Advanced Algorithms

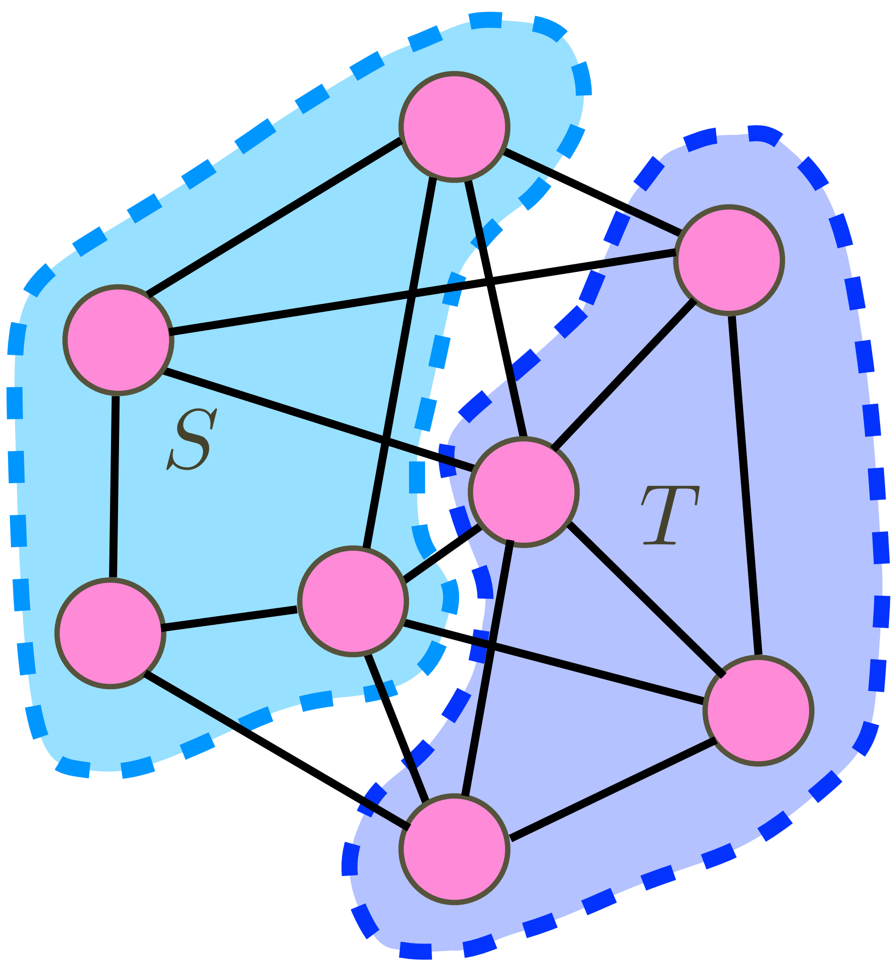
SDP-Based Algorithms

尹一通 Nanjing University, 2022 Fall

Max-Cut

Instance: An undirected graph $G(V, E)$.

Solution: A bipartition of V into S and T that maximizes the cut $E(S,T) = \big\{\{u,v\} \in E \mid u \in S \wedge v \in T\big\}$.



- **NP**-hard.
- One of Karp's 21 **NP**-complete problems (reduction from the *Partition* problem).
- a typical **Max-CSP** (Constraint Satisfaction Problem).
- *Greedy* is $1/2$-approximate.
- *Local search* is $1/2$-approximate.

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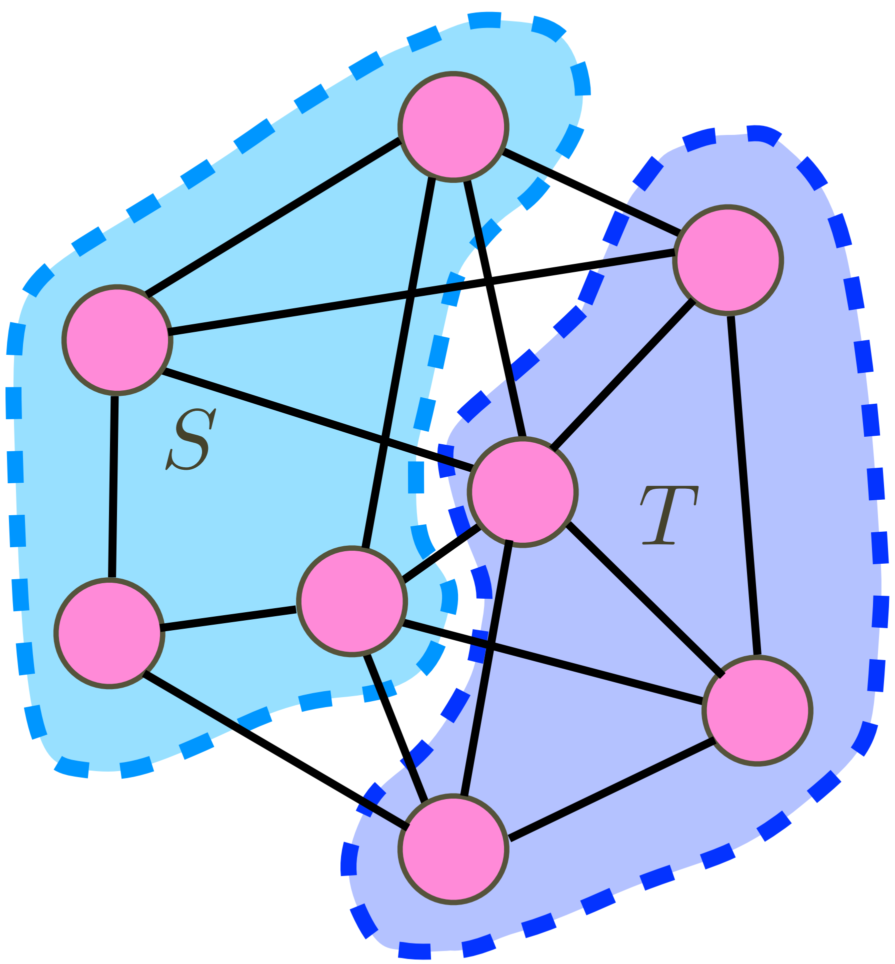

