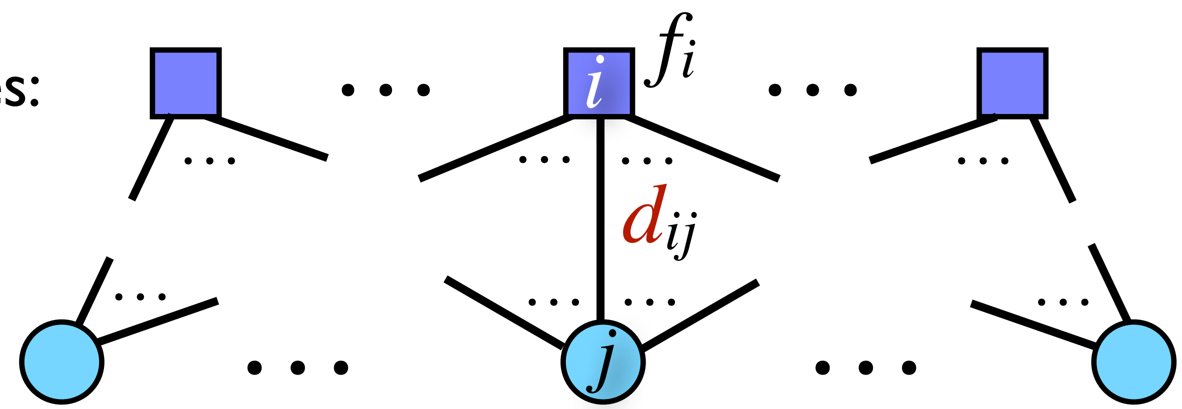


Instance: set F of facilities; set C of clients;
facility opening costs f: $F \to [0, \infty)$;
connection costs c: $F \times C \to [0, \infty)$;

Find a subset $I \subseteq F$ of *opening* facilities and a way $\phi: C \to I$ of *connecting* all clients to them such that the total cost $\sum_{j \in C} c_{\phi(j),j} + \sum_{i \in I} f_i$ is minimized.

- uncapacitated facility location;
- **NP**-hard; AP(*Approximation Preserving*)-reduction from Set Cover;
- [Dinur, Steuer 2014] no poly-time $(1\text{-}o(1))\ln n$-approx. algorithm unless **NP** = **P**.

Metric Facility Location

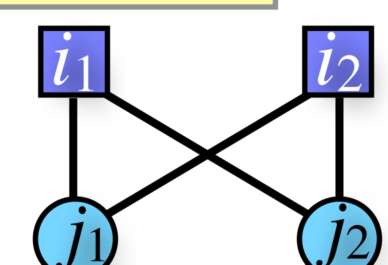


Instance: set F of facilities; set C of clients;
facility opening costs f: $F \to [0, \infty)$;
connection metric d: $F \times C \to [0, \infty)$;

Find a subset $I \subseteq F$ of *opening* facilities and a way ϕ: $C \to I$ of *connecting* all clients to them such that the total cost $\sum_{j \in C} d_{\phi(j),j} + \sum_{i \in I} f_i$ is minimized.

