Advanced Algorithms (Fall 2023)

Primal Dual

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Nanjing University
Outline

1 Duality of Linear Programming
   - Max-Flow Min-Cut Theorem Using LP Duality
   - 0-Sum Game and Nash Equilibrium

2 2-Approximation Algorithm for Weighted Vertex Cover Using Primal-Dual

3 3-Approximation Algorithm for Uncapacitated Facility Location Problem Using Primal Dual
\[
\begin{align*}
\text{min} & \quad 7x_1 + 4x_2 \\
& \quad x_1 + x_2 \geq 5 \\
& \quad x_1 + 2x_2 \geq 6 \\
& \quad 4x_1 + x_2 \geq 8 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

- optimum solution: \( x_1 = 1, x_2 = 4 \)
- optimum value = \( 7 \times 1 + 4 \times 4 = 23 \)

**Q:** How can we give a lower bound for the linear program, without solving it?
\[ \begin{align*} 
\min \quad & 7x_1 + 4x_2 \\
& x_1 + x_2 \geq 5 \\
& x_1 + 2x_2 \geq 6 \\
& 4x_1 + x_2 \geq 8 \\
& x_1, x_2 \geq 0 \\
\end{align*} \]

- \[ 7x_1 + 4x_2 \geq 2(x_1 + x_2) + (x_1 + 2x_2) \geq 2 \times 5 + 6 = 16 \]
- \[ 7x_1 + 4x_2 \geq (x_1 + x_2) + (x_1 + 2x_2) + (4x_1 + x_2) \geq 5 + 6 + 8 = 19 \]
- \[ 7x_1 + 4x_2 \geq 3(x_1 + x_2) + (4x_1 + x_2) \geq 3 \times 5 + 8 = 23 \]

**Q:** How can we obtain the best (i.e., largest) lower bound using this method?
### Primal LP

\[
\text{min } 7x_1 + 4x_2 \\
x_1 + x_2 \geq 5 \\
x_1 + 2x_2 \geq 6 \\
4x_1 + x_2 \geq 8 \\
x_1, x_2 \geq 0
\]

### Dual LP

\[
\text{max } 5y_1 + 6y_2 + 8y_3 \\
y_1 + y_2 + 4y_3 \leq 7 \\
y_1 + 2y_2 + y_3 \leq 4 \\
y_1, y_2, y_3 \geq 0
\]

### A general method to prove a lower bound

\[
7x_1 + 4x_2 \quad (\text{if } 7 \geq y_1 + y_2 + 4y_3 \text{ and } 4 \geq y_1 + 2y_2 + y_3)
\]

\[
\geq y_1(x_1 + x_2) + y_2(x_1 + 2x_2) + y_3(4x_1 + x_2) \quad (\text{if } y_1, y_2, y_3 \geq 0)
\]

\[
\geq 5y_1 + 6y_2 + 8y_3
\]

- to achieve the largest lower bound: maximize $5y_1 + 6y_2 + 8y_3$
\[
\begin{align*}
\text{min} & \quad 7x_1 + 4x_2 \\
x_1 + x_2 & \geq 5 \\
x_1 + 2x_2 & \geq 6 \\
4x_1 + x_2 & \geq 8 \\
x_1, x_2 & \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad 5y_1 + 6y_2 + 8y_3 \\
y_1 + y_2 + 4y_3 & \leq 7 \\
y_1 + 2y_2 + y_3 & \leq 4 \\
y_1, y_2, y_3 & \geq 0
\end{align*}
\]

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 4 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} \quad c = \begin{pmatrix} 7 \\ 4 \end{pmatrix}
\]

\[
\begin{align*}
\text{min} & \quad c^T x \\
Ax & \geq b \\
x & \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad b^T y \\
A^T y & \leq c \\
y & \geq 0
\end{align*}
\]
Primal LP
\[
\begin{align*}
\text{min} & \quad c^T x \\
Ax & \geq b \\
x & \geq 0
\end{align*}
\]

