Advanced Algorithms (Fall 2023)

Linear Programming Rounding

Lecturers: 尹一通, 刘景铖, 栗师
Nanjing University
Outline

1 Linear Programming and Rounding

2 Exact Algorithms Using LP: Integral Polytopes
   - Bipartite Matching Polytope
   - $s$-$t$ Flow Polytope
   - Weighted Interval Scheduling Problem

3 Approximation Algorithms Using LP: LP Rounding
   - 2-Approximation Algorithm for Weighted Vertex Cover
   - 2-Approximation Algorithm for Unrelated Machine Scheduling
Opti. Problem $X \iff$ Integer Program (IP) $\xrightarrow{\text{relax}}$ LP

- Integer programming is NP-hard; linear programming is in P
- For some problems LP $\equiv$ IP $\Rightarrow$ exact algorithms
- For some problems, LP $\not\equiv$ IP solve LP to obtain a fractional solution, round it to an integral solution $\Rightarrow$ approximation algorithms
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\[ \text{min } 7x_1 + 4x_2 \]
\[ x_1 + x_2 \geq 5 \]
\[ x_1 + 2x_2 \geq 6 \]
\[ 4x_1 + x_2 \geq 8 \]
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optimum solution:
\quad x_1 = 1, \quad x_2 = 4

optimum value =
\quad 7 \times 1 + 4 \times 4 = 23

general case: many variables and constraints, but objective and constraints are linear
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Standard Form of Linear Programs

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\begin{align*}
\text{min} & \quad c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n & \geq b_1 \\
a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,n} x_n & \geq b_2 \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \\
a_{m,1} x_1 + a_{m,2} x_2 + \cdots + a_{m,n} x_n & \geq b_m \\
x_1, x_2, \ldots, x_n & \geq 0
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- \( n \): number of variables
- \( m \): number of constraints
- \( \leq \) constraints? equalities?
- variables can be negative? maximization problem?
Standard Form of Linear Programs

\[ x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \]

\[ A := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{n \times m}, \]

\[ b := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m, \]

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\[ \geq: \text{coordinate-wise less than or equal to} \]
- [Fourier, 1827]: Fourier-Motzkin elimination method
- [Kantorovich, Koopmans 1939]: formulated the general linear programming problem
History

- [Fourier, 1827]: Fourier-Motzkin elimination method
- [Kantorovich, Koopmans 1939]: formulated the general linear programming problem
- [Dantzig 1946]: simplex method
- [Khachiyan 1979]: ellipsoid method, polynomial time, proved linear programming is in P
- [Karmarkar, 1984]: interior-point method, polynomial time, algorithm is practical
feasible region: the set of $x$’s satisfying $Ax \geq b$, $x \geq 0$

feasible region is a polyhedron

if every coordinate has an upper and lower bound in the polyhedron, then the polyhedron is a polytope
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\( x \) is a **convex combination** of \( x^{(1)}, x^{(2)}, \ldots, x^{(t)} \) if the following condition holds: there exist \( \lambda_1, \lambda_2, \ldots, \lambda_t \in [0, 1] \) such that

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\lambda_1 + \lambda_2 + \cdots + \lambda_t = 1, \quad \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_t x^{(t)} = x
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the set of convex combinations of \( x^{(1)}, x^{(2)}, \ldots, x^{(t)} \) is called the **convex hull** of these points
Preliminaries

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- let $P$ be polytope, $x \in P$. If there are no other points $x', x'' \in P$ such that $x$ is a convex combination of $x'$ and $x''$, then $x$ is called a **vertex/extreme point** of $P$
Lemma  A polytope has finite number of vertices, and it is the convex hull of the vertices.
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Lemma Let $x \in \mathbb{R}^n$ be an extreme point in a $n$-dimensional polytope. Then, there are $n$ constraints in the definition of the polytope, such that $x$ is the unique solution to the linear system obtained from the $n$ constraints by replacing inequalities to equalities.

Lemma If the feasible region of a linear program is a polytope, then the optimum value can be attained at some vertex of the polytope.

