Advanced Algorithms (Fall 2023)

Rounding Data and Dynamic Programming

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Outline

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Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

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- Motivation: you have budget W, and want to buy a subset of items of maximum total value

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- Optimum takes item 2 and greedy takes item 1.

Fractional Knapsack Problem

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Greedy Algorithm for Fractional Knapsack

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Theorem Greedy algorithm gives the optimum solution for fractional knapsack.

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W' - w_i] + v_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

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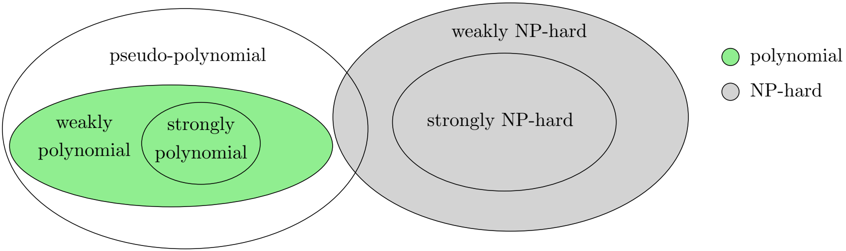
A: No.

- The input size is polynomial in n and $\log W$; running time is polynomial in n and W.
- The running time is pseudo-polynomial.

- n: number of integers $\quad W$: maximum value of all integers
- pseudo-polynomial time: $\mathrm{poly}(n, W)$ (e.g., DP for Knapsack)
- weakly polynomial time: $\mathrm{poly}(n, \log W)$ (e.g., Euclidean Algorithm for Greatest Common Divisor)
- strongly polynomial time: $\mathrm{poly}(n)$ time, assuming basic operations on integers taking $O(1)$ time (e.g., Kruskal's)
- weakly NP-hard: NP-hard to solve in time $\mathrm{poly}(n, \log W)$
- strongly NP-hard: NP-hard even if $W = \mathrm{poly}(n)$



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Idea for improving the running time to polynomial

- If we make weights upper bounded by $\mathrm{poly}(n)$, then pseudo-polynomial time becomes polynomial time
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- However, coarsening weights will change the problem.
- $\frac{\text{weight budget constraint} \quad : \quad \text{hard}}{\text{maximum value requirement} \quad : \quad \text{soft}}$
- We coarsen the values instead
- In the DP, we use values as parameters

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$$f[i, V'] = \begin{cases} 0 & V' \leq 0 \\ \infty & i = 0, V' > 0 \\ \min \left\{ \begin{array}{c} f[i-1, V'] \\ f[i-1, V' - v'_i] + w_i \end{array} \right\} & i > 0, V' > 0 \end{cases}$$

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- Output A times the largest V' such that $f[n, V'] \leq W$.

- Instance $\mathcal{I}$: $(v_1, v_2, \cdots, v_n)$ $\quad$ opt: optimum value of $\mathcal{I}$
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Theorem There is a $(1+\epsilon)$-approximation for the knapsack problem in time $O(\frac{n^3}{\epsilon})$.

Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms A_ϵ, where A_ϵ for every $\epsilon > 0$ is a (polynomial-time) $(1 \pm \epsilon)$-approximation algorithm.

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- Vertex cover? Maximum independent set?

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Input: n jobs index as $[n]$

each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$

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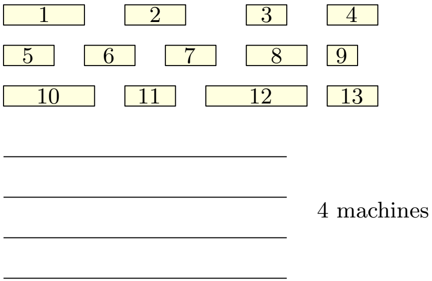

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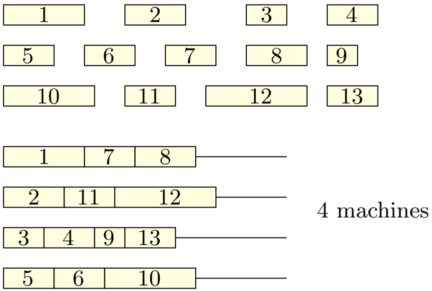