max $\sum_{uv \in E} y_{uv}$

not linear

s.t. $y_{uv} \leq |x_u - x_v|, \quad \forall uv \in E$

$x_v \in \{0, 1\}, \quad \forall v \in V$

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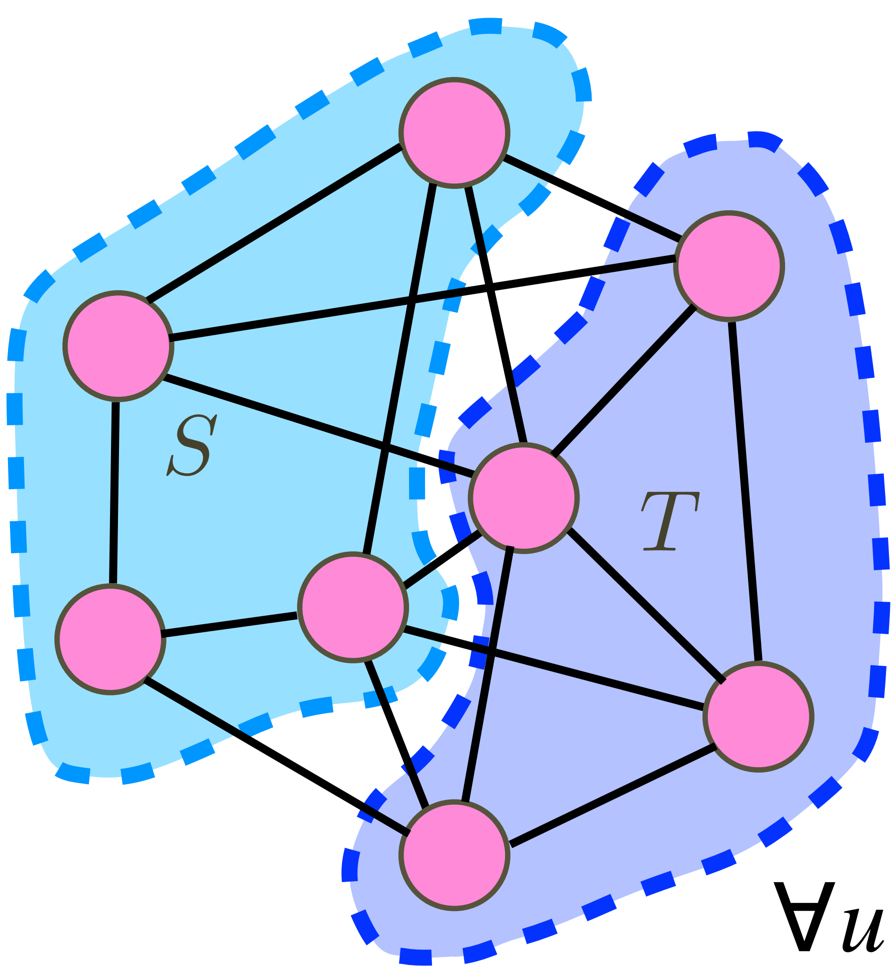

$$\max \quad \sum_{uv \in E} y_{uv}$$

$$\text{s.t.} \quad y_{uv} \le y_{uw} + y_{wv}, \qquad \forall u,v,w \in V$$

$$y_{uv} + y_{uw} + y_{wv} \le 2, \qquad \forall u,v,w \in V$$

$$y_{uv} \in \{0,1\}, \qquad \forall u,v \in V$$

$\forall u,v,w \in V$: 0 or 2 of $\{u,v\}, \{v,w\}, \{u,w\}$ are "crossing pairs"