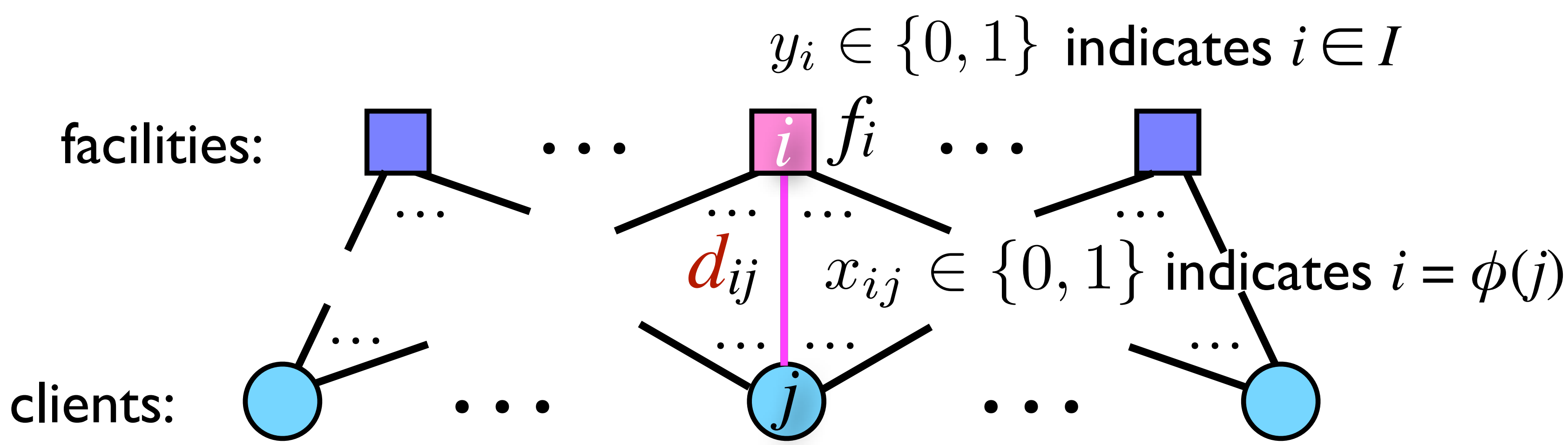
triangle inequality: $\forall i_1, i_2 \in F, \forall j_1, j_2 \in C$

$$d_{i_1 j_1} + d_{i_2 j_1} + d_{i_2 j_2} \geq d_{i_1 j_2}$$



Instance: set F of facilities; set C of clients;
facility opening costs f: $F \rightarrow [0, \infty)$;
connection metric d: $F \times C \rightarrow [0, \infty)$;

Find ϕ: $C \rightarrow I \subseteq F$ to minimize $\sum_{j \in C} d_{\phi(j),j} + \sum_{i \in I} f_i$

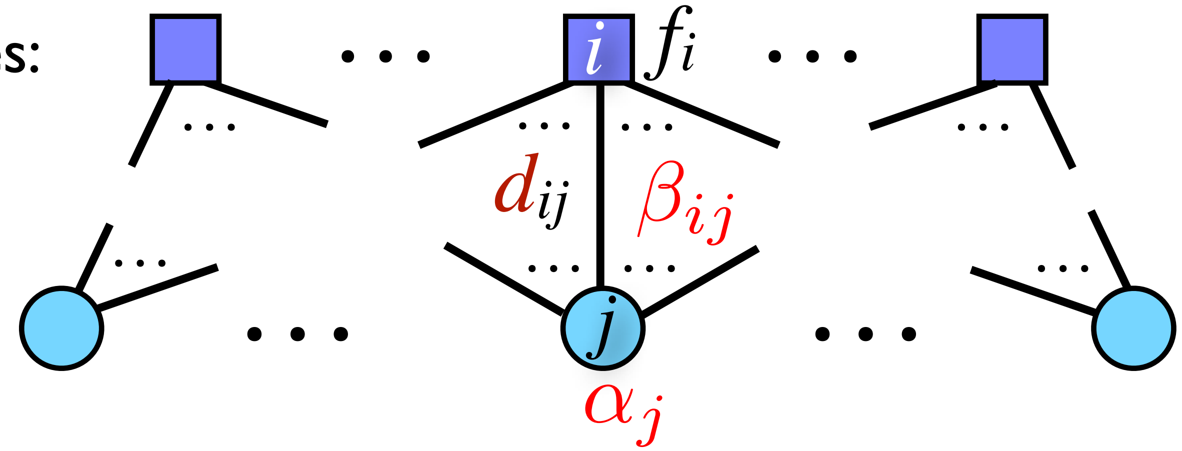


LP-relaxation: min $\sum_{i \in F, j \in C} d_{ij} x_{ij} + \sum_{i \in F} f_i y_i$

s.t. $y_i \geq x_{ij}, \quad \forall i \in F, j \in C$

$\sum_{i \in F} x_{ij} \geq 1, \quad \forall j \in C$

$x_{ij}, y_i \geq 0,$ ~~$x_{ij}, y_i \in \{0,1\},$~~ $\quad \forall i \in F, j \in C$



