Advanced Algorithms (Fall 2023)
Rounding Data and Dynamic Programming

Lecturers: 尹一通, 刘景铖, 栗师
Nanjing University
Outline

1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Outline

1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Knapsack Problem

**Input:** an integer bound \( W > 0 \)

a set of \( n \) items, each with an integer weight \( w_i > 0 \)

a value \( v_i > 0 \) for each item \( i \)

**Output:** a subset \( S \) of items that

maximizes \( \sum_{i \in S} v_i \) s.t. \( \sum_{i \in S} w_i \leq W \).
### Knapsack Problem

**Input:** an integer bound $W > 0$

- a set of $n$ items, each with an integer weight $w_i > 0$
- a value $v_i > 0$ for each item $i$

**Output:** a subset $S$ of items that

\[
\text{maximizes } \sum_{i \in S} v_i \quad \text{s.t. } \sum_{i \in S} w_i \leq W.
\]

- Motivation: you have budget $W$, and want to buy a subset of items of maximum total value
Greedy Algorithm

1: sort items according to non-increasing order of $\frac{v_i}{w_i}$
2: for each item in the ordering do
3:   take the item if we have enough budget
Greedy Algorithm

1: sort items according to non-increasing order of $v_i/w_i$
2: for each item in the ordering do
3: take the item if we have enough budget

Bad example: $W = 100$, $n = 2$, $w = (1, 100)$, $v = (1.1, 100)$. 
Greedy Algorithm

1: sort items according to non-increasing order of $v_i/w_i$
2: for each item in the ordering do
3: take the item if we have enough budget

- Bad example: $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$. 
- Optimum takes item 2 and greedy takes item 1.
Fractional Knapsack Problem

**Input:** integer bound $W > 0$,
a set of $n$ items, each with an integer weight $w_i > 0$
a value $v_i > 0$ for each item $i$

**Output:** a vector $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in [0, 1]^n$ that

maximizes $\sum_{i=1}^{n} \alpha_i v_i$ s.t. $\sum_{i=1}^{n} \alpha_i w_i \leq W$. 

Greedy Algorithm for Fractional Knapsack

1. sort items according to non-increasing order of $\frac{v_i}{w_i}$,
2. for each item according to the ordering, take as much fraction of the item as possible.

Theorem
Greedy algorithm gives the optimum solution for fractional knapsack.
**Fractional Knapsack Problem**

**Input:** integer bound \( W > 0 \),

a set of \( n \) items, each with an integer weight \( w_i > 0 \)

a value \( v_i > 0 \) for each item \( i \)

**Output:** a vector \((\alpha_1, \alpha_2, \cdots, \alpha_n) \in [0, 1]^n\) that

\[
\text{maximizes } \sum_{i=1}^{n} \alpha_i v_i \quad \text{s.t. } \sum_{i=1}^{n} \alpha_i w_i \leq W.
\]

**Greedy Algorithm for Fractional Knapsack**

1: sort items according to non-increasing order of \( v_i/w_i \),

2: for each item according to the ordering, take as much fraction of the item as possible.
Fractional Knapsack Problem

**Input:** integer bound \( W > 0 \),
a set of \( n \) items, each with an integer weight \( w_i > 0 \)
a value \( v_i > 0 \) for each item \( i \)

**Output:** a vector \((\alpha_1, \alpha_2, \ldots, \alpha_n) \in [0, 1]^n\) that maximizes
\[
\sum_{i=1}^{n} \alpha_i v_i \quad \text{s.t.} \quad \sum_{i=1}^{n} \alpha_i w_i \leq W.
\]

Greedy Algorithm for Fractional Knapsack

1. sort items according to non-increasing order of \( v_i/w_i \),
2. for each item according to the ordering, take as much fraction of the item as possible.

**Theorem** Greedy algorithm gives the optimum solution for fractional knapsack.
DP for Knapsack Problem

- \( opt[i, W'] \): the optimum value when budget is \( W' \) and items are \( \{1, 2, 3, \cdots, i\} \).

\[
\begin{align*}
\text{opt}[i, W'] = \begin{cases} 
0 & \text{if } i = 0 \\
\text{opt}[i-1, W'] & \text{if } i > 0, w_i > W' \\
\max \left\{ \text{opt}[i-1, W'], \text{opt}[i-1, W' - w_i] + v_i \right\} & \text{if } i > 0, w_i \leq W'
\end{cases}
\end{align*}
\]

Running time of the algorithm is \( O(nW) \).

Q: Is this a polynomial time?
A: No. The input size is polynomial in \( n \) and \( \log W \); running time is polynomial in \( n \) and \( W \). The running time is pseudo-polynomial.
DP for Knapsack Problem

- \( opt[i, W'] \): the optimum value when budget is \( W' \) and items are \( \{1, 2, 3, \cdots, i\} \).

\[
opt[i, W'] = \begin{cases} 
0 & i = 0 \\
opt[i - 1, W'] & i > 0, w_i > W' \\
\max \left\{ \begin{array}{l}
opt[i - 1, W'] \\
opt[i - 1, W' - w_i] + v_i 
\end{array} \right\} & i > 0, w_i \leq W'
\end{cases}
\]

- Running time of the algorithm is \( O(nW) \).
DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is $W'$ and items are $\{1, 2, 3, \cdots, i\}$.

$$opt[i, W'] = \begin{cases} 
0 & i = 0 \\
opt[i - 1, W'] & i > 0, w_i > W' \\
\max \left\{ \begin{array}{ll}
opt[i - 1, W'] \\
opt[i - 1, W' - w_i] + v_i
\end{array} \right. & i > 0, w_i \leq W'
\end{cases}$$

- Running time of the algorithm is $O(nW)$.

Q: Is this a polynomial time?
DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is $W'$ and items are $\{1, 2, 3, \cdots, i\}$.

$$
opt[i, W'] = \begin{cases} 
0 & i = 0 \\
opt[i - 1, W'] & i > 0, w_i > W' \\
\max \left\{ \begin{array}{l}
opt[i - 1, W'] \\
opt[i - 1, W' - w_i] + v_i
\end{array} \right. & i > 0, w_i \leq W'
\end{cases}
$$

- Running time of the algorithm is $O(nW)$.

Q: Is this a polynomial time?

A: No.
DP for Knapsack Problem

- **opt**\([i, W']\): the optimum value when budget is \(W'\) and items are \(\{1, 2, 3, \cdots, i\}\).

\[
\text{opt}[i, W'] = \begin{cases} 
0 & i = 0 \\
\text{opt}[i - 1, W'] & i > 0, w_i > W' \\
\max \left\{ \begin{array}{l}
\text{opt}[i - 1, W'] \\
\text{opt}[i - 1, W' - w_i] + v_i 
\end{array} \right. & i > 0, w_i \leq W'
\end{cases}
\]

- Running time of the algorithm is \(O(nW)\).

**Q:** Is this a polynomial time?

**A:** No.
- The input size is polynomial in \(n\) and \(\log W\); running time is polynomial in \(n\) and \(W\).
- The running time is pseudo-polynomial.
\( n \): number of integers \( W \): maximum value of all integers

- **pseudo-polynomial time**: \( \text{poly}(n, W) \) (e.g., DP for Knapsack)
- **weakly polynomial time**: \( \text{poly}(n, \log W) \) (e.g., Euclidean Algorithm for Greatest Common Divisor)
- **strongly polynomial time**: \( \text{poly}(n) \) time, assuming basic operations on integers taking \( O(1) \) time (e.g., Kruskal’s)

- **weakly NP-hard**: NP-hard to solve in time \( \text{poly}(n, \log W) \)
- **strongly NP-hard**: NP-hard even if \( W = \text{poly}(n) \)
Outline

1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Idea for improving the running time to polynomial

- If we make weights upper bounded by $\text{poly}(n)$, then pseudo-polynomial time becomes polynomial time.
- Coarsening the weights: $w'_i = \left\lfloor \frac{w_i}{A} \right\rfloor$ for some appropriately defined integer $A$. However, coarsening weights will change the problem. Weight budget constraint: hard, maximum value requirement: soft. We coarsen the values instead. In the DP, we use values as parameters.
Idea for improving the running time to polynomial

- If we make weights upper bounded by $\text{poly}(n)$, then pseudo-polynomial time becomes polynomial time.
- Coarsening the weights: $w'_i = \left\lfloor \frac{w_i}{A} \right\rfloor$ for some appropriately defined integer $A$.
- However, coarsening weights will change the problem.
Idea for improving the running time to polynomial

- If we make weights upper bounded by $\text{poly}(n)$, then pseudo-polynomial time becomes polynomial time.
- Coarsening the weights: $w'_i = \left\lfloor \frac{w_i}{A} \right\rfloor$ for some appropriately defined integer $A$.
- However, coarsening weights will change the problem.
  - weight budget constraint : hard
  - maximum value requirement : soft
Idea for improving the running time to polynomial

- If we make weights upper bounded by $\text{poly}(n)$, then pseudo-polynomial time becomes polynomial time.