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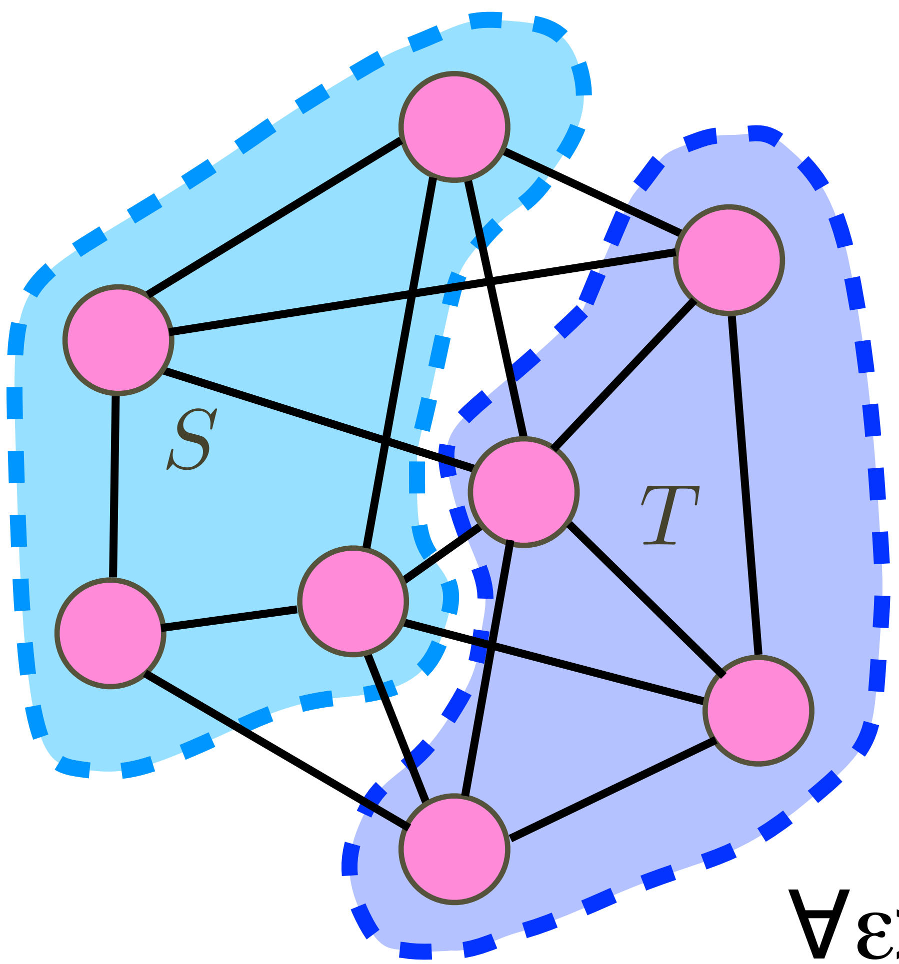


$$\max \quad \sum_{uv \in E} y_{uv}$$

integrality gap ≥ 2

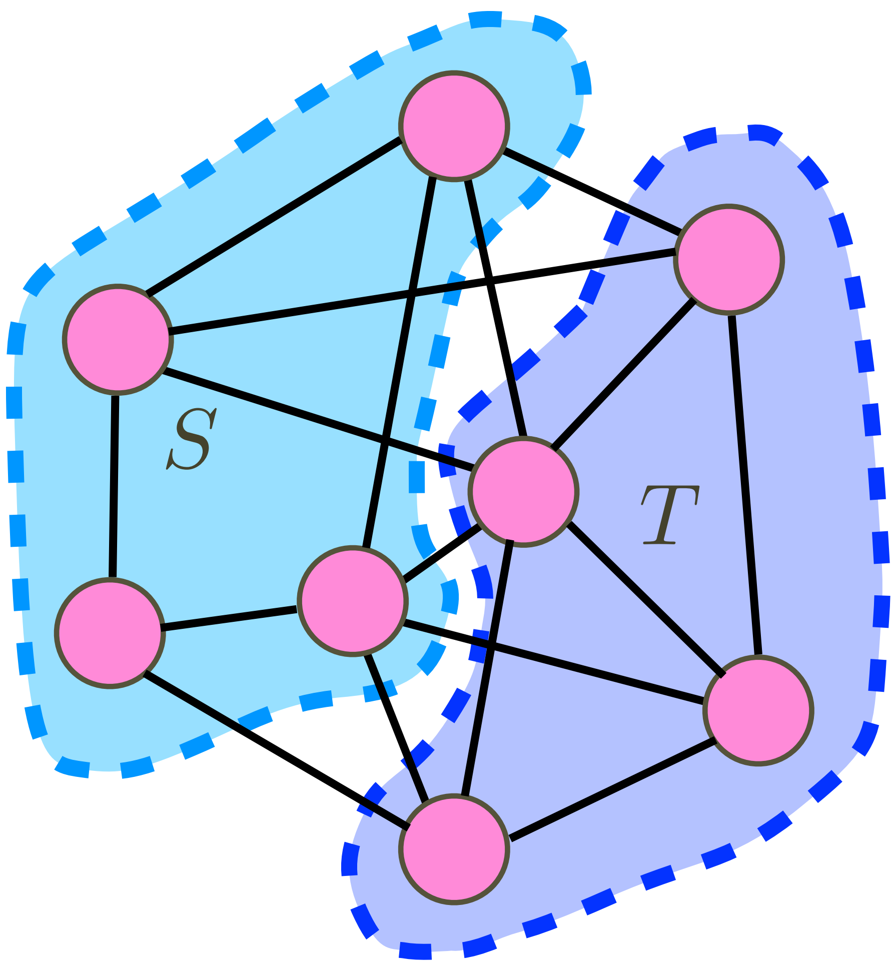
$$\text{s.t.} \quad y_{uv} \le y_{uw} + y_{wv}, \qquad \forall u, v, w \in V$$

$$y_{uv} + y_{uw} + y_{wv} \le 2, \qquad \forall u, v, w \in V$$

$$y_{uv} \in \{0, 1\}, \qquad \forall u, v \in V$$

$\forall \varepsilon > 0$: $\exists G$ s.t. $\mathrm{OPT}_{\mathrm{LP}}(G)/\mathrm{OPT}_{\mathrm{IP}}(G) > 2-\varepsilon$