Special cases (for minimization linear programs):
- if feasible region is empty, then its value is $\infty$
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<tr>
<th>Algorithm</th>
<th>Running Time</th>
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<tbody>
<tr>
<td>Simplex Method</td>
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<tr>
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Simplex Method

- [Dantzig, 1946]

- move from one vertex to another, so as to improve the objective
- repeat until we reach an optimum vertex

[Spielman-Teng, 2002]: smoothed analysis
Simplex Method

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Interior Point Method

- [Karmarkar, 1984]

- keep the solution inside the polytope
- design penalty function so that the solution is not too close to the boundary
- the final solution will be arbitrarily close to the optimum solution
- polynomial time
Ellipsoid Method

- [Khachiyan, 1979]
- used to decide if the feasible region is empty or not

- maintain an ellipsoid that contains the feasible region
- query a separation oracle if the center of ellipsoid is in the feasible region:
  - yes: then the feasible region is not empty
  - no: cut the ellipsoid in half, find smaller ellipsoid to enclose the half-ellipsoid, and repeat

- polynomial time, but impractical
Q: The exact running time of these algorithms?
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- It depends on many parameters: #variables, #constraints, #(non-zero coefficients), magnitude of integers
- Precision issue
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- it depends on many parameters: #variables, #constraints, #(non-zero coefficients), magnitude of integers
- precision issue

Open Problem

Can linear programming be solved in strongly polynomial time algorithm?
Applications of Linear Programming

- domain: computer science, mathematics, operations research, economics
- types of problems: transportation, scheduling, clustering, network routing, resource allocation, facility location
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Research Directions
- polynomial time exact algorithm
- polynomial time approximation algorithm
- sub-routines for the branch-and-bound method for integer programming
- other algorithmic models: online algorithm, distributed algorithms, dynamic algorithms, fast algorithms
Simple Example: Brewery Problem

- Small brewery produces ale and beer.
- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.
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- How can brewer maximize profits?

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\[
\text{max} \quad 13x + 23y
\]

\[
5x + 15y \leq 480 \quad \text{Corn}
\]

\[
4x + 4y \leq 160 \quad \text{Hops}
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\[
35x + 20y \leq 1190 \quad \text{Malt}
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\[
x, y \geq 0
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- For some combinatorial optimization problems, a polynomial-sized LP $Ax \leq b$ already defines an integral polytope, whose vertices correspond to valid integral solutions.
- Such a problem can be solved directly using the LP:
  \[
  \text{max / min } \quad c^T x \quad Ax \leq b.
  \]
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Example: Bipartite Matching Polytope

**Maximum Weight Bipartite Matching**

**Input:** bipartite graph \( G = (L \cup R, E) \)
edge weights \( w \in \mathbb{Z}^E_{>0} \)

**Output:** a matching \( M \subseteq E \) so as to
maximize \( \sum_{e \in M} w_e \)

---

LP Relaxation

\[
\begin{align*}
\max & \quad \sum_{e \in E} w_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in L \cup R \\
& \quad x_e \geq 0 \quad \forall e \in E
\end{align*}
\]

In IP:

\( x_e \in \{0, 1\} \):

\( \chi_M \in \{0, 1\} \) 

\( E: \chi_M e = 1 \iff e \in M \)

**Theorem**
The LP polytope is integral: It is the convex hull of \( \{\chi_M: M \text{ is a matching}\} \).
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- In IP: \( x_e \in \{0, 1\}: e \in M? \)
- \( \chi^M \in \{0, 1\}^E: \chi^M_e = 1 \iff e \in M \)

**Theorem** The LP polytope is integral: It is the convex hull of \( \{\chi^M : M \text{ is a matching}\} \).
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- case 1: fractional edges contain a cycle
  - color edges in cycle blue and red
  - \( x' \): \( +\epsilon \) for blue edges, \( -\epsilon \) for red edges
  - \( x'' \): \( -\epsilon \) for blue edges, \( +\epsilon \) for red edges
- case 2: fractional edges form a forest
  - color edges in a leaf-leaf path blue and red
  - \( x' \): \( +\epsilon \) for blue edges, \( -\epsilon \) for red edges
  - \( x'' \): \( -\epsilon \) for blue edges, \( +\epsilon \) for red edges
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- prove: \( x \) non integral \( \implies \) \( x \) non-vertex
- find \( x', x'' \in P: x' \neq x'', x = \frac{1}{2}(x' + x'') \)
Theorem  The LP polytope is integral: It is the convex hull of \( \{ \chi^M : M \text{ is a matching} \} \).

Proof.

- take \( x \) in the polytope \( P \)
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  - color edges in cycle blue and red
  - \( x' \): +\( \epsilon \) for blue edges, −\( \epsilon \) for red edges
  - \( x'' \): −\( \epsilon \) for blue edges, +\( \epsilon \) for red edges
**Theorem** The LP polytope is integral: It is the convex hull of \( \{ \chi^M : M \text{ is a matching} \} \).