- Coarsening the weights: $w_i' = \left\lfloor \frac{w_i}{A} \right\rfloor$ for some appropriately defined integer $A$.

- However, coarsening weights will change the problem.
  
  **weight budget constraint**: hard
  
  **maximum value requirement**: soft

- We coarsen the values instead.

- In the DP, we use values as parameters.
Let $A$ be some integer to be defined later
Let $A$ be some integer to be defined later

$v'_i := \left\lfloor \frac{v_i}{A} \right\rfloor$ be the scaled value of item $i$
Let $A$ be some integer to be defined later

$v'_i := \left\lfloor \frac{v_i}{A} \right\rfloor$ be the scaled value of item $i$

Definition of DP cells: $f[i, V'] = \min_{S \subseteq [i]}: \forall v'(S) \geq V', w(S)$

Output $A \times \max V'$ such that $f[n, V'] \leq W$. 
Let $A$ be some integer to be defined later

$v'_i := \left\lfloor \frac{v_i}{A} \right\rfloor$ be the scaled value of item $i$

Definition of DP cells: 

$$f[i, V'] = \min_{S \subseteq [i]: v'(S) \geq V', w(S)}$$

$$f[i, V'] = \begin{cases} 
0 & V' \leq 0 \\
\infty & i = 0, V' > 0 \\
\min \left\{ f[i - 1, V'], f[i - 1, V' - v'_i] + w_i \right\} & i > 0, V' > 0 
\end{cases}$$

Output $A$ times the largest $V'$ such that $f[n, V'] \leq W$. 
Let $A$ be some integer to be defined later

$v'_i := \left\lfloor \frac{v_i}{A} \right\rfloor$ be the scaled value of item $i$

Definition of DP cells: $f[i, V'] = \min_{S \subseteq [i]: v'(S) \geq V', w(S)}$

$$f[i, V'] = \begin{cases} 
0 & V' \leq 0 \\
\infty & i = 0, V' > 0 \\
\min \left\{ f[i - 1, V'], f[i - 1, V' - v'_i] + w_i \right\} & i > 0, V' > 0 
\end{cases}$$

Output $A$ times the largest $V'$ such that $f[n, V'] \leq W$. 
Instance $\mathcal{I}$: $(v_1, v_2, \cdots, v_n)$  \hspace{1cm} \text{opt}: \text{optimum value of } \mathcal{I}

Instance $\mathcal{I}'$: $(A v_1', \cdots, A V_n')$  \hspace{1cm} \text{opt}': \text{optimum value of } \mathcal{I}'
- Instance $\mathcal{I}$: $(v_1, v_2, \cdots, v_n)$  \quad opt: optimum value of $\mathcal{I}$
- Instance $\mathcal{I'}$: $(Av_1', \cdots, AV_n')$  \quad opt': optimum value of $\mathcal{I'}$

\[
v_i - A < Av_i' \leq v_i, \quad \forall i \in [n]
\]
\[
\implies opt - nA < opt' \leq opt
\]

- $opt \geq v_{\max} := \max_{i \in [n]} v_i$ (assuming $w_i \leq W, \forall i$)
- Instance $\mathcal{I}$: $(v_1, v_2, \cdots, v_n)$  \(\text{opt}:\) optimum value of $\mathcal{I}$

- Instance $\mathcal{I}'$: $(Av'_1, \cdots, AV'_n)$  \(\text{opt}'\): optimum value of $\mathcal{I}'$

\[
v_i - A < Av'_i \leq v_i, \quad \forall i \in [n]
\]

\[
\implies \text{opt} - nA < \text{opt}' \leq \text{opt}
\]

- \(\text{opt} \geq v_{\text{max}} := \max_{i \in [n]} v_i\) (assuming \(w_i \leq W, \forall i\))

- Setting \(A := \left\lfloor \frac{\epsilon \cdot v_{\text{max}}}{n} \right\rfloor\): \((1 - \epsilon)\text{opt} \leq \text{opt}' \leq \text{opt}\)
Instance $\mathcal{I}$: $(v_1, v_2, \cdots, v_n)$ \quad $\text{opt}$: optimum value of $\mathcal{I}$

Instance $\mathcal{I}'$: $(Av_1', \cdots, AV_n')$ \quad $\text{opt}'$: optimum value of $\mathcal{I}'$

$$v_i - A < Av_i' \leq v_i, \quad \forall i \in [n]$$

$$\implies \text{opt} - nA < \text{opt}' \leq \text{opt}$$

$\text{opt} \geq v_{\text{max}} := \max_{i \in [n]} v_i$ (assuming $w_i \leq W, \forall i$)

setting $A := \left\lfloor \frac{\epsilon \cdot v_{\text{max}}}{n} \right\rfloor$: $(1 - \epsilon)\text{opt} \leq \text{opt}' \leq \text{opt}$

$\forall i, v_i' = O\left(\frac{n}{\epsilon}\right)$ \quad $\implies$ \quad running time $= O\left(\frac{n^3}{\epsilon}\right)$
- Instance $\mathcal{I}$: $(v_1, v_2, \ldots, v_n)$ \quad opt: optimum value of $\mathcal{I}$
- Instance $\mathcal{I}'$: $(Av'_1, \ldots, AV'_n)$ \quad opt': optimum value of $\mathcal{I}'$

\[ v_i - A < Av'_i \leq v_i, \quad \forall i \in [n] \]
\[ \implies opt - nA < opt' \leq opt \]

- $\text{opt} \geq v_{\text{max}} := \max_{i \in [n]} v_i$ (assuming $w_i \leq W, \forall i$)
- setting $A := \left\lfloor \frac{\epsilon \cdot v_{\text{max}}}{n} \right\rfloor$: $(1 - \epsilon)\text{opt} \leq \text{opt}' \leq \text{opt}$

- $\forall i, v'_i = O\left(\frac{n}{\epsilon}\right) \implies$ running time $= O\left(\frac{n^3}{\epsilon}\right)$

**Theorem** There is a $(1 + \epsilon)$-approximation for the knapsack problem in time $O\left(\frac{n^3}{\epsilon}\right)$. 
Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms $A_\epsilon$, where $A_\epsilon$ for every $\epsilon > 0$ is a (polynomial-time) $(1 \pm \epsilon)$-approximation algorithm.

- Remark: the approximation ratio is $1 + \epsilon$ or $1 - \epsilon$, depending on whether the problem is a minimization/maximization problem.
Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms $A_\epsilon$, where $A_\epsilon$ for every $\epsilon > 0$ is a (polynomial-time) $(1 \pm \epsilon)$-approximation algorithm.

Remark: the approximation ratio is $1 + \epsilon$ or $1 - \epsilon$, depending on whether the problem is a minimization/maximization problem.

Def. A fully polynomial-time approximation scheme (FPTAS) is an approximation scheme $A_\epsilon$ such that the running time of $A_\epsilon$ is $\text{poly}(n, \frac{1}{\epsilon})$ for input instances of $n$. So, Knapsack admits an FPTAS.