Quadratic Program for Max-Cut



$$\max \sum_{uv \in E} y_{uv}$$

$$\text{s.t.} \quad y_{uv} \le |x_u - x_v|, \quad \forall uv \in E$$

$$x_v \in \{0,1\}, \quad \forall v \in V$$

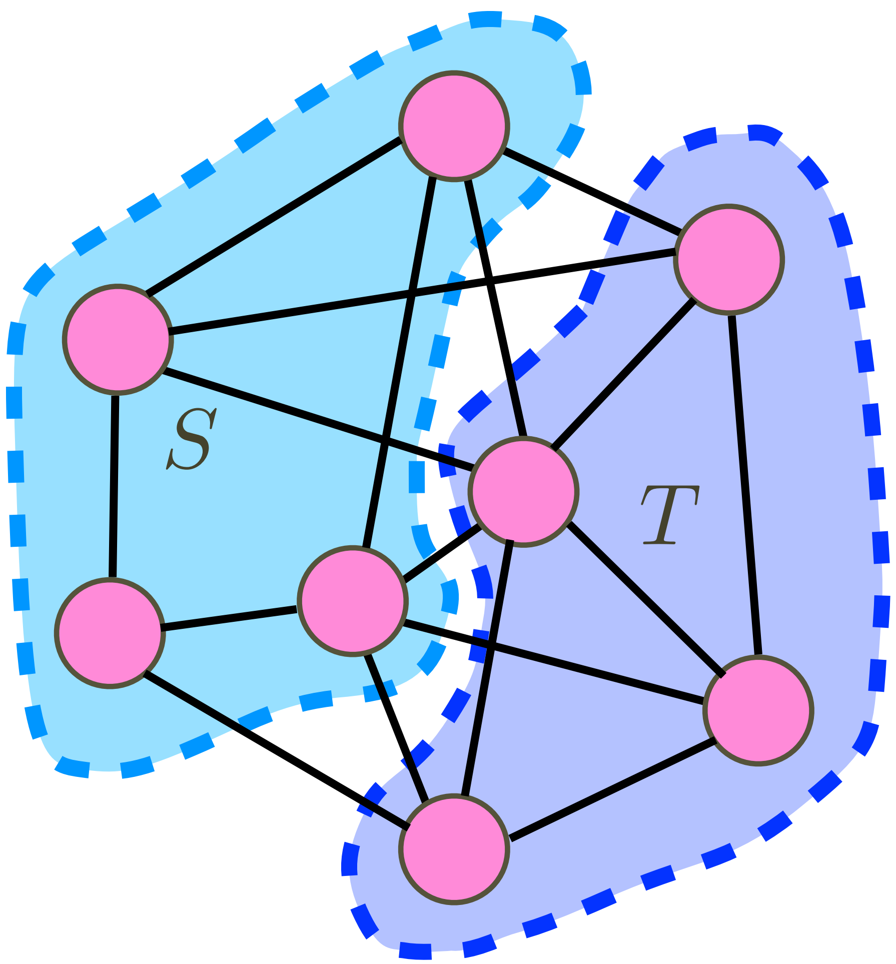
quadratic program:

$$\max \sum_{uv \in E} y_{uv}$$

$$\text{s.t.} \quad y_{uv} \le \tfrac{1}{2}(1 - x_u x_v), \quad \forall uv \in E$$

$$x_v \in \{-1,1\}, \quad \forall v \in V$$

Quadratic Program for Max-Cut



max $\sum_{uv \in E} y_{uv}$

s.t. $y_{uv} \leq |x_u - x_v|, \quad \forall uv \in E$

$x_v \in \{0, 1\}, \quad \forall v \in V$

strictly quadratic program:

max $\sum_{uv \in E} \frac{1}{2}(1 - x_u x_v)$

s.t. $x_v^2 = 1, \quad \forall v \in V$

Nonlinear, non-convex!

Relaxation

strictly quadratic program:

$$\begin{aligned} \max \quad & \sum_{uv \in E} \tfrac{1}{2}(1 - x_u x_v) \\ \text{s.t.} \quad & x_v^2 = 1, \qquad \forall v \in V \end{aligned}$$

relax to vector program: semidefinite program (SDP)

$$\begin{aligned} \max \quad & \frac{1}{2} \sum_{uv \in E} (1 - \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle) \\ \text{s.t.} \quad & \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle = 1, \qquad \forall v \in V \\ & \boldsymbol{x}_v \in \mathbb{R}^n, \qquad \forall v \in V \end{aligned}$$

inner-products:

$$\langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = \sum_{i=1}^{n} x_v(i) x_u(i)$$

$n = |V|$

Positive Semidefiniteness (PSD)

Definition (Positive Semidefiniteness):

A symmetric square matrix $A \in \mathbb{R}^{n \times n}$ is said to be **positive semidefinite**, denoted $A \succcurlyeq 0$, if $\forall x \in \mathbb{R}^n$, $x^{\mathrm{T}} A x \geq 0$.

Theorem:

For symmetric $A \in \mathbb{R}^{n \times n}$, the followings are equivalent:

- $A \succcurlyeq 0$
- all eigenvalues $\lambda(A) \geq 0$
- $A = B^{\mathrm{T}} B$ for some $B \in \mathbb{R}^{n \times n}$

Semidefinite Programing (SDP)

- Given $C, A_1, \dots, A_k \in \mathbb{R}^{n \times n}$ and $b_1, b_2, \dots, b_k \in \mathbb{R}$

maximize $\quad \mathrm{tr}(C^T Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} y_{ij}$

subject to $\quad \mathrm{tr}(A_r^T Y) \le b_r \,, \qquad \forall 1 \le r \le k$

$Y \succcurlyeq 0$

symmetric $Y \in \mathbb{R}^{n \times n}$

Semidefinite Programing (SDP)

- Given $C, A_1, \ldots, A_k \in \mathbb{R}^{n\times n}$ and $b_1, b_2, \ldots, b_k \in \mathbb{R}$

maximize $\quad \mathrm{tr}(C^T Y) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} y_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle$

subject to $\quad \mathrm{tr}(A_r^T Y) \le b_r\,, \qquad \forall 1 \le r \le k$

$\quad\quad\quad \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(r)}\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle \le b_r$

