Advanced Algorithms (Fall 2023)

Greedy and Local Search

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Nanjing University

Outline

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Maximum-Weight Spanning Tree Problem

Input: Graph $G = (V, E)$ and edge weights $w \in \mathbb{Z}_{>0}^{E}$

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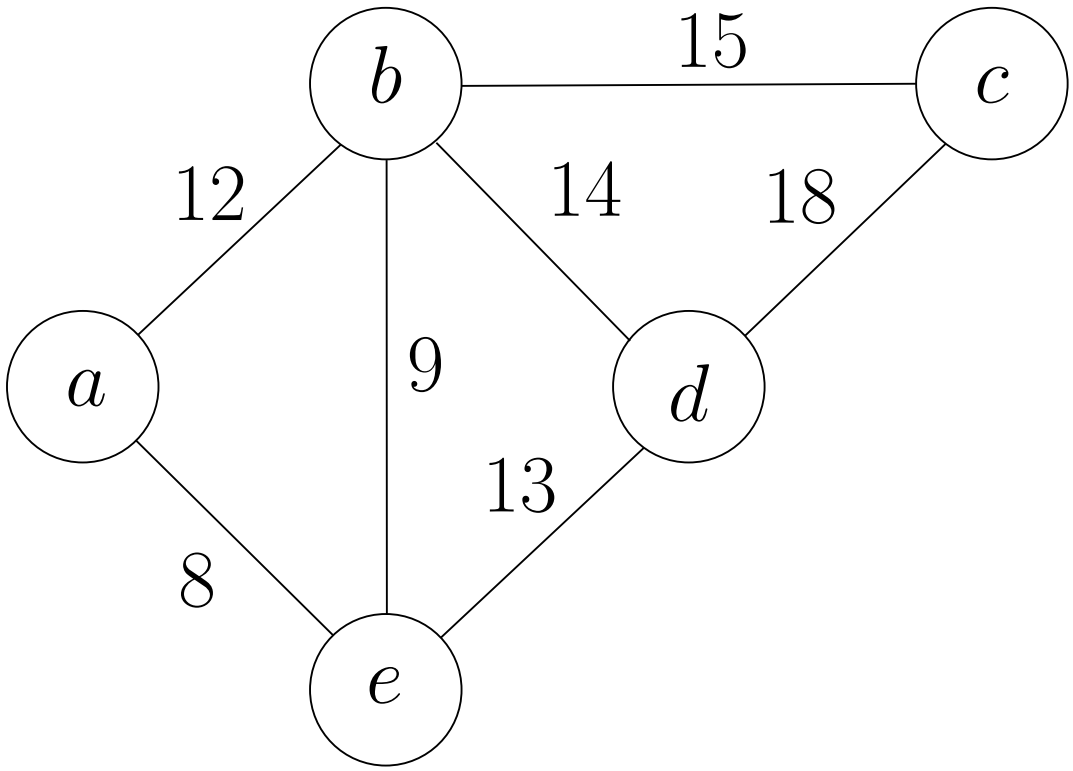

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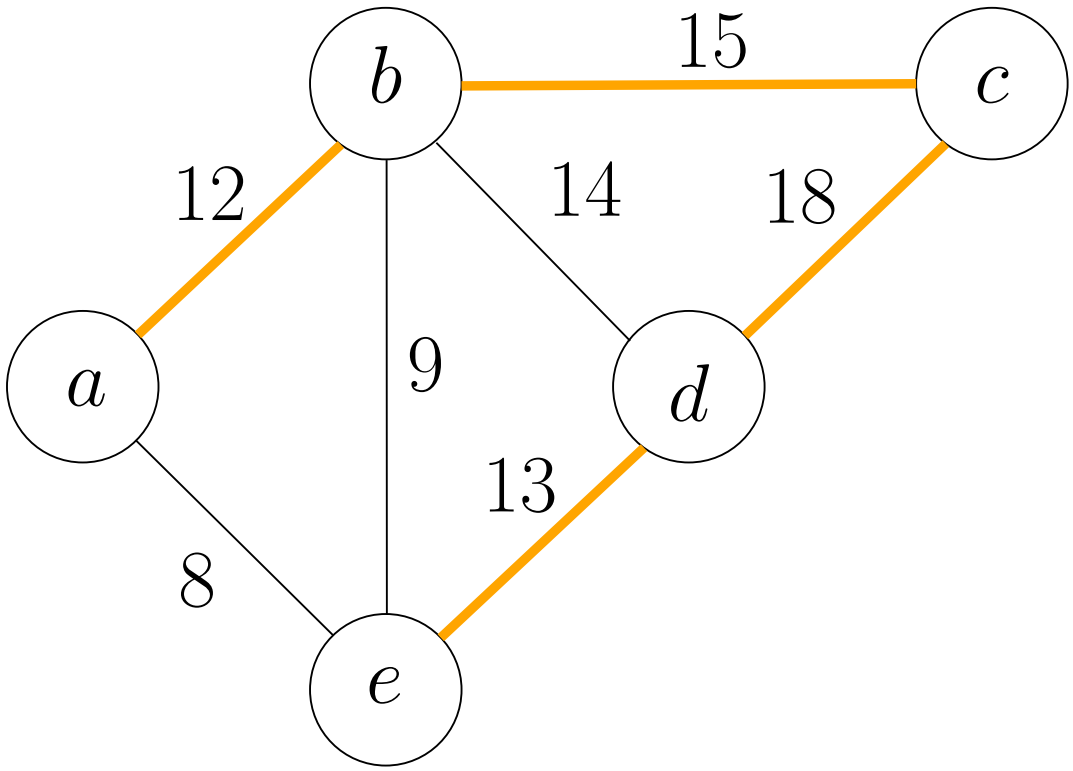

Kruskal's Algorithm for Maximum-Weight Spanning Tree

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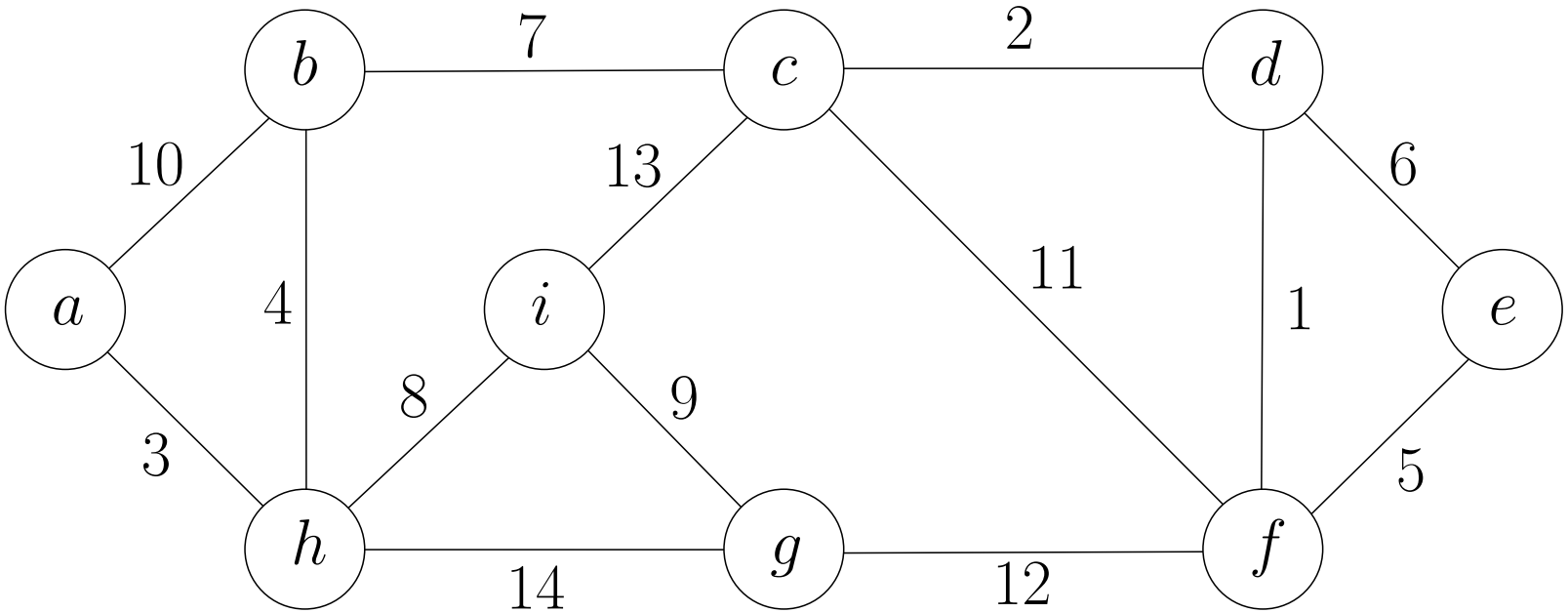

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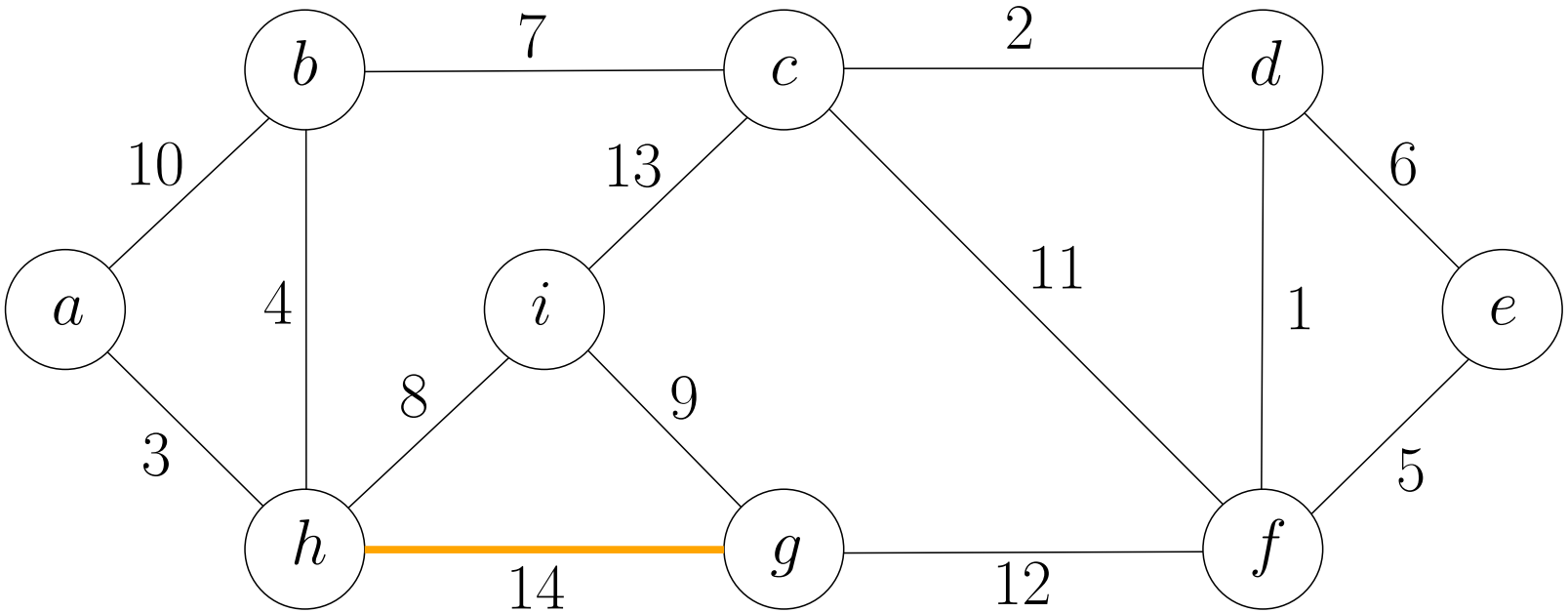

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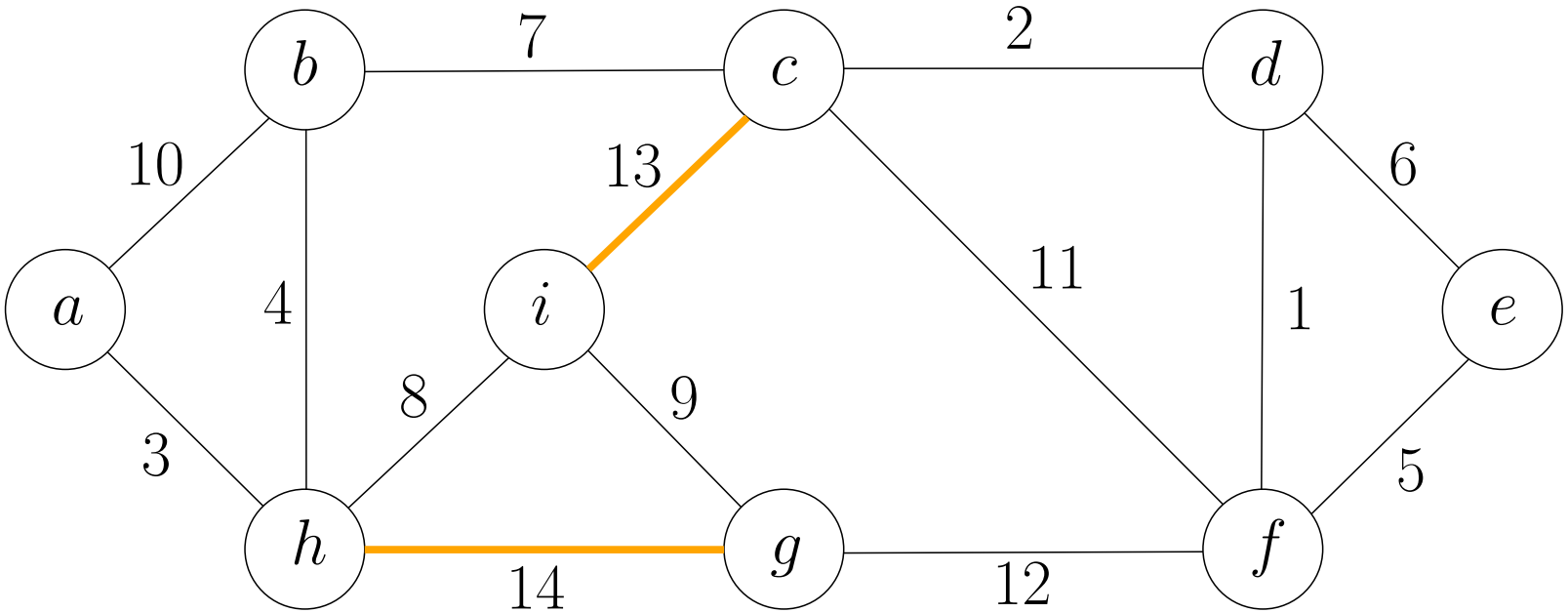

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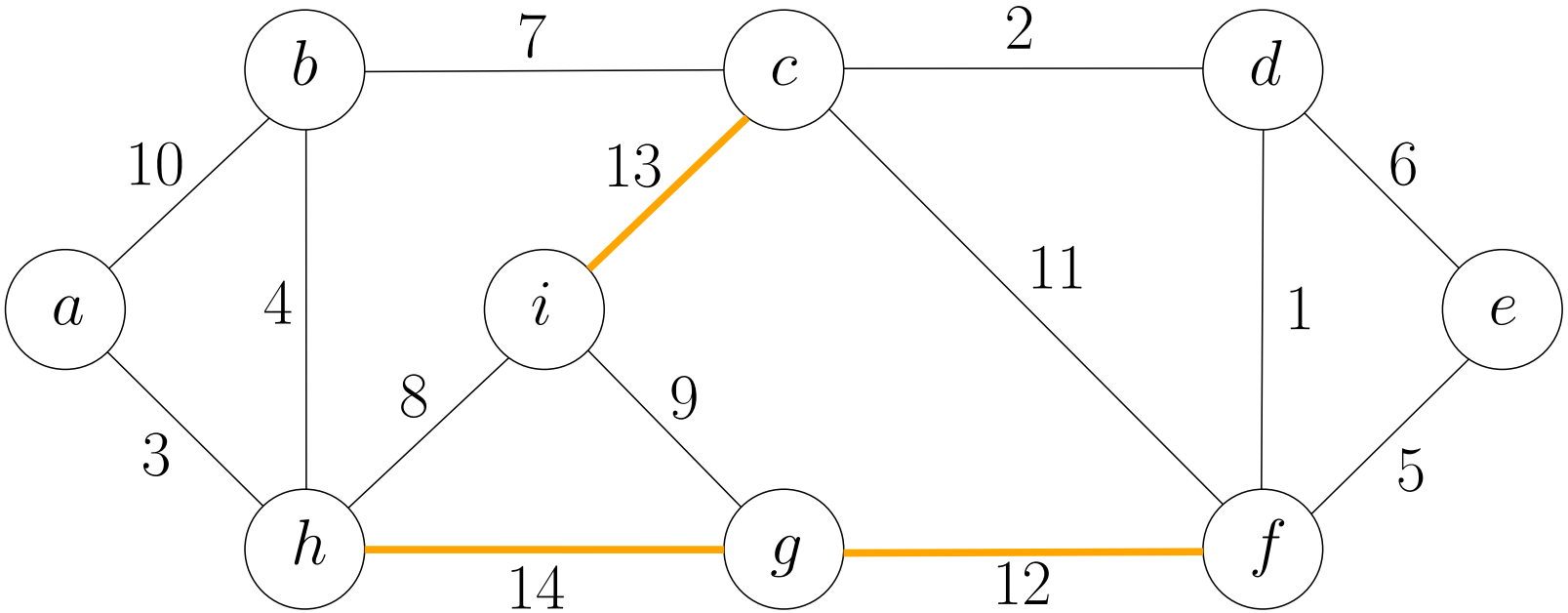