Q: Assume $P \neq NP$. What is a necessary condition for a NP-hard problem to admit an FPTAS? Vertex cover? Maximum independent set?
Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms \( A_\epsilon \), where \( A_\epsilon \) for every \( \epsilon > 0 \) is a (polynomial-time) \((1 \pm \epsilon)\)-approximation algorithm.

- Remark: the approximation ratio is \( 1 + \epsilon \) or \( 1 - \epsilon \), depending on whether the problem is a minimization/maximization problem.

Def. A fully polynomial-time approximation scheme (FPTAS) is an approximation scheme \( A_\epsilon \) such that the running time of \( A_\epsilon \) is \( \text{poly}(n, \frac{1}{\epsilon}) \) for input instances of \( n \).

- So, Knapsack admits an FPTAS.
Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms $A_\epsilon$, where $A_\epsilon$ for every $\epsilon > 0$ is a (polynomial-time) $(1 \pm \epsilon)$-approximation algorithm.

Remark: the approximation ratio is $1 + \epsilon$ or $1 - \epsilon$, depending on whether the problem is a minimization/maximization problem.

Def. A fully polynomial-time approximation scheme (FPTAS) is an approximation scheme $A_\epsilon$ such that the running time of $A_\epsilon$ is $\text{poly}(n, \frac{1}{\epsilon})$ for input instances of $n$.

So, Knapsack admits an FPTAS.

Q: Assume $P \neq NP$. What is a necessary condition for a NP-hard problem to admit an FPTAS?
Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms $A_\epsilon$, where $A_\epsilon$ for every $\epsilon > 0$ is a (polynomial-time) $(1 \pm \epsilon)$-approximation algorithm.

Remark: the approximation ratio is $1 + \epsilon$ or $1 - \epsilon$, depending on whether the problem is a minimization/maximization problem.

Def. A fully polynomial-time approximation scheme (FPTAS) is an approximation scheme $A_\epsilon$ such that the running time of $A_\epsilon$ is $\text{poly}(n, \frac{1}{\epsilon})$ for input instances of $n$.

So, Knapsack admits an FPTAS.

Q: Assume $P \neq NP$. What is a necessary condition for a NP-hard problem to admit an FPTAS?

Vertex cover? Maximum independent set?
Outline

1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Outline

1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Makespan Minimization on Identical Machines

**Input:** $n$ jobs index as $[n]$

- each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$
- $m$ machines
Makespan Minimization on Identical Machines

**Input:** \( n \) jobs index as \([n]\)

each job \( j \in [n] \) has a processing time \( p_j \in \mathbb{Z}_{>0} \)

\( m \) machines

**Output:** schedule of jobs on machines with minimum makespan
Makespan Minimization on Identical Machines

**Input:** $n$ jobs index as $[n]$

each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$

$m$ machines

**Output:** schedule of jobs on machines with minimum makespan

$\sigma : [n] \rightarrow [m]$ with minimum $\max_{i \in [m]} \sum_{j \in \sigma^{-1}(i)} p_j$
Makespan Minimization on Identical Machines

**Input:**  
$n$ jobs index as $[n]$  
each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$  
$m$ machines

**Output:** schedule of jobs on machines with minimum makespan  
$\sigma : [n] \rightarrow [m]$ with minimum $\max_{i \in [m]} \sum_{j \in \sigma^{-1}(i)} p_j$

```
1 2 3 4
5 6 7 8 9
10 11 12 13
```

4 machines
Makespan Minimization on Identical Machines

**Input:** $n$ jobs index as $[n]$

each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$

$m$ machines

**Output:** schedule of jobs on machines with minimum makespan $

\sigma : [n] \rightarrow [m]$ with minimum $\max_{i \in [m]} \sum_{j \in \sigma^{-1}(i)} p_j$

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

4 machines
Makespan Minimization on Identical Machines

**Input:** $n$ jobs index as $[n]$

each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$

$m$ machines

**Output:** schedule of jobs on machines with minimum makespan

$\sigma : [n] \rightarrow [m]$ with minimum $\max_{i \in [m]} \sum_{j \in \sigma^{-1}(i)} p_j$
Greedy Algorithm

1: start from an empty schedule
2: for $j = 1$ to $n$ do
3: put job $j$ on the machine with the smallest load
Greedy Algorithm

1: start from an empty schedule
2: for $j = 1$ to $n$ do
3: put job $j$ on the machine with the smallest load

Analysis of $\left(2 - \frac{1}{m}\right)$-Approximation for Greedy Algorithm
Greedy Algorithm

1: start from an empty schedule
2: for $j = 1$ to $n$ do
3: put job $j$ on the machine with the smallest load

Analysis of $(2 - \frac{1}{m})$-Approximation for Greedy Algorithm

$$p_{\text{max}} := \max_{j \in [n]} p_j$$

$$\text{alg} \leq p_{\text{max}} + \frac{1}{m} \cdot \left( \sum_{j \in [n]} p_j - p_{\text{max}} \right) = \left(1 - \frac{1}{m}\right) p_{\text{max}} + \frac{1}{m} \sum_{j \in [n]} p_j$$
Greedy Algorithm

1: start from an empty schedule
2: for $j = 1$ to $n$ do
3: put job $j$ on the machine with the smallest load

Analysis of $(2 - \frac{1}{m})$-Approximation for Greedy Algorithm

$$p_{\text{max}} := \max_{j \in [n]} p_j$$

$$\text{alg} \leq p_{\text{max}} + \frac{1}{m} \cdot \left( \sum_{j \in [n]} p_j - p_{\text{max}} \right) = (1 - \frac{1}{m}) p_{\text{max}} + \frac{1}{m} \sum_{j \in [n]} p_j$$

$$\begin{align*}
\text{opt} & \geq p_{\text{max}} \\
\text{opt} & \geq \frac{1}{m} \sum_{j \in [n]} p_j
\end{align*}$$

$\implies$ \quad $\text{alg} \leq (2 - \frac{1}{m}) \cdot \text{opt}$
Q: What happens if all items have size at most $\epsilon \cdot \text{opt}$?

A:

Let $p_j \in [n]$ be the size of item $j$. Then, $\text{alg} \leq \text{opt} + \epsilon \cdot \text{opt} = (1 + \epsilon) \text{opt}$.

Q: What can we do if all items have size at least $\epsilon \cdot \text{opt}$?

A: We can round the sizes, so that the number of distinct sizes is small.

**Overview of Algorithm**

1. Declare $j$ small if $p_j < \epsilon \cdot p_{\text{max}}$ and big otherwise.
2. Use truncation + DP to solve the instance defined by big jobs.
3. Use DP for instance $(p'_j)_{j \text{ big}}$ to schedule big jobs.
4. Add small jobs to the schedule greedily.
Q: What happens if all items have size at most $\epsilon \cdot \text{opt}$?

A: $\text{alg} \leq \frac{1}{m} \sum_{j \in [n]} p_j + \text{alg} \leq \text{opt} + \epsilon \cdot \text{opt} = (1 + \epsilon)\text{opt}$. 

Overview of Algorithm

1: declare $j$ small if $p_j < \epsilon \cdot p_{\text{max}}$ and big otherwise
2: use truncation + DP to solve the instance defined by big jobs
3: use DP for instance $(p_j')$ to schedule big jobs
4: add small jobs to schedule greedily
Q: What happens if all items have size at most $\epsilon \cdot \text{opt}$?

A: $\text{alg} \leq \frac{1}{m} \sum_{j \in [n]} p_j + \text{alg} \leq \text{opt} + \epsilon \cdot \text{opt} = (1 + \epsilon)\text{opt}$.

Q: What can we do if all items have size at least $\epsilon \cdot \text{opt}$?
Q: What happens if all items have size at most $\epsilon \cdot \text{opt}$?