Primal:

$$\min \sum_{i\in F, j\in C} d_{ij}x_{ij} + \sum_{i\in F} f_i y_i$$

$$\text{s.t.} \quad y_i - x_{ij} \ge 0, \quad \forall i \in F, j \in C$$

$$\sum_{i\in F} x_{ij} \ge 1, \quad \forall j \in C$$

$$x_{ij}, y_i \in \{0,1\}, \quad \forall i \in F, j \in C$$

Dual-relax:

$$\max \sum_{j\in C} \alpha_j$$

$$\text{s.t.} \quad \alpha_j - \beta_{ij} \le d_{ij}, \quad \forall i \in F, j \in C$$

$$\sum_{j\in C} \beta_{ij} \le f_i, \quad \forall i \in F$$

$$\alpha_j, \beta_{ij} \ge 0, \quad \forall i \in F, j \in C$$

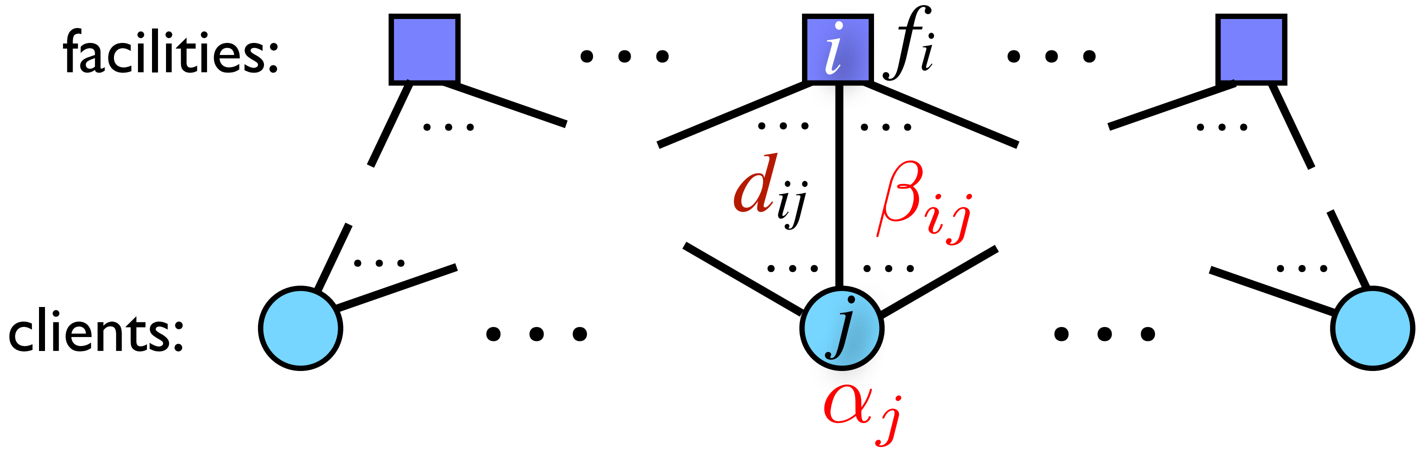
α_j: amount of value paid by client j to all facilities

$\beta_{ij} \ge \alpha_j - d_{ij}$: payment to facility i by client j (after deduction)

complimentary slackness conditions: (if *ideally* held)

$x_{ij} = 1 \Rightarrow \alpha_j - \beta_{ij} = d_{ij}$; $\quad \alpha_j > 0 \Rightarrow \sum_{i\in F} x_{ij} = 1$;