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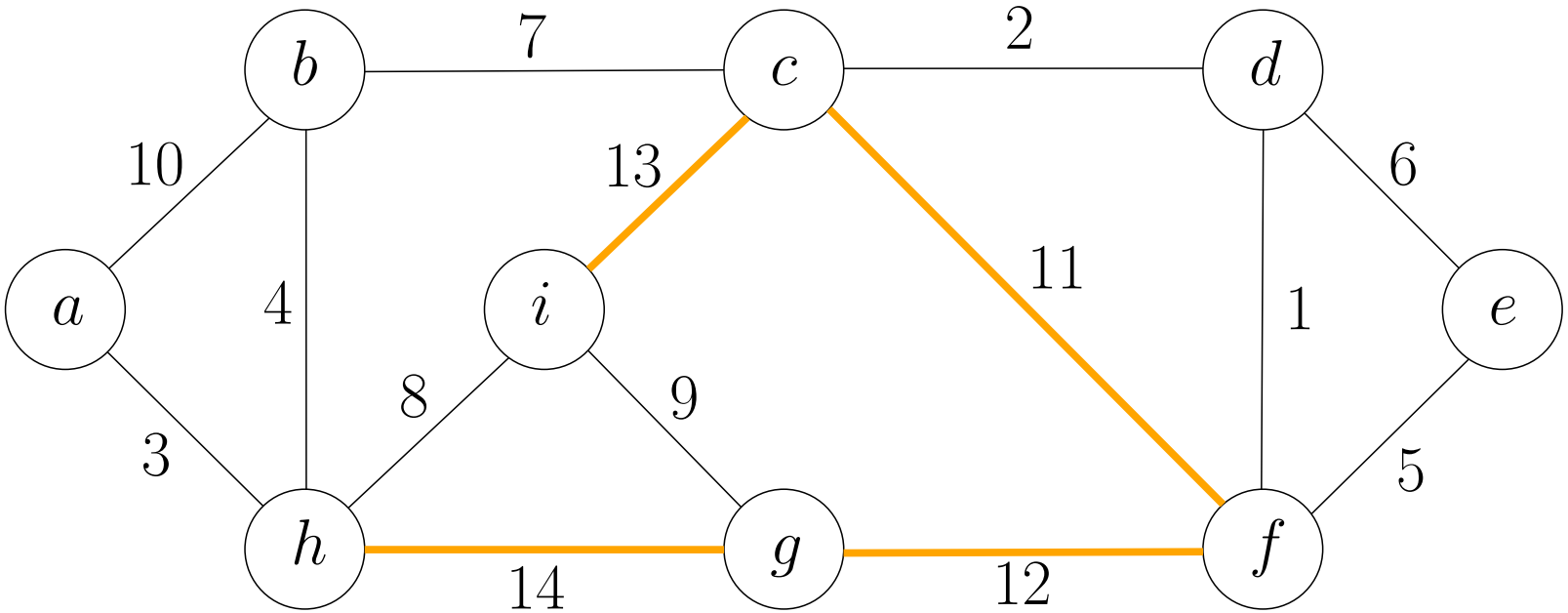

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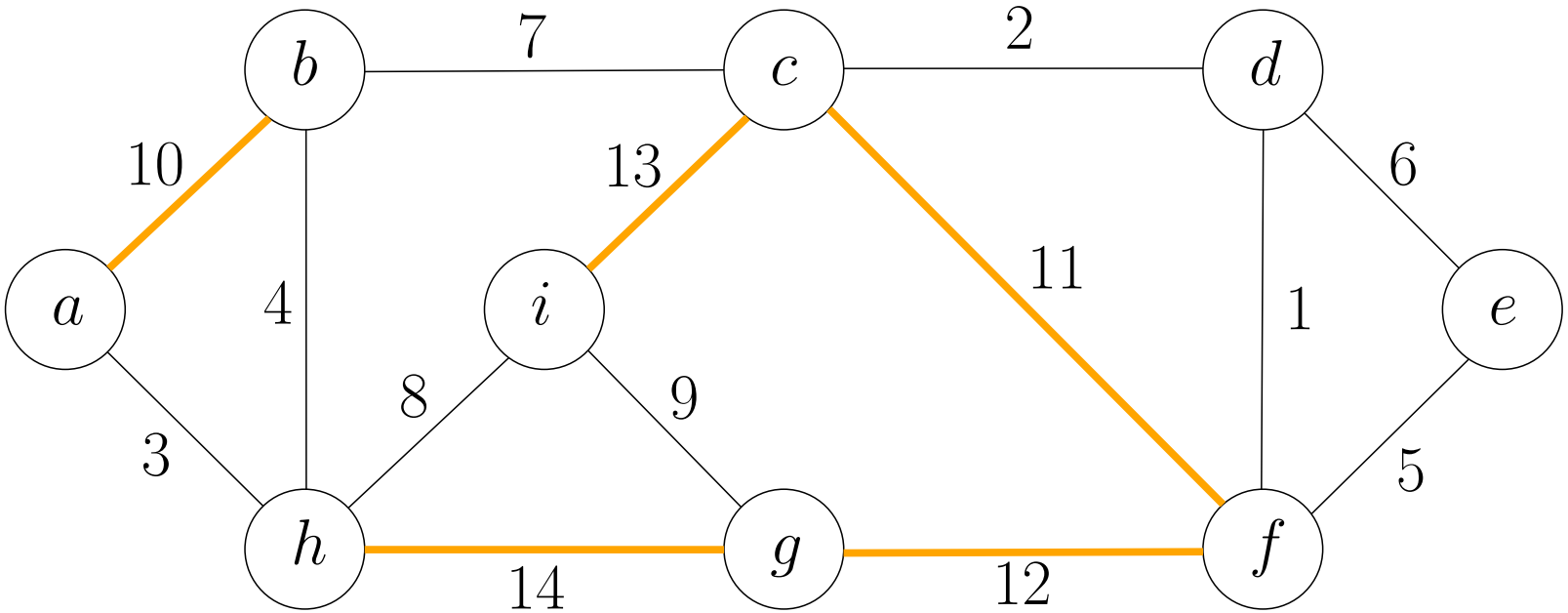

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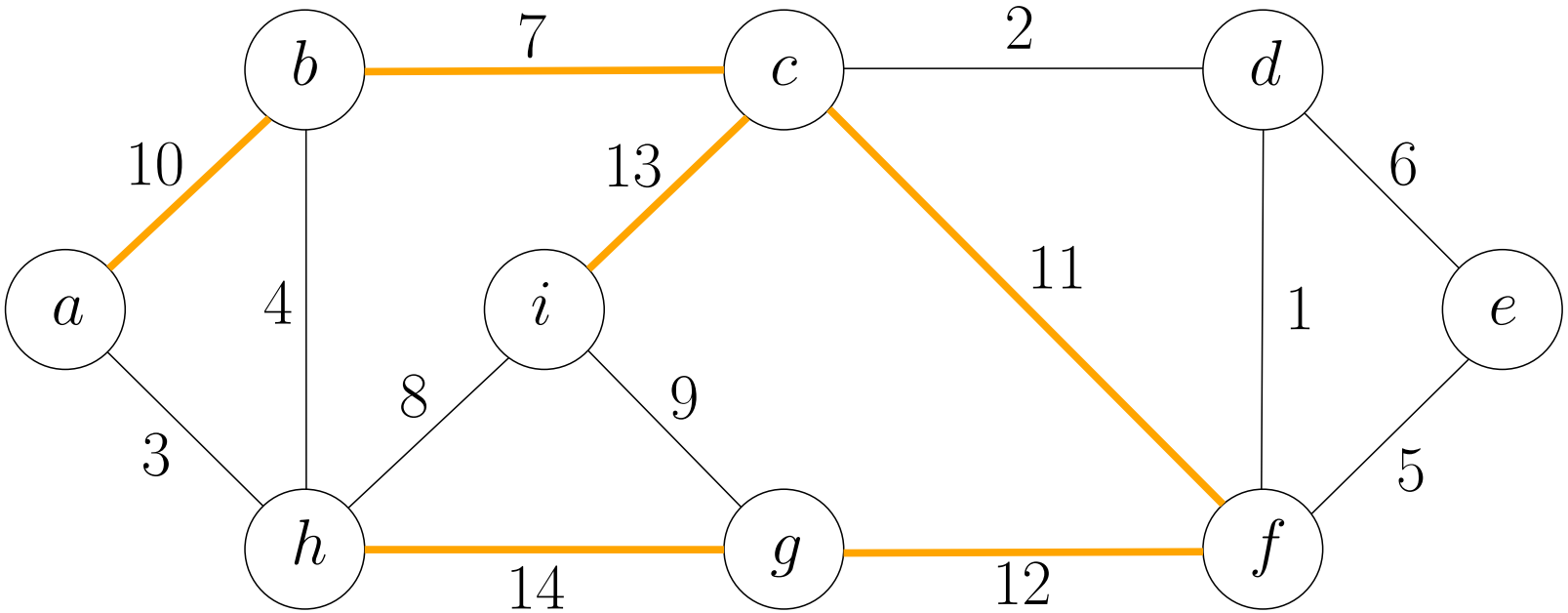

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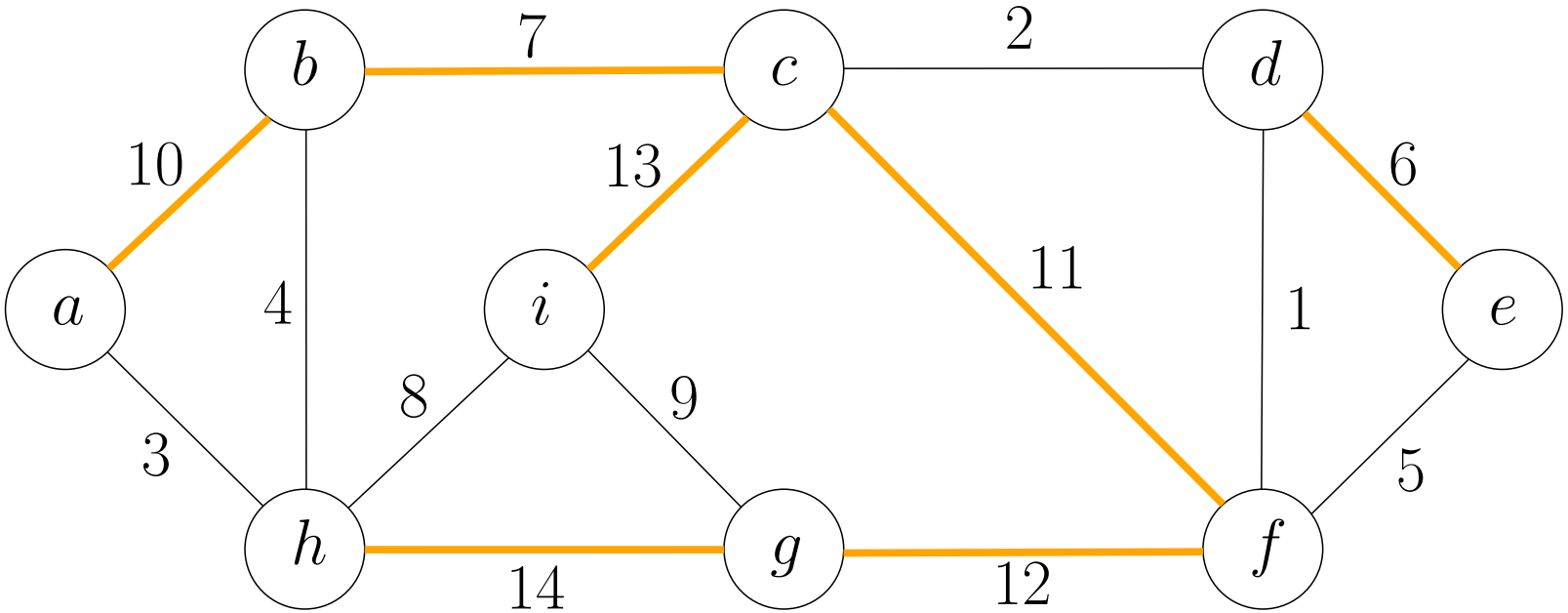

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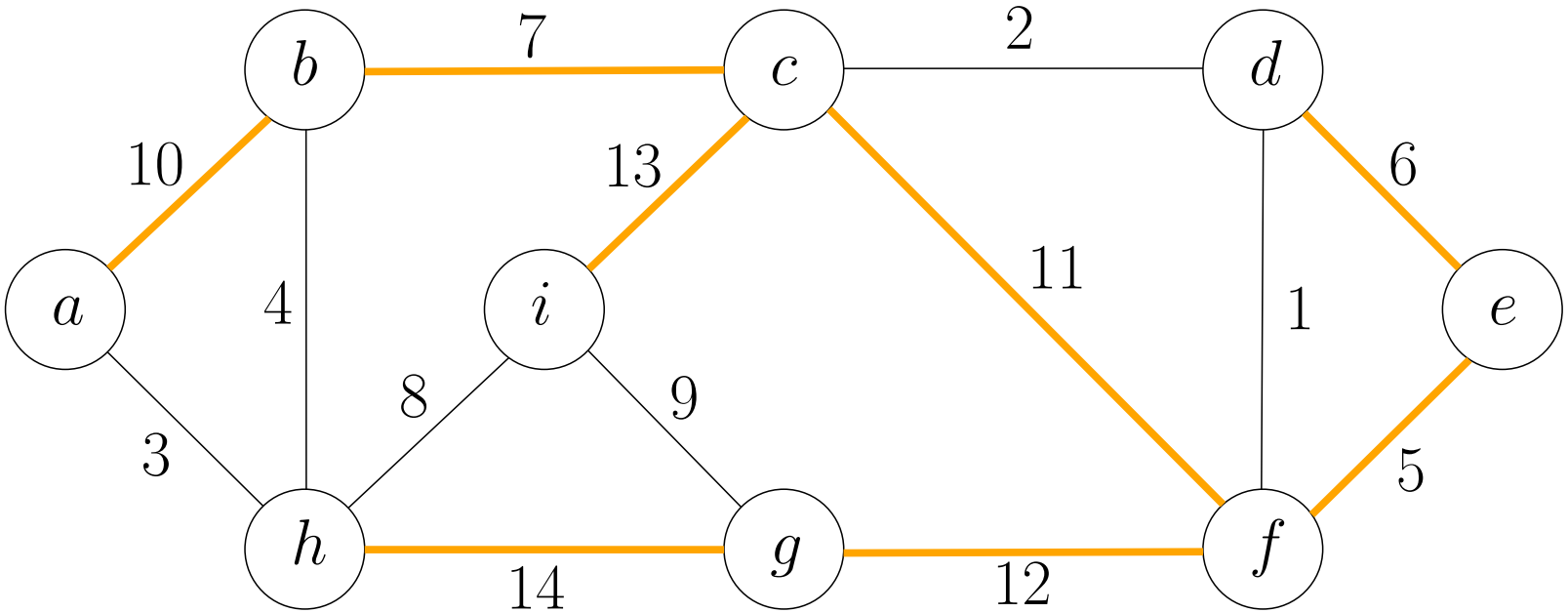

Proof of Correctness of Kruskal's Algorithm

Maximum-Weight Spanning Tree (MST) with Pre-Selected Edges

Input: Graph $G = (V, E)$ and edge weights $w \in \mathbb{Z}_{>0}^{E}$
a set $F_0 \subseteq E$ of edges, that does not contain a cycle

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Lemma (Key Lemma) Given an instance $(G = (V, E), w, F_0)$ of the MST with pre-selected edges problem, let e^* be the maximum weight edge in $E \setminus F_0$ such that $F_0 \cup \{e^*\}$ does not contain a cycle. Then there is an optimum solution $T = (V, E_T)$ to the instance with $e^* \in E_T$.

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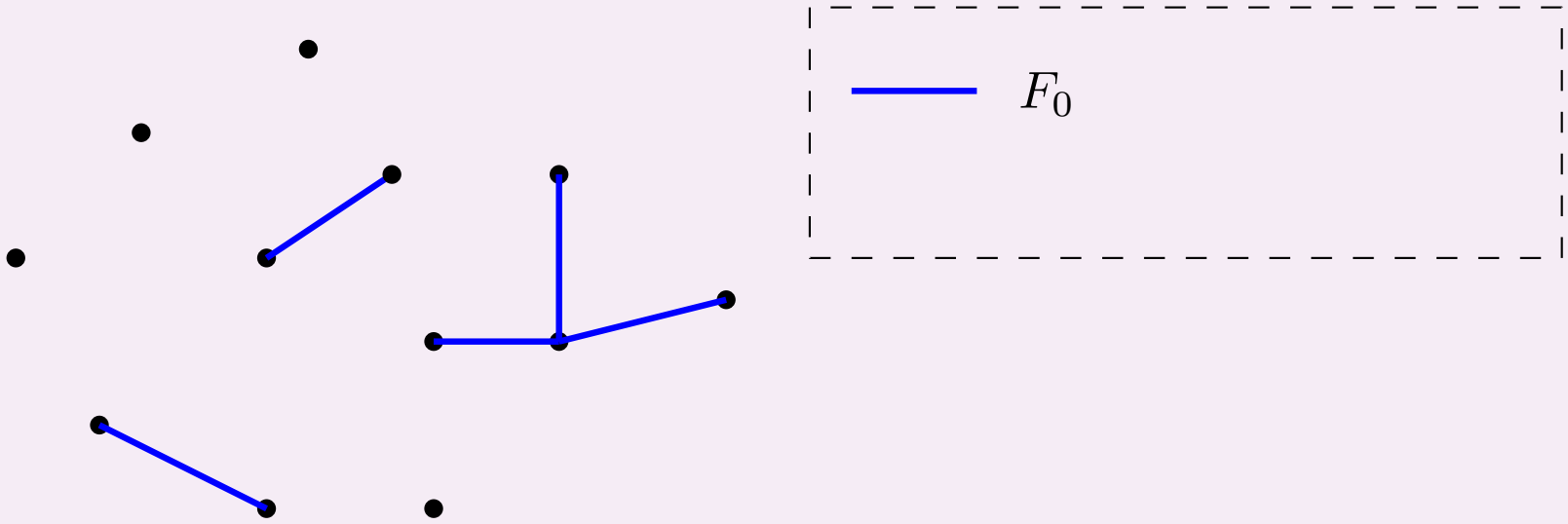
Proof of Key Lemma.