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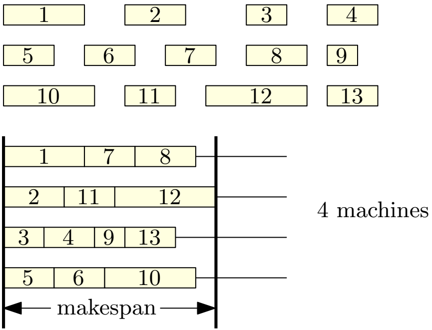

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$$\text{alg} \leq p_{\max} + \frac{1}{m} \cdot \Big(\sum_{j \in [n]} p_j - p_{\max}\Big) = \Big(1 - \frac{1}{m}\Big) p_{\max} + \frac{1}{m} \sum_{j \in [n]} p_j$$

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4: add small jobs to schedule greedily

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- $k := |\{p'_j : j \in B\}|$: #(distinct rounded sizes)

  $k \leq 1 + \log_{1+\epsilon} \frac{p_{\max}}{\epsilon p_{\max}} = O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)$

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- $\{q_1, q_2, \cdots, q_k\} := \{p'_j : j \in B\}$: the k distinct rounded sizes

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- $p'_j := \max\{p_{\max}(1+\epsilon)^t \leq p_j : t \in \mathbb{Z}\}, \forall j \in B$

  p'_j is the rounded size of j
- $k := |\{p'_j : j \in B\}|$: #(distinct rounded sizes)

  $k \leq 1 + \log_{1+\epsilon} \frac{p_{\max}}{\epsilon p_{\max}} = O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)$
- $\{q_1, q_2, \cdots, q_k\} := \{p'_j : j \in B\}$: the k distinct rounded sizes
- $n_1, \cdots, n_k$: #(big jobs) with rounded sizes being $q_1, \cdots, q_k$

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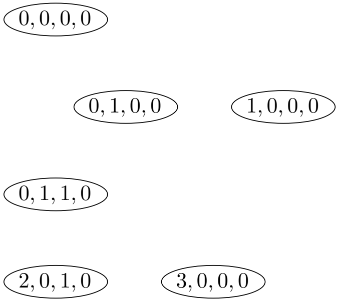
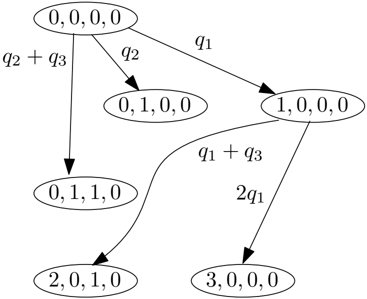
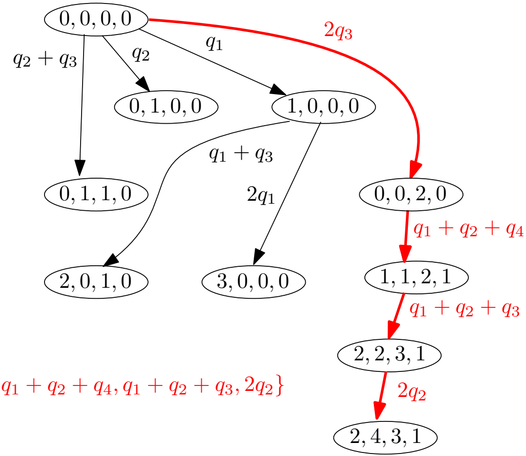
- problem can be solved in $O(m \cdot |E|)$ time using DP
- $O(m \cdot |E|) = O(m \cdot n^{2k}) = n^{O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)}$.


0, 0, 0, 0
0, 1, 0, 0
1, 0, 0, 0
0, 1, 1, 0
2, 0, 1, 0
3, 0, 0, 0



0, 0, 0, 0
q2 + q3
q2
q1
0, 1, 0, 0
1, 0, 0, 0
q1 + q3
0, 1, 1, 0
2q1
2, 0, 1, 0
3, 0, 0, 0



$\text{cost} = \max\{2q_3, q_1 + q_2 + q_4, q_1 + q_2 + q_3, 2q_2\}$

Analysis of Algorithm for Big Jobs

- $\mathcal{I}_B$: instance $(p_j)_{j\in B}$ $\quad \mathrm{opt}_B$: its optimum makespan
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Analysis of Algorithm for Big Jobs