**Proof.**
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- case 2: fractional edges form a forest
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- case 1: fractional edges contain a cycle
  - color edges in cycle blue and red
  - \( x' \): \( +\epsilon \) for blue edges, \( -\epsilon \) for red edges
  - \( x'' \): \( -\epsilon \) for blue edges, \( +\epsilon \) for red edges
- case 2: fractional edges form a forest
  - color edges in a leaf-leaf path blue and red
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- take \( x \) in the polytope \( P \)
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Outline

1. Linear Programming and Rounding

2. Exact Algorithms Using LP: Integral Polytopes
   - Bipartite Matching Polytope
   - $s$-$t$ Flow Polytope
   - Weighted Interval Scheduling Problem

3. Approximation Algorithms Using LP: LP Rounding
   - 2-Approximation Algorithm for Weighted Vertex Cover
   - 2-Approximation Algorithm for Unrelated Machine Scheduling
Example: $s$-$t$ Flow Polytope

**Flow Network**

- directed graph $G = (V, E)$, source $s \in V$, sink $t \in V$, edge capacities $c_e \in \mathbb{Z}_{>0}$, $\forall e \in E$
- $s$ has no incoming edges, $t$ has no outgoing edges
Def. A s-t flow is a vector \( f \in \mathbb{R}^E_{\geq 0} \) satisfying the following conditions:

- \( \forall e \in E, 0 \leq f_e \leq c_e \) (capacity constraints)
- \( \forall v \in V \setminus \{s, t\} \),
  \[
  \sum_{e \in \delta^\text{in}(v)} f_e = \sum_{e \in \delta^\text{out}(v)} f_e
  \] (flow conservation)

The value of flow \( f \) is defined as:

\[
\text{val}(f) := \sum_{e \in \delta^\text{out}(s)} f_e = \sum_{e \in \delta^\text{in}(t)} f_e
\]
Maximum Flow Problem

**Input:** flow network \((G = (V, E), c, s, t)\)

**Output:** maximum value of a \(s-t\) flow \(f\)
Maximum Flow Problem

**Input:** flow network \((G = (V, E), c, s, t)\)

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- Ford-Fulkerson method

---

The Maximum Flow Problem involves finding the maximum amount of flow that can be sent from a source node \(s\) to a sink node \(t\) in a given flow network. The network is defined by a directed graph \(G = (V, E)\), where \(V\) is the set of nodes and \(E\) is the set of directed edges. Each edge has a capacity \(c(e)\), which represents the maximum flow that can pass through the edge.

The goal is to find a flow \(f\) such that the total flow out of \(s\) is equal to the total flow into \(t\), and the flow on each edge \(e\) does not exceed its capacity \(c(e)\).

An algorithm that can be used to solve this problem is the Ford-Fulkerson method, which iteratively finds augmenting paths between the source and sink nodes and increases the flow along these paths until no more augmenting paths can be found.
Maximum Flow Problem

**Input:** flow network \((G = (V, E), c, s, t)\)

**Output:** maximum value of a \(s-t\) flow \(f\)

- Ford-Fulkerson method
- Maximum-Flow Min-Cut Theorem: value of the maximum flow is equal to the value of the minimum \(s-t\) cut
Maximum Flow Problem

**Input:** flow network \((G = (V, E), c, s, t)\)

**Output:** maximum value of a \(s-t\) flow \(f\)

![Diagram of a flow network with labels and capacities]

- **Ford-Fulkerson method**
- **Maximum-Flow Min-Cut Theorem:** value of the maximum flow is equal to the value of the minimum \(s-t\) cut
- **[Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva, 2022]:** nearly linear-time algorithm
Theorem

The LP polytope is integral.

Sketch of Proof.

Take any \( s \)-\( t \) flow \( x \); consider fractional edges \( E' \).

Every \( v \) \( \not\in \{s, t\} \) must be incident to 0 or 2 edges in \( E' \).

Ignoring the directions of \( E' \), it contains a cycle, or a \( s \)-\( t \) path.

We can increase/decrease flow values along cycle/path.
LP for Maximum Flow

\[
\text{max } \sum_{e \in \delta^\text{in}(t)} x_e
\]

\[
x_e \leq c_e \quad \forall e \in E
\]

\[
\sum_{e \in \delta^\text{out}(v)} x_e - \sum_{e \in \delta^\text{in}(v)} x_e = 0 \quad \forall v \in V \setminus \{s, t\}
\]

\[
x_e \geq 0 \quad \forall e \in E
\]

