Advanced Algorithms (Fall 2023)
Primal Dual

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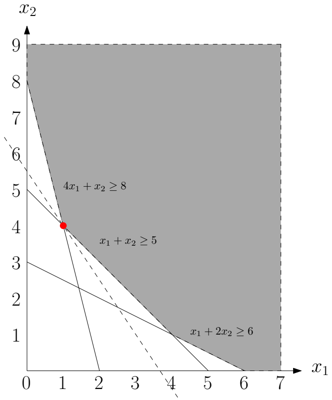
Outline

1. Duality of Linear Programming
   - Max-Flow Min-Cut Theorem Using LP Duality
   - 0-Sum Game and Nash Equilibrium
2. 2-Approximation Algorithm for Weighted Vertex Cover Using Primal-Dual
3. 3-Approximation Algorithm for Uncapacitated Facility Location Problem Using Primal Dual

$$\min \quad 7x_1 + 4x_2$$
$$x_1 + x_2 \geq 5$$
$$x_1 + 2x_2 \geq 6$$
$$4x_1 + x_2 \geq 8$$
$$x_1, x_2 \geq 0$$



- optimum solution: $x_1 = 1, x_2 = 4$
- optimum value $= 7 \times 1 + 4 \times 4 = 23$

Q: How can we give a lower bound for the linear program, without solving it?

$$\begin{aligned} \min \quad 7x_1 + 4x_2 \\ x_1 + x_2 \geq 5 \\ x_1 + 2x_2 \geq 6 \\ 4x_1 + x_2 \geq 8 \\ x_1, x_2 \geq 0 \end{aligned}$$

- $7x_1 + 4x_2 \geq 2(x_1 + x_2) + (x_1 + 2x_2) \geq 2 \times 5 + 6 = 16$
- $7x_1 + 4x_2 \geq (x_1 + x_2) + (x_1 + 2x_2) + (4x_1 + x_2) \geq 5 + 6 + 8 = 19$
- $7x_1 + 4x_2 \geq 3(x_1 + x_2) + (4x_1 + x_2) \geq 3 \times 5 + 8 = 23$

Q: How can we obtain the best (i.e., largest) lower bound using this method?

Primal LP

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & 5y_1 + 6y_2 + 8y_3 \\ & y_1 + y_2 + 4y_3 \leq 7 \\ & y_1 + 2y_2 + y_3 \leq 4 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

A general method to prove a lower bound

$$\begin{aligned} & 7x_1 + 4x_2 && (\text{if } 7 \geq y_1 + y_2 + 4y_3 \text{ and } 4 \geq y_1 + 2y_2 + y_3) \\ \geq\ & y_1(x_1 + x_2) + y_2(x_1 + 2x_2) + y_3(4x_1 + x_2) && (\text{if } y_1, y_2, y_3 \geq 0) \\ \geq\ & 5y_1 + 6y_2 + 8y_3 \end{aligned}$$

- to achieve the largest lower bound: maximize $5y_1 + 6y_2 + 8y_3$

$$
\begin{aligned}
\min \quad & 7x_1 + 4x_2 \\
x_1 + x_2 &\geq 5 \\
x_1 + 2x_2 &\geq 6 \\
4x_1 + x_2 &\geq 8 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & 5y_1 + 6y_2 + 8y_3 \\
y_1 + y_2 + 4y_3 &\leq 7 \\
y_1 + 2y_2 + y_3 &\leq 4 \\
y_1, y_2, y_3 &\geq 0
\end{aligned}
$$

$$
A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 4 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} \quad c = \begin{pmatrix} 7 \\ 4 \end{pmatrix}
$$

$$
\begin{aligned}
\min \quad & c^{\mathrm{T}} x \\
Ax &\geq b \\
x &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & b^{\mathrm{T}} y \\
A^{\mathrm{T}} y &\leq c \\
y &\geq 0
\end{aligned}
$$