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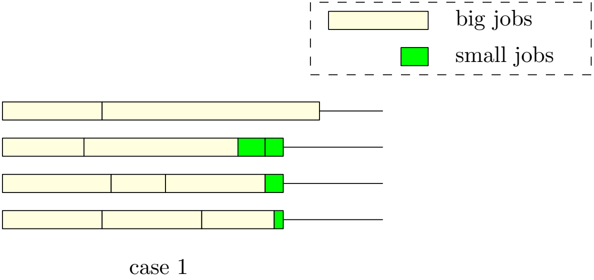
Adding small jobs to schedule

```
starting from the schedule for big jobs
for every small job j do
    add j to the machine with the smallest load
```

Outline

Analysis of the Final Algorithm



- Case 1: makespan is not increased by small jobs

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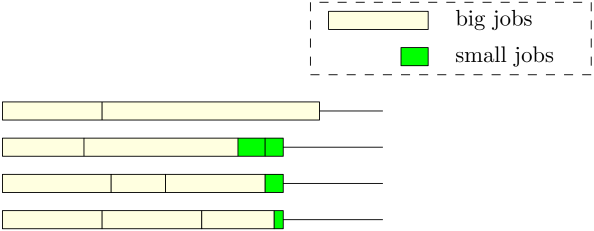
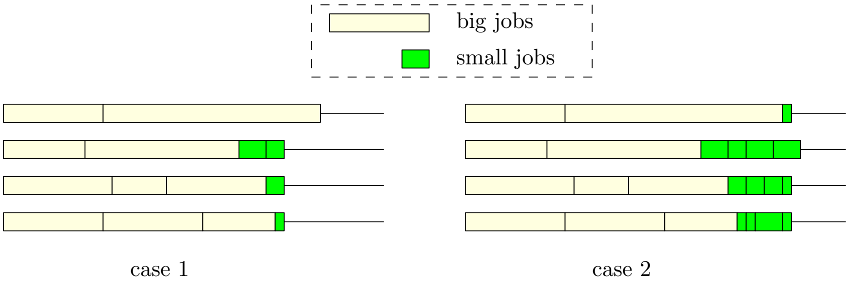


case 1

- Case 1: makespan is not increased by small jobs

$$\text{alg} \leq (1+\epsilon)\text{opt}_B \leq (1+\epsilon)\text{opt}.$$



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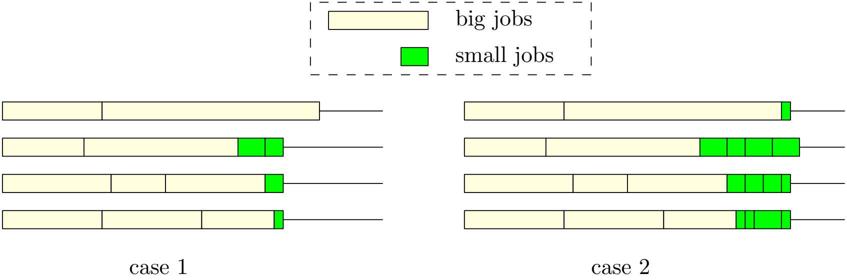

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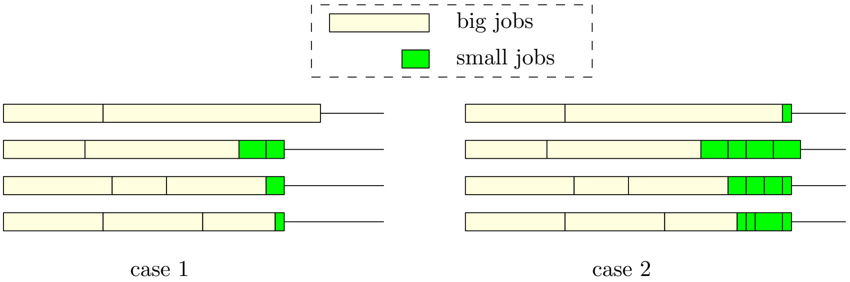

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$$\text{alg} \leq \epsilon \cdot p_{\max} + \frac{1}{m}\sum_{j\in[n]} p_j \leq \epsilon \cdot \text{opt} + \text{opt} = (1+\epsilon)\cdot \text{opt}.$$

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Bin Packing

Input: n items indexed by $[n]$, with sizes $s_1, s_2, \cdots, s_n \in (0, 1]$

Output: a packing of items into smallest number of bins of capacity 1.