$y_i = 1 \Rightarrow \sum_{j\in C} \beta_{ij} = f_i$; $\quad \beta_{ij} > 0 \Rightarrow y_i = x_{ij}$;



$$\min \sum_{i\in F, j\in C} d_{ij}x_{ij} + \sum_{i\in F} f_i y_i$$

$$\text{s.t.} \quad y_i - x_{ij} \geq 0, \quad \forall i \in F, j \in C$$

$$\sum_{i\in F} x_{ij} \geq 1, \quad \forall j \in C$$

$$x_{ij}, y_i \in \{0,1\}, \quad \forall i \in F, j \in C$$

$$\max \sum_{j\in C} \alpha_j$$

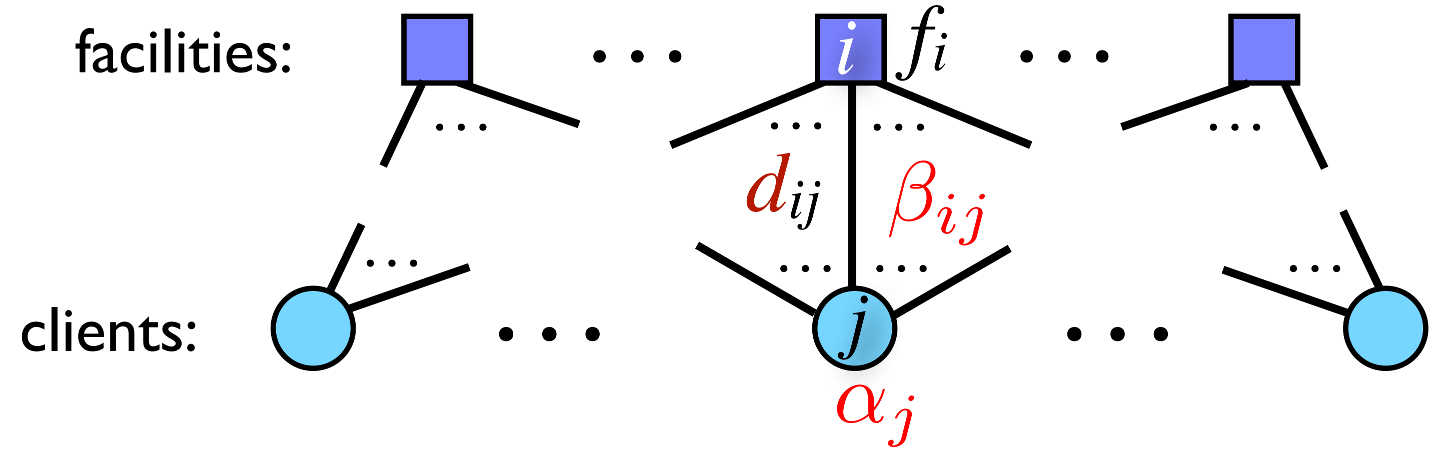
$$\text{s.t.} \quad \alpha_j - \beta_{ij} \leq d_{ij}, \quad \forall i \in F, j \in C$$

$$\sum_{j\in C} \beta_{ij} \leq f_i, \quad \forall i \in F$$

$$\alpha_j, \beta_{ij} \geq 0, \quad \forall i \in F, j \in C$$

Initially $\boldsymbol{\alpha} = \mathbf{0}$, $\boldsymbol{\beta} = \mathbf{0}$, no facility is ***open***, no client is ***connected***;
raise α_j for all client j *simultaneously* at a *uniform continuous* rate:

- upon $\alpha_j = d_{ij}$ for a closed facility i: edge (i,j) is *paid*; fix $\beta_{ij} = \alpha_j - d_{ij}$ as α_j being raised;
- upon $\sum_{j\in C} \beta_{ij} = f_i$: tentatively open facility i; all ***unconnected*** clients j with *paid* edge (i,j) to facility i are declared ***connected*** to facility i and stop raising α_j;
- upon $\alpha_j = d_{ij}$ for an ***unconnected*** client j and tentatively open facility i: client j is declared ***connected*** to facility i: and stop raising α_j;