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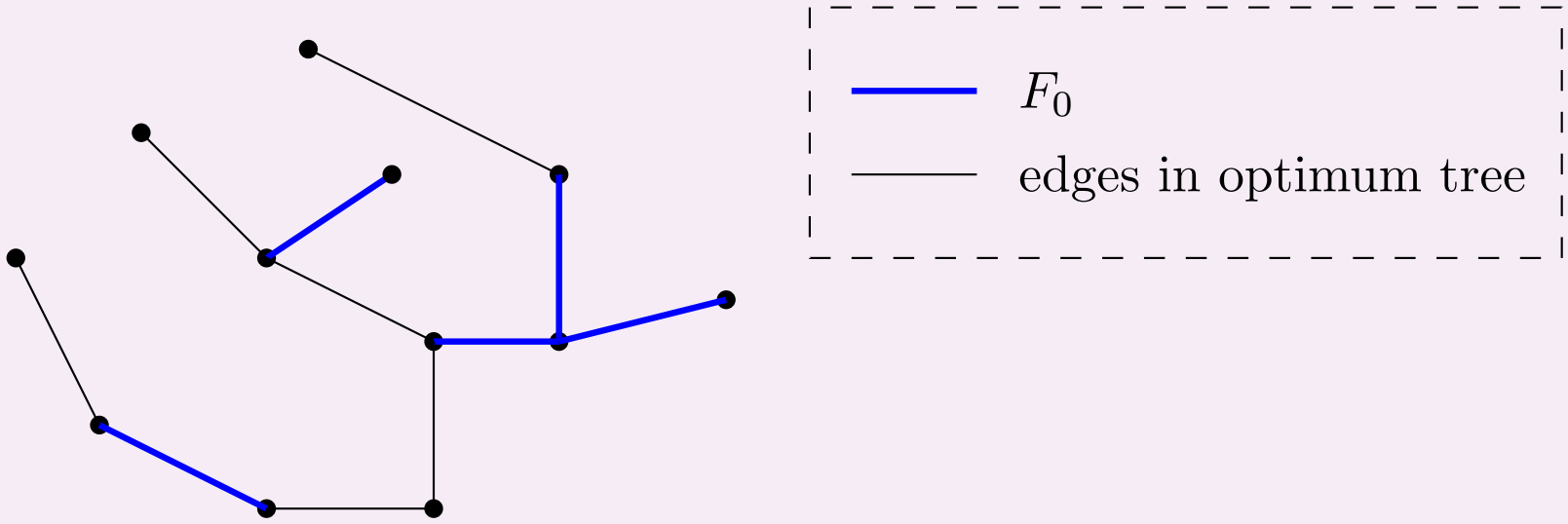

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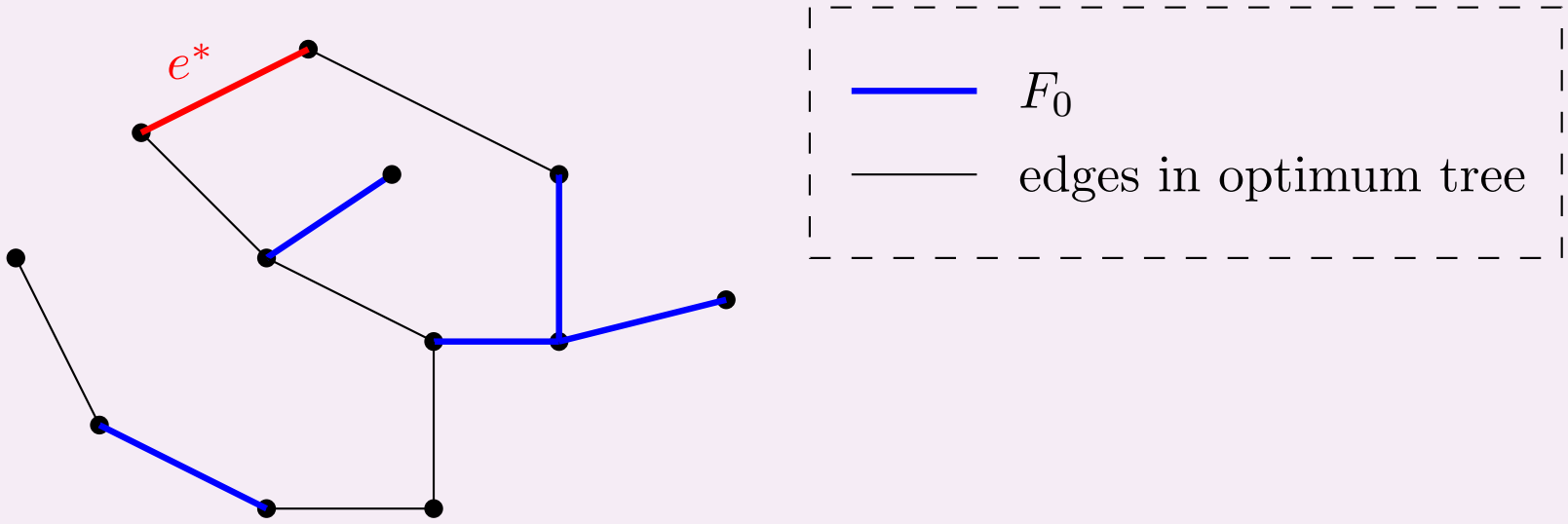

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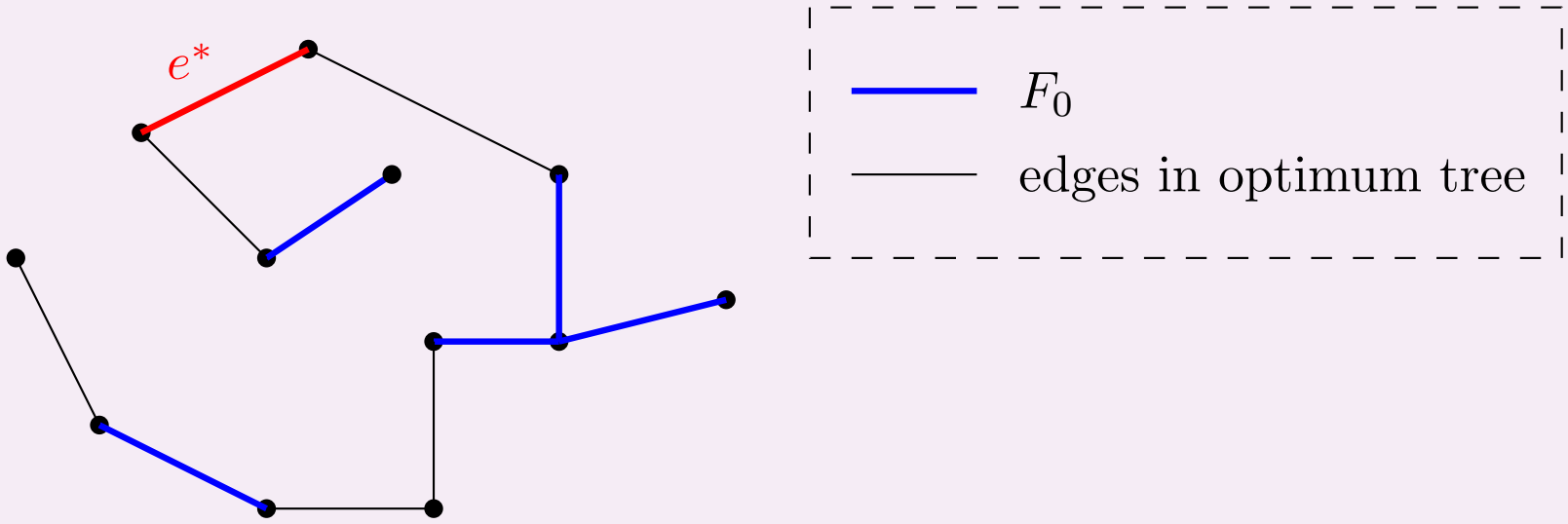

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A General Maximization Problem

Input: E: the ground set of elements

$w \in \mathbb{Z}^E_{>0}$: weight vector on elements

$\mathcal{S}$: an (implicitly given) family of subsets of E

- $\emptyset \in \mathcal{S}$
- $\mathcal{S}$ is downward closed: if $A \in \mathcal{S}, B \subsetneq A$, then $B \in \mathcal{S}$.

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- maximum-weight spanning tree: $\mathcal{S}$ = family of forests

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- Matroids: cases where greedy algorithm is optimum

Def. A (finite) matroid $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where E is a finite set (called the ground set) and $\mathcal{I}$ is a family of subsets of E (called independent sets) with the following properties:

1. $\emptyset \in \mathcal{I}$.
2. (downward-closed property) If $B \subsetneq A \in \mathcal{I}$, then $B \in \mathcal{I}$.
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Proof of Exchange Property.

- $|B| < |A| \Rightarrow (V, B)$ has more CC than (V, A).
- Some edge in A connects two different CC of (V, B). $\square$

Feasible Family for Knapsack Packing Does Not Satisfy Augmentation Property

- $c_1 = c_2 = 10, c_3 = 20, C = 20.$
- $\{1, 2\}, \{3\} \in \mathcal{I}$, but $\{1, 3\}, \{2, 3\} \notin \mathcal{I}$.

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Theorem The greedy algorithm gives optimum solution for the maximum-weight independent set problem in a matroid.

Lemma (Key Lemma)

- given: matroid $\mathcal{M} = (E, \mathcal{I})$, weights $w \in \mathbb{Z}^E_{>0}$, $A \in \mathcal{I}$,
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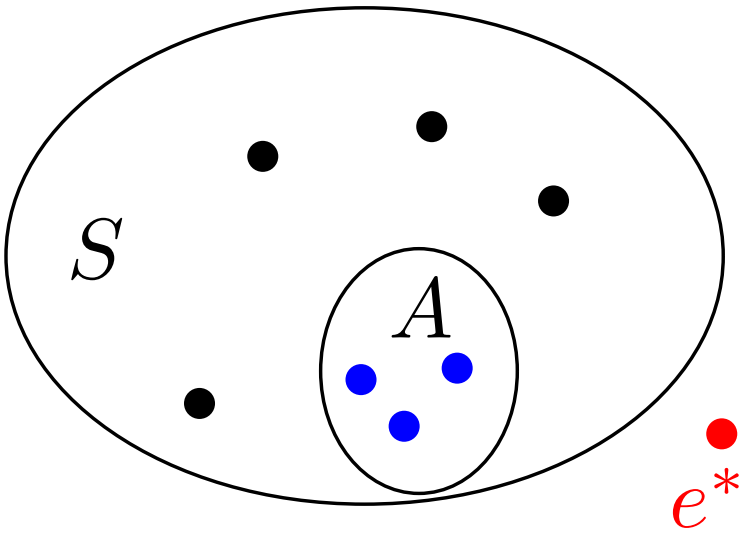

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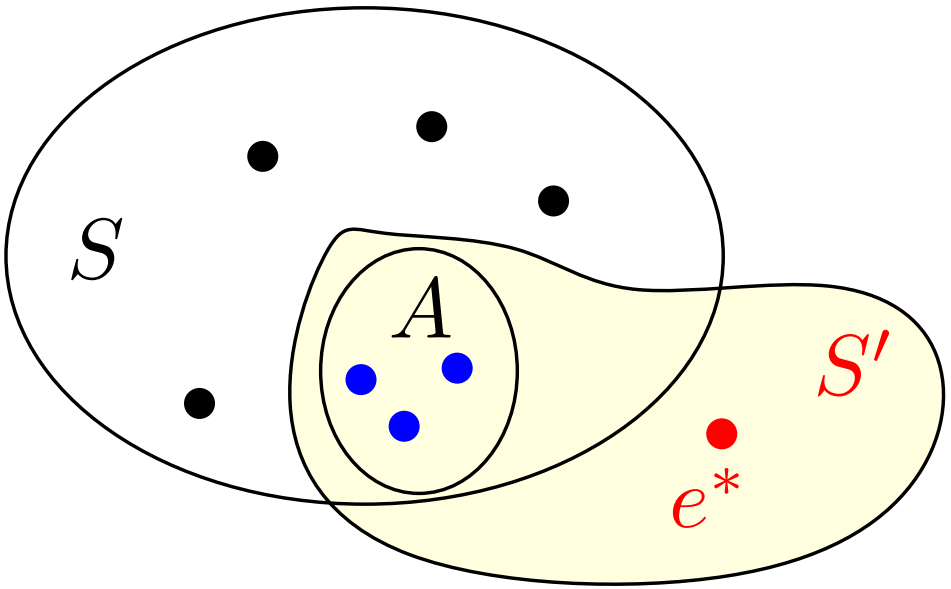

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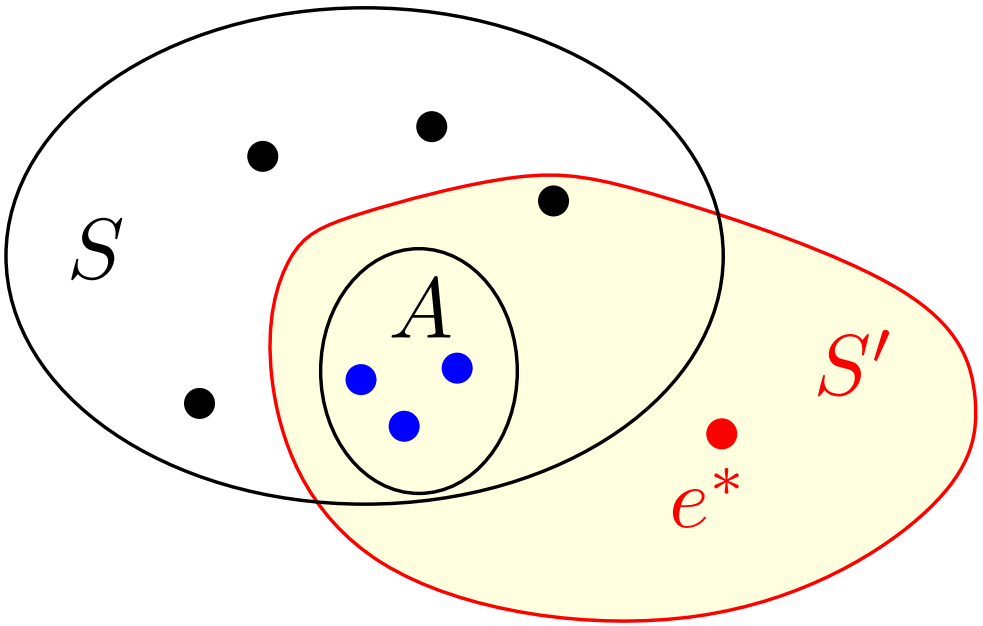

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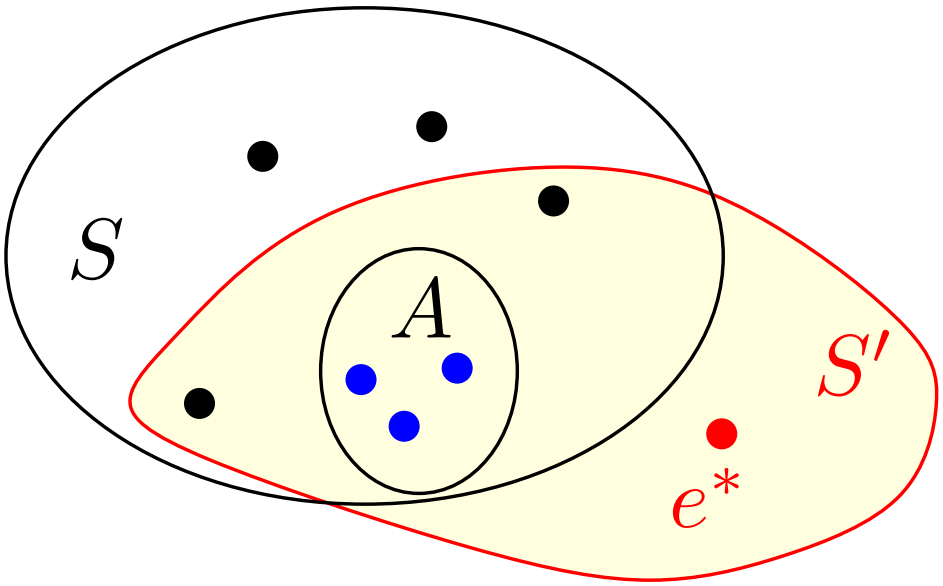

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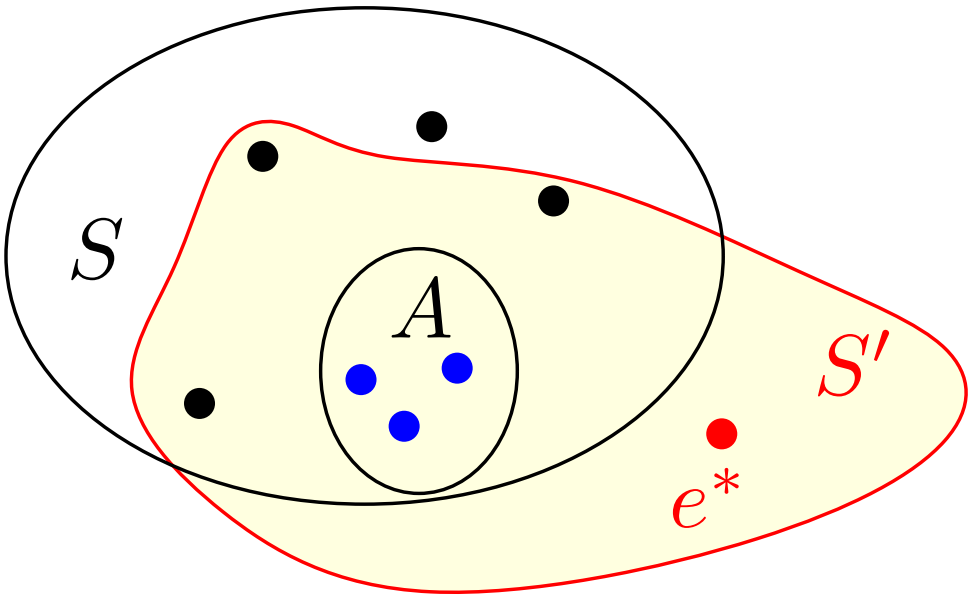

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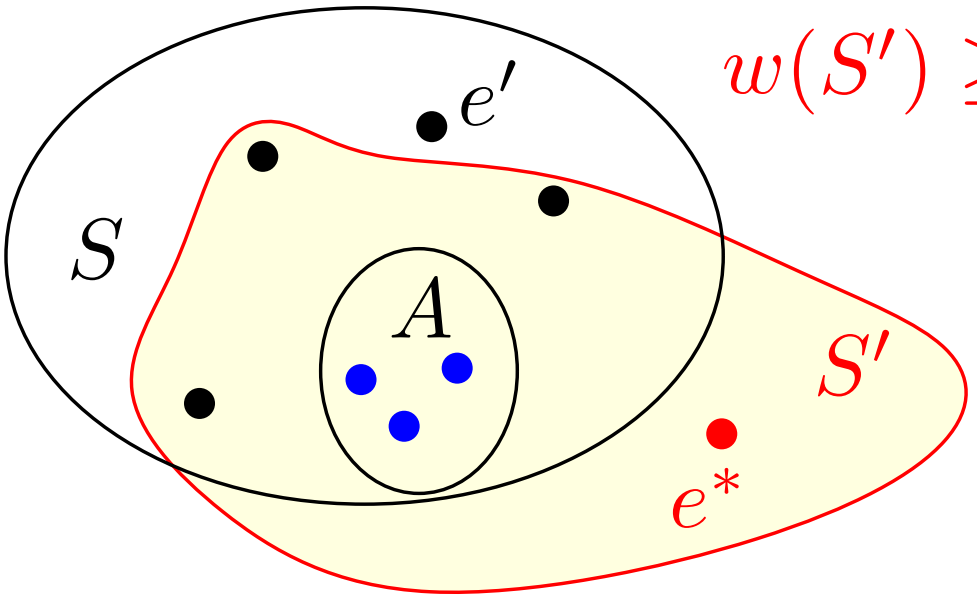
$$w_{e^*} \geq w_{e'}$$
$$w(S') \geq w(S)$$



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- S' and S differ by exactly one element

Lemma (Key Lemma)

- given: matroid $\mathcal{M} = (E, \mathcal{I})$, weights $w \in \mathbb{Z}^E_{>0}$, $A \in \mathcal{I}$,
- goal: find a maximum weight independent set containing A
- $e^* = \arg\max_{e \in E \setminus A : A \cup \{e\} \in \mathcal{I}} w_e$, assuming e^* exists
- Then, some optimum solution contains e^*

Proof.