Primal LP

$$\min \quad c^{\mathrm{T}}x$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^{\mathrm{T}}y$$
$$A^{\mathrm{T}}y \leq c$$
$$y \geq 0$$

Relationships

Primal LP	dual LP
variables	constraints
constraints	variables
obj. coefficients	RHS constants
RHS constants	obj. coefficients

More Relationships

Primal LP	Dual LP
variable in $\mathbb{R}$	equlities
equlities	variable in $\mathbb{R}$

Primal LP

$$\min \quad c^{\mathrm{T}} x$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^{\mathrm{T}} y$$
$$A^{\mathrm{T}} y \leq c$$
$$y \geq 0$$

- $P :=$ value of primal LP
- $D :=$ value of dual LP

Theorem (Weak Duality Theorem)
$D \leq P$.

Proof.

- x: an arbitrary solution to Primal LP
- y: an arbitrary solution to Dual LP
- $b^{\mathrm{T}} y \leq (Ax)^{\mathrm{T}} y = x^{\mathrm{T}} A^{\mathrm{T}} y \leq x^{\mathrm{T}} c = c^{\mathrm{T}} x$.

□

Theorem (Strong Duality Theorem) $D = P$.

Proof of Strong Duality Theorem

Lemma (Variant of Farkas Lemma) $Ax \le b, x \ge 0$ is infeasible, if and only if $y^{\mathrm{T}}A \ge 0, y^{\mathrm{T}}b < 0, y \ge 0$ is feasible.

- $\forall \epsilon > 0, \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} x \le \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix}, x \ge 0$ is infeasible
- There exists $y \in \mathbb{R}^m_{\ge 0}, \alpha \ge 0$, such that $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} \ge 0$, $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix} < 0$
- we can prove $\alpha \ne 0$; assume $\alpha = 1$
- $-y^{\mathrm{T}}A + c^{\mathrm{T}} \ge 0, -y^{\mathrm{T}}b + P - \epsilon < 0 \iff A^{\mathrm{T}}y \le c, b^{\mathrm{T}}y > P - \epsilon$
- $\forall \epsilon > 0, D > P - \epsilon \quad \implies \quad D = P$ (since $D \le P$) □

- duality is mutual: the dual of the dual of an LP is the LP itself.

Primal LP

$$\max \quad b^{\mathrm{T}} y$$
$$A^{\mathrm{T}} y \leq c$$
$$y \geq 0$$

Dual LP

$$\min \quad c^{\mathrm{T}} x$$
$$Ax \geq b$$
$$x \geq 0$$

- Duality theorem holds when one LP is infeasible:
- Minimization LP is infeasible $\implies$ value $= \infty$ $\iff$ dual LP value $= \infty$ $\implies$ feasible region of dual LP is unbounded

Complementary Slackness

Primal LP

$$\begin{aligned} \min \quad & c^{\mathrm{T}} x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & b^{\mathrm{T}} y \\ & A^{\mathrm{T}} y \leq c \\ & y \geq 0 \end{aligned}$$

- x^* and y^*: optimum primal and dual solutions
- $D = b^{\mathrm{T}} y^* \leq (Ax^*)^{\mathrm{T}} y^* = (x^*)^{\mathrm{T}} A^{\mathrm{T}} y^* \leq (x^*)^{\mathrm{T}} c = c^{\mathrm{T}} x^* = P$.
- $P = D$: all the inequlaities hold with equalities.

Complementary Slackness

- $y_i^* > 0 \implies \sum_j a_{ij} x_j^* = b_i$.
- $x_j^* > 0 \implies \sum_i a_{ij} y_i^* = c_j$.