**Theorem**  The LP polytope is integral.
LP for Maximum Flow

\[ \text{max} \quad \sum_{e \in \delta^{\text{in}}(t)} x_e \]

\[ \sum_{e \in \delta^{\text{out}}(v)} x_e - \sum_{e \in \delta^{\text{in}}(v)} x_e = 0 \quad \forall v \in V \setminus \{s, t\} \]

\[ \sum_{e \in \delta^{\text{out}}(v)} x_e - \sum_{e \in \delta^{\text{in}}(v)} x_e \leq c_e \quad \forall e \in E \]

\[ x_e \geq 0 \quad \forall e \in E \]

**Theorem**  The LP polytope is integral.

**Sketch of Proof.**

- Take any \( s-t \) flow \( x \); consider fractional edges \( E' \)
- Every \( v \notin \{ s, t \} \) must be incident to 0 or \( \geq 2 \) edges in \( E' \)
- Ignoring the directions of \( E' \), it contains a cycle, or a \( s-t \) path
- We can increase/decrease flow values along cycle/path
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Weighted Interval Scheduling Problem

**Input:** $n$ activities, activity $i$ starts at time $s_i$, finishes at time $f_i$, and has weight $w_i > 0$

$i$ and $j$ can be scheduled together iff $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

**Output:** maximum weight subset of jobs that can be scheduled

- optimum value $= 220$
Weighted Interval Scheduling Problem

**Input:** \( n \) activities, activity \( i \) starts at time \( s_i \), finishes at time \( f_i \), and has weight \( w_i > 0 \)

\( i \) and \( j \) can be scheduled together iff \([s_i, f_i)\) and \([s_j, f_j)\) are disjoint

**Output:** maximum weight subset of jobs that can be scheduled

- optimum value = 220
- Classic Problem for Dynamic Programming
Weighted Interval Scheduling Problem

**Linear Program**

\[
\begin{align*}
\text{max} & \quad \sum_{j \in [n]} x_j w_j \\
\sum_{j \in [n]: t \in [s_j, f_j]} x_j & \leq 1 \quad \forall t \in [T] \\
x_j & \geq 0 \quad \forall j \in [n]
\end{align*}
\]
Weighted Interval Scheduling Problem

Linear Program

\[
\begin{align*}
\text{max} & \quad \sum_{j \in [n]} x_j w_j \\
\sum_{j \in [n]: t \in [s_j, f_j]} x_j & \leq 1 \quad \forall t \in [T] \\
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Weighted Interval Scheduling Problem

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**Theorem** The LP polytope is integral.

**Def.** A matrix \( A \in \mathbb{R}^{m \times n} \) is said to be totally unimodular (TUM), if every sub-square of \( A \) has determinant in \( \{-1, 0, 1\} \).
Weighted Interval Scheduling Problem

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**Theorem** If a polytope \( P \) is defined by \( Ax \leq b, x \geq 0 \) with a totally unimodular matrix \( A \) and integral \( b \), then \( P \) is integral.
Weighted Interval Scheduling Problem

**Linear Program**

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\begin{align*}
\text{max} & \quad \sum_{j \in [n]} x_j w_j \\
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**Definition**  A matrix \( A \in \mathbb{R}^{m \times n} \) is said to be totally unimodular (TUM), if every sub-square of \( A \) has determinant in \([-1, 0, 1]\).

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**Lemma**  A matrix \( A \in \{0, 1\}^{m \times n} \) where the 1’s on every column form an interval is TUM.

- So, the matrix for the LP is TUM, and the polytope is integral.
Theorem  If a polytope $P$ is defined by $Ax \leq b$, $x \geq 0$ with a totally unimodular matrix $A$ and integral $b$, then $P$ is integral.
**Theorem**  If a polytope $P$ is defined by $Ax \leq b, x \geq 0$ with a totally unimodular matrix $A$ and integral $b$, then $P$ is integral.

**Proof.**

- Every vertex $x \in P$ is the unique solution to the linear system (after permuting coordinates): $\begin{pmatrix} A' & 0 \\ 0 & I \end{pmatrix} x = \begin{pmatrix} b' \\ 0 \end{pmatrix}$, where
- $A'$ is a square submatrix of $A$ with $\det(A') = \pm 1$, $b'$ is a sub-vector of $b$,
- and the rows for $b'$ are the same as the rows for $A'$.
Theorem. If a polytope $P$ is defined by $Ax \leq b, x \geq 0$ with a totally unimodular matrix $A$ and integral $b$, then $P$ is integral.

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- Let $x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$, so that $A'x^1 = b'$ and $x^2 = 0$. 