$Y \succcurlyeq 0$

symmetric $Y \in \mathbb{R}^{n\times n}$

$\iff \begin{cases} Y = V^{\mathrm{T}} V \\ V \in \mathbb{R}^{n\times n} \end{cases}$

V's column vectors: $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^n$

Semidefinite Programing (SDP)

- Given $C, A_1, \ldots, A_k \in \mathbb{R}^{n \times n}$ and $b_1, b_2, \ldots, b_k \in \mathbb{R}$

$$\text{maximize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$$

$$\text{subject to} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{(r)} \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle \leq b_r \,, \qquad \forall 1 \leq r \leq k$$

$$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^n$$

- SDPs are LPs for inner products and generalize LPs
- SDPs are convex programs:
  - poly-time solvable by ellipsoid method

SDP Relaxation

- Quadratic program for max-cut in $G(V, E)$ on $n = |V|$ vertices:

$$\begin{aligned} \text{max} \quad & \sum_{uv \in E} \tfrac{1}{2}(1 - x_u x_v) \\ \text{s.t.} \quad & x_v^2 = 1, \qquad \forall v \in V \end{aligned}$$

- SDP relaxation:

$$\begin{aligned} \text{max} \quad & \frac{1}{2} \sum_{uv \in E} (1 - \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle) \\ \text{s.t.} \quad & \|\boldsymbol{x}_v\|_2 = 1, \qquad \forall v \in V \\ & \boldsymbol{x}_v \in \mathbb{R}^n, \qquad \forall v \in V \end{aligned}$$

Random Hyperplane Rounding

- max-cut in $G(V, E)$ with $n = |V|$:

$$\max \quad \sum_{uv \in E} \tfrac{1}{2}(1 - x_u x_v)$$

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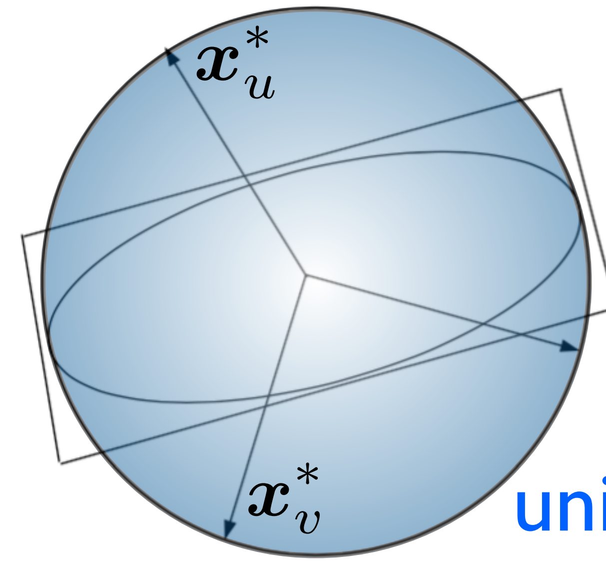

Goemans-Williamson'95:

SDP relaxation:

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$$\boldsymbol{x}_v \in \mathbb{R}^n, \qquad \forall v \in V$$

Optima SDP solution: $\boldsymbol{x}_v^*$

Rounding:

uniform random unit vector

$$\boldsymbol{u} \in \mathbb{R}^n, \quad \|\boldsymbol{u}\|_2 = 1$$

$$\hat{x}_v = \operatorname{sgn}(\langle \boldsymbol{x}_v^*, \boldsymbol{u} \rangle)$$

Random Hyperplane Rounding

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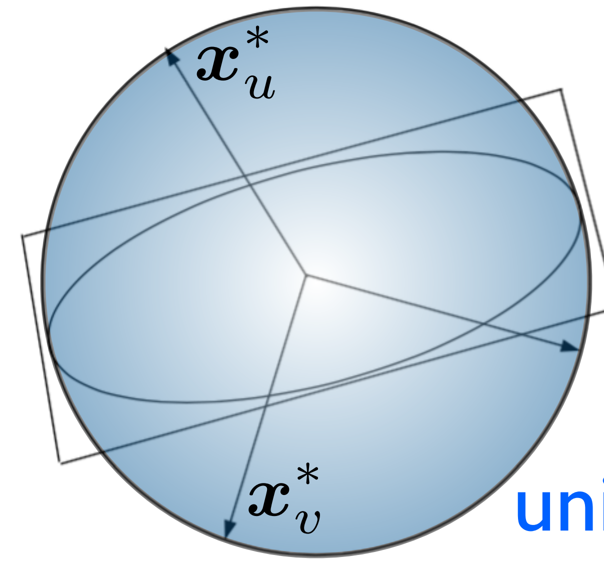

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Rounding:

random
$\boldsymbol{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$

each $r_i \sim N(0,1)$ *i.i.d.*
normal distribution

$$\hat{x}_v = \mathrm{sgn}(\langle \boldsymbol{x}_v^*, \boldsymbol{r}\rangle)$$

$$\max \sum_{uv \in E} \frac{1}{2}(1 - x_u x_v)$$

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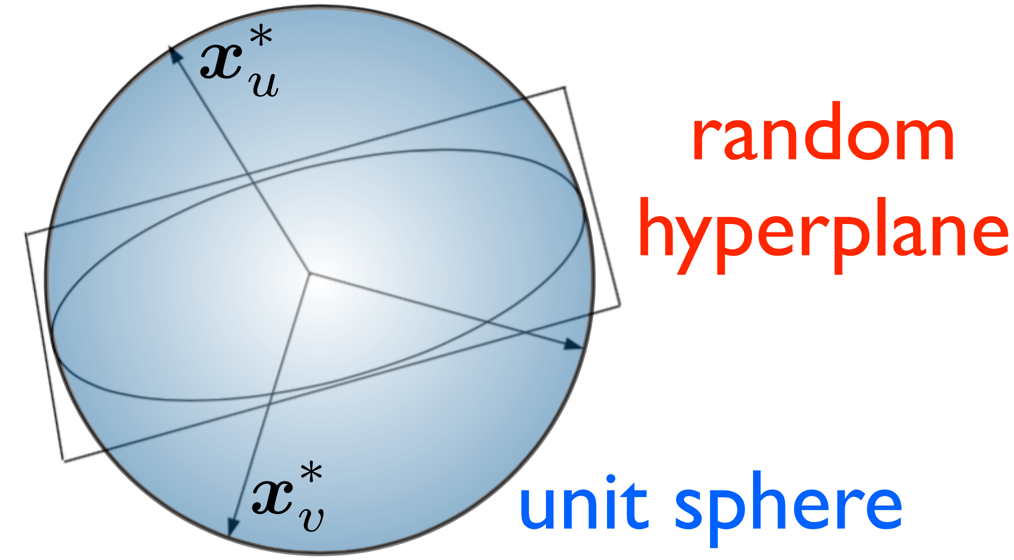

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normal distribution

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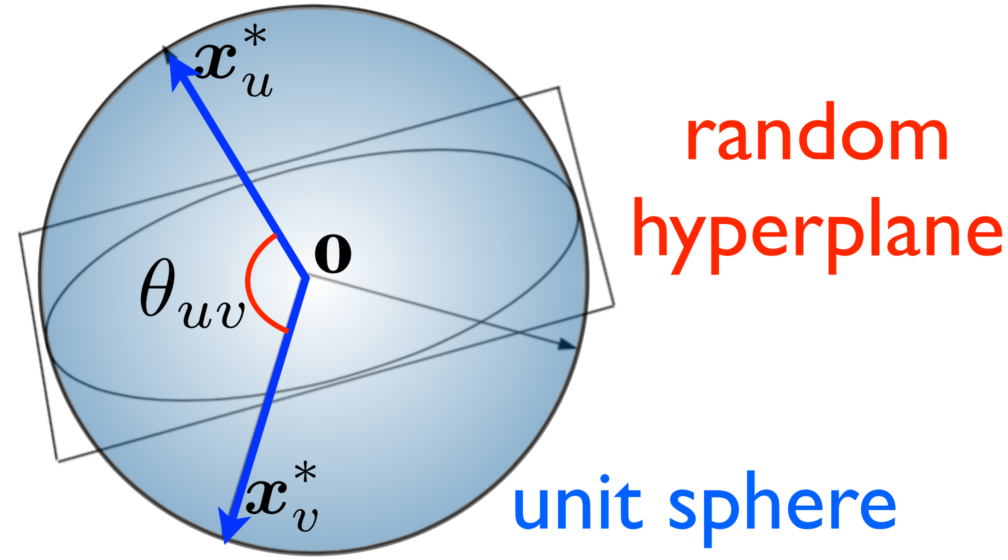
$\boldsymbol{u} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|_2}$ is uniform random unit vector

spherically symmetric:

$$\Pr[(r_1, \ldots, r_n)] = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-r_i^2/2} = (2\pi)^{-d/2} e^{-\|\boldsymbol{r}\|^2/2}$$

$$\max \sum_{uv\in E} \frac{1}{2}(1 - x_u x_v)$$

$$\text{s.t.} \quad x_v^2 = 1, \qquad \forall v \in V$$



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normal distribution

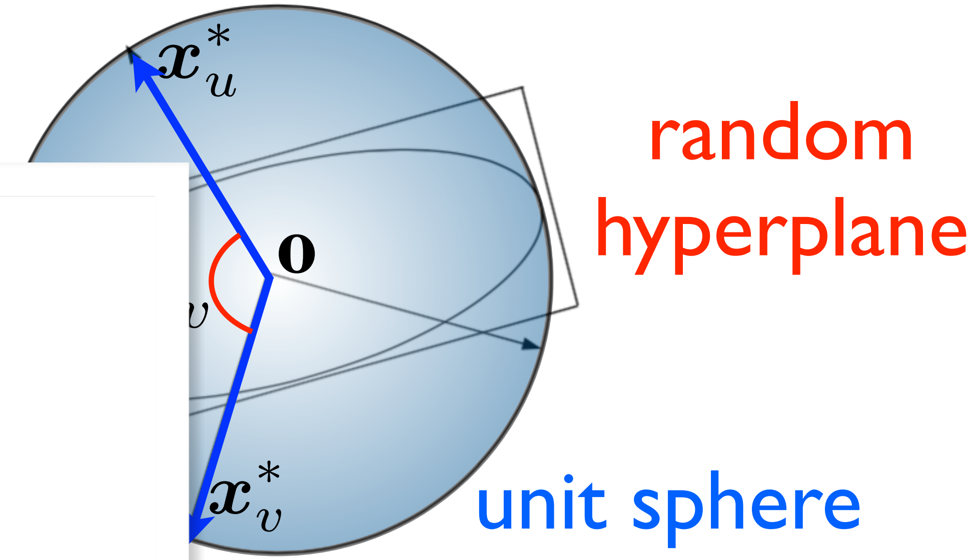
$$\hat{x}_v = \operatorname{sgn}(\langle \boldsymbol{x}_v^*, \boldsymbol{r}\rangle)$$