Dual LP
\[
\begin{align*}
\text{max} & \quad b^T y \\
A^T y & \leq c \\
y & \geq 0
\end{align*}
\]

Relationships

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>dual LP</th>
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</thead>
<tbody>
<tr>
<td>variables</td>
<td>constraints</td>
</tr>
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</tr>
<tr>
<td>obj. coefficients</td>
<td>RHS constants</td>
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</tr>
</tbody>
</table>

More Relationships

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable in ( \mathbb{R} )</td>
<td>equalities</td>
</tr>
<tr>
<td>equalities</td>
<td>variable in ( \mathbb{R} )</td>
</tr>
</tbody>
</table>
**Primal LP**

\[
\begin{align*}
\text{min} & \quad c^T x \\
Ax & \geq b \\
\end{align*}
\]

\[x \geq 0\]

- **P** := value of primal LP
- **D** := value of dual LP

**Theorem** (Weak Duality Theorem)

\[D \leq P.\]

**Proof.**

- \(x\): an arbitrary solution to Primal LP
- \(y\): an arbitrary solution to Dual LP

\[b^T y \leq (Ax)^T y = x^T A^T y \leq x^T c = c^T x.\]
**Theorem** (Strong Duality Theorem) $D = P$. 

**Proof of Strong Duality Theorem**

**Lemma** (Variant of Farkas Lemma) $Ax \leq b$ is infeasible, if and only if $A^Ty = 0$, $y^Tb < 0$, $y \geq 0$ is feasible

- $\forall \epsilon > 0, -Ax \leq -b, x \geq 0, c^Tx \leq P - \epsilon$ is in feasible
- There exists $y \in \mathbb{R}^m, \alpha$, such that $(y^T, \alpha)\begin{pmatrix} -A \\ c^T \end{pmatrix} \geq 0$, $(y^T, \alpha)\begin{pmatrix} -b \\ P - \epsilon \end{pmatrix} < 0$
- we can prove $\alpha > 0$; assume $\alpha = 1$
- $-y^TA + c^T \geq 0$, $-y^Tb + P - \epsilon < 0 \iff A^Ty \leq c, b^Ty > P - \epsilon$
- $\forall \epsilon > 0, D > P - \epsilon \implies D = P$ (since $D \leq P$)
- duality is mutual: the dual of the dual of an LP is the LP itself.

Primal LP

\[
\begin{align*}
\text{max} & \quad b^T y \\
A^T y & \leq c \\
y & \geq 0
\end{align*}
\]

Dual LP

\[
\begin{align*}
\text{min} & \quad c^T x \\
Ax & \geq b \\
x & \geq 0
\end{align*}
\]

- Duality theorem holds when one LP is infeasible:
- Minimization LP is infeasible \(\implies\) value = \(\infty\)
  \(\iff\) dual LP value = \(\infty\) \(\implies\) feasible region of dual LP is unbounded
Complementary Slackness

**Primal LP**

\[
\begin{align*}
\min & \quad c^T x \\
Ax & \geq b \\
x & \geq 0
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\max & \quad b^T y \\
A^T y & \leq c \\
y & \geq 0
\end{align*}
\]

- \(x^*\) and \(y^*\): optimum primal and dual solutions
- \(D = b^T y^* \leq (Ax^*)^T y^* = (x^*)^T A^T y^* \leq (x^*)^T c = c^T x^* = P\).
- \(P = D\): all the inequalities hold with equalities.