Simple Example for Duality: Brewery problem

Beverage	Corn (pounds)	Hops (pounds)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
Constraint	480	160	1190	

Primal LP

$$\begin{aligned} \max \quad & 13x + 23y \\ & 5x + 15y \le 480 \\ & 4x + 4y \le 160 \\ & 35x + 20y \le 1190 \\ & x, y \ge 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & 480\alpha + 160\beta + 1190\gamma \\ & 5\alpha + 4\beta + 35\gamma \ge 13 \\ & 15\alpha + 4\beta + 20\gamma \ge 23 \\ & \alpha, \beta, \gamma \ge 0 \end{aligned}$$

α, β, γ: the value of 1 pound of corn, hops and malt respectively.

$$\begin{aligned} \min \quad & 480\alpha + 160\beta + 1190\gamma \\ & 5\alpha + 4\beta + 35\gamma \geq 13 \\ & 15\alpha + 4\beta + 20\gamma \geq 23 \\ & \alpha, \beta, \gamma \geq 0 \end{aligned}$$

Covering LP

- $\min\ c^{\mathrm{T}}x$, s.t. $Ax \geq b, x \geq 0$, A, b, c are non-negative
- increasing values of variables can not make the solution feasible

$$\begin{aligned} \max \quad & 13x + 23y \\ & 5x + 15y \leq 480 \\ & 4x + 4y \leq 160 \\ & 35x + 20y \leq 1190 \\ & x, y \geq 0 \end{aligned}$$

Packing LP

- $\max c^{\mathrm{T}}x$, s.t. $Ax \leq b, x \geq 0$, A, b, c are non-negative
- decreasing values of variables (still guarnateeing the non-negativity) can not make the solution infeasible

The dual of a covering LP is a packing LP, and vice versa.

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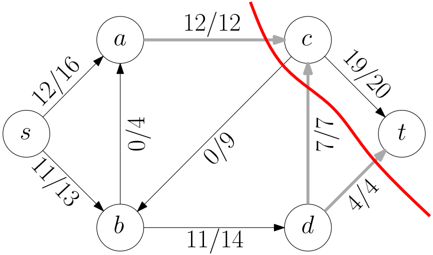
Maximum Flow Problem

Input: flow network $(G = (V, E), c, s, t)$

Output: maximum value of a s-t flow f



LP for Maximum Flow

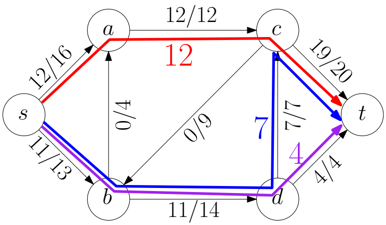
$$\max \sum_{e \in \delta^{\text{in}}(t)} x_e$$

$$x_e \leq c_e \qquad \forall e \in E$$

$$\sum_{e \in \delta^{\text{out}}(v)} x_e - \sum_{e \in \delta^{\text{in}}(v)} x_e = 0 \qquad \forall v \in V \setminus \{s, t\}$$

$$x_e \geq 0 \qquad \forall e \in E$$

An Equivalent Packing LP



- $\mathcal{P}$: the set of all simple paths from s to t
- $f_P, P \in \mathcal{P}$: the flow on P

$$\max \quad \sum_{P \in \mathcal{P}} f_P$$

$$\sum_{P \in \mathcal{P}: e \in P} f_P \leq c_e \qquad \forall e \in E$$

$$f_P \geq 0 \qquad \forall P \in \mathcal{P}$$

$$\min \quad \sum_{e \in E} c_e y_e$$

$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$

$$y_e \geq 0 \qquad \forall e \in E$$

- dual constraints: the shortest s-t path w.r.t weights y has length ≥ 1

Dual LP

$$\min \quad \sum_{e\in E} c_e y_e$$

$$\sum_{e\in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$

$$y_e \geq 0 \qquad \forall e \in E$$

Theorem The optimum value can be attained at an integral point y.

Maximum Flow Minimum Cut Theorem The value of the maximum flow equals the value of the minimum cut.

Proof of Theorem.