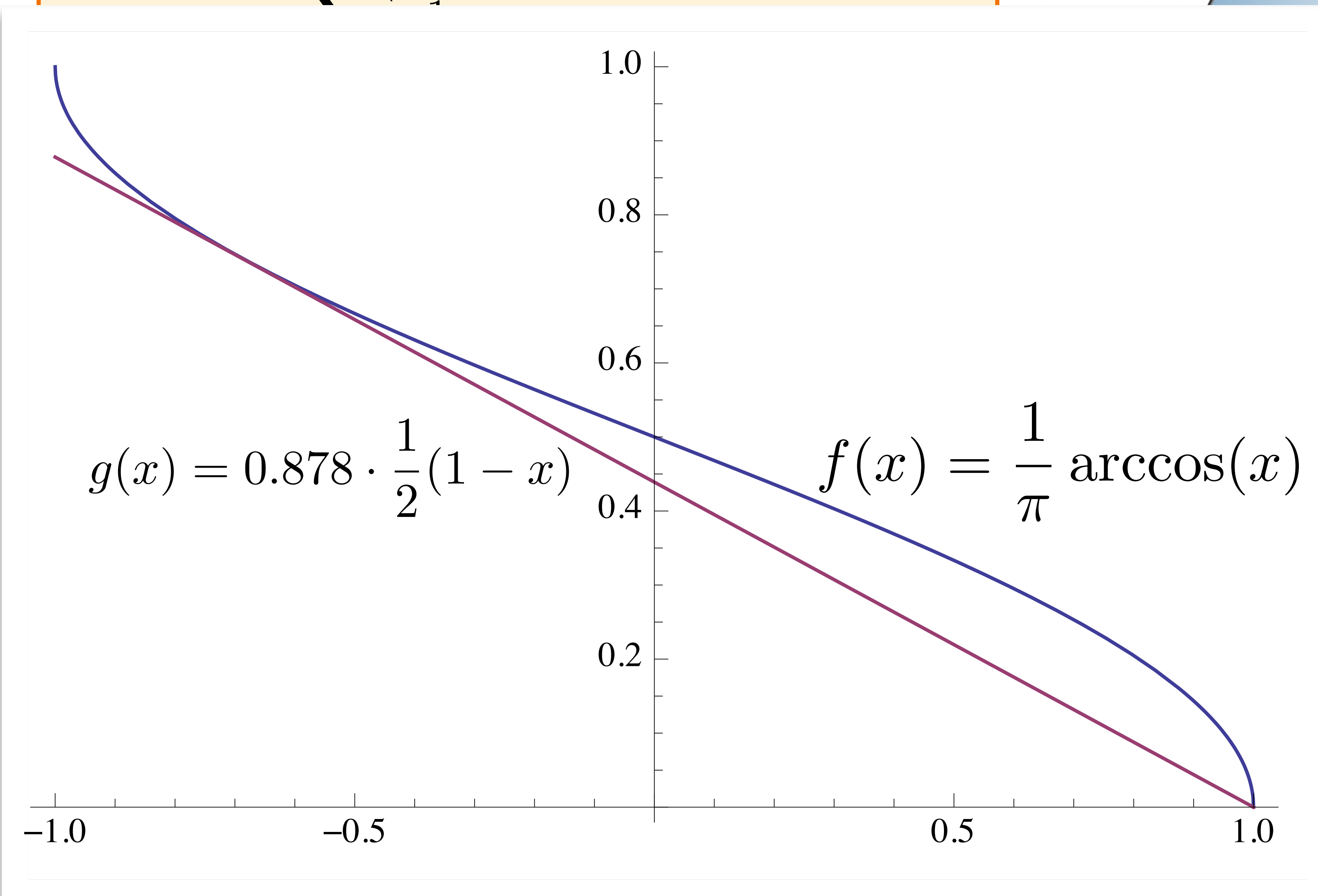
$$\mathbf{E}[\text{cut}] = \sum_{uv\in E} \Pr[\operatorname{sgn}(\langle \boldsymbol{x}_u^*, \boldsymbol{r}\rangle) \neq \operatorname{sgn}(\langle \boldsymbol{x}_v^*, \boldsymbol{r}\rangle)] = \sum_{uv\in E} \frac{\theta_{uv}}{\pi}$$

$$= \sum_{uv\in E} \frac{\arccos\langle \boldsymbol{x}_u^*, \boldsymbol{x}_v^*\rangle}{\pi}$$

where $\theta_{uv} = \angle \boldsymbol{x}_u^* \boldsymbol{o} \boldsymbol{x}_v^*$

$$\|\boldsymbol{x}_u^*\| \cdot \|\boldsymbol{x}_v^*\| \cdot \cos\theta_{uv} = \langle \boldsymbol{x}_u^*, \boldsymbol{x}_v^*\rangle$$





$r_2, \ldots, r_n) \in \mathbb{R}^n$

$N(0,1)$ *i.i.d.*

mal distribution

Optima SDP solution: $\boldsymbol{x}_v^*$

$\hat{x}_v = \mathrm{sgn}(\langle \boldsymbol{x}_v^*, \boldsymbol{r} \rangle)$

$$\mathbf{E}[\text{cut}] = \sum_{uv \in E} \frac{\arccos\langle \boldsymbol{x}_u^*, \boldsymbol{x}_v^* \rangle}{\pi} \geq \alpha \sum_{uv \in E} \frac{1}{2}(1 - \langle \boldsymbol{x}_u^*, \boldsymbol{x}_v^* \rangle) = \alpha \, \mathrm{OPT}_{\mathrm{SDP}}$$

$$\geq \alpha \, \mathrm{OPT}$$

where $\alpha = \inf_{x \in [-1,1]} \frac{2\arccos(x)}{\pi(1-x)} = 0.87856\ldots$