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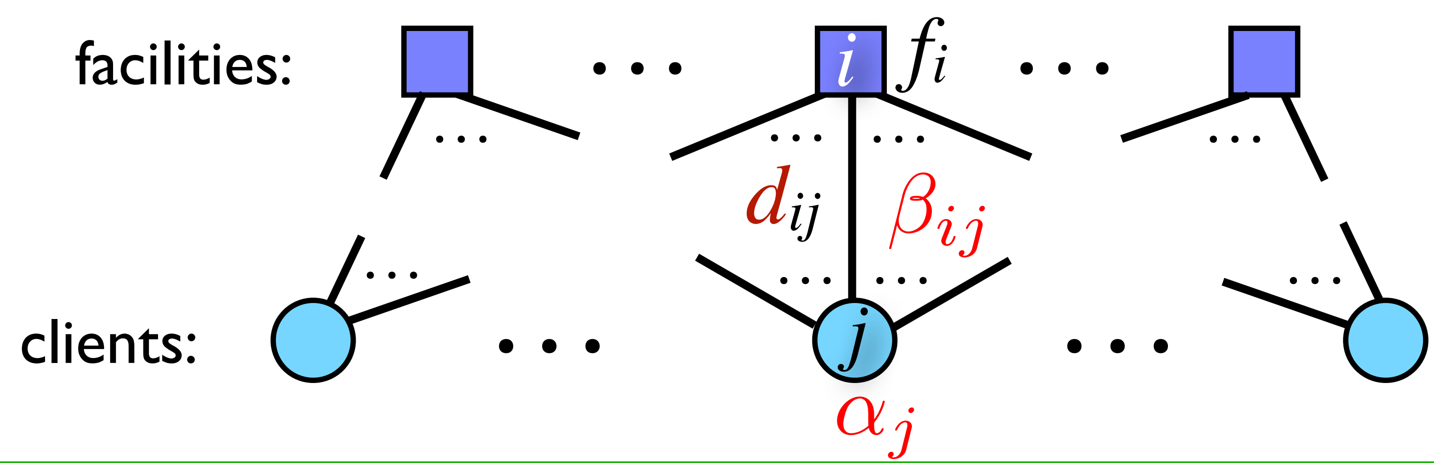
- upon $\alpha_j = d_{ij}$ for an ***unconnected*** client j and tentatively open facility i: client j is declared ***connected*** to facility i: and stop raising α_j;

- The events that occur at the same time are processed in arbitrary order.
- Fully paid facilities are tentatively open: $\sum_{j \in C} \beta_{ij} = f_i$
- Each client is connected through a tight edge ($\alpha_j - \beta_{ij} = d_{ij}$) to an open facility.
- Eventually all clients connect to tentatively opening facilities.

A client may have tight edges to more than one facilities:
We might have opened more facilities than necessary!



Phase I:

Initially $\boldsymbol{\alpha} = \mathbf{0}$, $\boldsymbol{\beta} = \mathbf{0}$, no facility is ***open***, no client is ***connected***;

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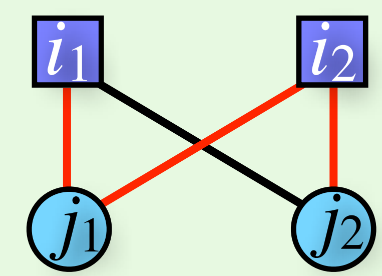
- upon $\sum_{j \in C} \beta_{ij} = f_i$: tentatively open facility i; all ***unconnected*** clients j with *paid* edge (i, j) to facility i are declared ***connected*** to facility i and stop raising α_j;
- upon $\alpha_j = d_{ij}$ for an ***unconnected*** client j and tentatively open facility i: client j is declared ***connected*** to facility i and stop raising α_j;

"i is j's connecting witness"