- let $S \supseteq A, S \in \mathcal{I}$ be an optimum solution, $e^* \notin S$

```
S' ← A ∪ {e*}
while |S'| < |S| do
    let e be any element in S \ S' with S' ∪ {e} ∈ I
                                    ▷ e exists due to exchange property
    S' ← S' ∪ {e}
```

- S' and S differ by exactly one element
- $w(S') := \sum_{e \in S'} w_e \geq w(S) \implies S'$ is also optimum $\square$

Examples of Matroids

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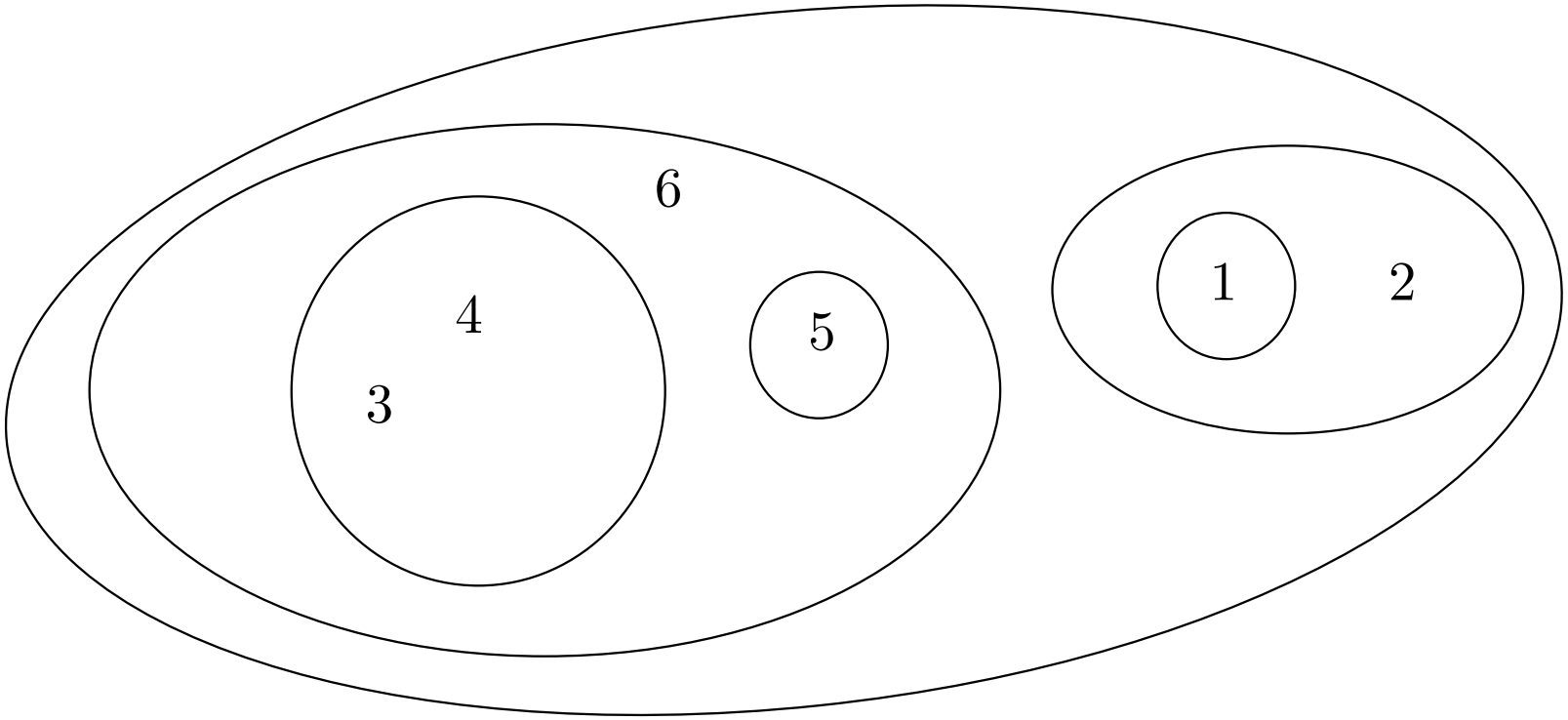
- Laminar Matroid: laminar family of subsets of E $\{E_1, E_2, \cdots, E_t\}$, positive integers $k_1, k_2, \cdots, k_t$

$$\mathcal{I} = \{A \subseteq E : |A \cap E_i| \leq k_i, \forall i \in [t]\}.$$

Def. A family $\{E_1, E_2, \cdots, E_t\}$ of subsets of E is said to be laminar if for every two distinct subsets E_i, E_j in the family, we have $E_i \cap E_j = \emptyset$ or $E_i \subsetneq E_j$ or $E_j \subsetneq E_i$.

- $\Big\{\{1\}, \{1,2\}, \{3,4\}, \{5\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\Big\}$ is a laminar family.



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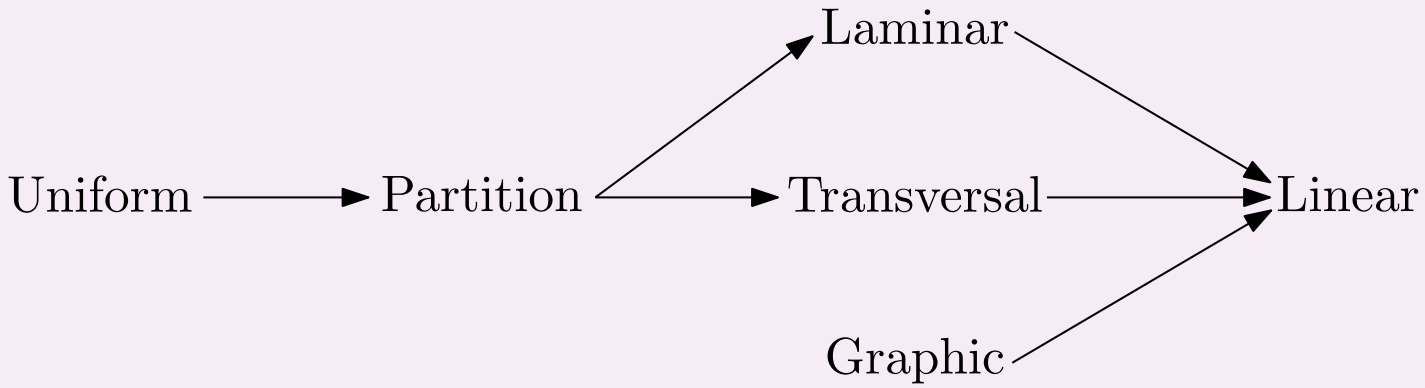
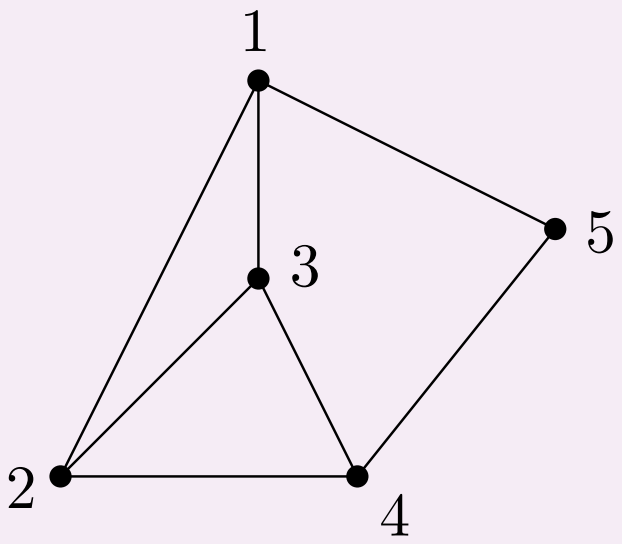
Relationship between matroids



A Graphic Matroid is A Linear Matroid



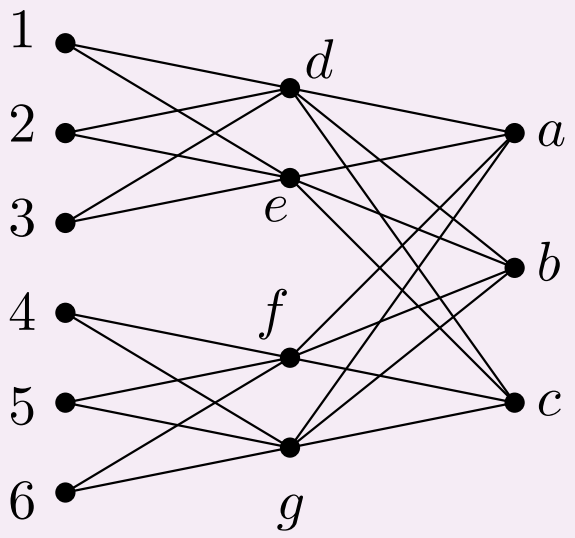
edges	vectors
$(1,2)$	$(1,-1,0,0,0)$
$(1,3)$	$(1,0,-1,0,0)$
$(1,5)$	$(1,0,0,0,-1)$
$(2,3)$	$(0,1,-1,0,0)$
$(2,4)$	$(0,1,0,-1,0)$
$(3,4)$	$(0,0,1,-1,0)$
$(4,5)$	$(0,0,0,1,-1)$

A Laminar Matroid is A Linear Matroid

Example

sets	upper bounds
$\{1, 2, 3\}$	2
$\{3, 4, 5\}$	2
$\{1, 2, 3, 4, 5, 6\}$	3

A DAG (left to right)



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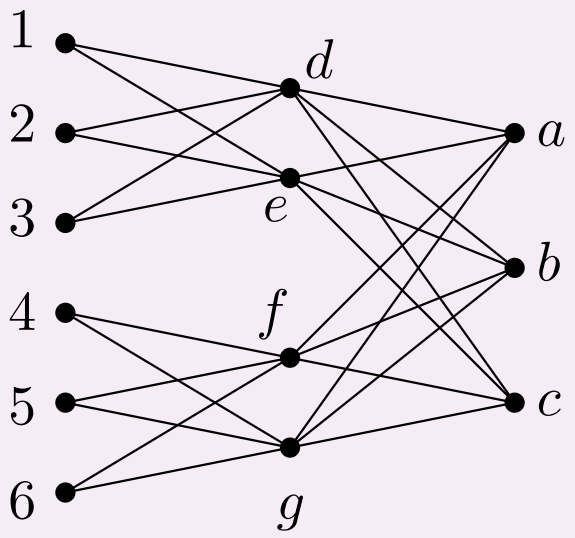

- $x^a, x^b, x^c \in \mathbb{R}^3$ are linearly independent
- x^d, x^e, x^f, x^g: $\text{rand}(0,1) \cdot x^a + \text{rand}(0,1)x^b + \cdot\text{rand}(0,1)x^c$
- x^1, x^2, x^3: $\text{rand}(0,1) \cdot x^d + \text{rand}(0,1)x^e$
- x^4, x^5, x^6: $\text{rand}(0,1) \cdot x^f + \text{rand}(0,1)x^g$
- each $\text{rand}(0,1)$ gives an independent random real in $[0,1]$

Other Terminologies Related To a Matroid $\mathcal{M} = (E, \mathcal{I})$

- A subset of E that is not independent is dependent.
- A maximal indepent set is called a basis (plural: bases)
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Proof.

By exchange property. $\square$

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Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the rank of a subset A of E, denoted as $r_{\mathcal{M}}(A)$, is defined as the size of the maximum independent subset of A. $r_{\mathcal{M}} : 2^E \to \mathbb{Z}_{\geq 0}$ is called the rank function of $\mathcal{M}$.

Outline

Recap: Approximation Algorithms

- For minimization problems:

$$\text{approximation ratio} := \frac{\text{cost of our solution}}{\text{cost of optimum solution}} \geq 1$$

- For maximization problems:

$$\text{approximation ratio} := \frac{\text{value of our solution}}{\text{value of optimum solution}} \leq 1$$

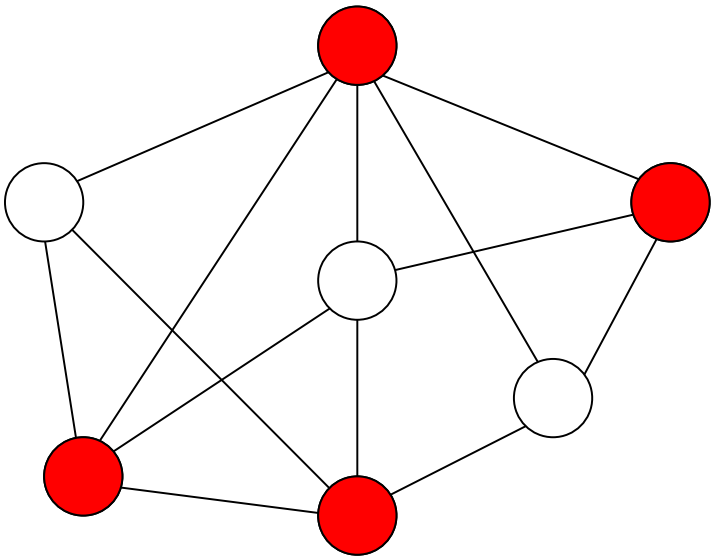
or

$$\text{approximation ratio} := \frac{\text{value of optimum solution}}{\text{value of our solution}} \geq 1$$

Outline

Vertex Cover Problem

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $C \subseteq V$ such that for every $(u, v) \in E$ then $u \in C$ or $v \in C$.



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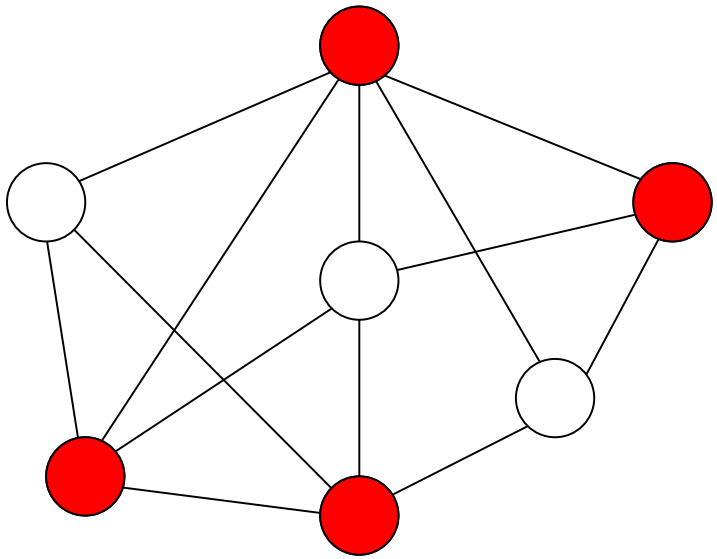

Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$

First Try: A "Natural" Greedy Algorithm

Natural Greedy Algorithm for Vertex-Cover

```
E' ← E, C ← ∅
while E' ≠ ∅ do
    let v be the vertex of the maximum degree in (V, E')
    C ← C ∪ {v},
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return C
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- We prove it for the more general set cover problem
- The logarithmic factor is tight for this algorithm

2-Approximation Algorithm for Vertex Cover

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E' ← E, C ← ∅
while E' ≠ ∅ do
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- To cover E', the optimum solution needs $|E'|$ vertices □

Outline

Set Cover

Input: $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

Output: minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

Set Cover with Bounded Frequency f

Input: $U, |U| = n$: ground set

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every $j \in U$ appears in at most f subsets in $\{S_1, S_2, \cdots, S_m\}$

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Vertex Cover = Set Cover with Frequency 2

- edges $\Leftrightarrow$ elements
- vertices $\Leftrightarrow$ sets
- every edge (element) can be covered by 2 vertices (sets)

f-Approximation Algorithm for Set Cover with Frequency f

```
C ← ∅
while ⋃_{i∈C} S_i ≠ U do
    let e be any element in U \ ⋃_{i∈C} S_i
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Theorem The algorithm is a f-approximation algorithm.

f-Approximation Algorithm for Set Cover with Frequency f

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2: **while** $\bigcup_{i \in C} S_i \neq U$ **do**
3: let e be any element in $U \setminus \bigcup_{i \in C} S_i$
4: $C \leftarrow C \cup \{i \in [m] : e \in S_i\}$
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Theorem The algorithm is a f-approximation algorithm.

Proof.

- Let U' be the set of all elements e considered in Step 3
- Observation: no set S_i contains two elements in U'
- To cover U', the optimum solution needs $|U'|$ sets
- $C \leq f \cdot |U'|$ □

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Input: $U, |U| = n$: ground set

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Greedy Algorithm for Set Cover

1: $C \leftarrow \emptyset, U' \leftarrow U$
2: **while** $U' \neq \emptyset$ **do**
3: choose the i that maximizes $|U' \cap S_i|$
4: $C \leftarrow C \cup \{i\}, U' \leftarrow U' \setminus S_i$
5: **return** C

- g: minimum number of sets needed to cover U

Lemma Let $u_t, t \in \mathbb{Z}_{\geq 0}$ be the number of uncovered elements after t steps. Then for every $t \geq 1$, we have

$$u_t \leq \left(1 - \frac{1}{g}\right) \cdot u_{t-1}.$$

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- at beginning of step t, some set in $S_1^*, S_2^*, \cdots, S_g^*$ must contain $\geq \frac{u_{t-1}}{g}$ uncovered elements
- $u_t \leq u_{t-1} - \frac{u_{t-1}}{g} = \left(1 - \frac{1}{g}\right) u_{t-1}$. $\square$

Proof of $(\ln n + 1)$-approximation.

- Let $t = \lceil g \cdot \ln n \rceil$. $u_0 = n$. Then
$$u_t \leq \left(1 - \frac{1}{g}\right)^{g \cdot \ln n} \cdot n < e^{-\ln n} \cdot n = n \cdot \frac{1}{n} = 1.$$
- So $u_t = 0$, approximation ratio $\leq \frac{\lceil g \cdot \ln n \rceil}{g} \leq \ln n + 1$. $\square$

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- A more careful analysis gives a H_n-approximation, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the n-th harmonic number.
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$(1-c)\ln n$-hardness for any $c = \Omega(1)$

Let $c > 0$ be any constant. There is no polynomial-time $(1-c)\ln n$-approximation algorithm for set-cover, unless

- NP $\subseteq$ quasi-poly-time, [Lund, Yannakakis 1994; Feige 1998]
- P $=$ NP. [Dinur, Steuer 2014]

Outline

- set cover: use smallest number of sets to cover all elements.
- maximum coverage: use k sets to cover maximum number of elements

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Output: $C \subseteq [m], |C| = k$ with the maximum $\bigcup_{i \in C} S_i$

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Greedy Algorithm for Maximum Coverage

```
C ← ∅, U' ← U
for t ← 1 to k do
    choose the i that maximizes |U' ∩ S_i|
    C ← C ∪ {i}, U' ← U' \ S_i
return C
```

Theorem Greedy algorithm gives $(1 - \frac{1}{e})$-approximation for maximum coverage.