**Complementary Slackness**

- \(y_i^* > 0 \implies \sum_j a_{ij} x_j^* = b_i\).
- \(x_j^* > 0 \implies \sum_i a_{ij} y_i^* = c_j\).
## Simple Example for Duality: Brewery problem

<table>
<thead>
<tr>
<th>Beverage</th>
<th>Corn (pounds)</th>
<th>Hops (pounds)</th>
<th>Malt (pounds)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ale (barrel)</td>
<td>5</td>
<td>4</td>
<td>35</td>
<td>13</td>
</tr>
<tr>
<td>Beer (barrel)</td>
<td>15</td>
<td>4</td>
<td>20</td>
<td>23</td>
</tr>
<tr>
<td>Constraint</td>
<td>480</td>
<td>160</td>
<td>1190</td>
<td></td>
</tr>
</tbody>
</table>

### Primal LP

\[
\begin{align*}
\text{max} \quad & 13x + 23y \\
5x + 15y & \leq 480 \\
4x + 4y & \leq 160 \\
35x + 20y & \leq 1190 \\
x, y & \geq 0
\end{align*}
\]

### Dual LP

\[
\begin{align*}
\text{min} \quad & 480\alpha + 160\beta + 1190\gamma \\
5\alpha + 4\beta + 35\gamma & \geq 13 \\
15\alpha + 4\beta + 20\gamma & \geq 23 \\
\alpha, \beta, \gamma & \geq 0
\end{align*}
\]

\(\alpha, \beta, \gamma\): the value of 1 pound of corn, hops and malt respectively.
\[
\begin{align*}
\text{min} & \quad 480\alpha + 160\beta + 1190\gamma \\
& \quad 5\alpha + 4\beta + 35\gamma \geq 13 \\
& \quad 15\alpha + 4\beta + 20\gamma \geq 23 \\
& \quad \alpha, \beta, \gamma \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad 13x + 23y \\
& \quad 5x + 15y \leq 480 \\
& \quad 4x + 4y \leq 160 \\
& \quad 35x + 20y \leq 1190 \\
& \quad x, y \geq 0
\end{align*}
\]

The dual of a covering LP is a packing LP, and vice versa.
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**Maximum Flow Problem**

**Input:** flow network 
\((G = (V, E), c, s, t)\)

**Output:** maximum value of a 
\(s-t\) flow \(f\)

**LP for Maximum Flow**

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \delta^{\text{in}}(t)} x_e \\
\text{subject to} & \quad x_e \leq c_e \quad \forall e \in E \\
& \quad \sum_{e \in \delta^{\text{out}}(v)} x_e - \sum_{e \in \delta^{\text{in}}(v)} x_e = 0 \quad \forall v \in V \setminus \{s, t\} \\
& \quad x_e \geq 0 \quad \forall e \in E
\end{align*}
\]
An Equivalent Packing LP

- $\mathcal{P}$: the set of all simple paths from $s$ to $t$
- $f_P, P \in \mathcal{P}$: the flow on $P$

\[
\text{max} \sum_{P \in \mathcal{P}} f_P
\]
\[
\text{min} \sum_{e \in E} c_e y_e
\]

\[
\sum_{P \in \mathcal{P}: e \in P} f_P \leq c_e \quad \forall e \in E
\]
\[
\sum_{e \in P} y_e \geq 1 \quad \forall P \in \mathcal{P}
\]
\[
f_P \geq 0 \quad \forall P \in \mathcal{P}
\]
\[
y_e \geq 0 \quad \forall e \in E
\]

- dual constraints: the shortest $s$-$t$ path w.r.t weights $y$ has length $\geq 1$
**Dual LP**

\[
\min \sum_{e \in E} c_e y_e
\]

\[
\sum_{e \in P} y_e \geq 1 \quad \forall P \in \mathcal{P}
\]

\[
y_e \geq 0 \quad \forall e \in E
\]