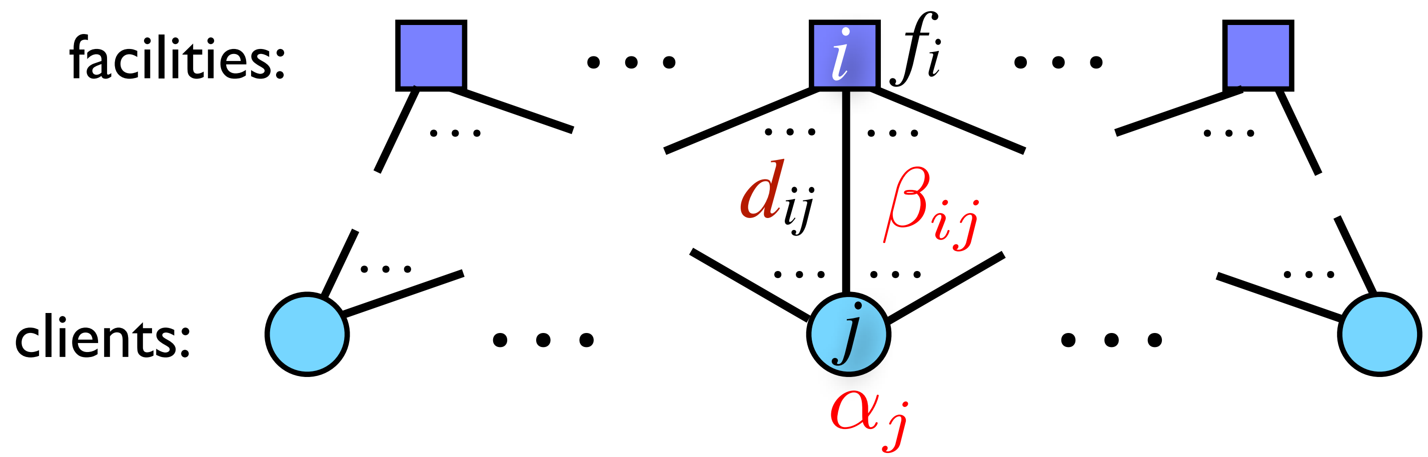
Phase II:

construct graph $G(V, E)$ where V={tentatively open facilities}

and $\{i_1, i_2\} \in E$ if $\exists$client j s.t. both $\beta_{i_1 j} > 0$ and $\beta_{i_2 j} > 0$ in **Phase I**;

find a maximal independent set I of G and permanently open facilities in I;

For each client j: if the facility i with $\beta_{ij} > 0$ has $i \in I$ or j's connecting witness i has $i \in I$, then j is connected to i (directly connected); otherwise, client j is connected to an arbitrary $i' \in I$ that is adjacent (in G) to j's connecting witness i (indirectly connected);



Primal:

$$\min \sum_{i\in F, j\in C} d_{ij}x_{ij} + \sum_{i\in F} f_i y_i$$

s.t.

$$y_i - x_{ij} \geq 0, \quad \forall i \in F, j \in C$$

$$\sum_{i\in F} x_{ij} \geq 1, \quad \forall j \in C$$

$$x_{ij}, y_i \in \{0,1\}, \quad \forall i \in F, j \in C$$

Dual-relax:

$$\max \sum_{j\in C} \alpha_j$$

s.t.

$$\alpha_j - \beta_{ij} \leq d_{ij}, \quad \forall i \in F, j \in C$$

$$\sum_{j\in C} \beta_{ij} \leq f_i, \quad \forall i \in F$$

$$\alpha_j, \beta_{ij} \geq 0, \quad \forall i \in F, j \in C$$

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$y_i = 1 \Rightarrow \sum_{j\in C} \beta_{ij} = f_i$; $\quad \beta_{ij} > 0 \Rightarrow y_i = x_{ij}$;

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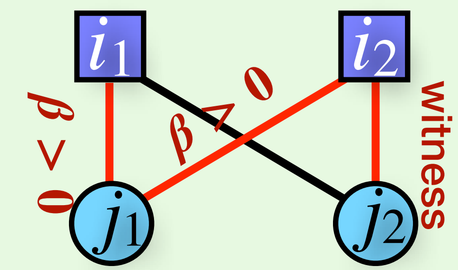