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- The $\left(1 - \frac{1}{e}\right)$-approximation extends to a more general problem.

Outline

Def. Let $n \in \mathbb{Z}_{>0}$. A set function $f : 2^{[n]} \to \mathbb{R}$ is called submodular if it satisfies one of the following three equivalent conditions:

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- $M := (E, \mathcal{I})$: a matroid, $A \subsetneq E, i, j \in E \setminus A, i \neq j$
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$(1 - \frac{1}{e})$-Approximation for Submodular Maximization with Cardinality Constraint

Submodular Maximization under a Cardinality Constraint

Input: An oracle to a non-negative monotone submodular function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$, $\quad k \in [n]$

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Greedy Algorithm for the Problem

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S ← ∅
for t ← 1 to k do
    choose the i that maximizes f(S ∪ {i})
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For $A, B \subseteq [n]$, we have $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$. So, $f(A \cup B) \leq f(A) + f(B)$ as $f(A \cap B) \geq 0$. $\square$

Lemma Let $f : 2^{[n]} \to \mathbb{R}$ be submodular. Let $S \subseteq [n]$, and $f_S(A) = f(S \cup A) - f(S)$ for every $A \subseteq [n]$. (f_S is the marginal value function for set S.) Then f_S is also submodular.

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Proof.

- Let $A, B \subseteq [n] \setminus S$; it suffices to consider ground set $[n] \setminus S$.

$$\begin{aligned}
& f_S(A \cup B) + f_S(A \cap B) - f_S(A) + f_S(B) \\
&= f(S \cup A \cup B) - f(S) + f(S \cup (A \cap B)) - f(S) \\
&\quad - \Big(f(S \cup A) - f(S) + f(S \cup B) - f(S) \Big) \\
&= f(S \cup A \cup B) + f(S \cup (A \cap B)) - f(S \cup A) - f(S \cup B) \\
&\leq 0
\end{aligned}$$

- The last inequality is by $S \cup A \cup B = (S \cup A) \cup (S \cup B)$, $S \cup (A \cap B) = (S \cup A) \cap (S \cup B)$ and submodularity of f. $\square$

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Local Search for Maximum-Cut

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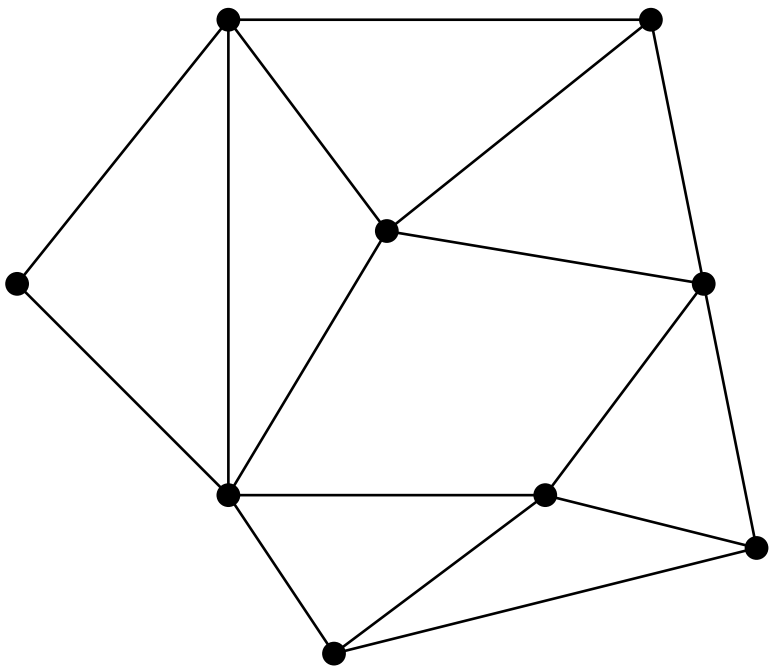
Input: Graph $G = (V, E)$

Output: partition of V into $(S, T = V \setminus S)$ so as to maximize $|E(S,T)|$, $E(S,T) = \{uv \in E : u \in S \wedge v \in T\}$.



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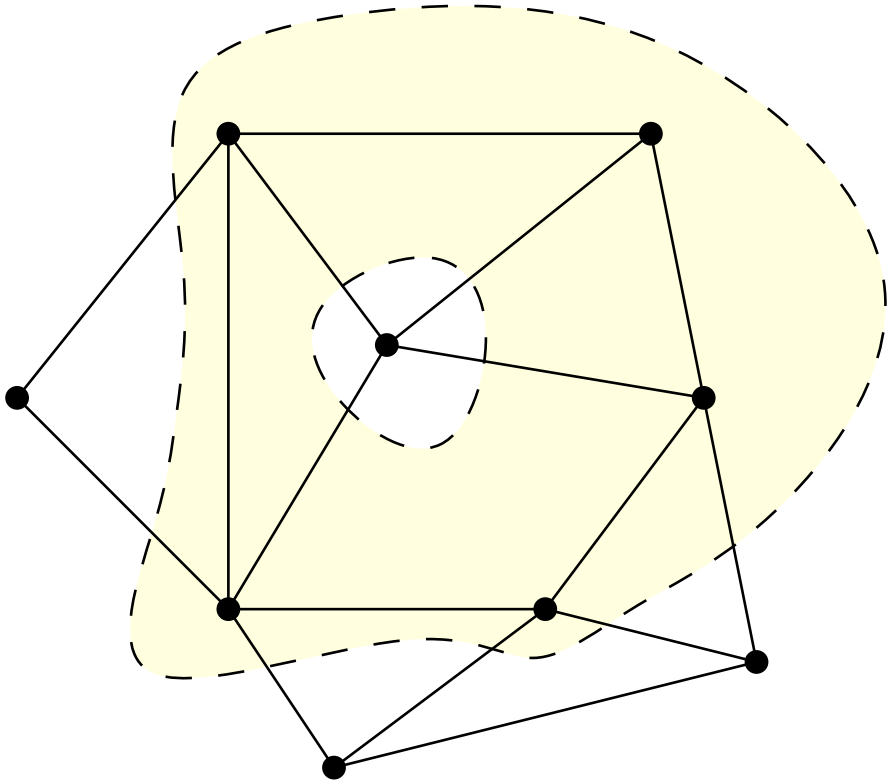

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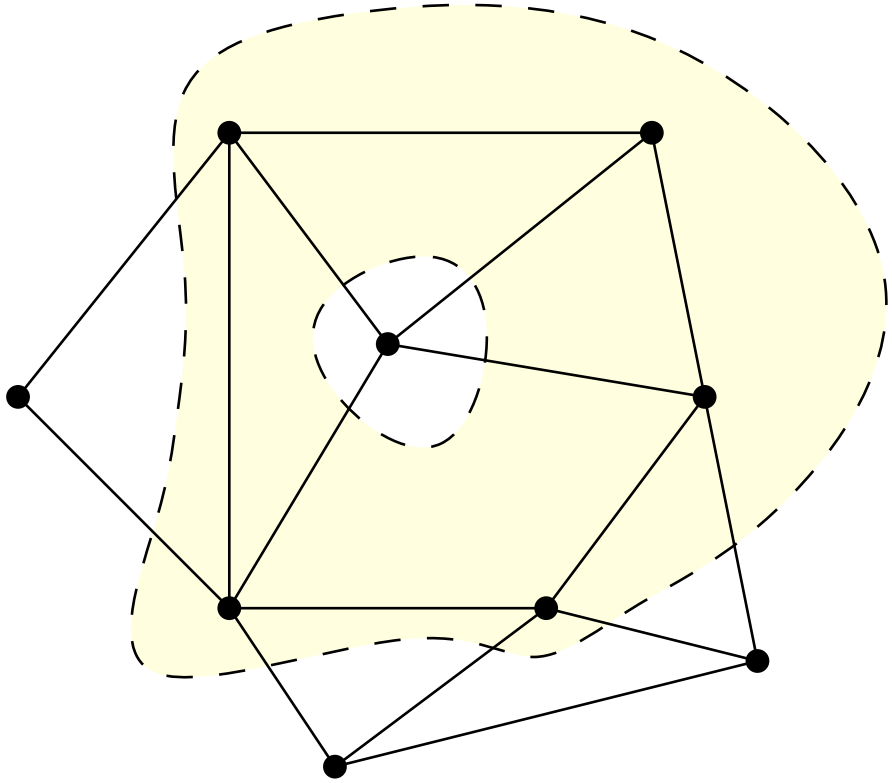

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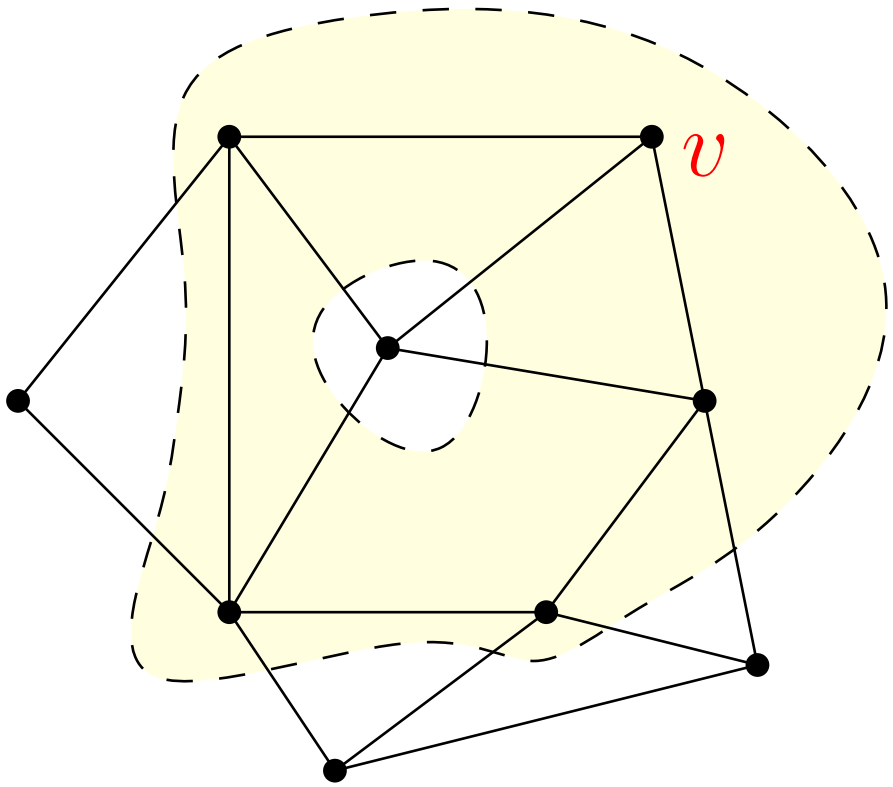

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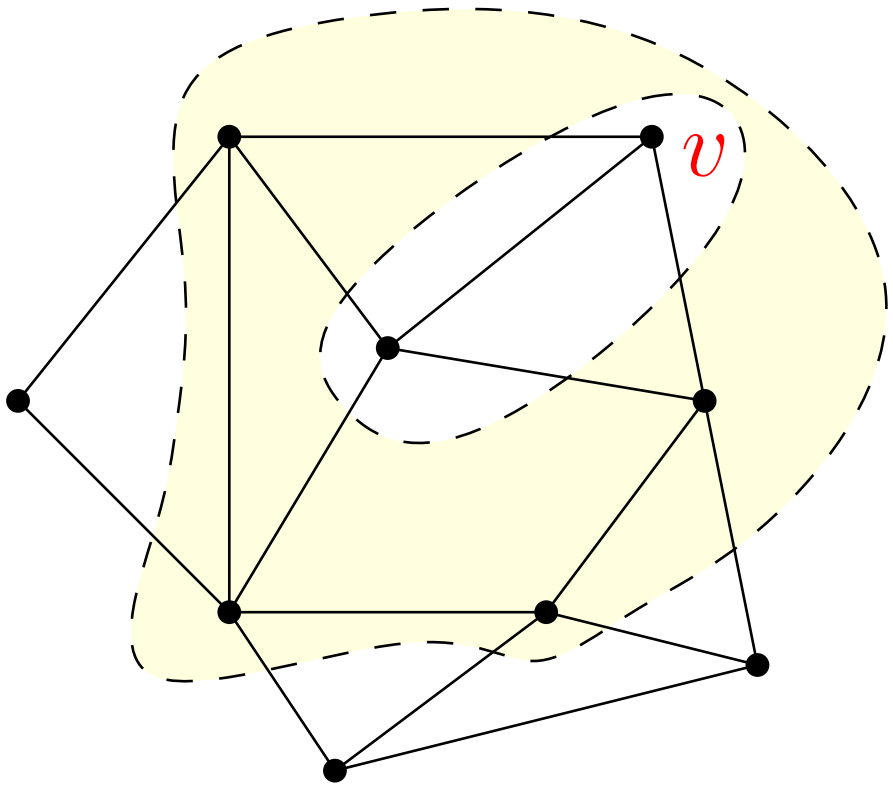

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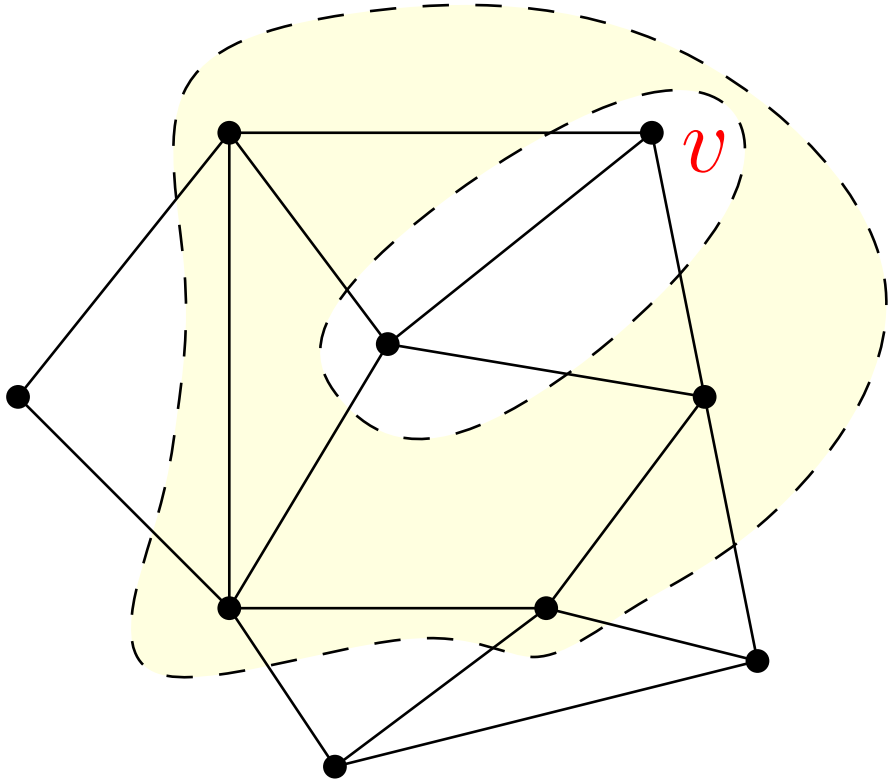

Local-Search for Maximum-Cut

1: $(S, T) \leftarrow$ any cut
2: **while** $\exists v \in V$, changing side of v increases cut value **do**
3: switch v to the other side in (S, T)
4: **return** (S, T)

Lemma Local search gives a 2-approximation for maximum-cut.

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- d_v: degree of v

Proof.

- $\forall v \in S : E(v, S) \leq E(v, T) \Rightarrow |E(v, T)| \geq \frac{1}{2} d_v$
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- adding all inequalities:

$$2|E(S,T)| \geq \frac{1}{2} \sum_{v \in V} d_v = |E|.$$

- So $|E(S,T)| \geq \frac{1}{2}|E| \geq \frac{1}{2}$(value of optimum cut). $\square$

- The following algorithm also gives a 2-approximation

Greedy Algorithm for Maximum-Cut

1: $S \leftarrow \emptyset, T \leftarrow \emptyset$
2: **for** every $v \in V$, in arbitrary order **do**
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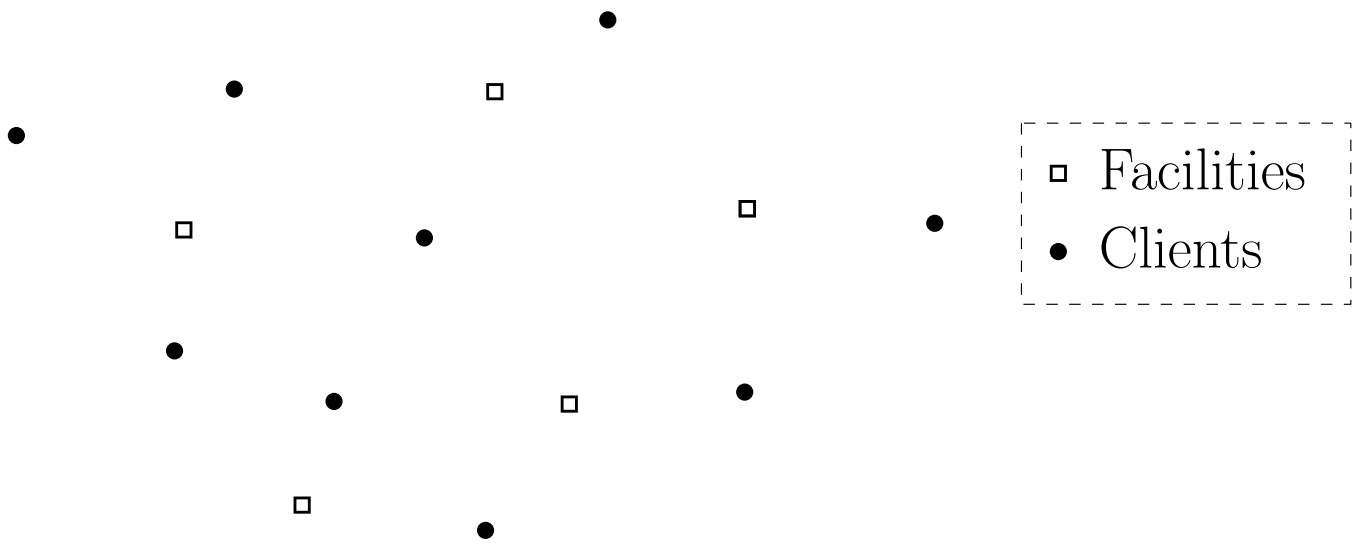
- [Goemans-Williamson] 0.878-approximation via Semi-definite programming (SDP)
- Under Unique-Game-Conjecture (UGC), the ratio is best possible