**Theorem** The optimum value can be attained at an integral point \( y \).

**Maximum Flow Minimum Cut Theorem** The value of the maximum flow equals the value of the minimum cut.

**Proof of Theorem.**

- Given any optimum \( y \), let \( d_v \) be the length of shortest path from \( s \) to \( v \), for every \( v \in V \). \( d_s = 0, d_t = 1 \)
- Randomly choose \( \theta \in (0, 1) \), and output cut \( (S := \{v : d_v \leq \theta\}, T := \{v : d_v > \theta\}) \)
- Lemma: \( \mathbb{E}[\text{cut value of}(S, T)] \leq \sum_{e \in E} c_e y_e \)
- Any cut \( (S, T) \) in the support is optimum
\[
\begin{align*}
\max & \quad \sum_{P \in \mathcal{P}} f_P \\
\sum_{P \in \mathcal{P} : e \in P} f_P & \leq c_e \quad \forall e \in E \\
\sum_{P \in \mathcal{P}} f_P & \geq 0 \quad \forall P \in \mathcal{P}
\end{align*}
\]
\[
\begin{align*}
\min & \quad \sum_{e \in E} c_e y_e \\
\sum_{e \in P} y_e & \geq 1 \quad \forall P \in \mathcal{P} \\
y_e & \geq 0 \quad \forall e \in E
\end{align*}
\]

- **pros of new LP:** it is a packing LP, dual is a covering LP, easier to understand and analyze

- **cons of new LP:** exponential size, can not be solved directly
  - when we only need to do non-algorithmic analysis
  - ellipsoid method with separation oracle can solve some exponential size LP
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**0-Sum Game**

**Input:** a payoff matrix \( M \in \mathbb{R}^{m \times n}, m, n \geq 1 \),

- two players: row player R, column player C

**Output:** R plays a row \( i \in [m] \), C plays a column \( j \in [n] \)

- payoff of game is \( M_{ij} \)
- R wants to minimize \( M_{ij} \), C wants to maximize \( M_{ij} \)

---

**Rock-Scissor-Paper Game**

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>S</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>S</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>P</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- game depends on who plays first

By allowing **mixed strategies**, each player has a best strategy, regardless of who plays first.
<table>
<thead>
<tr>
<th>Strategy Type</th>
<th>Player R</th>
<th>Player C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure Strategy</td>
<td>Row $i \in [m]$</td>
<td>Column $j \in [n]$</td>
</tr>
<tr>
<td>Mixed Strategy</td>
<td>Distribution $x$ over $[m]$</td>
<td>Distribution $y$ over $[n]$</td>
</tr>
<tr>
<td></td>
<td>$x \in [0, 1]^m, \sum_{i=1}^m x_i = 1$</td>
<td>$y \in [0, 1]^n, \sum_{j=1}^n y_j = 1$</td>
</tr>
</tbody>
</table>

$$M(x, y) := \sum_{i=1}^m \sum_{j=1}^n x_i y_j M_{ij}$$

$$M(x, j) := \sum_{i=1}^m x_i M_{ij}, \quad M(i, y) := \sum_{j=1}^n y_j M_{ij}$$

- If R plays a mixed strategy $y$ first, then it is the best for C to play a pure strategy $j$. Value of game is $\inf_x \max_{j \in [n]} M(x, j)$.
- If C plays a mixed strategy $x$ first, then it is the best for R to play a pure strategy $i$. Value of game is $\sup_y \min_{i \in [m]} M(i, y)$. 


Theorem (Von Neumann (1928), Nash’s Equilibrium)

\[ \inf_x \max_{j \in [n]} M(x, j) = \sup_{y} \min_{i \in [m]} M(i, y). \]

Coro. \[ \inf_x \sup_y M(x, y) = \sup_y \inf_x M(x, y). \]

Coro. There are mixed strategies \( x^* \) and \( y^* \) satisfying
\[ M(x, y^*) \geq M(x^*, y^*), \forall x \] and
\[ M(x^*, y) \leq M(x^*, y^*), \forall y. \]

Proof.
- \( V := \inf_x \sup_y M(x, y) = \sup_y \inf_x M(x, y) \)
- \( x^* \): the strategy \( x \) that minimizes \( \sup_y M(x, y) \)
- \( y^* \): the strategy \( y \) that maximizes \( \inf_x M(x, y) \)
- \( M(x^*, y^*) \leq V, M(x^*, y^*) \geq V \implies M(x^*, y^*) = V \)
- \( M(x^*, y) \leq V, \forall y \) and \( M(x, y^*) \geq V, \forall x. \)
As long as the first player can play a mixed strategy, then he will not be at a disadvantage.