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- Cramer’s rule: $x^1_i = \frac{\det(A'_i|b)}{\det(A')} \text{ for every } i \implies x^1_i \text{ is integer}$

  - $A'_i|b$: the matrix of $A'$ with the $i$-th column replaced by $b$
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\begin{pmatrix}
A' & 0 \\
0 & I
\end{pmatrix} x = \begin{pmatrix} b' \\
0
\end{pmatrix},
\]

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  $A'_i|b$: the matrix of $A'$ with the $i$-th column replaced by $b$

**Obs.** If $A$ is TUM, then $A^T$ is also TUM.
Lemma  Let $A' \in \{0, \pm 1\}^{n \times n}$ such that every row of $A'$ contains at most one 1 and one $-1$. Then $\det(A') \in \{0, \pm 1\}$.

Proof.
**Lemma** Let $A' \in \{0, \pm 1\}^{n \times n}$ such that every row of $A'$ contains at most one 1 and one $-1$. Then $\det(A') \in \{0, \pm 1\}$.

**Proof.**

- wlog assume every row of $A'$ contains one 1 and one $-1$
**Lemma** Let $A' \in \{0, \pm 1\}^{n \times n}$ such that every row of $A'$ contains at most one 1 and one $-1$. Then $\det(A') \in \{0, \pm 1\}$.

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- wlog assume every row of $A'$ contains one 1 and one $-1$
- otherwise, we can reduce the matrix
Lemma Let $A' \in \{0, \pm 1\}^{n \times n}$ such that every row of $A'$ contains at most one $1$ and one $-1$. Then $\det(A') \in \{0, \pm 1\}$.

Proof.

- wlog assume every row of $A'$ contains one $1$ and one $-1$
- otherwise, we can reduce the matrix
- treat $A'$ as a directed graph: columns $\equiv$ vertices, rows $\equiv$ arcs
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- wlog assume every row of $A'$ contains one 1 and one $-1$
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**Lemma** Let $A \in \{0, \pm 1\}^{m \times n}$ such that every row of $A$ contains at most one 1 and one $-1$. Then $A$ is TUM.
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**Lemma** Let $A \in \{0, \pm 1\}^{m \times n}$ such that every row of $A$ contains at most one 1 and one $-1$. Then $A$ is TUM.

**Coro.** The matrix for $s-t$ flow polytope is TUM; thus, the polytope is integral.
**Lemma** Let $A' \in \{0, \pm 1\}^{n \times n}$ such that every row of $A'$ contains at most one 1 and one $-1$. Then $\det(A') \in \{0, \pm 1\}$.

**Lemma** A matrix $A \in \{0, 1\}^{m \times n}$ where the 1’s on every row form an interval is TUM.

**Proof.**
Lemma  Let $A' \in \{0, \pm 1\}^{n \times n}$ such that every row of $A'$ contains at most one 1 and one $-1$. Then $\det(A') \in \{0, \pm 1\}$.

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- take any square submatrix $A'$ of $A$, 

\[ \det(A') \in \{0, \pm 1\} \]
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Proof.

- take any square submatrix $A'$ of $A$,
- the 1's on every row of $A'$ form an interval.
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Lemma A matrix $A \in \{0, 1\}^{m\times n}$ where the 1's on every row form an interval is TUM.

Proof.

- take any square submatrix $A'$ of $A$,
- the 1's on every row of $A'$ form an interval.
- $A'M$ is a matrix satisfying condition of first lemma, where
- $\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$. $\det(M) = 1$. 

$\det(A'M) \in \{0, \pm 1\}$.
Lemma Let $A' \in \{0, \pm 1\}^{n \times n}$ such that every row of $A'$ contains at most one $1$ and one $-1$. Then $\det(A') \in \{0, \pm 1\}$.

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- the $1$'s on every row of $A'$ form an interval.
- $A'M$ is a matrix satisfying condition of first lemma, where

\[
M = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -1 \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix} \quad \text{det}(M) = 1.
\]

- $\det(A'M) \in \{0, \pm 1\} \implies \det(A') \in \{0, \pm 1\}$.
Lemma  The edge-vertex incidence matrix $A$ of a bipartite graph is totally-unimodular.

Proof. 

Remark: Bipartiteness is needed. The edge-vertex incidence matrix

$$
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
$$

of a triangle has determinant $2$. 

Coro. Bipartite matching polytope is integral.
Lemma The edge-vertex incidence matrix $A$ of a bipartite graph is totally-unimodular.

Proof.

1. $G = (L \cup R, E)$: the bipartite graph
Lemma  The edge-vertex incidence matrix $A$ of a bipartite graph is totally-unimodular.

Proof.

- $G = (L \cup R, E)$: the bipartite graph
- $A'$: obtained from $A$ by negating columns correspondent to $R$
**Lemma** The edge-vertex incidence matrix $A$ of a bipartite graph is totally-unimodular.