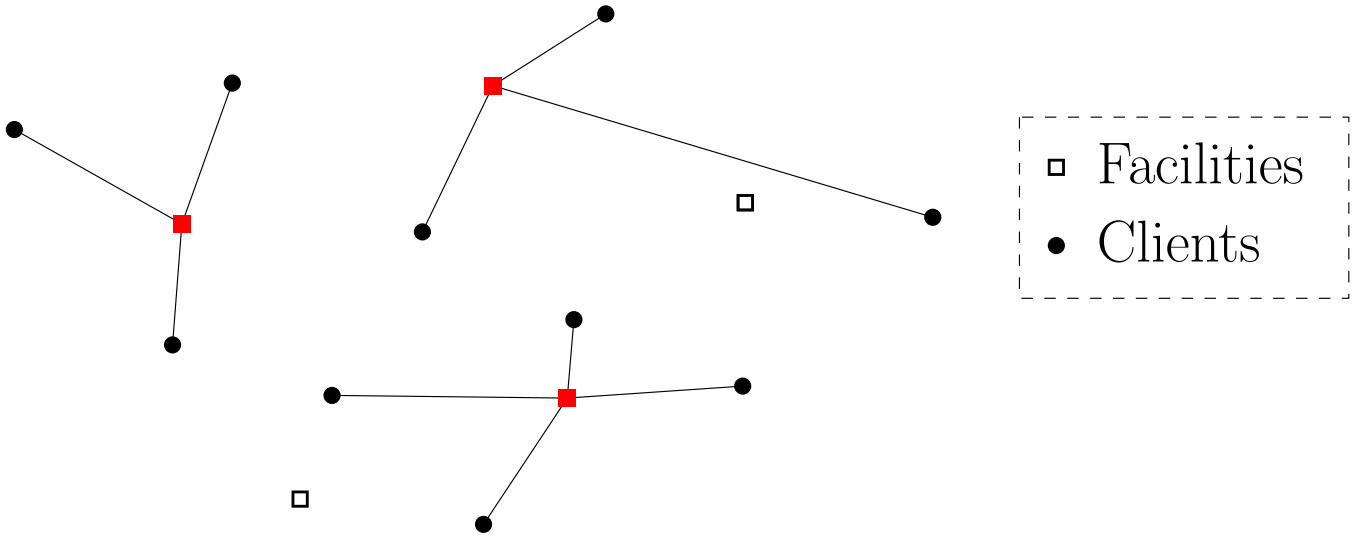
Outline



Uncapacitated Facility Location

Input: F: Facilities D: Clients

c: metric over $F \cup D$ $(f_i)_{i \in F}$: facility costs



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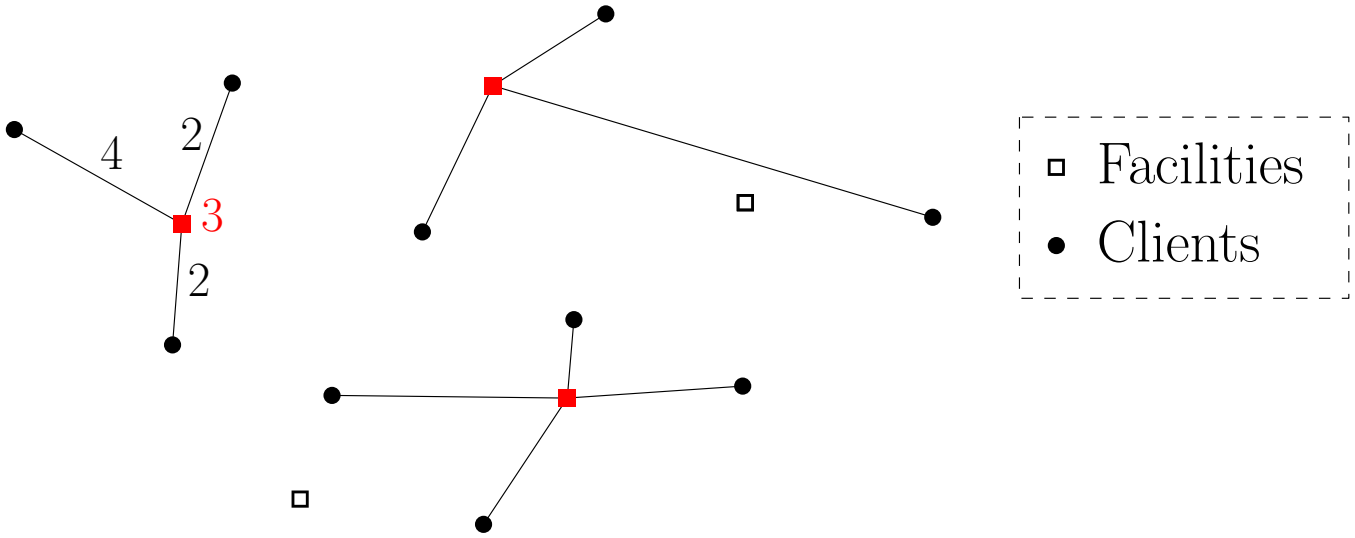
$c(j, S)$: smallest distance between j and a facility in S



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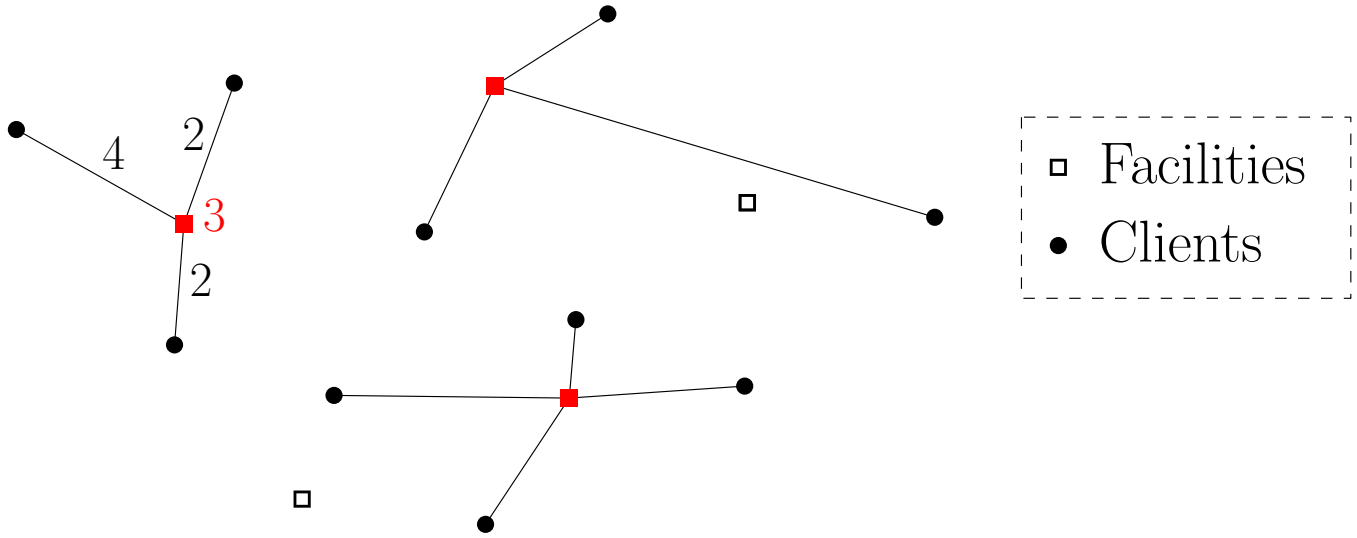

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- Best-approximation ratio: 1.488-Approximation [Li, 2011]
- 1.463-hardness, $1.463 \approx$ root of $x = 1 + 2e^{-x}$

- $\text{cost}(S) := \sum_{i \in S} f_i + \sum_{j \in D} c(j, S), \forall S \subseteq F$

Local Search Algorithm for Uncapacitated Facility Location

1: $S \leftarrow$ arbitrary set of facilities
2: **while** exists $S' \subseteq F$ with $|S \setminus S'| \leq 1$, $|S' \setminus S| \leq 1$ and $\text{cost}(S') < \text{cost}(S)$ **do**
3: $\quad S' \leftarrow S$
4: **return** S

- The algorithm runs in pseodu-polynomial time, but we ignore the issue for now.

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- The algorithm runs in pseodu-polynomial time, but we ignore the issue for now.

S is a local optimum, under the following local operations

- add$(i), i \notin S$: $S \leftarrow S \cup \{i\}$
- delete$(i), i \in S$: $S \leftarrow S \setminus \{i\}$
- swap$(i, i'), i \in S, i' \notin S$: $S \leftarrow S \setminus \{i\} \cup \{i'\}$

- S: the local optimum returned by the algorithm
- S^*: the (unknown) optimum solution

$$F := \sum_{i \in S} f_i \qquad \sigma_j : \text{closest facility in } S \text{ to } j \qquad C := \sum_{j \in D} c_{j\sigma_j}$$

$$F^* := \sum_{i \in S^*} f_i \qquad \sigma_j^* : \text{closest facility in } S^* \text{ to } j \qquad C^* := \sum_{j \in D} c_{j\sigma_j^*}$$

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Lemma (analysis for connection cost) $C \leq F^* + C^*$

Lemma (analysis for facility cost) $F \leq F^* + 2C^*$

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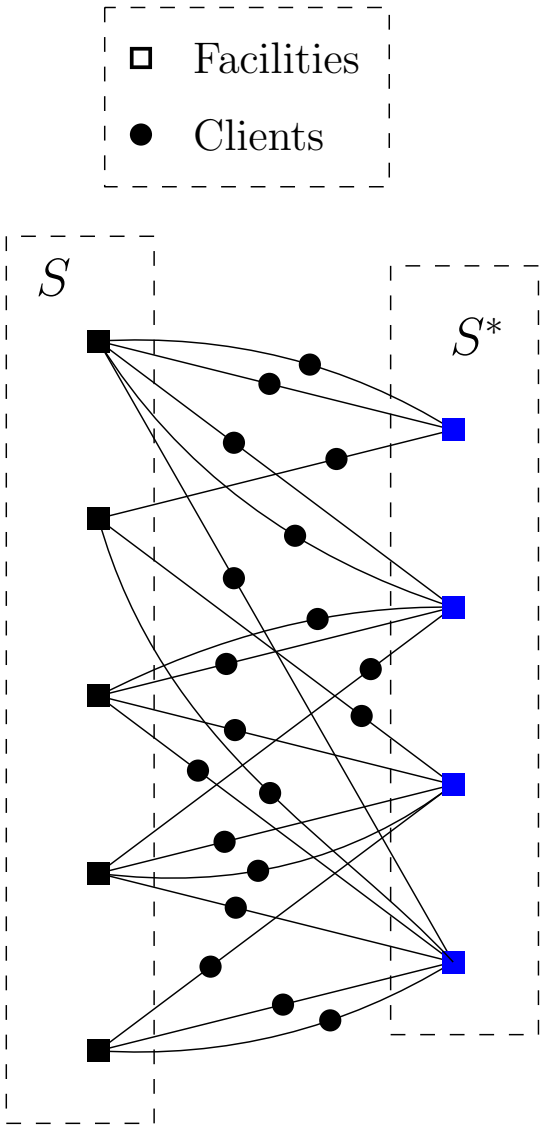
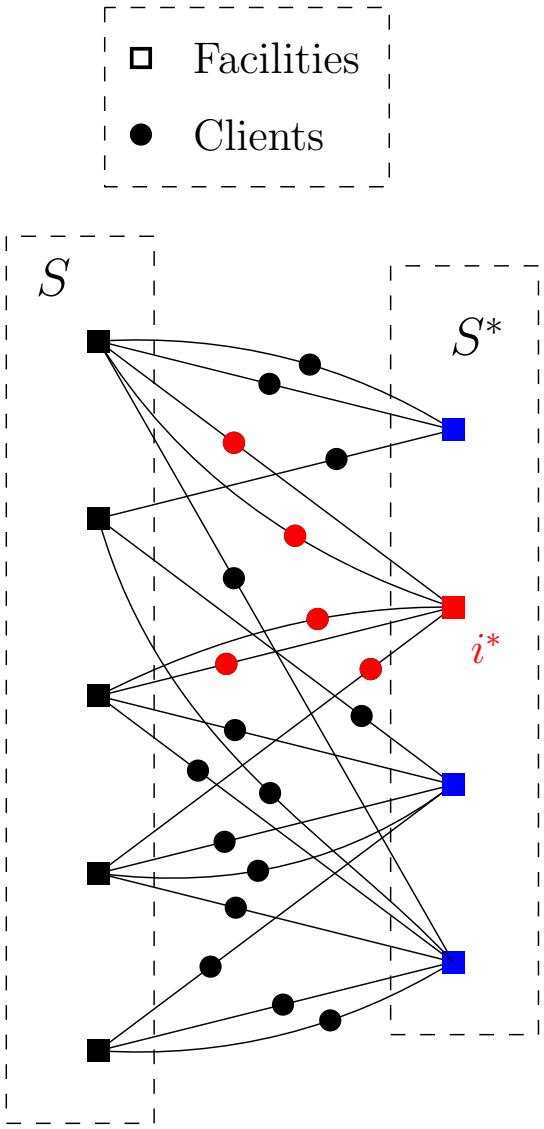
So, $F + C \leq 2F^* + 3C^* \leq 3(F^* + C^*)$

Outline

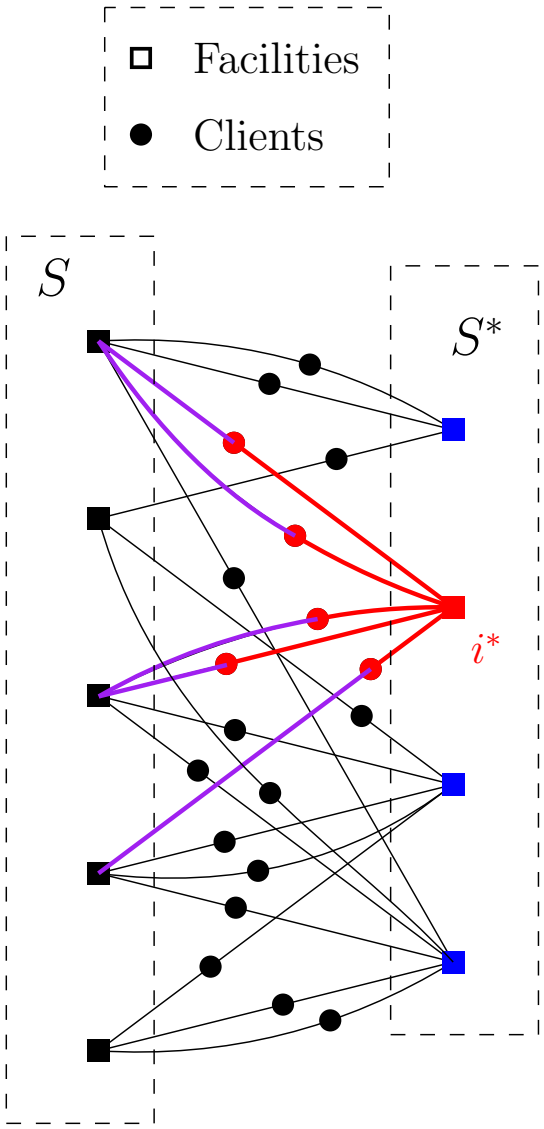


Analysis of C



Analysis of C

- adding i^* does not increase the cost:



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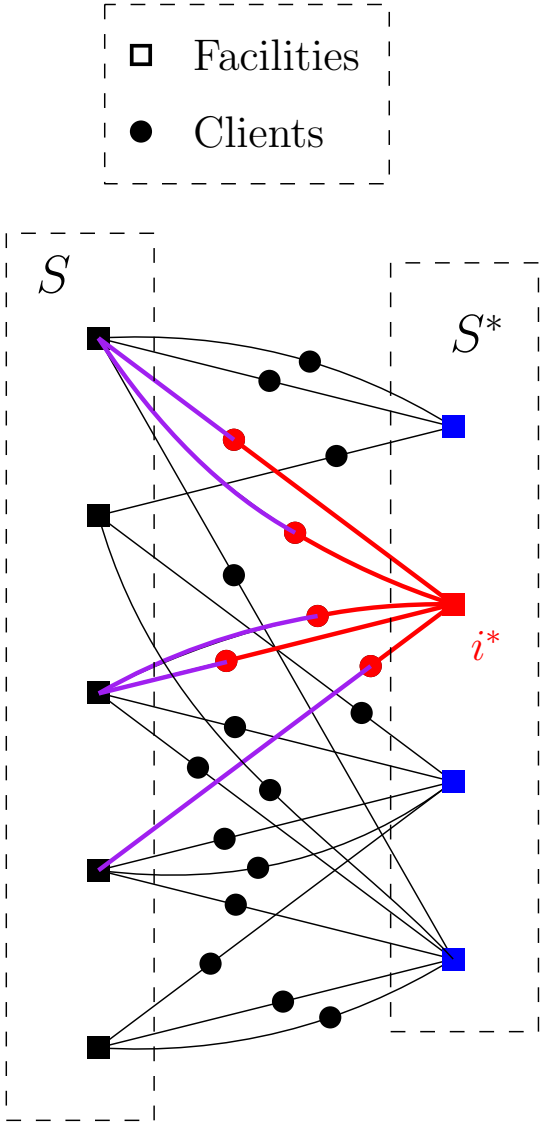
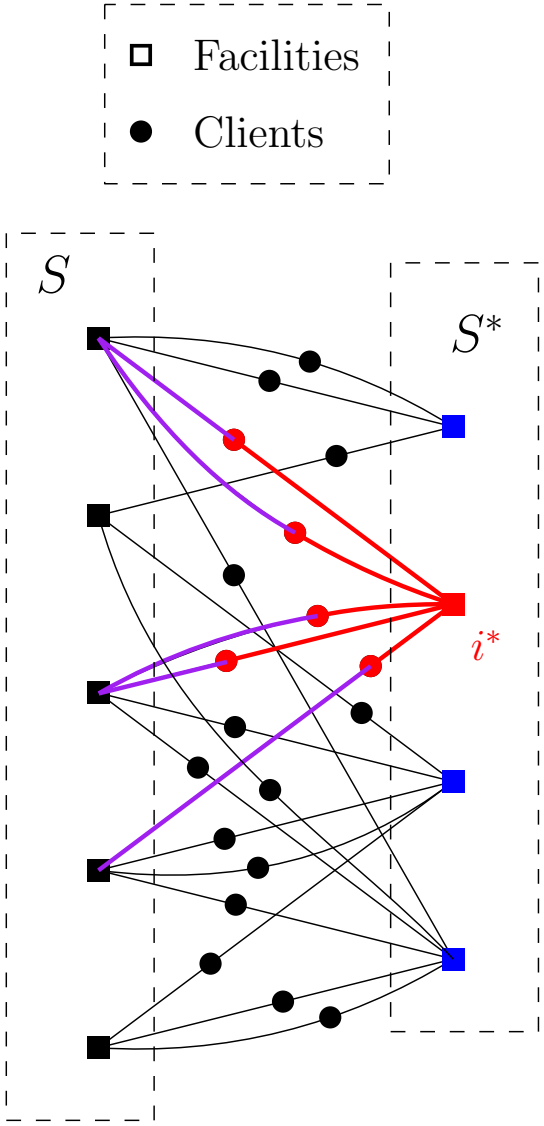