- Given any optimum y, let d_v be the length of shortest path from s to v, for every $v \in V$. $\quad d_s = 0, d_t = 1$
- Randomly choose $\theta \in (0,1)$, and output cut $(S := \{v : d_v \leq \theta\}, T := \{v : d_v > \theta\})$
- Lemma: $\mathbb{E}[\text{cut value of}(S,T)] \leq \sum_{e\in E} c_e y_e$
- Any cut (S,T) in the support is optimum $\square$

$$\max \quad \sum_{P \in \mathcal{P}} f_P$$
$$\sum_{P \in \mathcal{P}: e \in P} f_P \leq c_e \qquad \forall e \in E$$
$$f_P \geq 0 \qquad \forall P \in \mathcal{P}$$

$$\min \quad \sum_{e \in E} c_e y_e$$
$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$
$$y_e \geq 0 \qquad \forall e \in E$$

- pros of new LP: it is a packing LP, dual is a covering LP, easier to understand and analyze
- cons of new LP: exponential size, can not be solved directly
  - when we only need to do non-algorithmic analysis
  - ellipsoid method with separation oracle can solve some exponential size LP

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0-Sum Game

Input: a payoff matrix $M \in \mathbb{R}^{m\times n}, m, n \geq 1$,
two players: row player R, column player C

Output: R plays a row $i \in [m]$, C plays a column $j \in [n]$
payoff of game is M_{ij}
R wants to minimize M_{ij}, C wants to maximize M_{ij}

Rock-Scissor-Paper Game

payoff	R	S	P
R	0	-1	1
S	1	0	- 1
P	-1	1	0

- game depends on who plays first

By allowing mixed strategies, each player has a best strategy, regardless of who plays first

	row player R	column player C
pure strategy	row $i \in [m]$	column $j \in [n]$
mixed strategy	distribution x over $[m]$ $x \in [0,1]^m, \sum_{i=1}^m x_i = 1$	distribution y over $[n]$ $y \in [0,1]^n, \sum_{j=1}^n y_j = 1$

$$M(x,y) := \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j M_{ij}$$

$$M(x,j) := \sum_{i=1}^{m} x_i M_{ij}, \qquad M(i,y) := \sum_{j=1}^{n} y_j M_{ij}$$

- If R plays a mixed strategy y first, then it is the best for C to play a pure strategy j. Value of game is $\inf_x \max_{j\in[n]} M(x,j)$.
- If C plays a mixed strategy x first, then it is the best for R to play a pure strategy i. Value of game is $\sup_y \min_{i\in[m]} M(i,y)$.

Theorem (Von Neumann (1928), Nash's Equilibrium)

$$\inf_x \max_{j\in[n]} M(x,j) = \sup_y \min_{i\in[m]} M(i,y).$$

Coro. $\inf_x \sup_y M(x,y) = \sup_y \inf_x M(x,y)$.

Coro. There are mixed strategies x^* and y^* satisfying $M(x,y^*) \geq M(x^*,y^*), \forall x$ and $M(x^*,y) \leq M(x^*,y^*), \forall y$.

Proof.

- $V := \inf_x \sup_y M(x,y) = \sup_y \inf_x M(x,y)$
- x^*: the strategy x that minimizes $\sup_y M(x,y)$
- y^*: the strategy y that maximizes $\inf_x M(x,y)$
- $M(x^*,y^*) \leq V, M(x^*,y^*) \geq V \implies M(x^*,y^*) = V$
- $M(x^*,y) \leq V, \forall y$ and $M(x,y^*) \geq V, \forall x$. □

- As long as the first player can play a mixed strategy, then he will not be at a disadvantage.
- If both players can play mixed strategies, then they do not need to know the strategy of the other player.

Def. $\inf_x \sup_y M(x,y) = \sup_y \inf_x M(x,y)$ is called the value of the game. The two strategies x^* and y^* in the corollary are called the optimum strategies for R and C respectively.

Theorem (Von Neumann (1928), Nash's Equilibrium)

$$\inf_x \max_{j\in[n]} M(x,j) = \sup_y \min_{i\in[m]} M(i,y).$$

- Can be proved by LP duality.