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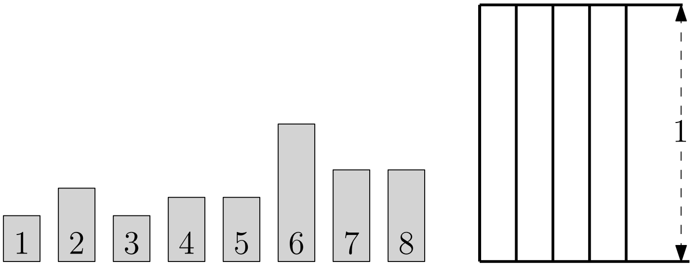

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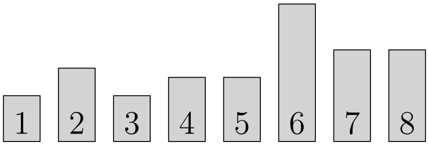

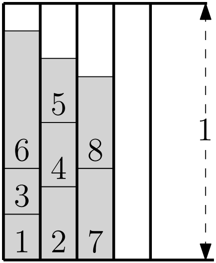

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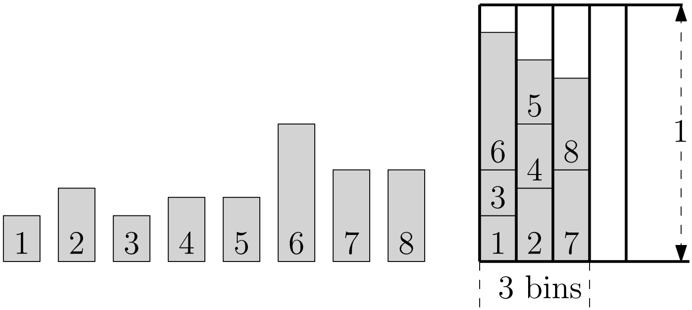

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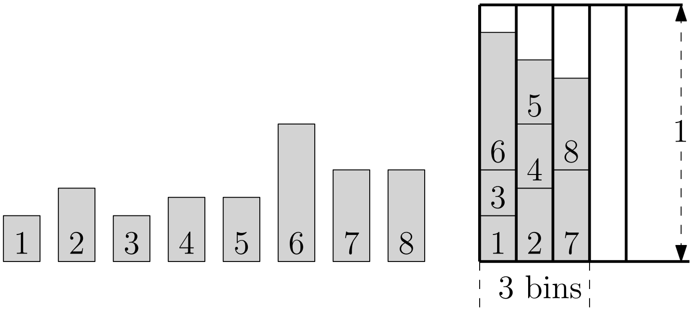

	#containers	container capacity
bin packing	objective	fixed
scheduling	fixed	objective