A: $\text{alg} \leq \frac{1}{m} \sum_{j \in [n]} p_j + \text{alg} \leq \text{opt} + \epsilon \cdot \text{opt} = (1 + \epsilon)\text{opt}$.

Q: What can we do if all items have size at least $\epsilon \cdot \text{opt}$?

A: We can round the sizes, so that \#(distinct sizes) is small.
Q: What happens if all items have size at most $\epsilon \cdot \text{opt}$?

A: \[ \text{alg} \leq \frac{1}{m} \sum_{j \in [n]} p_j + \text{alg} \leq \text{opt} + \epsilon \cdot \text{opt} = (1 + \epsilon)\text{opt}. \]

Q: What can we do if all items have size at least $\epsilon \cdot \text{opt}$?

A: We can round the sizes, so that $\#$(distinct sizes) is small.

Overview of Algorithm

1: declare $j$ small if $p_j < \epsilon \cdot p_{\text{max}}$ and big otherwise.
Q: What happens if all items have size at most $\epsilon \cdot \text{opt}$?

A: $\text{alg} \leq \frac{1}{m} \sum_{j \in [n]} p_j + \text{alg} \leq \text{opt} + \epsilon \cdot \text{opt} = (1 + \epsilon)\text{opt}$.

Q: What can we do if all items have size at least $\epsilon \cdot \text{opt}$?

A: We can round the sizes, so that $\#$(distinct sizes) is small.

Overview of Algorithm

1. declare $j$ small if $p_j < \epsilon \cdot p_{\max}$ and big otherwise
2. use trunction + DP to solve the instance defined by big jobs
Q: What happens if all items have size at most $\epsilon \cdot \text{opt}$?

A: $\text{alg} \leq \frac{1}{m} \sum_{j \in [n]} p_j + \text{alg} \leq \text{opt} + \epsilon \cdot \text{opt} = (1 + \epsilon)\text{opt}$.

Q: What can we do if all items have size at least $\epsilon \cdot \text{opt}$?

A: We can round the sizes, so that $\#(\text{distinct sizes})$ is small.

Overview of Algorithm

1: declare $j$ small if $p_j < \epsilon \cdot p_{\text{max}}$ and big otherwise
2: use truncation + DP to solve the instance defined by big jobs
3: use DP for instance $(p'_j)_j \text{ big}$ to schedule big jobs
Q: What happens if all items have size at most $\epsilon \cdot \text{opt}$?

A: \[ \text{alg} \leq \frac{1}{m} \sum_{j \in [n]} p_j + \text{alg} \leq \text{opt} + \epsilon \cdot \text{opt} = (1 + \epsilon)\text{opt}. \]

Q: What can we do if all items have size at least $\epsilon \cdot \text{opt}$?

A: We can round the sizes, so that $\#$(distinct sizes) is small

Overview of Algorithm

1. declare $j$ small if $p_j < \epsilon \cdot p_{\text{max}}$ and big otherwise
2. use truncation + DP to solve the instance defined by big jobs
3. use DP for instance $(p'_j)_{j \text{ big}}$ to schedule big jobs
4. add small jobs to schedule greedily
Outline

1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
\[ B := \{ j \in [n] : p_j \geq \epsilon p_{\max} \} : \text{set of big jobs} \]
\begin{itemize}
  \item \( B := \{ j \in [n] : p_j \geq \epsilon p_{\text{max}} \} \): set of big jobs
  \item \( p_j' := \max \{ p_{\text{max}}(1 + \epsilon)^t \leq p_j : t \in \mathbb{Z} \}, \forall j \in B \)
    
    \( p_j' \) is the \textbf{rounded size of} \( j \)
\end{itemize}
Dynamic Programming for Big Jobs

- $B := \{ j \in [n] : p_j \geq \epsilon p_{\text{max}} \}$: set of big jobs
- $p'_j := \max \{ p_{\text{max}} (1 + \epsilon)^t \leq p_j : t \in \mathbb{Z} \}, \forall j \in B$
  - $p'_j$ is the rounded size of $j$
- $k := |\{ p'_j : j \in B \}|$: #(distinct rounded sizes)
  - $k \leq 1 + \log_{1+\epsilon} \frac{p_{\text{max}}}{\epsilon p_{\text{max}}} = O\left( \frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon} \right)$
\( B := \{ j \in [n] : p_j \geq \varepsilon p_{\max} \} \): set of big jobs

\( p'_j := \max\{ p_{\max}(1 + \varepsilon)^t \leq p_j : t \in \mathbb{Z} \}, \forall j \in B \)

\( p'_j \) is the rounded size of \( j \)

\( k := |\{ p'_j : j \in B \}|: \#(\text{distinct rounded sizes}) \)

\[ k \leq 1 + \log_{1+\varepsilon} \frac{p_{\max}}{\varepsilon p_{\max}} = O\left( \frac{1}{\varepsilon} \cdot \log \frac{1}{\varepsilon} \right) \]

\( \{ q_1, q_2, \cdots, q_k \} := \{ p'_j : j \in B \} \): the \( k \) distinct rounded sizes
Dynamic Programming for Big Jobs

- $B := \{j \in [n] : p_j \geq \epsilon p_{\text{max}}\}$: set of big jobs
- $p'_j := \max\{p_{\text{max}}(1 + \epsilon)^t \leq p_j : t \in \mathbb{Z}\}, \forall j \in B$
  - $p'_j$ is the rounded size of $j$
- $k := |\{p'_j : j \in B\}|$: #(distinct rounded sizes)
  - $k \leq 1 + \log_{1+\epsilon} \frac{p_{\text{max}}}{\epsilon p_{\text{max}}} = O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)$
- $\{q_1, q_2, \cdots, q_k\} := \{p'_j : j \in B\}$: the $k$ distinct rounded sizes
- $n_1, \cdots, n_k$: #(big jobs) with rounded sizes being $q_1, \cdots, q_k$
Constructing a Directed Acyclic Graph $G = (V, E)$

Let $a_1, \ldots, a_k$ denote the instance with $a_1$ jobs of size $q_1$, $a_2$ jobs of size $q_2$, \ldots, $a_k$ jobs of size $q_k$. An arc $(a_1, \ldots, a_k) \rightarrow (b_1, \ldots, b_k)$ of weight $P_{ki} = 1(b_i - a_i)q_i$, if $a_i \leq b_i$, $\forall i \in [k]$, and $a_i < b_i$ for some $i \in [k]$ reduces instance $(b_1, \ldots, b_k)$ to $(a_1, \ldots, a_k)$ requiring 1 machine of load $P_{ki} = 1(b_i - a_i)q_i$.

Goal: find a path from $(0, \ldots, 0)$ to $(n_1, \ldots, n_k)$ of at most $m$ edges, so as to minimize the maximum weight on the path. The problem can be solved in $O(m \cdot |E|)$ time using DP.
Constructing a Directed Acyclic Graph $G = (V, E)$

- A vertex $(a_1, \cdots, a_k)$, $a_i \in [0, n_i], \forall i \in [k]$ denotes the instance with $a_1$ jobs of size $q_1$, $a_2$ jobs of size $q_2$, $\cdots$, $a_k$ jobs of size $q_k$.
Constructing a Directed Acyclic Graph $G = (V, E)$

- a vertex $(a_1, \cdots, a_k)$, $a_i \in [0, n_i], \forall i \in [k]$ denotes the instance with $a_1$ jobs of size $q_1$, $a_2$ jobs of size $q_2$, \ldots, $a_k$ jobs of size $q_k$
- an arc $(a_1, \cdots, a_k) \rightarrow (b_1, \cdots b_k)$ of weight $\sum_{i=1}^{k} (b_i - a_i)q_i$, if $a_i \leq b_i, \forall i \in [k]$, and $a_i < b_i$ for some $i \in [k]$
- reducing instance $(b_1, \cdots b_k)$ to $(a_1, \cdots, a_k)$ requires 1 machine of load $\sum_{i=1}^{k} (b_i - a_i)q_i$
Constructing a Directed Acyclic Graph $G = (V, E)$

- A vertex $(a_1, \cdots, a_k)$, $a_i \in [0, n_i], \forall i \in [k]$ denotes the instance with $a_1$ jobs of size $q_1$, $a_2$ jobs of size $q_2$, $\cdots$, $a_k$ jobs of size $q_k$.