If both players can play mixed strategies, then they do not need to know the strategy of the other player.

**Def.** $\inf_x \sup_y M(x, y) = \sup_y \inf_x M(x, y)$ is called the **value** of the game. The two strategies $x^*$ and $y^*$ in the corollary are called the **optimum strategies** for R and C respectively.

**Theorem** (Von Neumann (1928), Nash’s Equilibrium)

$$\inf_x \max_{j \in [n]} M(x, j) = \sup_y \min_{i \in [m]} M(i, y).$$

Can be proved by LP duality.
LP for Row Player

\[
\begin{align*}
\text{min} & \quad R \\
& \sum_{i=1}^{m} x_i = 1 \\
& R - \sum_{i=1}^{m} M_{ij} x_i \geq 0 \quad \forall j \in [n] \\
& x_i \geq 0 \quad \forall i \in [m]
\end{align*}
\]

LP for Column Player

\[
\begin{align*}
\text{max} & \quad C \\
& \sum_{j=1}^{n} y_j = 1 \\
& C - \sum_{j=1}^{n} M_{ij} y_j \leq 0 \quad \forall i \in [m] \\
& y_j \geq 0 \quad \forall j \in [n]
\end{align*}
\]

- The two LPs are dual to each other.

| \( x_i, i \in [m] \) | primal variable (\( \in \mathbb{R}_{\geq 0} \)) | dual constraint (\( \leq \)) |
| \( y_j, j \in [n] \) | dual variable (\( \in \mathbb{R}_{\geq 0} \)) | primal constraint (\( \geq \)) |
| \( R \) | primal variable (\( \in \mathbb{R} \)) | dual constraint (\( = \)) |
| \( C \) | dual variable (\( \in \mathbb{R} \)) | primal constraint (\( = \)) |
Let $V$ be the value of the game, $x^*$ and $y^*$ be the two optimum strategies. Complementary slackness implies:

- If $x_i^* > 0$, then $M(i, y^*) = V$.
- If $y_j^* > 0$, then $M(x^*, j) = V$.

The game is called 0-sum game as the payoff for R is the negative of the payoff for C.
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LP Relaxation

\[
\begin{align*}
\text{min} & \quad \sum_{v \in V} w_v x_v \\
x_u + x_v & \geq 1 \quad \forall (u, v) \in E \\
x_v & \geq 0 \quad \forall v \in V
\end{align*}
\]

Dual LP

\[
\begin{align*}
\text{max} & \quad \sum_{e \in E} y_e \\
\sum_{e \in \delta(v)} y_e & \leq w_v \quad \forall v \in V \\
y_e & \geq 0 \quad \forall e \in E
\end{align*}
\]

- Algorithm constructs integral primal solution \(x\) and dual solution \(y\) simultaneously.
Primal-Dual Algorithm for Weighted Vertex Cover Problem

1: $x \leftarrow 0, y \leftarrow 0$, all edges said to be uncovered
2: while there exists at least one uncovered edge do
3: take such an edge $e$ arbitrarily
4: increasing $y_e$ until the dual constraint for one end-vertex $v$ of $e$ becomes tight
5: $x_v \leftarrow 1$, claim all edges incident to $v$ are covered
6: return $x$

Lemma
1. $x$ satisfies all primal constraints
2. $y$ satisfies all dual constraints
3. $P \leq 2D \leq 2D^* \leq 2 \cdot \text{opt}$
   
   \[ P := \sum_{v \in V} x_v : \text{value of } x \]
   
   \[ D := \sum_{e \in E} y_e : \text{value of } y \]
   
   $D^*$: dual LP value
Proof of $P \leq 2D$.