First-Fit

```
initially there are 0 bins
for i ← 1 to n do
    if item i fits into an existing bin then put i into the bin
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Lemma The greedy algorithm gives a 2-approximation.

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Proof.

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Equal Partition

Input: n numbers $x_1, x_2, \cdots, x_n \in \mathbb{Z}_{>0}$

Output: decide if there is a partition of $[n]$ into A and B such that $\sum_{i \in A} x_i = \sum_{i \in B} x_i$

Theorem Equal Partition is (weakly) NP-hard.

- The approximation ratio is bad only when opt is small
- NP-hard to decide between $\mathrm{opt} \leq 2$ and $\mathrm{opt} \geq 3$.
- Open: NP-hard to decide between $\mathrm{opt} \leq 100$ and $\mathrm{opt} \geq 102$?

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Theorem First-Fit-Decreasing algorithm outputs a solution using at most $(11/9) \cdot \text{opt} + 4$ bins. That is, it is an asymptotic $11/9$-approximation.

Def. An asymptotic polynomial-time approximation scheme (APTAS) for minimization problems is a family of algorithms A_ϵ along with a constant $c \geq 0$, where algorithm A_ϵ for every $\epsilon > 0$ returns a solution of value at most $(1+\epsilon)\mathrm{opt} + c$ in polynomial time.