**Proof.**
- $G = (L \cup R, E)$: the bipartite graph
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remark: bipartiteness is needed. The edge-vertex incidence matrix
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Linear Programming and Rounding

Exact Algorithms Using LP: Integral Polytopes
- Bipartite Matching Polytope
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Approximation Algorithms Using LP: LP Rounding
- 2-Approximation Algorithm for Weighted Vertex Cover
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Approximation Algorithm based on LP Rounding

- Opti. Problem $X \iff 0/1$ Integer Program (IP) $\xrightarrow{\text{relax}}$ LP
Approximation Algorithm based on LP Rounding

Opti. Problem $X \leftrightarrow 0/1$ Integer Program (IP) $\xrightarrow{\text{relax}} \text{LP}$

0/1 Integer Program

\[
\begin{align*}
\min & \quad c^T x \\
Ax & \geq b \\
x & \in \{0, 1\}^n
\end{align*}
\]

Linear Program Relaxation

\[
\begin{align*}
\min & \quad c^T x \\
Ax & \geq b \\
x & \in [0, 1]^n
\end{align*}
\]
Approximation Algorithm based on LP Rounding

- Opti. Problem $X \iff 0/1$ Integer Program (IP) $\overset{\text{relax}}{\Longrightarrow}$ LP

0/1 Integer Program

$$\begin{align*}
\text{min} & \quad c^T x \\
Ax & \geq b \\
x & \in \{0, 1\}^n
\end{align*}$$

Linear Program Relaxation

$$\begin{align*}
\text{min} & \quad c^T x \\
Ax & \geq b \\
x & \in [0, 1]^n
\end{align*}$$

- LP $\leq$ IP

Integer programming is NP-hard, linear programming is in P

Solve LP to obtain a fractional $x \in [0, 1]^n$.

Round it to an integral $\tilde{x} \in \{0, 1\}^n$ $\iff$ solution for $X$

Prove $c^T \tilde{x} \leq \alpha \cdot c^T x$, then $c^T \cdot \tilde{x} \leq \alpha \cdot \text{LP} \leq \alpha \cdot \text{IP} = \alpha \cdot \text{opt} \implies \alpha$-approximation
Approximation Algorithm based on LP Rounding

- Opti. Problem $X \iff 0/1$ Integer Program (IP) $\overset{\text{relax}}{\Longrightarrow} \text{LP}$

0/1 Integer Program

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\begin{align*}
\text{min} & \quad c^T x \\
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x & \in \{0, 1\}^n
\end{align*}
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Approximation Algorithm based on LP Rounding

Opti. Problem $X \iff \text{0/1 Integer Program (IP)} \xrightarrow{\text{relax}} \text{LP}

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\text{0/1 Integer Program} & \\
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### 0/1 Integer Program

- $\min c^T x$
- $Ax \geq b$
- $x \in \{0, 1\}^n$

### Linear Program Relaxation

- $\min c^T x$
- $Ax \geq b$
- $x \in [0, 1]^n$

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**0/1 Integer Program**

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- \(\iff\) \(\alpha\)-approximation
IP

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\[
\begin{align*}
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\end{align*}
\]

The ratio between IP = opt and LP is called the integrality gap of the LP relaxation. The approximation ratio based on this analysis cannot be better than the worst integrality gap.
\begin{align*}
\text{IP} & \quad \min c^T x \\
& \quad Ax \geq b \\
& \quad x \in \{0, 1\}^n
\end{align*}

\begin{align*}
\text{LP Relaxation} & \quad \min c^T x \\
& \quad Ax \geq b \\
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Bullet point:
The approximation ratio based on this analysis can not be better than the worst integrality gap.
**Def.** The ratio between $\text{IP} = \text{opt}$ and $\text{LP}$ is called the *integrality gap* of the LP relaxation.

- The approximation ratio based on this analysis can not be better than the worst integrality gap.
Outline

1. Linear Programming and Rounding

2. Exact Algorithms Using LP: Integral Polytopes
   - Bipartite Matching Polytope
   - s-t Flow Polytope
   - Weighted Interval Scheduling Problem

3. Approximation Algorithms Using LP: LP Rounding
   - 2-Approximation Algorithm for Weighted Vertex Cover
   - 2-Approximation Algorithm for Unrelated Machine Scheduling
Weighted Vertex Cover Problem

**Input:** graph $G = (V, E)$, vertex weights $w \in \mathbb{Z}_{>0}^V$

**Output:** vertex cover $S$ of $G$, to minimize $\sum_{v \in S} w_v$
\( x_v \in \{0, 1\}, \forall v \in V \): indicate if we include \( v \) in the vertex cover