$$P = \sum_{v \in V} w_v x_v \leq \sum_{v \in V} x_v \sum_{e \in \delta(v)} y_e = \sum_{(u,v) \in E} y(u,v)(x_u + x_v)$$

$$\leq 2 \sum_{e \in E} y_e = 2D.$$  

- a more general framework: construct an arbitrary maximal dual solution $y$; choose the vertices whose dual constraints are tight
- $y$ is maximal: increasing any coordinate $y_e$ makes $y$ infeasible
- primal-dual algorithms do not need to solve LPs
- LPs are used in analysis only
- faster than LP-rounding algorithm in general
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2. 2-Approximation Algorithm for Weighted Vertex Cover Using Primal-Dual

3. 3-Approximation Algorithm for Uncapacitated Facility Location Problem Using Primal Dual
Uncapacitated Facility Location Problem

**Input:** \( F \): potential facilities \( C \): clients

\( d \): (symmetric) metric over \( F \cup C \)

\( (f_i)_{i \in F} \): facility opening costs

**Output:** \( S \subseteq F \), so as to minimize \( \sum_{i \in S} f_i + \sum_{j \in C} d(j, S) \)

- 1.488-approximation [Li, 2011]
- 1.463-hardness of approximation, \( 1.463 \approx \text{root of } x = 1 + 2e^{-x} \)
- \( y_i \): open facility \( i \)?
- \( x_{i,j} \): connect client \( j \) to facility \( i \)?

### Basic LP Relaxation

\[
\min \sum_{i \in F} f_i y_i + \sum_{i \in F, j \in C} d(i,j) x_{i,j}
\]

\[
\sum_{i \in F} x_{i,j} \geq 1 \quad \forall j \in C
\]

\[
x_{i,j} \leq y_i \quad \forall i \in F, j \in C
\]

\[
x_{i,j} \geq 0 \quad \forall i \in F, j \in C
\]

\[
y_i \geq 0 \quad \forall i \in F
\]

**Obs.** When \((y_i)_{i \in F}\) is determined, \((x_{i,j})_{i \in F, j \in C}\) can be determined automatically.
Basic LP Relaxation

\[
\begin{align*}
\min & \quad \sum_{i \in F} f_i y_i + \sum_{i \in F, j \in C} d(i, j) x_{i,j} \\
\sum_{i \in F} x_{i,j} & \geq 1 \quad \forall j \in C \\
x_{i,j} & \leq y_i \quad \forall i \in F, j \in C \\
x_{i,j} & \geq 0 \quad \forall i \in F, j \in C \\
y_i & \geq 0 \quad \forall i \in F
\end{align*}
\]

- LP is not of covering type
- harder to understand the dual
- consider an equivalent covering LP
- idea: treat a solution as a set of stars
- \((i, J), i \in F, J \subseteq C\): star with center \(i\) and leaves \(J\)
- \(\text{cost}(i, J) := f_i + \sum_{j \in J} d(i, j)\): cost of star \((i, J)\)
- \(x_{i,J} \in \{0, 1\}\): if star \((i, J)\) is chosen

**Equivalent LP**

\[
\begin{align*}
\min & \sum_{(i,J)} \text{cost}(i, J) \cdot x_{i,J} \\
\sum_{(i,J): j \in J} x_{i,J} & \geq 1 \quad \forall j \in C \\
x_{i,J} & \geq 0 \quad \forall (i, J)
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\max & \sum_{j \in C} \alpha_j \\
\sum_{j \in J} \alpha_j & \leq \text{cost}(j, J) \quad \forall (i, J) \\
\alpha_j & \geq 0 \quad \forall j \in C
\end{align*}
\]