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Theorem For any fixed $\epsilon > 0$, there is a polynomial time algorithm that, given a bin-packing instance $\mathcal{I}$, outputs a solution with at most $(1+\epsilon)\mathrm{opt} + 1$ bins.

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- truncate item sizes to obtain $\mathcal{I}'$, using DP to solve $\mathcal{I}'$
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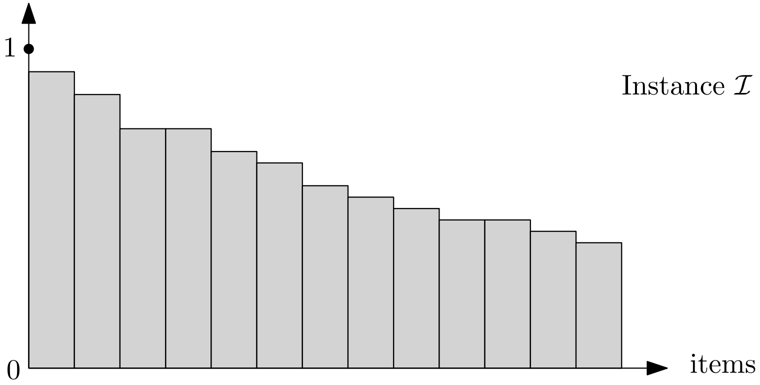
- general instance: pack big items using truncation + DP, then use First-Fit to pack small items

Outline

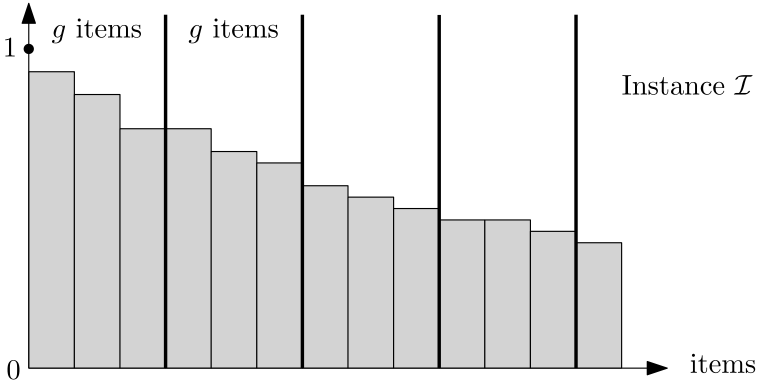
Construction of Instance $\mathcal{I}'$

1: sort items in non-increasing sizes



Construction of Instance $\mathcal{I}'$

1: sort items in non-increasing sizes
2: partition items into groups of size g



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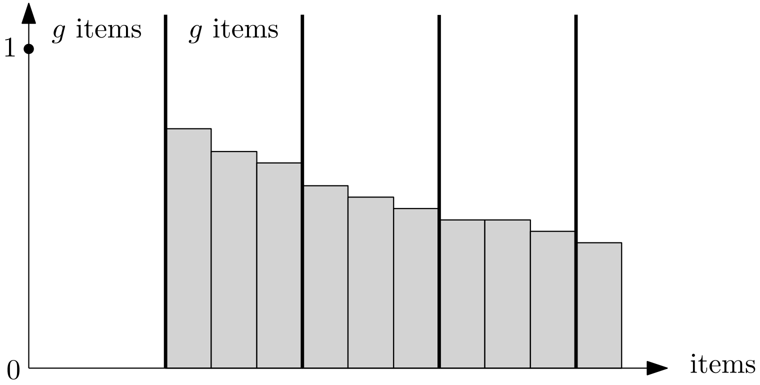

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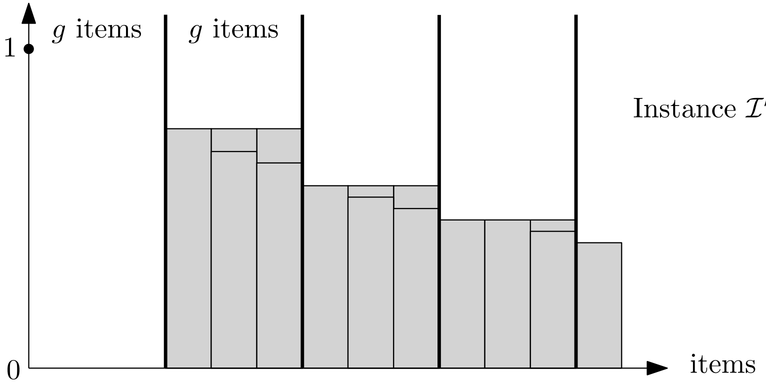

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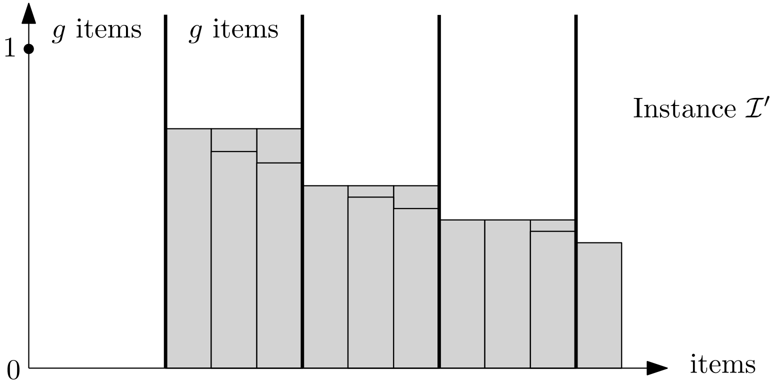

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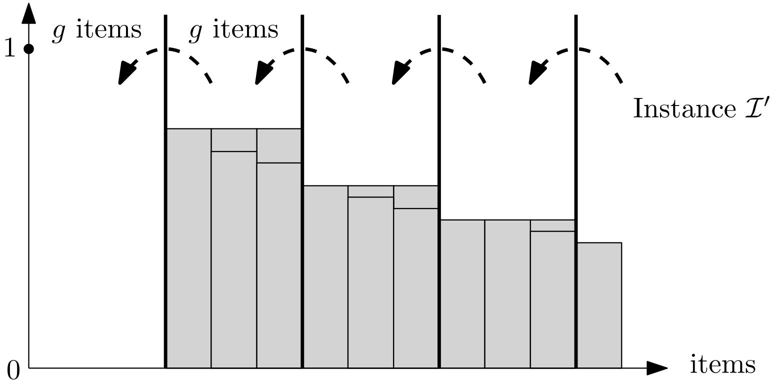

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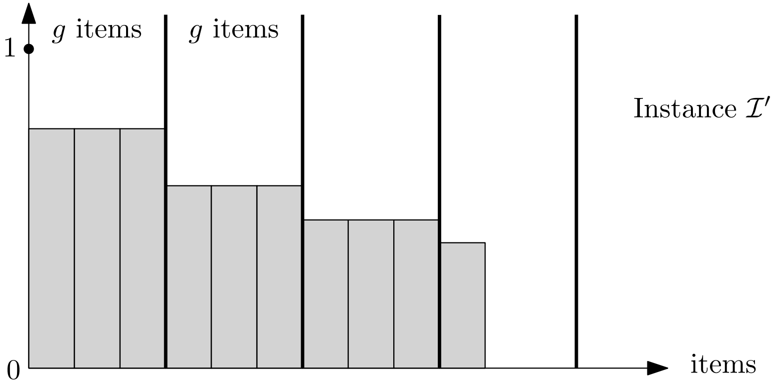