- An arc $(a_1, \cdots, a_k) \rightarrow (b_1, \cdots b_k)$ of weight $\sum_{i=1}^{k} (b_i - a_i)q_i$, if $a_i \leq b_i, \forall i \in [k]$, and $a_i < b_i$ for some $i \in [k]$.

- Reducing instance $(b_1, \cdots b_k)$ to $(a_1, \cdots, a_k)$ requires 1 machine of load $\sum_{i=1}^{k} (b_i - a_i)q_i$.

- Goal: find a path from $(0, \cdots, 0)$ to $(n_1, \cdots, n_k)$ of at most $m$ edges, so as to minimize the maximum weight on the path.
Constructing a Directed Acyclic Graph $G = (V, E)$

- a vertex $(a_1, \cdots, a_k)$, $a_i \in [0, n_i], \forall i \in [k]$ denotes the instance with $a_1$ jobs of size $q_1$, $a_2$ jobs of size $q_2$, \ldots, $a_k$ jobs of size $q_k$
- an arc $(a_1, \cdots, a_k) \rightarrow (b_1, \cdots, b_k)$ of weight $\sum_{i=1}^{k} (b_i - a_i)q_i$, if $a_i \leq b_i, \forall i \in [k]$, and $a_i < b_i$ for some $i \in [k]$
- reducing instance $(b_1, \cdots, b_k)$ to $(a_1, \cdots, a_k)$ requires 1 machine of load $\sum_{i=1}^{k} (b_i - a_i)q_i$

Goal: find a path from $(0, \cdots, 0)$ to $(n_1, \cdots, n_k)$ of at most $m$ edges, so as to minimize the maximum weight on the path.

- problem can be solved in $O(m \cdot |E|)$ time using DP
- $O(m \cdot |E|) = O(m \cdot n^{2k}) = n^{O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)}$. 
Theorem

The dynamic programming algorithm gives a schedule of makespan at most $(1 + \epsilon) \text{opt}_B$ in time $n^{O(1/\epsilon \log 1/\epsilon)}$. 

Adding small jobs to schedule

1. starting from the schedule for big jobs
2. for every small job $j$
3. add $j$ to the machine with the smallest load
Analysis of Algorithm for Big Jobs

- $\mathcal{I}_B$: instance $(p_j)_{j \in B}$, $\text{opt}_B$: its optimum makespan
- $\mathcal{I}'_B$: instance $(p'_j)_{j \in B}$, $\text{opt}'_B$: its optimum makespan
- $\text{opt}'_B \leq \text{opt}_B$

Adding small jobs to schedule:
1. Starting from the schedule for big jobs
2. For every small job $j$
3. Add $j$ to the machine with the smallest load
## Analysis of Algorithm for Big Jobs

- $\mathcal{I}_B$: instance $(p_j)_{j \in B}$ \quad $\text{opt}_B$: its optimum makespan
- $\mathcal{I}'_B$: instance $(p'_j)_{j \in B}$ \quad $\text{opt}'_B$: its optimum makespan
- $\text{opt}'_B \leq \text{opt}_B$
- schedule for $\mathcal{I}'_B \Rightarrow$ schedule for $\mathcal{I}_B$: $(1 + \epsilon)$-blowup in makespan
Analysis of Algorithm for Big Jobs

- $\mathcal{I}_B$: instance $(p_j)_{j \in B}$  \hspace{1cm} $\text{opt}_B$: its optimum makespan
- $\mathcal{I}'_B$: instance $(p'_j)_{j \in B}$  \hspace{1cm} $\text{opt}'_B$: its optimum makespan
- $\text{opt}'_B \leq \text{opt}_B$
- schedule for $\mathcal{I}'_B \Rightarrow$ schedule for $\mathcal{I}_B$: \hspace{1cm} $(1 + \epsilon)$-blowup in makespan

**Theorem**  The dynamic programming algorithm gives a schedule of makespan at most $(1 + \epsilon)\text{opt}_B$ in time $n^{O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)}$. 
Analysis of Algorithm for Big Jobs

- $\mathcal{I}_B$: instance $(p_j)_{j \in B}$  $\text{opt}_B$: its optimum makespan
- $\mathcal{I}'_B$: instance $(p'_j)_{j \in B}$  $\text{opt}'_B$: its optimum makespan
- $\text{opt}'_B \leq \text{opt}_B$
- schedule for $\mathcal{I}'_B \Rightarrow$ schedule for $\mathcal{I}_B$: $(1 + \epsilon)$-blowup in makespan

**Theorem**  The dynamic programming algorithm gives a schedule of makespan at most $(1 + \epsilon)\text{opt}_B$ in time $n^{O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)}$.

Adding small jobs to schedule

1: starting from the schedule for big jobs
2: for every small job $j$ do
3: add $j$ to the machine with the smallest load
1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Analysis of the Final Algorithm

---

**Case 1:** makespan is not increased by small jobs
Case 1: makespan is not increased by small jobs

\[ \text{alg} \leq (1 + \epsilon)\text{opt}_B \leq (1 + \epsilon)\text{opt}. \]
Case 1: makespan is not increased by small jobs

$$alg \leq (1 + \epsilon)opt_B \leq (1 + \epsilon)opt.$$ 

Case 2: makespan is increased by small jobs
Case 1: makespan is not increased by small jobs

\[ \text{alg} \leq (1 + \epsilon) \text{opt}_B \leq (1 + \epsilon) \text{opt}. \]

Case 2: makespan is increased by small jobs
- loads between any two machines differ by at most size of a small job, which is at most \( \epsilon \cdot p_{\text{max}} \)
Analysis of the Final Algorithm

Case 1: makespan is not increased by small jobs

\[ \text{alg} \leq (1 + \epsilon) \text{opt}_B \leq (1 + \epsilon) \text{opt}. \]

Case 2: makespan is increased by small jobs

- loads between any two machines differ by at most size of a small job, which is at most \( \epsilon \cdot p_{\text{max}} \)

\[ \text{alg} \leq \epsilon \cdot p_{\text{max}} + \frac{1}{m} \sum_{j \in [n]} p_j \leq \epsilon \cdot \text{opt} + \text{opt} = (1 + \epsilon) \cdot \text{opt}. \]
1. **Knapsack Problem**
   - Introduction
   - FPTAS for Knapsack Problem

2. **PTAS for Makespan Minimization on Identical Machines**
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. **An asymptotical PTAS for Bin Packing**
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Outline

1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Bin Packing

Input: $n$ items indexed by $[n]$, with sizes $s_1, s_2, \cdots, s_n \in (0, 1]$

Output: a packing of items into smallest number of bins of capacity 1.
Bin Packing

**Input:** $n$ items indexed by $[n]$, with sizes $s_1, s_2, \cdots, s_n \in (0, 1]$

**Output:** a packing of items into smallest number of bins of capacity 1.
**Bin Packing**

**Input:** \( n \) items indexed by \([n]\), with sizes \( s_1, s_2, \cdots, s_n \in (0, 1] \)

**Output:** a packing of items into smallest number of bins of capacity 1.
Bin Packing

**Input:** \( n \) items indexed by \([n]\), with sizes \( s_1, s_2, \ldots, s_n \in (0, 1] \)

**Output:** a packing of items into smallest number of bins of capacity 1.
Bin Packing

**Input:** $n$ items indexed by $[n]$, with sizes $s_1, s_2, \ldots, s_n \in (0, 1]$.

**Output:** a packing of items into smallest number of bins of capacity 1.

<table>
<thead>
<tr>
<th>bin packing</th>
<th>objective</th>
<th>container capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>scheduling</td>
<td>fixed</td>
<td>fixed</td>
</tr>
<tr>
<td></td>
<td></td>
<td>objective</td>
</tr>
</tbody>
</table>
First-Fit

1: initially there are 0 bins
2: \textbf{for} i \leftarrow 1 \textbf{ to } n \textbf{ do}
3: \textbf{if} item i \textbf{ fits into an existing bin then} put i \textbf{ into the bin}
4: \textbf{else} open a new bin and put i \textbf{ into the bin}
First-Fit

1: initially there are 0 bins
2: for $i \leftarrow 1$ to $n$ do
3: if item $i$ fits into an existing bin then put $i$ into the bin
4: else open a new bin and put $i$ into the bin

Obs. In the output, at most one bin has total size $\leq 1/2$. 
First-Fit

1: initially there are 0 bins
2: for $i \leftarrow 1$ to $n$ do
3: if item $i$ fits into an existing bin then put $i$ into the bin
4: else open a new bin and put $i$ into the bin

Obs. In the output, at most one bin has total size $\leq 1/2$.