LP for Row Player

$$\min \quad R$$
$$\sum_{i=1}^{m} x_i = 1$$
$$R - \sum_{i=1}^{m} M_{ij} x_i \geq 0 \quad \forall j \in [n]$$
$$x_i \geq 0 \quad \forall i \in [m]$$

LP for Column Player

$$\max \quad C$$
$$\sum_{j=1}^{n} y_j = 1$$
$$C - \sum_{j=1}^{n} M_{ij} y_j \leq 0 \quad \forall i \in [m]$$
$$y_j \geq 0 \quad \forall j \in [n]$$

- The two LPs are dual to each other.

$x_i, i \in [m]$	primal variable ($\in \mathbb{R}_{\geq 0}$)	dual constraint ($\leq$)
$y_j, j \in [n]$	dual variable ($\in \mathbb{R}_{\geq 0}$)	primal constraint ($\geq$)
R	primal variable ($\in \mathbb{R}$)	dual constraint ($=$)
C	dual variable ($\in \mathbb{R}$)	primal constraint ($=$)

- Let V be the value of the game, x^* and y^* be the two optimum strategies. Complementrary slackness implies:
  - If $x_i^* > 0$, then $M(i, y^*) = V$.
  - If $y_j^* > 0$, then $M(x^*, j) = V$.
- The game is called 0-sum game as the payoff for R is the negative of the payoff for C.

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LP Relaxation

$$\min \quad \sum_{v \in V} w_v x_v$$

$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$

$$x_v \geq 0 \qquad \forall v \in V$$

Dual LP

$$\max \quad \sum_{e \in E} y_e$$

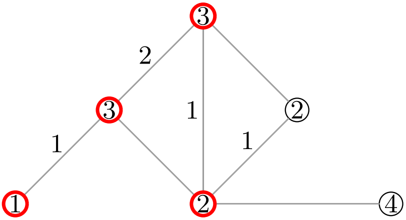
$$\sum_{e \in \delta(v)} y_e \leq w_v \qquad \forall v \in V$$

$$y_e \geq 0 \qquad \forall e \in E$$

- Algorithm constructs integral primal solution x and dual solution y simultaneously.

Primal-Dual Algorithm for Weighted Vertex Cover Problem

$x \leftarrow 0, y \leftarrow 0$, all edges said to be uncovered
while there exists at least one uncovered edge **do**
    take such an edge e arbitrarily
    increasing y_e until the dual constraint for one end-vertex v of e becomes tight
    $x_v \leftarrow 1$, claim all edges incident to v are covered
return x



Lemma

1. x satisfies all primal constraints
2. y satisfies all dual constraints
3. $P \le 2D \le 2D^* \le 2 \cdot \text{opt}$

   $P := \sum_{v \in V} x_v$: value of x

   $D := \sum_{e \in E} y_e$: value of y

   D^* : dual LP value

Proof of $P \le 2D$.

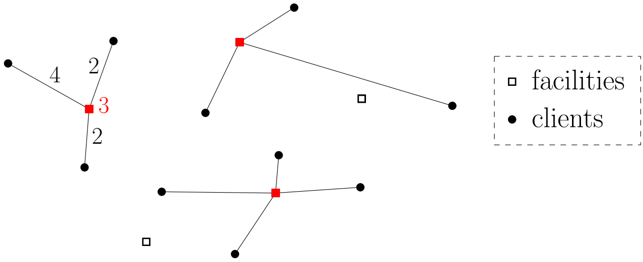
$$P = \sum_{v\in V} w_v x_v \le \sum_{v\in V} x_v \sum_{e\in\delta(v)} y_e = \sum_{(u,v)\in E} y_{(u,v)}(x_u + x_v)$$

$$\le 2\sum_{e\in E} y_e = 2D. \qquad \square$$

- a more general framework: construct an arbitrary maximal dual solution y; choose the vertices whose dual constraints are tight
- y is maximal: increasing any coordinate y_e makes y infeasible