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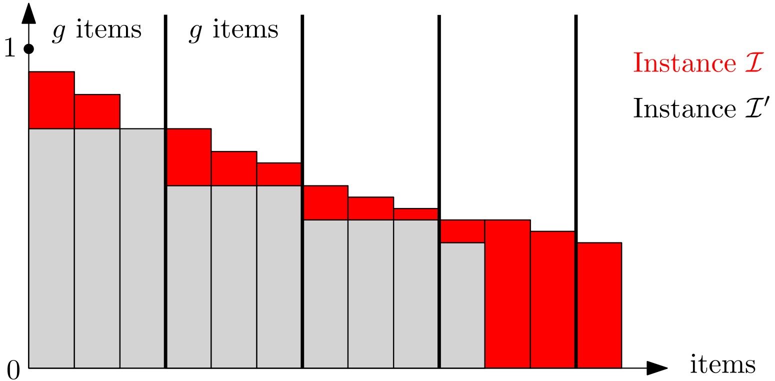

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```

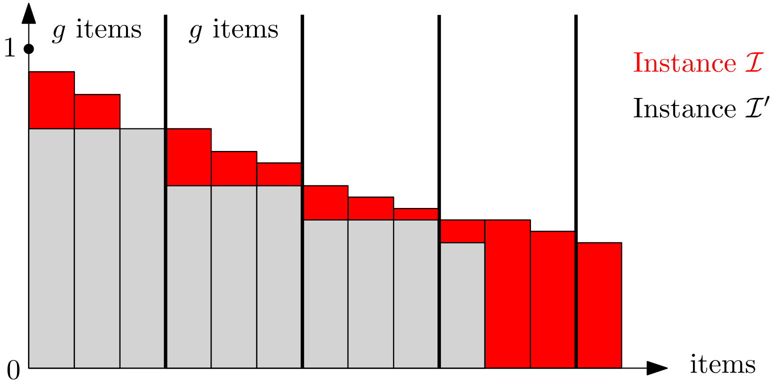

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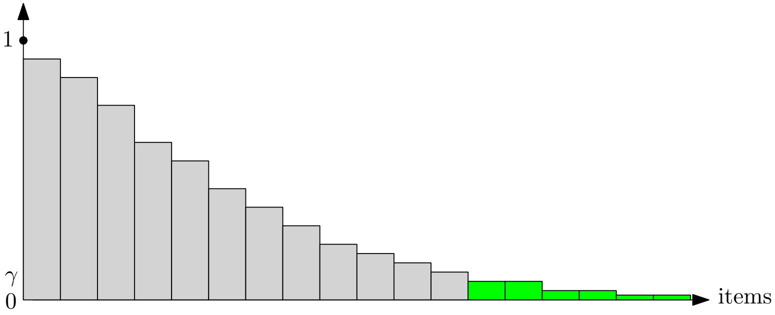
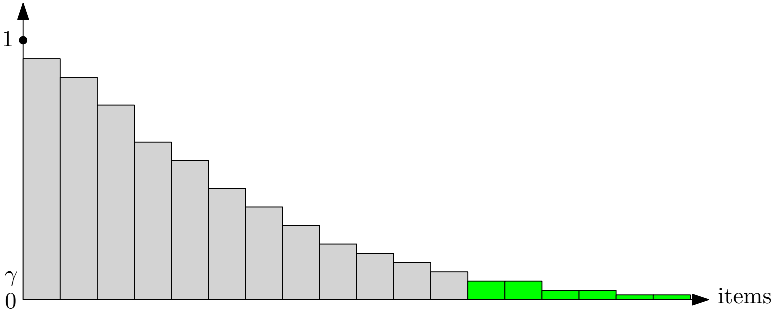
Theorem There is an $O(n^{2/(\epsilon\gamma)})$-time $(1+\epsilon)$-approximation algorithm for the bin-packing problem when all items have size at least γ,

Outline



Combining Algorithms for Small and Big Items



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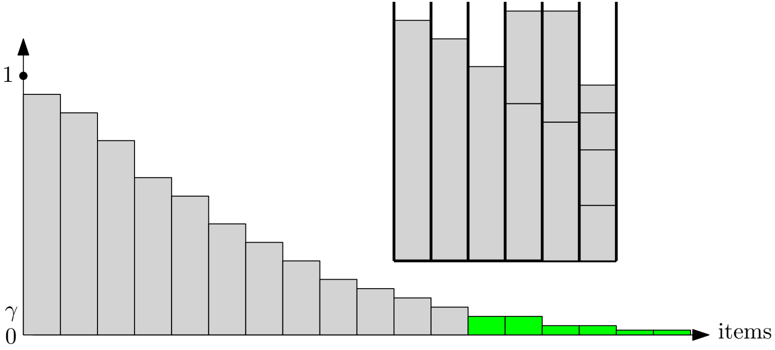
1: Use truncation + DP to obtain solution $\mathcal{S}$ for big items



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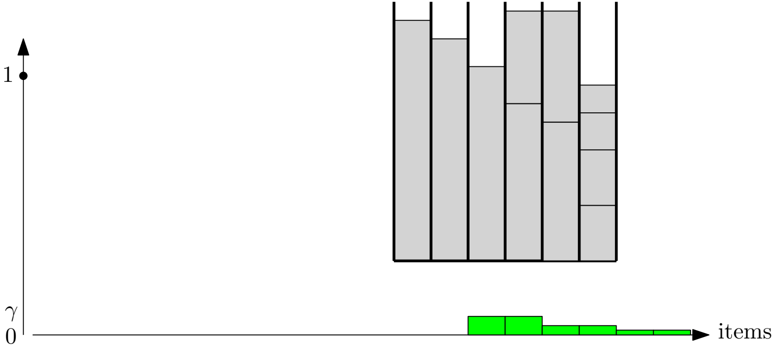

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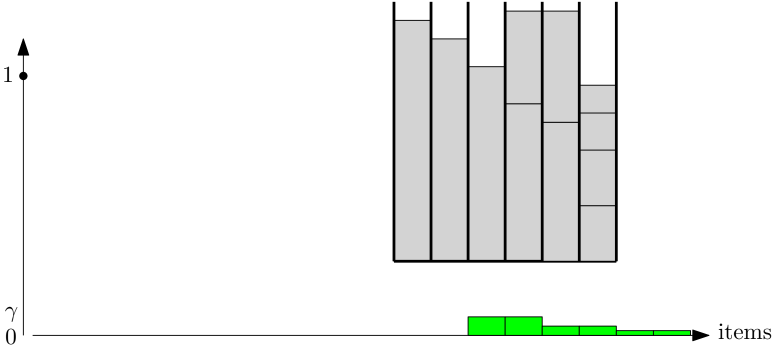

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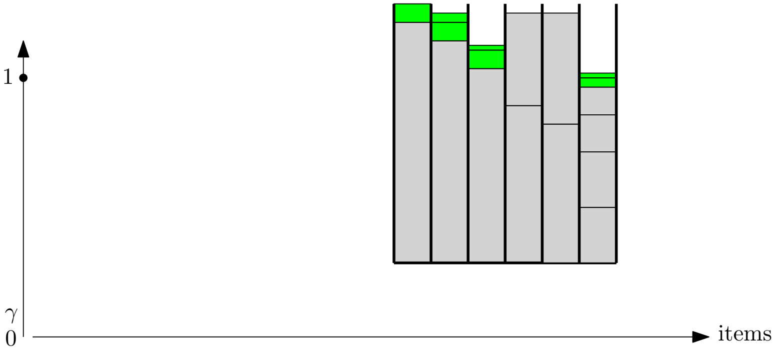

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Analysis of the Combined Algorithm

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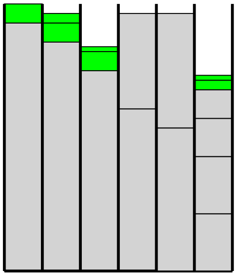

case 1

- Case 1: no new bins are used to pack small items