- both LPs have exponential size, but the final algorithm can run in polynomial time
\[
\begin{align*}
\text{min} & \quad \sum_{(i,J)} \text{cost}(i, J) \cdot x_{i,J} \\
\sum_{(i,J): j \in J} x_{i,J} & \geq 1 \quad \forall j \in C \\
x_{i,J} & \geq 0 \quad \forall (i, J)
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad \sum_{j \in C} \alpha_j \\
\sum_{j \in J} \alpha_j & \leq \text{cost}(j, J) \quad \forall (i, J) \\
\alpha_j & \geq 0 \quad \forall j \in C
\end{align*}
\]

- \(\alpha_j\): budget of \(j\)
- dual constraints: total budget in any star is \(\leq\) its cost
- \(\Rightarrow\) opt \(\geq\) total budget = dual value
Construction of Dual Solution $\alpha$

- $\alpha_j$'s can only increase
- $\alpha$ is always feasible
- if a dual constraint becomes tight, freeze all clients in star
- unfrozen clients are called active clients

Construction of Dual Solution $\alpha$

1: $\alpha_j \leftarrow 0, \forall j \in C$
2: while exists at least one active client do
3: increase the budgets $\alpha_j$ for all active clients $j$ at uniform rate, until (at least) one new client is frozen
Construction of Dual Solution $\alpha$

- □: tight facilities; they are temporarily open
- ■: permanently closed
- $t_i$: time when facility $i$ becomes tight
- construct a bipartite graph: $(i, j)$ exists $\iff \alpha_j > d(i, j)$,
  - $\alpha_j > d(i, j)$: $j$ contributes to $i$, (solid lines)
  - $\alpha_j = d(i, j)$: $j$ does not contribute to $i$, but its budget is just enough for it to connect to $i$ (dashed lines)
  - $\alpha_j < d(i, j)$: budget of $j$ is not enough to connect to $i$
Construction of Integral Primal Solution

1: $S \leftarrow \emptyset$, all clients are unowned
2: for every temporarily open facility $i$, in increasing order of $t_i$ do
3:   if all (solid-line) neighbors of $i$ are unowned then
4:     $S \leftarrow S \cup \{i\}$, open facility $i$
5:   connect to all its neighbors to $i$
6:   let $i$ own them
7: connect unconnected clients to their nearest facilities in $S$
- $S$: set of open facilities
- $C_1$: clients that make contributions
- $C_2$: clients that do not make contributions

- $f$: total facility cost
- $c_j$: connection cost of client $j$
- $c = \sum_{j \in C} c_j$: total connection cost
- $D = \sum_{j \in C} \alpha_j$: value of $\alpha$

**Lemma**

- $f + \sum_{j \in C_1} c_j \leq \sum_{j \in C_1} \alpha_j$
- for any client $j \in C_2$, we have $c_j \leq 3\alpha_j$
Lemma

- $f + \sum_{j \in C_1} c_j \leq \sum_{j \in C_1} \alpha_j$
- For any client $j \in C_2$, we have $c_j \leq 3\alpha_j$

So, $f + c = f + \sum_{j \in C} c_j \leq 3 \sum_{j \in C} \alpha_j = 3D \leq 3 \cdot \text{opt}$.

Stronger statement:

$$3f + c = 3f + \sum_{j \in C} c_j \leq 3 \sum_{j \in C} \alpha_j = 3D \leq 3 \cdot \text{opt}.$$
Proof of $\forall j \in C_2, c_j \leq 3\alpha_j$

- at time $\alpha_j$, $j$ is frozen.
- let $i$ be the temporarily open facility it connects to
- $i \in S$: then $c_j \leq \alpha_j$. assume $i \notin S$.
- there exists a client $j'$, which made contribution to $i$, and owned by another facility $i' \in S$
- $d(j, i) \leq \alpha_j$
- $d(j', i) < \alpha_{j'}, d(j', i') < \alpha_{j'}$
- $\alpha_{j'} = t'_i \leq t_i \leq \alpha_j$
- $d(j, i') \leq d(j, i) + d(i, j') + d(j', i') \leq \alpha_j + \alpha_{j'} + \alpha_j = 3\alpha_j$