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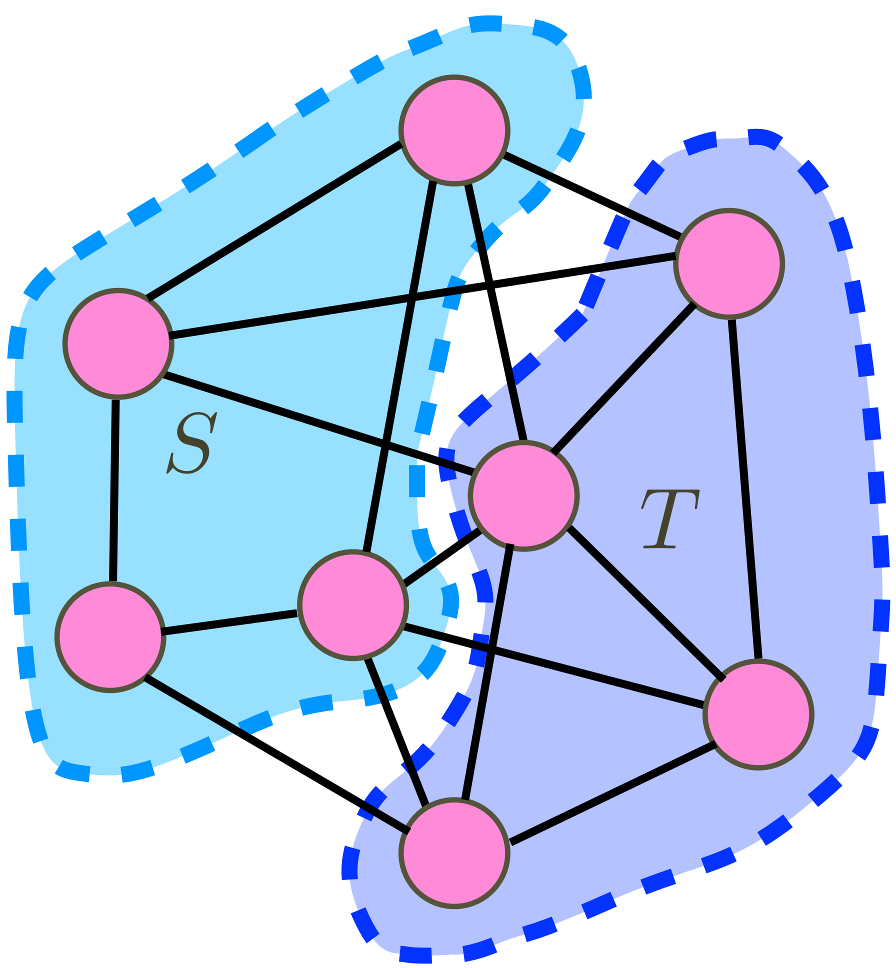


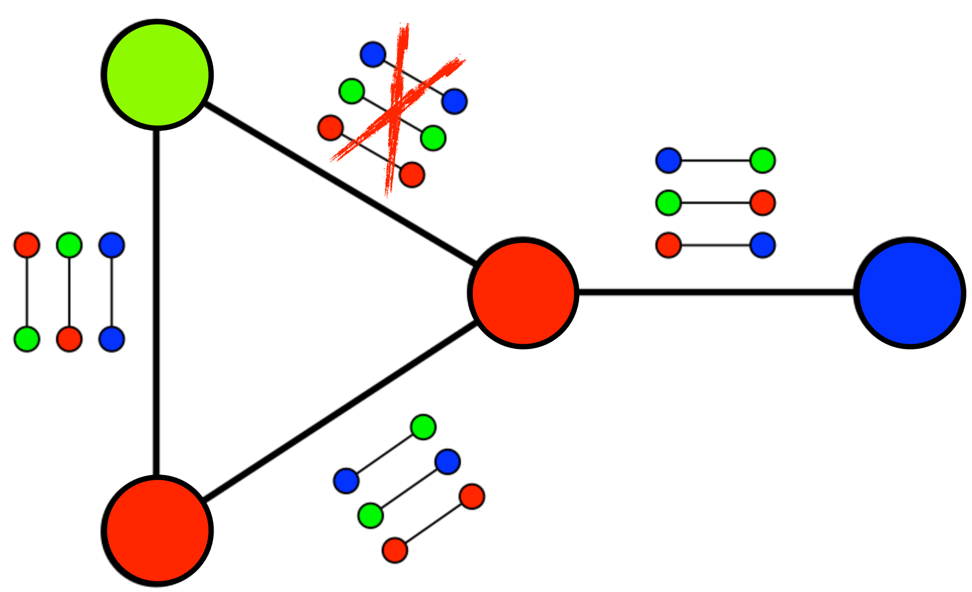
- A typical **Max-CSP**.
- Rounding **SDP relaxation** is 0.878~-approximate.
- Can be derandomized via conditional expectations.
- Assuming the **unique games conjecture**: no poly-time algorithm has approximation ratio < 0.878~

Unique Games Conjecture

Unique Label Cover (ULC):

Instance: An undirected graph $G(V,E)$; q colors; each edge e associated with a bijection ϕ_e: $[q]\to[q]$.

A coloring $\sigma\in[q]^V$ satisfies the constraint of the edge $e=(u,v)\in E$ if $\phi_e(\sigma_u)=\phi_e(\sigma_v)$.



Unique Games Conjecture:
(Khot 2002)

$\forall\varepsilon$, $\exists q$ such that it is ~~NP~~ ??? -hard to distinguish between ULC instances:

- $>1-\varepsilon$ fraction of edges satisfied by a coloring;
- no more than ε fraction of edges satisfied by any coloring;