**Integer Program**

\[
\begin{align*}
\text{min} \quad & \sum_{v \in V} w_v x_v \\
x_u + x_v & \geq 1 \quad \forall (u, v) \in E \\
x_v & \in \{0, 1\} \quad \forall v \in V
\end{align*}
\]

**LP Relaxation**

\[
\begin{align*}
\text{min} \quad & \sum_{v \in V} w_v x_v \\
x_u + x_v & \geq 1 \quad \forall (u, v) \in E \\
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\]
- $x_v \in \{0, 1\}, \forall v \in V$: indicate if we include $v$ in the vertex cover

<table>
<thead>
<tr>
<th>Integer Program</th>
<th>LP Relaxation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\min \sum_{v \in V} w_v x_v$</td>
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</tr>
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</tr>
</tbody>
</table>

- IP := value of integer program, LP := value of linear program
- LP $\leq$ IP $=$ opt
Rounding Algorithm

1: Solve LP to obtain solution \( \{x_u^*\}_{u \in V} \)

\[ \triangleright \text{So, } \text{LP} = \sum_{u \in V} w_u x_u^* \leq \text{IP} \]
Rounding Algorithm

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   \[ \triangleright \text{So, } LP = \sum_{u \in V} w_u x_u^* \leq \text{IP} \]

2: return \( S := \{ u \in V : x_u \geq 1/2 \} \)
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Lemma $S$ is a vertex cover of $G$.

Proof.
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- Consider any \((u, v) \in E\): we have \( x_u^* + x_v^* \geq 1 \)
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2: return \( S := \{u \in V : x_u \geq 1/2\} \)

Lemma \( S \) is a vertex cover of \( G \).

Proof.
- Consider any \((u, v) \in E\): we have \( x_u^* + x_v^* \geq 1 \)
- So, \( x_u^* \geq 1/2 \) or \( x_v^* \geq 1/2 \) \( \implies u \in S \) or \( v \in S \).
Rounding Algorithm

1: Solve LP to obtain solution \( \{x_u^*\}_{u \in V} \)

\[ \triangleright \text{So, LP } = \sum_{u \in V} w_u x_u^* \leq \text{IP} \]

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**Lemma** \( S \) is a vertex cover of \( G \).

**Lemma** \( \text{cost}(S) := \sum_{u \in S} w_u \leq 2 \cdot LP \).
Rounding Algorithm

1: Solve LP to obtain solution \( \{x_u^*\}_{u \in V} \)
   \[ \triangleright \text{So, LP} = \sum_{u \in V} w_u x_u^* \leq \text{IP} \]
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\textbf{Lemma} \ S \text{ is a vertex cover of } G.

\textbf{Lemma} \ \text{cost}(S) := \sum_{u \in S} w_u \leq 2 \cdot \text{LP}.

\textbf{Proof.}

\[
\text{cost}(S) = \sum_{u \in S} w_u \leq \sum_{u \in S} w_u \cdot 2 x_u^* = 2 \sum_{u \in S} w_u \cdot x_u^* \\
\leq 2 \sum_{u \in V} w_u \cdot x_u^* = 2 \cdot \text{LP}. \]

\[\square\]
Rounding Algorithm

1: Solve LP to obtain solution \( \{x_u^*\}_{u \in V} \)

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**Theorem** The algorithm is a 2-approximation algorithm for weighted vertex cover.
Rounding Algorithm

1: Solve LP to obtain solution \( \{x_u^*\}_{u \in V} \)

\[ \Rightarrow \text{So, } \text{LP} = \sum_{u \in V} w_u x_u^* \leq \text{IP} \]

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\[ \text{Proof. } \]

\[ \text{cost}(S) \leq 2 \cdot \text{LP} \leq 2 \cdot \text{IP} = 2 \cdot (\text{optimum value}) \]
Outline

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**Unrelated Machine Scheduling**

**Input:** \( J, |J| = n \): jobs
- \( M, |M| = m \): machines
- \( p_{ij} \): processing time of job \( j \) on machine \( i \)

**Output:** assignment \( \sigma : J \leftrightarrow M \), so as to minimize makespan:

\[
\max_{i \in M} \sum_{j \in \sigma^{-1}(i)} p_{ij}
\]
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![Diagram showing job assignments and machine loads](image-url)
Assumption: we are given a target makespan $T$, and $p_{ij} \in [0, T] \cup \{\infty\}$
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$x_{ij}$: fraction of $j$ assigned to $i$

\[\sum_{i} x_{ij} = 1 \quad \forall j \in J\]
\[\sum_{j} p_{ij} x_{ij} \leq T \quad \forall i \in M\]
\[x_{ij} \geq 0 \quad \forall i, j\]
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$x_{ij} \geq 0 \quad \forall ij$
2-Approximate Rounding Algorithm of Shmoys-Tardos