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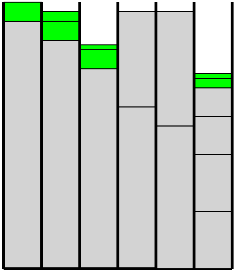

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$$\#(\text{bins used}) \leq (1+\epsilon)\cdot \mathrm{opt}(\mathcal{I}_{\mathrm{big}}) \leq (1+\epsilon)\cdot \mathrm{opt}(\mathcal{I})$$

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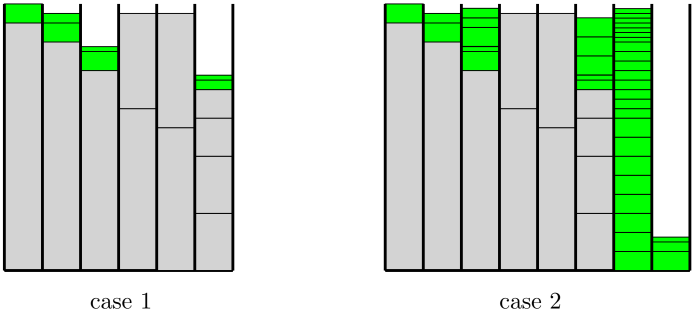

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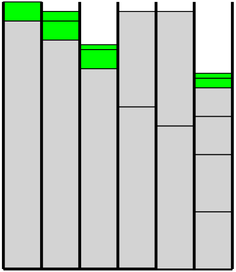

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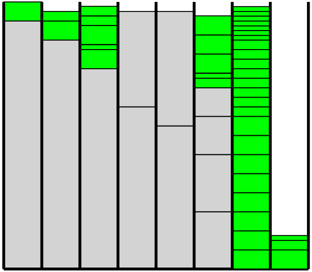

case 2

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$$\#(\text{bins used}) \leq (1+\epsilon)\cdot \mathrm{opt}(\mathcal{I}_{\mathrm{big}}) \leq (1+\epsilon)\cdot \mathrm{opt}(\mathcal{I})$$

- Case 2: new bins are used
  at most one bin has total size $\leq 1-\gamma$

$$\#(\text{bins used}) < \frac{\mathrm{opt}(\mathcal{I})}{1-\gamma} + 1$$

- Setting $\gamma = \epsilon/2 \quad \implies$
  #(bins used) $< \frac{\mathrm{opt}(\mathcal{I})}{1-\epsilon/2} + 1 \leq (1+\epsilon)\mathrm{opt}(\mathcal{I}) + 1$

Theorem There is an $O(n^{2/(\epsilon^2)})$-time algorithmn that outputs a solution with at most $(1+\epsilon)\mathrm{opt}(\mathcal{I}) + 1$ bins.

Theorem There is an APTAS for bin-packing.