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construct graph $G(V,E)$ where V={tentatively open facilities}

and $\{i_1, i_2\} \in E$ if $\exists$client j s.t. both $\beta_{i_1 j} > 0$ and $\beta_{i_2 j} > 0$ in **Phase I**;

find a maximal independent set I of G and permanently open facilities in I;

For each client j: if the facility i with $\beta_{ij} > 0$ has $i \in I$ or j's connecting witness i has $i \in I$, then j is connected to i (directly connected); otherwise, client j is connected to an arbitrary $i' \in I$ that is adjacent (in G) to j's connecting witness i (indirectly connected);

Denote by ϕ the output mapping from clients to facilities.

$$SOL = \sum_{i\in I} f_i + \sum_{j\text{: directly connected}} d_{\phi(j)j} + \sum_{j\text{: indirectly connected}} d_{\phi(j)j} \le 3\sum_{j\in C}\alpha_j \le 3\ OPT$$

$$\le \sum_{j\text{: directly connected}} \alpha_j$$

triangle inequality + maximality of I

$$\le 3 \sum_{j\text{: indirectly connected}} \alpha_j$$

Phase I:

Initially $\boldsymbol{\alpha} = \mathbf{0}$, $\boldsymbol{\beta} = \mathbf{0}$, no facility is ***open***, no client is ***connected***;

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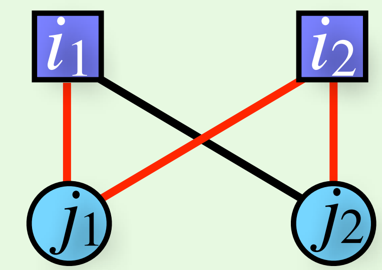


Phase II:

construct graph $G(V, E)$ where V={tentatively open facilities}

and $\{i_1, i_2\} \in E$ if $\exists$client j s.t. both $\beta_{i_1 j} > 0$ and $\beta_{i_2 j} > 0$ in **Phase I**;

find a maximal independent set I of G and permanently open facilities in I;

For each client j: if the facility i with $\beta_{ij} > 0$ has $i \in I$ or j's connecting witness i has $i \in I$, then j is connected to i (directly connected); otherwise, client j is connected to an arbitrary $i' \in I$ that is adjacent (in G) to j's connecting witness i (indirectly connected);