Analysis of C

- adding i^* does not increase the cost:

$$\sum_{j \in \sigma^{*-1}(i^*)} c_{\sigma(j)j} \leq f_{i^*} + \sum_{j \in \sigma^{*-1}(i^*)} c_{i^*j}$$



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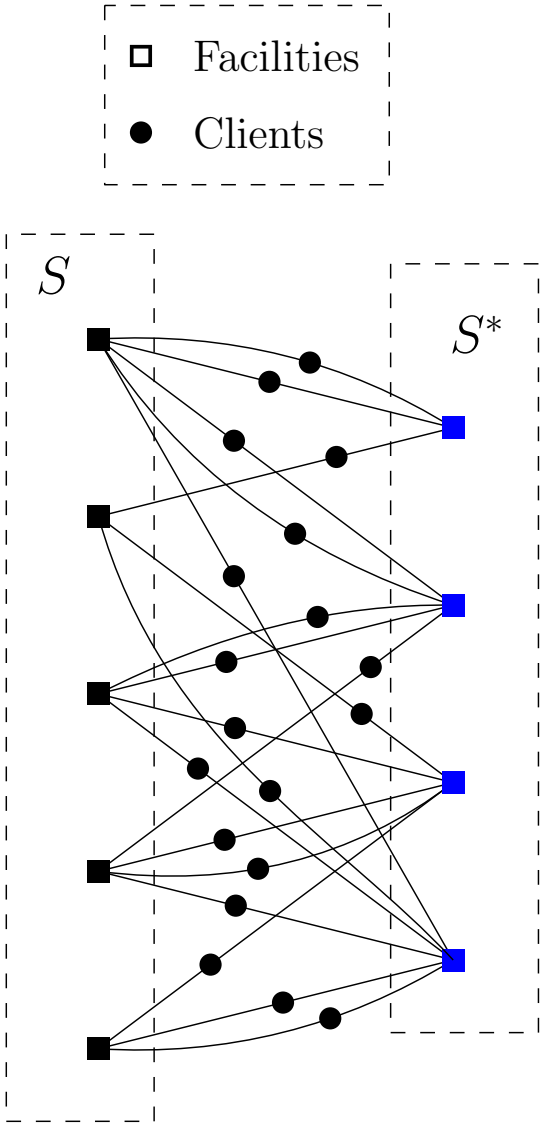
- summing up over all $i^* \in S^*$, we get

$$\sum_{j \in D} c_{\sigma(j)j} \leq \sum_{i^* \in S^*} f_{i^*} + \sum_{j \in D} c_{\sigma^*(j)j}$$

$$C \leq F^* + C^*$$

Outline



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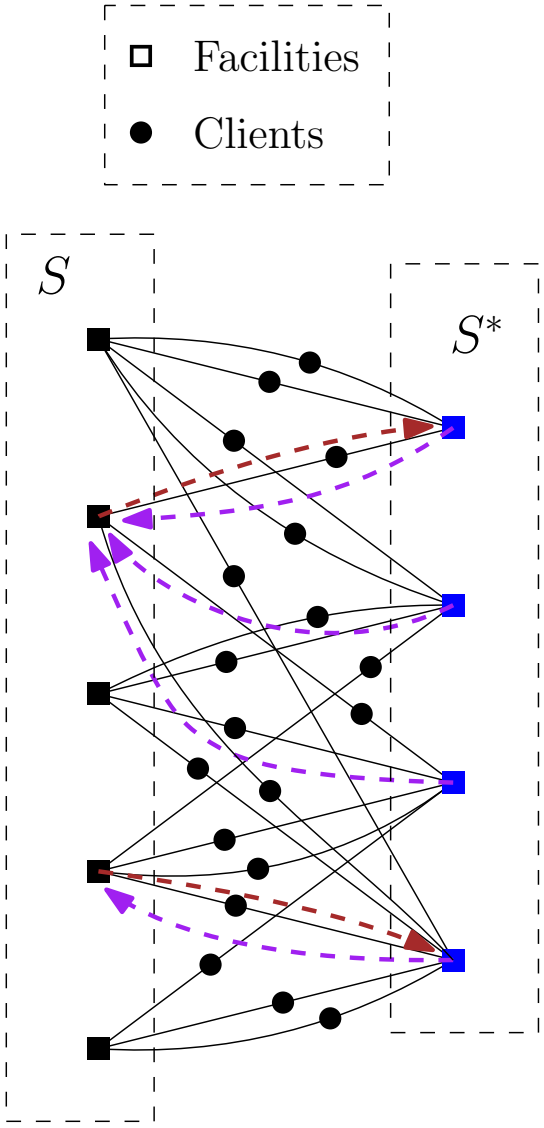

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- $\phi(i^*), i^* \in S^*$: closest facility in S to i^*
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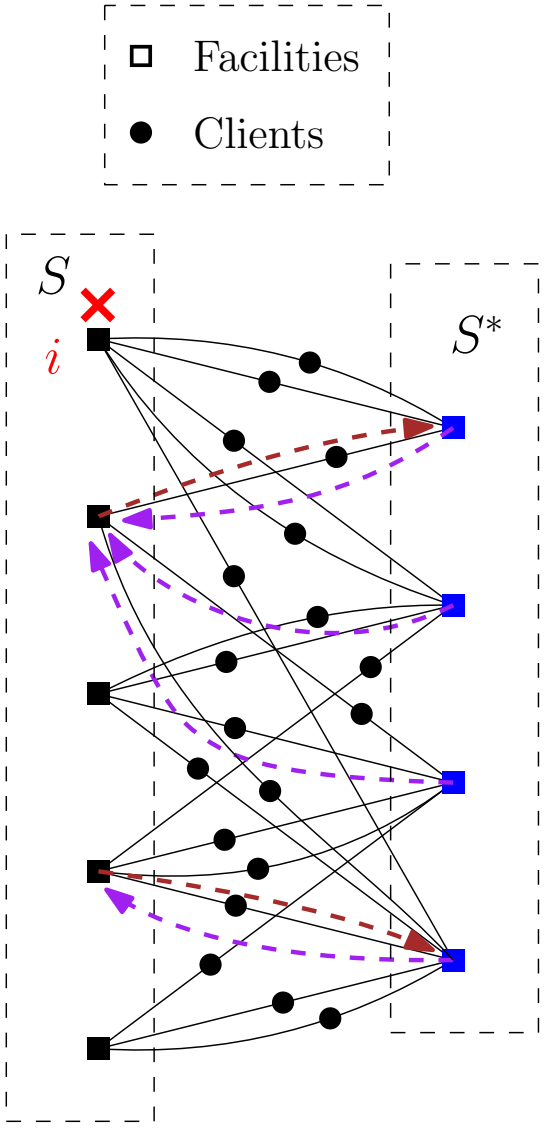

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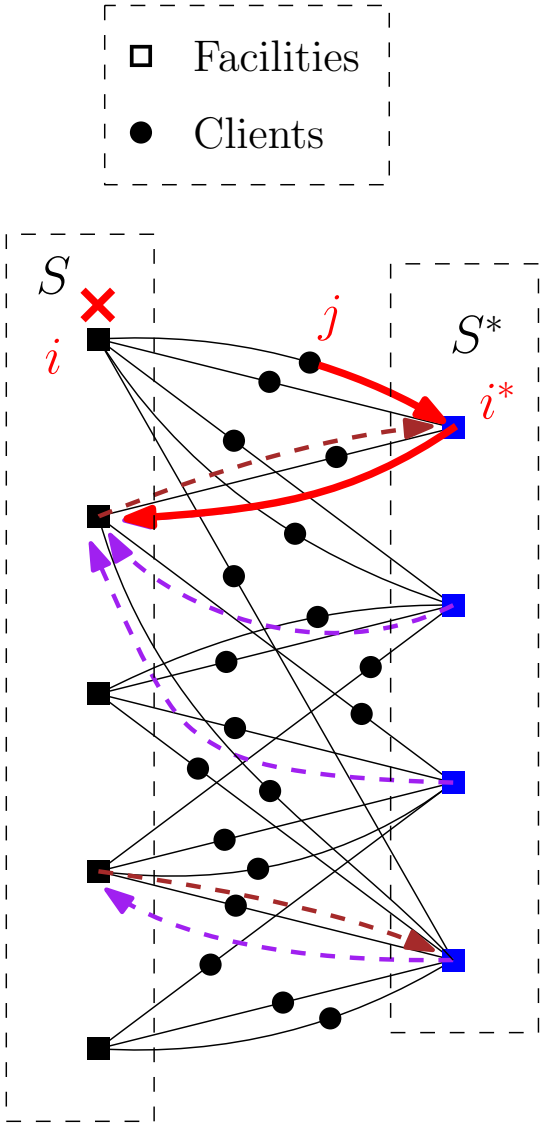

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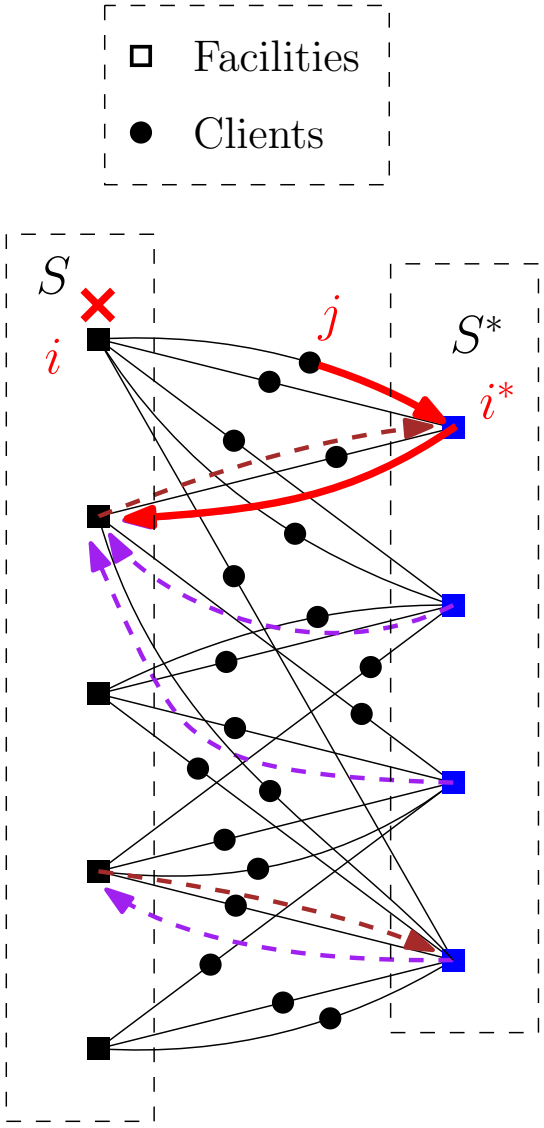

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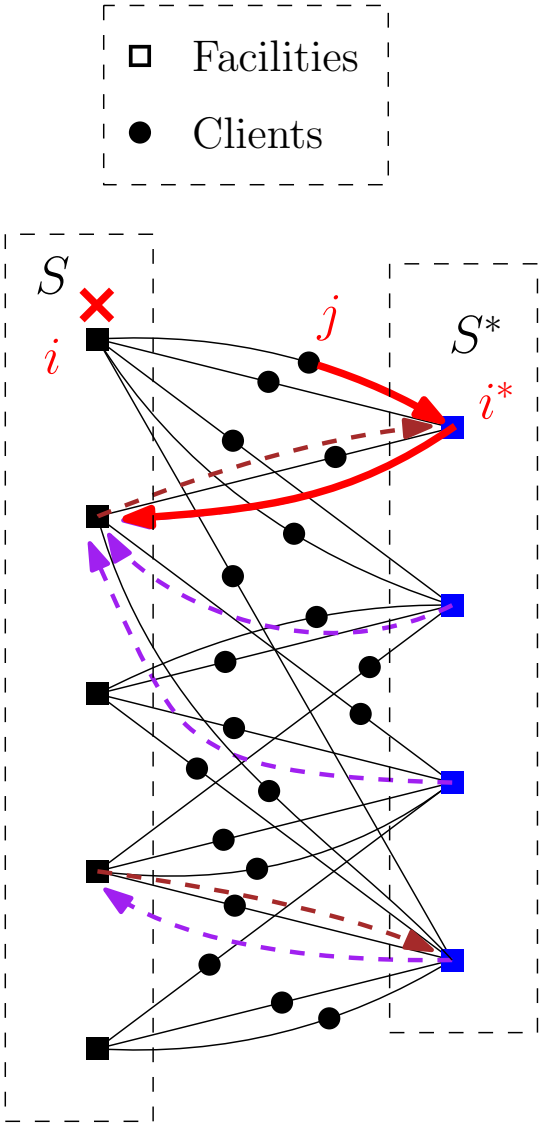

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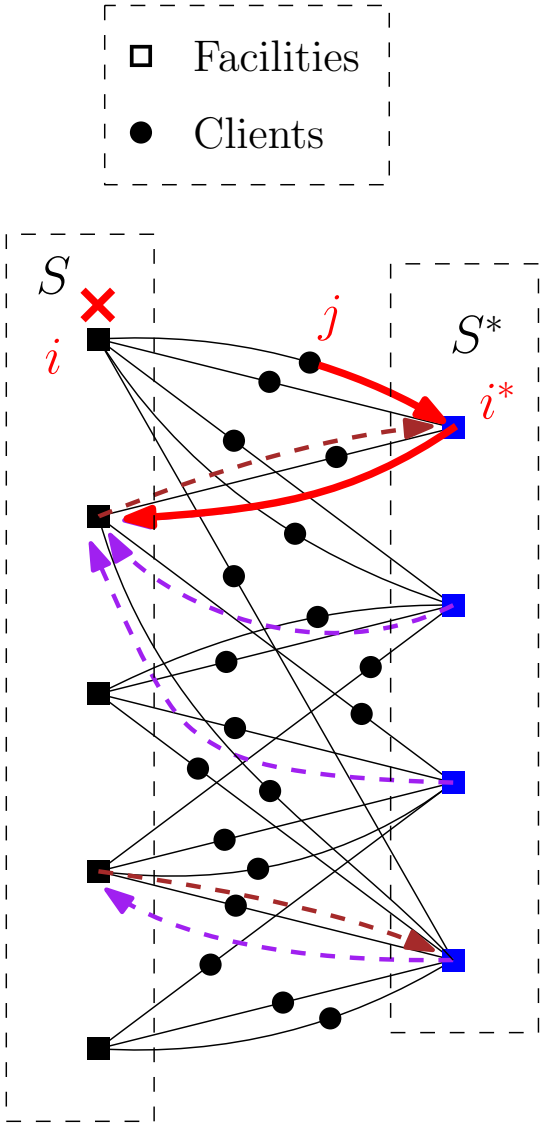

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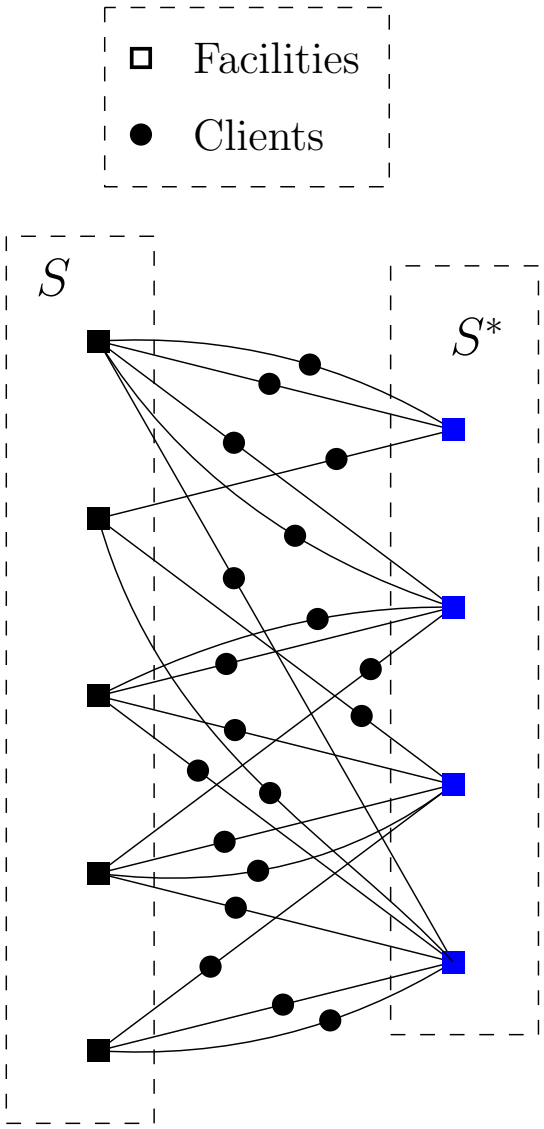

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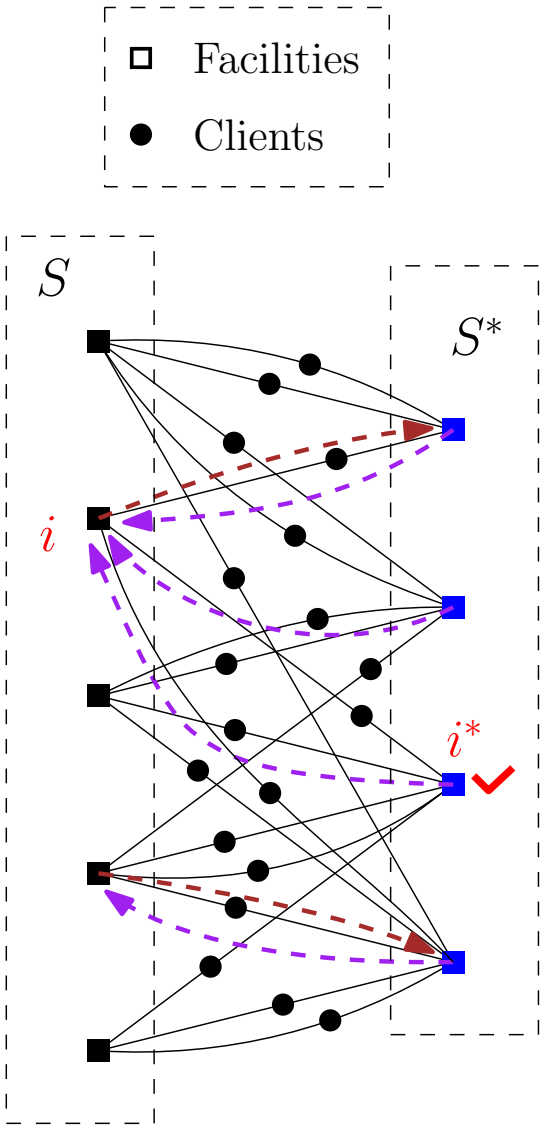