The figure illustrates the bipartite matching polytope with $x_{ij}$ representing the decision variable for assigning job $j$ to machine $M$. The constraint $x_{ij}$ between $J$ and $M$ is a point in the bipartite-matching polytope, where all jobs in $J$ are matched.
\[ \sum_i x_{ij} = 1 \]

The diagram shows a bipartite graph with nodes labeled \( J \) and \( M \). Each node in the \( J \) set is connected to every node in the \( M \) set. This represents a constraint that the total assignment of jobs to machines must be unity, as stated by \( \sum_i x_{ij} = 1 \).
2-Approximate Rounding Algorithm of Shmoys-Tardos

\[ \sum_i x_{ij} = 1 \quad x_{ij} \quad \sum_i p_{ij} x_{ij} \leq T \]

\[ J \quad M \]
2-Approximate Rounding Algorithm of Shmoys-Tardos

\[ p_{ij_1} \geq p_{ij_2} \geq \cdots \geq p_{ij_5} \]
2-Approximate Rounding Algorithm of Shmoys-Tardos

\[ x_{ij} \geq x_{ij_2} \geq \cdots \geq x_{ij_5} \]

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2-Approximate Rounding Algorithm of Shmoys-Tardos

\[ x_{ij} \]

between jobs and sub-machines is a point in the bipartite-matching polytope, where all jobs in \( J \) are matched.
2-Approximate Rounding Algorithm of Shmoys-Tardos

\( x_{ij} \)

\( J \)  \( M \)

\( \sum_g x_{gj} = 1 \)

\( \sum_j x_{gj} \leq 1 \)

Obs. \( x \) between \( J \) and sub-machines is a point in the bipartite-matching polytope, where all jobs in \( J \) are matched.
2-Approximate Rounding Algorithm of Shmoys-Tardos

\[
x_{ij}
\]

\[
J \quad M
\]

\[
\sum_g x_{gj} = 1
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\[
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\textbf{Obs.} \ x \ between \ J \ and \ sub-machines \ is \ a \ point \ in \ the \ bipartite-matching \ polytope, \ where \ all \ jobs \ in \ J \ are \ matched.
Recall bipartite matching polytope is integral.
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$x$ is a **convex combination** of matchings.
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Any matching in the combination covers all jobs $J$. 
Recall bipartite matching polytope is integral.

- $x$ is a convex combination of matchings.
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**Lemma** Any matching in the combination gives an schedule of makespan $\leq 2T$. 
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Proof. 

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**Proof.**
- focus on machine $i$, let $i_1, i_2, \cdots, i_a$ be the sub-machines for $i$
**Lemma**  Any matching in the combination gives an schedule of makespan \( \leq 2T \).

**Proof.**

- focus on machine \( i \), let \( i_1, i_2, \cdots, i_a \) be the sub-machines for \( i \)
- assume job \( k_t \) is assigned to sub-machine \( i_t \).
**Lemma** Any matching in the combination gives an schedule of makespan $\leq 2T$. 

**Proof.**

- focus on machine $i$, let $i_1, i_2, \cdots, i_a$ be the sub-machines for $i$
- assume job $k_t$ is assigned to sub-machine $i_t$. 

\[
\text{(load on } i) = \sum_{t=1}^{a} p_{ik_t} \leq p_{ik_1} + \sum_{t=2}^{a} \sum_j x_{i_{t-1}j} \cdot p_{ij} \\
\leq p_{ik_1} + \sum_j x_{ij}p_{ij} \leq T + T = 2T. 
\]
Summary

- linear programming, simplex method, interior point method, ellipsoid method
- integral LP polytopes: bipartite matching polytope, $s$-$t$ flow polytope, weighted interval scheduling polytope
linear programming, simplex method, interior point method, ellipsoid method

integral LP polytopes: bipartite matching polytope, $s$-$t$ flow polytope, weighted interval scheduling polytope

approximation algorithm using LP rounding
  - 2-approximation algorithm for weighted vertex cover
  - 2-approximation for unrelated machine scheduling
Linear Program : 线性规划
Integer Program : 整数规划
Feasible Region : 解域
Polyhedron : 凸多面体
Polytope : 有界凸多面体
Vertex/Extreme Point : 顶点
Convex Combination : 凸组合
Convex Hull : 凸包
Dual : 对偶
Totally Unimodular : 完全单位模的