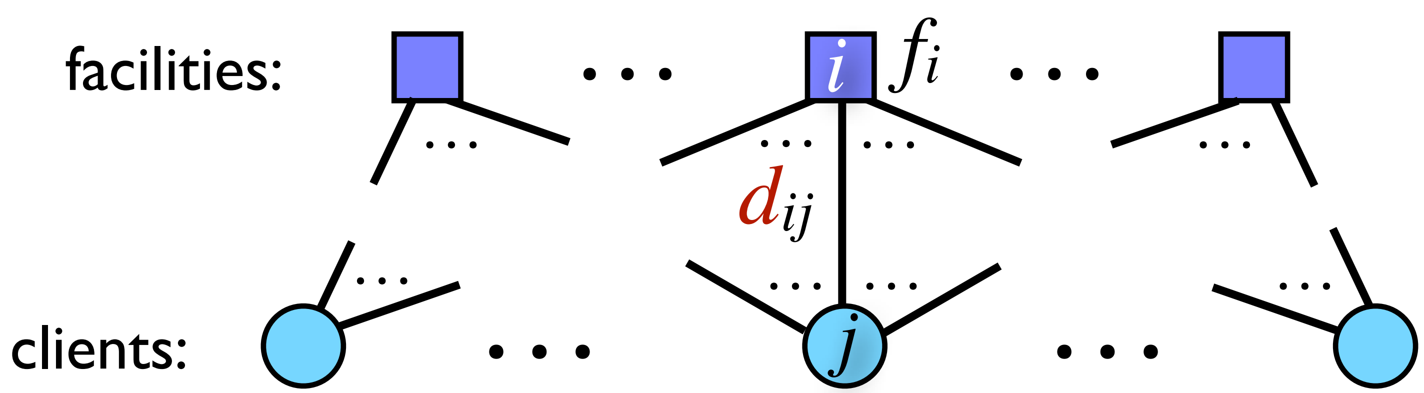
$$SOL \leq 3\ OPT$$

can be implemented discretely: in $O(m \log m)$ time, $m=|F||C|$

- sort all edges $(i,j) \in F \times C$ by non-decreasing d_{ij}
- dynamically maintain the time of next event by heap

Instance: set F of facilities; set C of clients;
facility opening costs f: $F \rightarrow [0, \infty)$;
connection metric d: $F \times C \rightarrow [0, \infty)$;

Find ϕ: $C \rightarrow I \subseteq F$ to minimize $\sum_{j \in C} d_{\phi(j),j} + \sum_{i \in I} f_i$



$$\min \quad \sum_{i \in F, j \in C} d_{ij} x_{ij} + \sum_{i \in F} f_i y_i$$

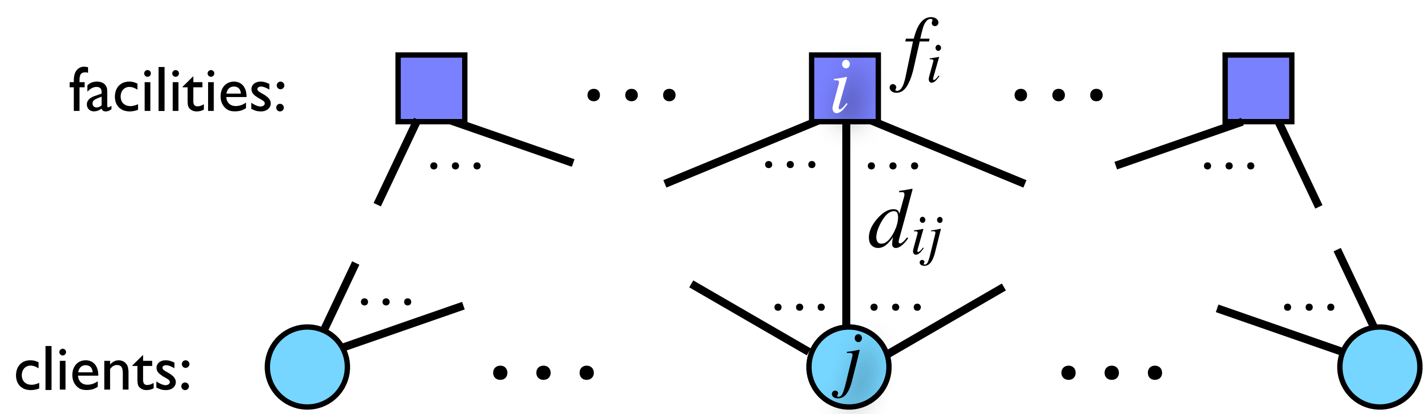
$$\text{s.t.} \quad y_i - x_{ij} \geq 0, \qquad \forall i \in F, j \in C$$

$$\sum_{i \in F} x_{ij} \geq 1, \qquad \forall j \in C$$

$$x_{ij}, y_i \in \{0, 1\}, \qquad \forall i \in F, j \in C$$

- Integrality gap = 3
- no poly-time <1.463-approx. algorithm unless **NP**=**P**
- [Li 2011] 1.488-approx. algorithm

k-Median



Instance: set F of facilities; set C of clients;
connection metric d: $F \times C \to [0, \infty)$;

Find a subset $I \subseteq F$ of $\leq k$ opening facilities and a way ϕ: $C \to I$ of connecting all clients to them such that the total cost $\sum_{j \in C} d_{\phi(j),j}$ is minimized.