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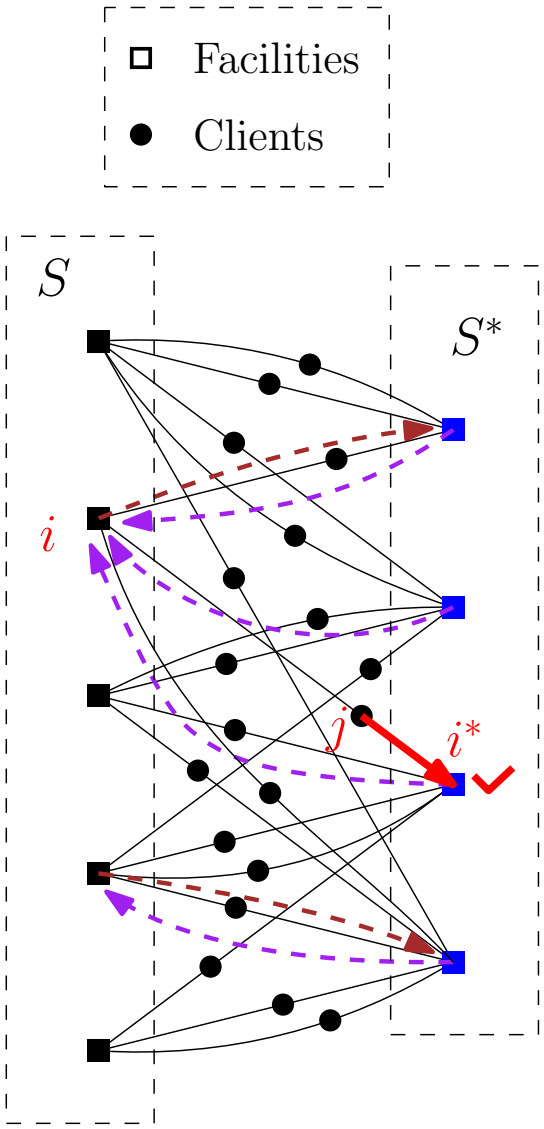

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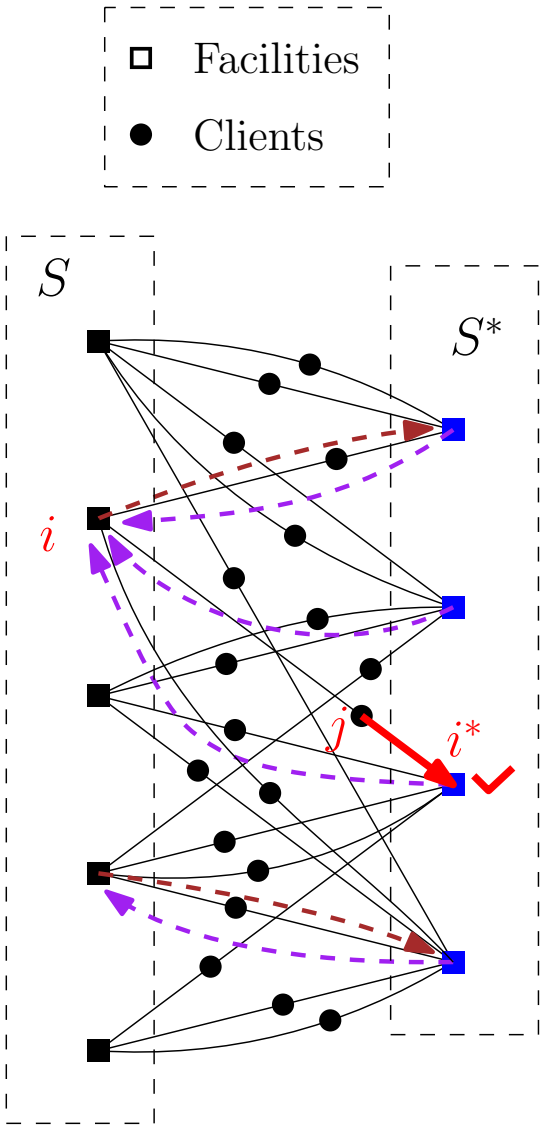

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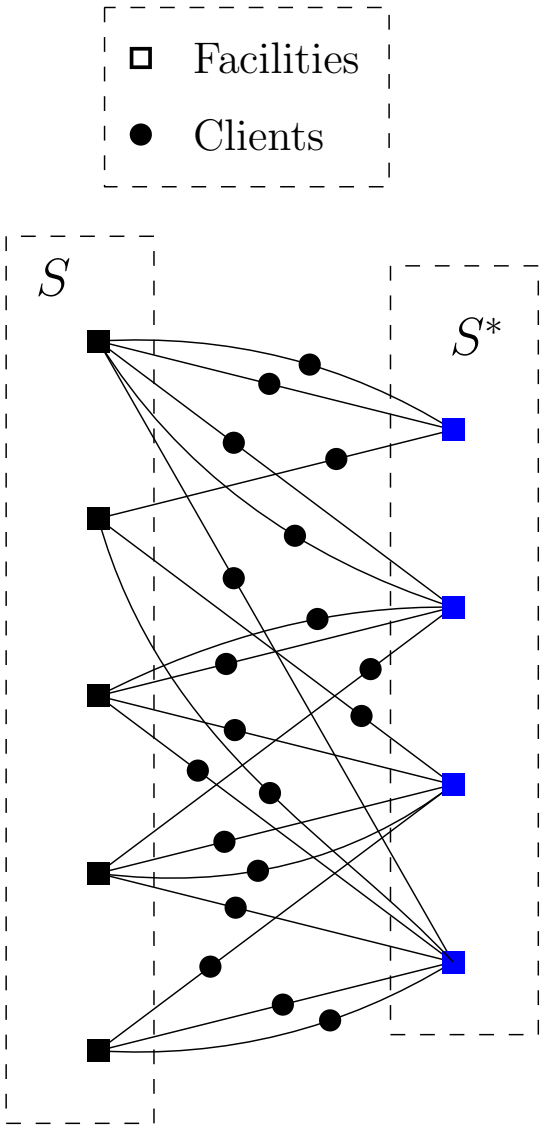

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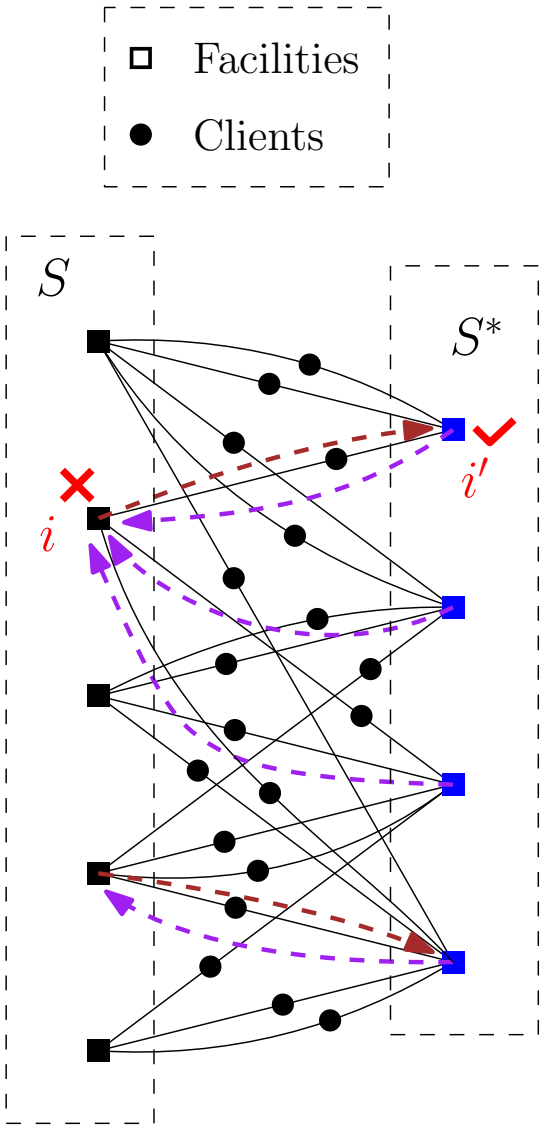

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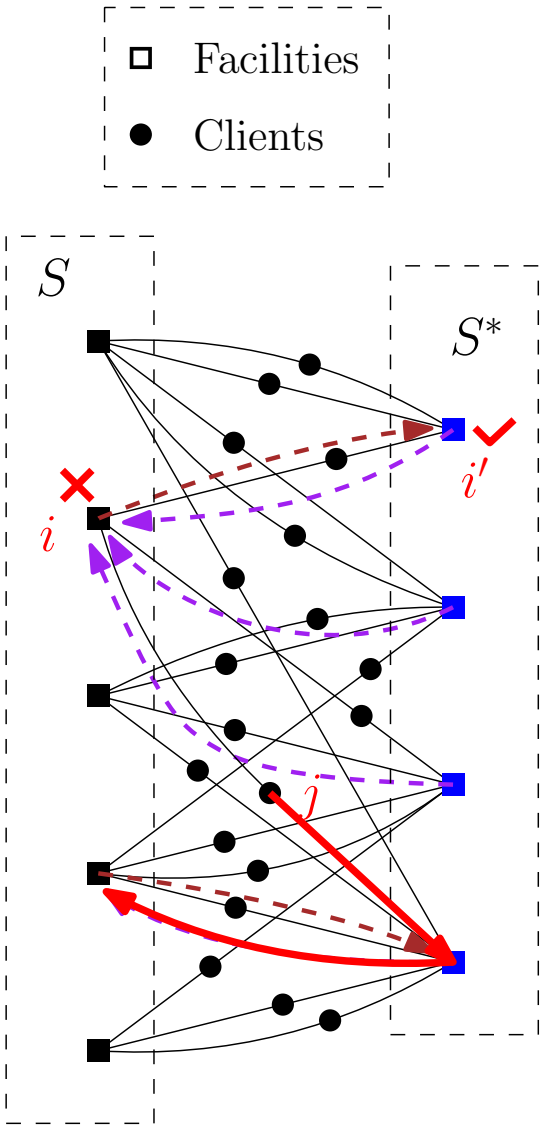

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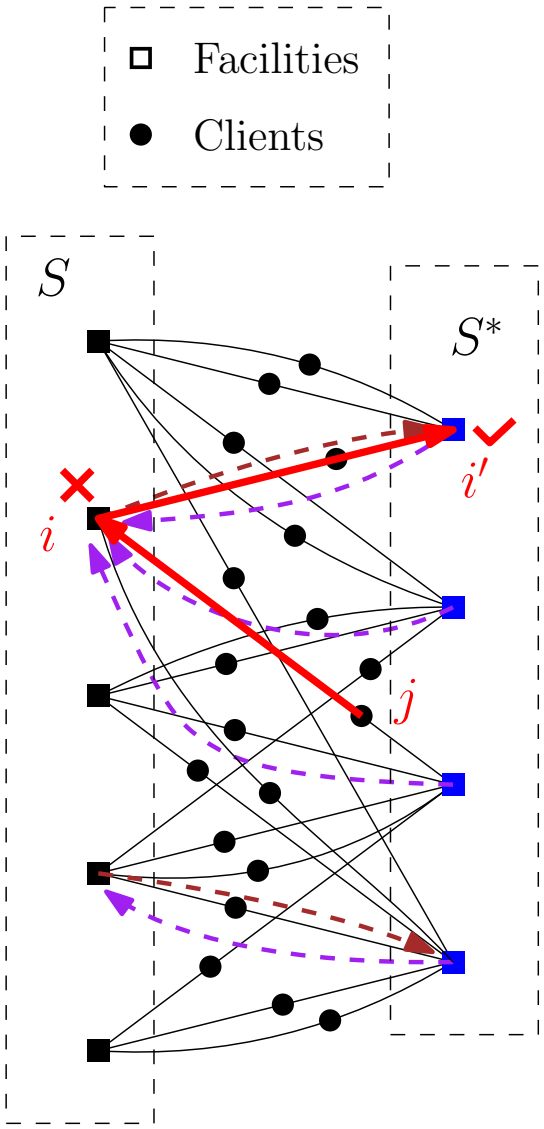

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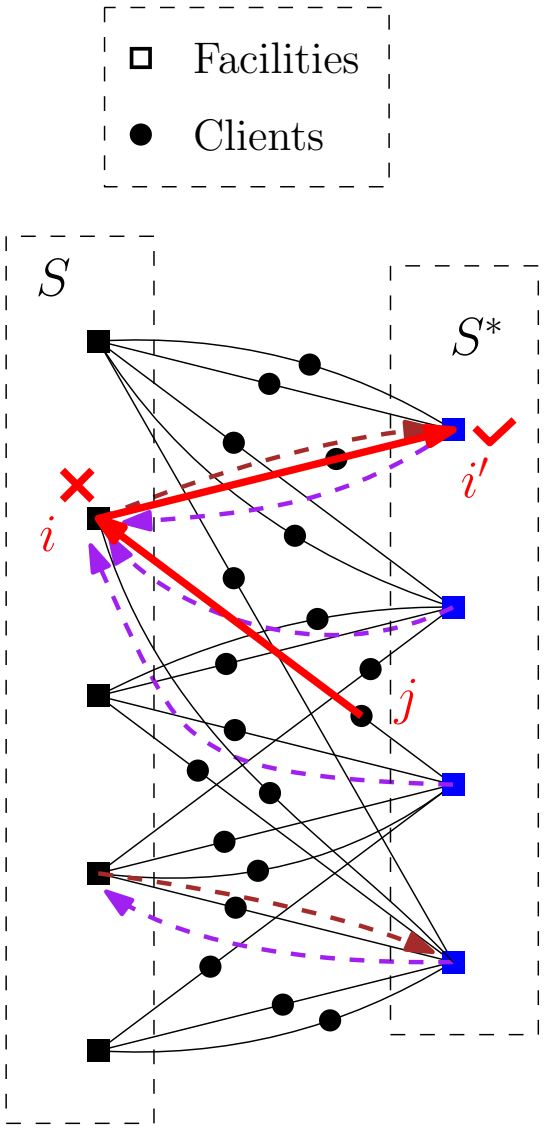

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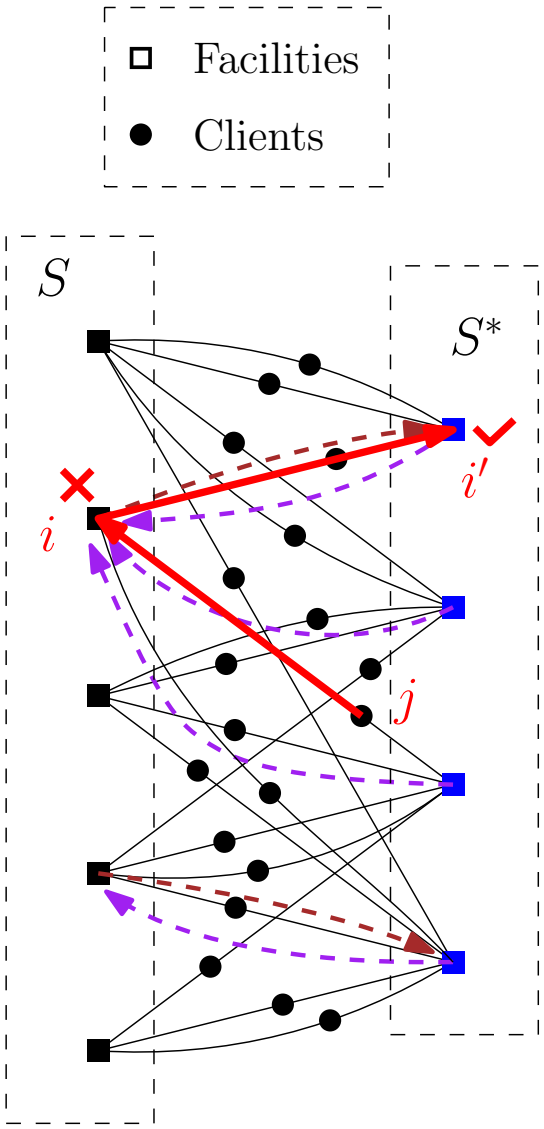
    $c_{ij} + c_{ii'} - c_{ij} = c_{ii'} \leq c_{i\sigma^*(j)} \leq c_{ij} + c_{\sigma^*(j)j}$



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    $c_{ij} + c_{ii'} - c_{ij} = c_{ii'} \leq c_{i\sigma^*(j)} \leq c_{ij} + c_{\sigma^*(j)j}$
- $$f_i \leq f_{i'} + 2 \sum_{j \in \sigma^{-1}(i) : \phi(\sigma^*(j)) \neq i} c_{\sigma^*(j)j} + \sum_{j \in \sigma^{-1}(i) : \phi(\sigma^*(j)) = i} (c_{ij} + c_{\sigma^*(j)j})$$

- $i \in S$ is not paired: $f_i \leq 2 \sum_{j \in \sigma^{-1}(i)} c_{\sigma^*(j)j}$
- $i^* \in S^*$ is not paired: $0 \leq f_{i^*} + \sum_{j \in \sigma^{-1}(\phi(i^*)) \cap \sigma^{*-1}(i^*)} \left(c_{i^*j} - c_{\sigma(j)j}\right)$
- $i \in S$ and $i' \in S^*$ are paired:

$$f_i \leq f_{i'} + 2 \sum_{j \in \sigma^{-1}(i) : \phi(\sigma^*(j)) \neq i} c_{\sigma^*(j)j} + \sum_{j \in \sigma^{-1}(i) : \phi(\sigma^*(j)) = i} \left(c_{ij} + c_{\sigma^*(j)j}\right)$$

- summing all the inequalities:

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Exercise: scaling facility costs by some $\lambda > 1$ can give a $(1 + \sqrt{2})$-approximation.

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- Handling pseudo-polynomial running time issue:

Local Search Algorithm for Uncapacitated Facility Location

1: $S \leftarrow$ arbitrary set of facilities, $\delta \leftarrow \frac{\epsilon}{4|F|}$
2: **while** exists $S' \subseteq F$ with $|S \setminus S'| \leq 1$, $|S' \setminus S| \leq 1$ and $\mathrm{cost}(S') < (1-\delta)\mathrm{cost}(S)$ **do**
3: $\quad S' \leftarrow S$
4: **return** S