- If our algorithm uses $t$ bins, then $\text{opt} > \frac{t-1}{2}$ and $\text{opt} \in \mathbb{Z}_{>0}$
- $t$ is even: $\text{opt} \geq \frac{t}{2}$
- $t$ is odd: $\text{opt} \geq \frac{t+1}{2}$. 
First-Fit

1: initially there are 0 bins
2: for $i \leftarrow 1$ to $n$ do
3: if item $i$ fits into an existing bin then put $i$ into the bin
4: else open a new bin and put $i$ into the bin

Obs. In the output, at most one bin has total size $\leq 1/2$.

- If our algorithm uses $t$ bins, then $\text{opt} > \frac{t-1}{2}$ and $\text{opt} \in \mathbb{Z}_{>0}$
- $t$ is even: $\text{opt} \geq \frac{t}{2}$  $t$ is odd: $\text{opt} \geq \frac{t+1}{2}$.

Lemma  The greedy algorithm gives a 2-approximation.
**Theorem**  Unless $P=NP$, there is no poly-time approximation algorithm for bin packing with approximation ratio $< \frac{3}{2}$. 

**Proof.** It is NP-hard to decide if whether the items can be packed into two bins or not, using the reduction from equal partition.

**Equal Partition**

- **Input:** $n$ numbers $x_1, x_2, \ldots, x_n \in \mathbb{Z} > 0$
- **Output:** decide if there is a partition of $[n]$ into $A$ and $B$ such that $\sum_{i \in A} x_i = \sum_{i \in B} x_i$

**Theorem** Equal Partition is (weakly) NP-hard.
**Theorem**  Unless $P=NP$, there is no poly-time approximation algorithm for bin packing with approximation ratio $< 3/2$.

**Proof.**

- It is NP-hard to decide if whether the items can be packed into 2 bins or not, using the reduction from equal partition. □
**Theorem**  Unless \(P=NP\), there is no poly-time approximation algorithm for bin packing with approximation ratio \(< \frac{3}{2}\).

**Proof.**
- It is NP-hard to decide if whether the items can be packed into 2 bins or not, using the reduction from equal partition. \(\square\)

**Equal Partition**

**Input:** \(n\) numbers \(x_1, x_2, \cdots, x_n \in \mathbb{Z}_{>0}\)

**Output:** decide if there is a partition of \([n]\) into \(A\) and \(B\) such that \(\sum_{i \in A} x_i = \sum_{i \in B} x_i\)

**Theorem**  Equal Partition is (weakly) NP-hard.
The approximation ratio is bad only when $\text{opt}$ is small

NP-hard to decide between $\text{opt} \leq 2$ and $\text{opt} \geq 3$.

Open: NP-hard to decide between $\text{opt} \leq 100$ and $\text{opt} \geq 102$?
- The approximation ratio is bad only when $\text{opt}$ is small.
- NP-hard to decide between $\text{opt} \leq 2$ and $\text{opt} \geq 3$.
- Open: NP-hard to decide between $\text{opt} \leq 100$ and $\text{opt} \geq 102$?
- The conjecture has not been disproved (assuming $P \neq NP$):

**Conjecture:** There is an efficient algorithm that outputs a solution with $\text{opt} + 1$ bins.
The approximation ratio is bad only when $\text{opt}$ is small.
NP-hard to decide between $\text{opt} \leq 2$ and $\text{opt} \geq 3$.
Open: NP-hard to decide between $\text{opt} \leq 100$ and $\text{opt} \geq 102$?
The conjecture has **not** been disproved (assuming $P \neq NP$):

**Conjecture**: There is an efficient algorithm that outputs a solution with $\text{opt} + 1$ bins.

- **asymptotic $\alpha$-approximation**: an efficient algorithm that finds solution with $\alpha \cdot \text{opt} + c$ bins, with $c = O(1)$.
The approximation ratio is bad only when $\text{opt}$ is small.

NP-hard to decide between $\text{opt} \leq 2$ and $\text{opt} \geq 3$.

Open: NP-hard to decide between $\text{opt} \leq 100$ and $\text{opt} \geq 102$?

The conjecture has not been disproved (assuming $\text{P} \neq \text{NP}$):

**Conjecture:** There is an efficient algorithm that outputs a solution with $\text{opt} + 1$ bins.

Asymptotic $\alpha$-approximation: an efficient algorithm that finds solution with $\alpha \cdot \text{opt} + c$ bins, with $c = O(1)$.

**Theorem** First-Fit-Decreasing algorithm outputs a solution using at most $(11/9) \cdot \text{opt} + 4$ bins. That is, it is an asymptotic $11/9$-approximation.
**Def.** An asymptotic polynomial-time approximation scheme (APTAS) for minimization problems is a family of algorithms $A_\epsilon$ along with a constant $c \geq 0$, where algorithm $A_\epsilon$ for every $\epsilon > 0$ returns a solution of value at most $(1 + \epsilon)\text{opt} + c$ in polynomial time.
**Def.** An asymptotic polynomial-time approximation scheme (APTAS) for minimization problems is a family of algorithms $A_\epsilon$ along with a constant $c \geq 0$, where algorithm $A_\epsilon$ for every $\epsilon > 0$ returns a solution of value at most $(1 + \epsilon)\text{opt} + c$ in polynomial time.

**Theorem** For any fixed $\epsilon > 0$, there is a polynomial time algorithm that, given a bin-packing instance $\mathcal{I}$, outputs a solution with at most $(1 + \epsilon)\text{opt} + 1$ bins.

That is, there is an APTAS for bin-packing.
• \( \gamma > 0 \) a small constant: item \( i \) is \( \begin{cases} \text{small} & \text{if } s_i < \gamma \\ \text{big} & \text{if } s_i \geq \gamma \end{cases} \)

What to do if all items are small?

First-Fit: all but at most 1 bin has total size \( \leq 1 - \gamma \)

\( \text{alg} \leq l_{\text{opt}} \)

\( m < 1 - \gamma \cdot \text{opt} + 1 \), \( \gamma := \epsilon/2 \)

\( 1 - \gamma < 1 + \epsilon \)

What to do if all items are big?

truncate item sizes to obtain \( I' \), using DP to solve \( I' \)

two essential properties:

\( \text{opt}(I') \approx \text{opt}(I) \)

#(item sizes in \( I' \)) is small

general instance: pack big items using truncation + DP, then use First-Fit to pack small items
\[ \gamma > 0 \text{ a small constant: item } i \text{ is } \begin{cases} \text{small} & \text{if } s_i < \gamma \\ \text{big} & \text{if } s_i \geq \gamma \end{cases} \]

What to do if all items are small?
\( \gamma > 0 \) a small constant: item \( i \) is \( \begin{cases} 
small & \text{if } s_i < \gamma \\
big & \text{if } s_i \geq \gamma 
\end{cases} \)

**What to do if all items are small?**

- First-Fit: all but at most 1 bin has total size \( \leq 1 - \gamma \)

\[
\text{alg} \leq \left\lceil \frac{\text{opt}}{1-\gamma} \right\rceil < \frac{1}{1-\gamma} \cdot \text{opt} + 1, \quad \gamma := \frac{\epsilon}{2} \Rightarrow \frac{1}{1-\gamma} < 1 + \epsilon
\]
• $\gamma > 0$ a small constant: item $i$ is \[ \begin{cases} \text{small} & \text{if } s_i < \gamma \\ \text{big} & \text{if } s_i \geq \gamma \end{cases} \]

What to do if all items are small?

- **First-Fit**: all but at most 1 bin has total size $\leq 1 - \gamma$
- $\text{alg} \leq \left\lceil \frac{\text{opt}}{1-\gamma} \right\rceil < \frac{1}{1-\gamma} \cdot \text{opt} + 1, \quad \gamma := \epsilon/2 \quad \Rightarrow \quad \frac{1}{1-\gamma} < 1 + \epsilon$

What to do if all items are big?

- Truncate item sizes to obtain $I'$, using DP to solve $I'$
- Two essential properties: $\text{opt}(I') \approx \text{opt}(I)$
- #(item sizes in $I'$) is small
- General instance: pack big items using truncation + DP, then use First-Fit to pack small items
$\gamma > 0$ a small constant: item $i$ is
\[
\begin{cases}
\text{small} & \text{if } s_i < \gamma \\
\text{big} & \text{if } s_i \geq \gamma
\end{cases}
\]

What to do if all items are small?

- First-Fit: all but at most 1 bin has total size $\leq 1 - \gamma$
- $\text{alg} \leq \left\lceil \frac{\text{opt}}{1-\gamma} \right\rceil < \frac{1}{1-\gamma} \cdot \text{opt} + 1$, $\gamma := \epsilon/2 \implies \frac{1}{1-\gamma} < 1 + \epsilon$

What to do if all items are big?

- truncate item sizes to obtain $\mathcal{I}'$, using DP to solve $\mathcal{I}'$
- two essential properties:
  \[
  \text{opt}(\mathcal{I}') \approx \text{opt}(\mathcal{I}) \quad \#(\text{item sizes in } \mathcal{I}') \text{ is small}
  \]
• \( \gamma > 0 \) a small constant: item \( i \) is
\[
\begin{cases} 
\text{small} & \text{if } s_i < \gamma \\
\text{big} & \text{if } s_i \geq \gamma
\end{cases}
\]

### What to do if all items are small?
- **First-Fit**: all but at most 1 bin has total size \( \leq 1 - \gamma \)
- \( \text{alg} \leq \left\lceil \frac{\text{opt}}{1-\gamma} \right\rceil < \frac{1}{1-\gamma} \cdot \text{opt} + 1, \quad \gamma := \epsilon/2 \quad \Rightarrow \quad \frac{1}{1-\gamma} < 1 + \epsilon \)

### What to do if all items are big?
- truncate item sizes to obtain \( \mathcal{I}' \), using DP to solve \( \mathcal{I}' \)
- two essential properties:
  \[ \text{opt}(\mathcal{I}') \approx \text{opt}(\mathcal{I}) \quad \#(\text{item sizes in } \mathcal{I}') \text{ is small} \]
- general instance: pack big items using truncation + DP, then use First-Fit to pack small items
1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Construction of Instance $\mathcal{I}'$

1. sort items in non-increasing sizes

Instance $\mathcal{I}$
Construction of Instance $I'$

1. sort items in non-increasing sizes
2. partition items into groups of size $g$
Construction of Instance $\mathcal{I}'$

1. sort items in non-increasing sizes
2. partition items into groups of size $g$
3. discard the first group
Construction of Instance $\mathcal{I}'$

1: sort items in non-increasing sizes
2: partition items into groups of size $g$
3: discard the first group
4: **for** each of the other groups **do**
5: change item size to the biggest size in group
Construction of Instance $I'$

1. sort items in non-increasing sizes
2. partition items into groups of size $g$
3. discard the first group
4. for each of the other groups do
5. change item size to the biggest size in group

Instance $I'$

$\text{opt}(I) - g \leq \text{opt}(I')$
Construction of Instance $\mathcal{I}'$

1: sort items in non-increasing sizes
2: partition items into groups of size $g$
3: discard the first group
4: for each of the other groups do
5: change item size to the biggest size in group

\[ \text{opt}(\mathcal{I}) - g \leq \text{opt}(\mathcal{I}') \]
Construction of Instance $I'$

1. sort items in non-increasing sizes
2. partition items into groups of size $g$
3. discard the first group
4. for each of the other groups do
5. change item size to the biggest size in group

$$\text{opt}(I) - g \leq \text{opt}(I')$$
Construction of Instance $\mathcal{I}'$

1. sort items in non-increasing sizes
2. partition items into groups of size $g$
3. discard the first group
4. for each of the other groups do
5. change item size to the biggest size in group

$\text{opt}(\mathcal{I}) - g \leq \text{opt}(\mathcal{I}')$
Construction of Instance $\mathcal{I}'$

1: sort items in non-increasing sizes
2: partition items into groups of size $g$
3: discard the first group
4: for each of the other groups do
5: change item size to the biggest size in group

Instance $\mathcal{I}$

Instance $\mathcal{I}'$

$\text{opt}(\mathcal{I}) - g \leq \text{opt}(\mathcal{I}') \leq \text{opt}(\mathcal{I})$
every group in $\mathcal{I}'$ has the same size.
every group in \( \mathcal{I}' \) has the same size.

\( k := \) the number of distinct sizes in \( \mathcal{I}' \), \( k \leq \left\lfloor \frac{n}{g} \right\rfloor \)
every group in $\mathcal{I}'$ has the same size.

$k :=$ the number of distinct sizes in $\mathcal{I}'$, $k \leq \left\lfloor \frac{n}{g} \right\rfloor$

$\mathcal{I}'$ can be solved exactly by DP in $O(n^{2k})$-time
every group in $I'$ has the same size.

$k := \text{the number of distinct sizes in } I', \quad k \leq \left\lfloor \frac{n}{g} \right\rfloor$

$I'$ can be solved exactly by DP in $O(n^{2k})$-time

**Dynamic Programming for $I'$ in $O(n^{2k})$-time**

- let $s^{(1)} \geq s^{(2)} \geq \cdots \geq s^{(k)}$ be the $k$ distinct sizes
every group in $\mathcal{I}'$ has the same size.

$k := \text{the number of distinct sizes in } \mathcal{I}', \quad k \leq \left\lfloor \frac{n}{g} \right\rfloor$

$\mathcal{I}'$ can be solved exactly by DP in $O(n^{2k})$-time

---

**Dynamic Programming for $\mathcal{I}'$ in $O(n^{2k})$-time**

- let $s^{(1)} \geq s^{(2)} \geq \cdots \geq s^{(k)}$ be the $k$ distinct sizes
- let $n_1, n_2, \cdots, n_k$ be the number of items of each size
• every group in $\mathcal{I}'$ has the same size.
• $k :=$ the number of distinct sizes in $\mathcal{I}'$, $k \leq \left\lfloor \frac{n}{g} \right\rfloor$
• $\mathcal{I}'$ can be solved exactly by DP in $O(n^{2k})$-time

Dynamic Programming for $\mathcal{I}'$ in $O(n^{2k})$-time

• let $s^{(1)} \geq s^{(2)} \geq \cdots \geq s^{(k)}$ be the $k$ distinct sizes
• let $n_1, n_2, \cdots, n_k$ be the number of items of each size
• vertex $(a_1, a_2, \cdots, a_k)$: the instance with $a_1$ items of size $s^{(1)}$, $a_2$ items of size $s^{(2)}$, $\cdots$, and $a_k$ items of size $s^{(k)}$
• every group in $\mathcal{I}'$ has the same size.
• $k :=$ the number of distinct sizes in $\mathcal{I}'$, $k \leq \left\lfloor \frac{n}{g} \right\rfloor$
• $\mathcal{I}'$ can be solved exactly by DP in $O(n^{2k})$-time

**Dynamic Programming for $\mathcal{I}'$ in $O(n^{2k})$-time**

• let $s^{(1)} \geq s^{(2)} \geq \cdots \geq s^{(k)}$ be the $k$ distinct sizes
• let $n_1, n_2, \cdots, n_k$ be the number of items of each size
• vertex $(a_1, a_2, \cdots, a_k)$: the instance with $a_1$ items of size $s^{(1)}$, $a_2$ items of size $s^{(2)}$, $\cdots$, and $a_k$ items of size $s^{(k)}$
• an arc $(a_1, a_2, \cdots, a_k) \to (b_1, b_2, \cdots, b_k)$ if
  • $a_i \geq b_i$ for every $i \in [k]$ and,
  • $s^{(1)}(b_1 - a_1) + s^{(2)}(b_2 - a_2) + \cdots + s^{(k)}(b_k - a_k) \leq 1$
every group in $I'$ has the same size.

$k :=$ the number of distinct sizes in $I'$, $k \leq \left\lfloor \frac{n}{g} \right\rfloor$

$I'$ can be solved exactly by DP in $O(n^{2k})$-time

---

**Dynamic Programming for $I'$ in $O(n^{2k})$-time**

- let $s^{(1)} \geq s^{(2)} \geq \cdots \geq s^{(k)}$ be the $k$ distinct sizes
- let $n_1, n_2, \ldots, n_k$ be the number of items of each size
- vertex $(a_1, a_2, \cdots, a_k)$: the instance with $a_1$ items of size $s^{(1)}$, $a_2$ items of size $s^{(2)}$, \ldots, and $a_k$ items of size $s^{(k)}$
- an arc $(a_1, a_2, \cdots, a_k) \rightarrow (b_1, b_2, \cdots, b_k)$ if
  - $a_i \geq b_i$ for every $i \in [k]$ and,
  - $s^{(1)}(b_1 - a_1) + s^{(2)}(b_2 - a_2) + \cdots + s^{(k)}(b_k - a_k) \leq 1$
- DP: computing the shortest path from $(0, 0, \cdots, 0)$ to $(n_1, n_2, \cdots, n_k)$
opt(\mathcal{I}) - g \leq opt(\mathcal{I}') \leq opt(\mathcal{I}).
\[ \text{opt}(\mathcal{I}) - g \leq \text{opt}(\mathcal{I}') \leq \text{opt}(\mathcal{I}). \]

- solving \( \mathcal{I}' \) \( \Rightarrow \) packing for \( \mathcal{I} \) with \( \leq \text{opt}(\mathcal{I}) + g \) bins
\[ \text{opt}(\mathcal{I}) - g \leq \text{opt}(\mathcal{I}') \leq \text{opt}(\mathcal{I}). \]

- solving \( \mathcal{I}' \) \( \Rightarrow \) packing for \( \mathcal{I} \) with \( \leq \text{opt}(\mathcal{I}) + g \) bins
- \( s_i \geq \gamma, \forall i \in [n] \) \( \implies \) \( \text{opt}(\mathcal{I}) \geq \gamma n. \)
\[
\text{opt}(\mathcal{I}) - g \leq \text{opt}(\mathcal{I}') \leq \text{opt}(\mathcal{I}).
\]

- solving \(\mathcal{I}'\) \(\Rightarrow\) packing for \(\mathcal{I}\) with \(\leq \text{opt}(\mathcal{I}) + g\) bins
- \(s_i \geq \gamma, \forall i \in [n]\) \(\Rightarrow\) \(\text{opt}(\mathcal{I}) \geq \gamma n\).
- setting \(g := \epsilon \gamma n\) \(\Rightarrow\) \(g \leq \epsilon \cdot \text{opt}(\mathcal{I})\) and \(k \leq \frac{n}{g} \leq \frac{1}{\epsilon \gamma}\)
\[ \text{opt}(I) - g \leq \text{opt}(I') \leq \text{opt}(I). \]

- solving \( I' \) \( \Rightarrow \) packing for \( I \) with \( \leq \text{opt}(I) + g \) bins
- \( s_i \geq \gamma, \forall i \in [n] \) \( \implies \) \( \text{opt}(I) \geq \gamma n. \)
- setting \( g := \epsilon\gamma n \) \( \implies \) \( g \leq \epsilon \cdot \text{opt}(I) \) and \( k \leq \frac{n}{g} \leq \frac{1}{\epsilon\gamma} \)

**Theorem** There is an \( O(n^2/(\epsilon\gamma)) \)-time \((1 + \epsilon)\)-approximation algorithm for the bin-packing problem when all items have size at least \( \gamma \),
1. Knapsack Problem
   - Introduction
   - FPTAS for Knapsack Problem

2. PTAS for Makespan Minimization on Identical Machines
   - Introduction
   - Dynamic Programming to Schedule Big Jobs
   - Analysis of Combined Algorithm

3. An asymptotical PTAS for Bin Packing
   - Introduction
   - Algorithm for Big Items
   - Combination of Algorithms for Big and Small Items
Combining Algorithms for Small and Big Items

1: Use truncation + DP to obtain solution $S$ for big items.
2: Starting from $S$, use First-Fit to pack small items.
Combining Algorithms for Small and Big Items

1: Use truncation + DP to obtain solution $S$ for big items
Combining Algorithms for Small and Big Items

1: Use truncation + DP to obtain solution $S$ for big items
Combining Algorithms for Small and Big Items

1. Use truncation + DP to obtain solution $S$ for big items
Combining Algorithms for Small and Big Items

1. Use truncation + DP to obtain solution $S$ for big items
2. Starting from $S$, use First-Fit to pack small items
Combining Algorithms for Small and Big Items

1. Use truncation + DP to obtain solution $\mathcal{S}$ for big items
2. Starting from $\mathcal{S}$, use First-Fit to pack small items
Analysis of the Combined Algorithm

Case 1: no new bins are used to pack small items

\[
\text{#(bins used)} \leq (1 + \epsilon) \cdot \text{opt}(I_{\text{big}}) \leq (1 + \epsilon) \cdot \text{opt}(I)
\]

Case 2: new bins are used

\[
\text{at most one bin has total size } \leq 1 - \gamma \quad \text{#(bins used)} < \text{opt}(I) + 1
\]
Analysis of the Combined Algorithm

Case 1: no new bins are used to pack small items

- Case 1: \( \text{bins used} \leq (1 + \epsilon) \cdot \text{opt}(I_{\text{big}}) \leq (1 + \epsilon) \cdot \text{opt}(I) \)
- Case 2: new bins are used at most one bin has total size \( \leq 1 - \gamma \)
  \( \#(\text{bins used}) < \text{opt}(I_{\text{big}}) + 1 \)
Case 1: no new bins are used to pack small items

\[
\#(\text{bins used}) \leq (1 + \epsilon) \cdot \text{opt}(I_{\text{big}}) \leq (1 + \epsilon) \cdot \text{opt}(I)
\]
Analysis of the Combined Algorithm

- **Case 1**: no new bins are used to pack small items
  \[
  \#(\text{bins used}) \leq (1 + \epsilon) \cdot \text{opt}(I_{\text{big}}) \leq (1 + \epsilon) \cdot \text{opt}(I)
  \]

- **Case 2**: new bins are used
Analysis of the Combined Algorithm

Case 1: no new bins are used to pack small items

\[ \#(\text{bins used}) \leq (1 + \epsilon) \cdot \text{opt}(I_{\text{big}}) \leq (1 + \epsilon) \cdot \text{opt}(I) \]

Case 2: new bins are used
at most one bin has total size \( \leq 1 - \gamma \)

\[ \#(\text{bins used}) < \frac{\text{opt}(I)}{1 - \gamma} + 1 \]