- primal-dual algorithms do not need to solve LPs
- LPs are used in analysis only
- faster than LP-rounding algorithm in general

Outline



Uncapacitated Facility Location Problem

Input: F: pontential facilities C: clients

d: (symmetric) metric over $F \cup C$ $(f_i)_{i \in F}$: facility opening costs

Output: $S \subseteq F$, so as to minimize $\sum_{i \in S} f_i + \sum_{j \in C} d(j, S)$

- 1.488-approximation [Li, 2011]
- 1.463-hardness of approximation, 1.463 $\approx$ root of $x = 1 + 2e^{-x}$

- y_i: open facility i?
- $x_{i,j}$: connect client j to facility i?

Basic LP Relaxation

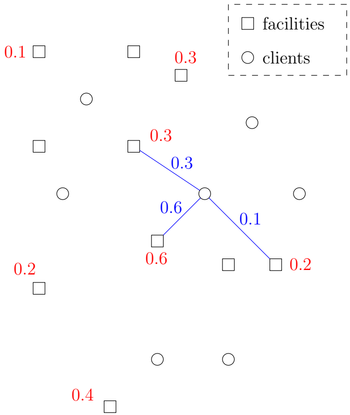
$$\min \quad \sum_{i\in F} f_i y_i + \sum_{i\in F, j\in C} d(i,j) x_{i,j}$$

$$\sum_{i\in F} x_{i,j} \geq 1 \qquad \forall j \in C$$

$$x_{i,j} \leq y_i \qquad \forall i \in F, j \in C$$

$$x_{i,j} \geq 0 \qquad \forall i \in F, j \in C$$

$$y_i \geq 0 \qquad \forall i \in F$$



Obs. When $(y_i)_{i\in F}$ is determined, $(x_{i,j})_{i\in F, j\in C}$ can be determined automatically.

Basic LP Relaxation

$$\min \quad \sum_{i\in F} f_i y_i + \sum_{i\in F, j\in C} d(i,j) x_{i,j}$$

$$\sum_{i\in F} x_{i,j} \geq 1 \qquad \forall j \in C$$

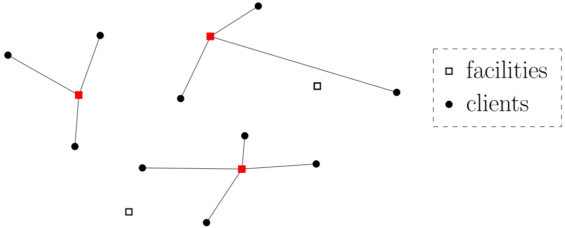
$$x_{i,j} \leq y_i \qquad \forall i \in F, j \in C$$

$$x_{i,j} \geq 0 \qquad \forall i \in F, j \in C$$

$$y_i \geq 0 \qquad \forall i \in F$$

- LP is not of covering type
- harder to understand the dual

- consider an equivalent covering LP
- idea: treat a solution as a set of stars



- $(i, J), i \in F, J \subseteq C$: star with center i and leaves J
- $\mathrm{cost}(i, J) := f_i + \sum_{j \in J} d(i, j)$: cost of star (i, J)
- $x_{i,J} \in \{0, 1\}$: if star (i, J) is chosen

Equivalent LP

$$\min \quad \sum_{(i,J)} \mathrm{cost}(i, J) \cdot x_{i,J}$$

$$\sum_{(i,J): j \in J} x_{i,J} \geq 1 \qquad \forall j \in C$$

$$x_{i,J} \geq 0 \qquad \forall (i, J)$$

Dual LP

$$\max \quad \sum_{j \in C} \alpha_j$$

$$\sum_{j \in J} \alpha_j \leq \mathrm{cost}(j, J) \qquad \forall (i, J)$$

$$\alpha_j \geq 0 \qquad \forall j \in C$$

- both LPs have exponential size, but the final algorithm can run in polynomial time

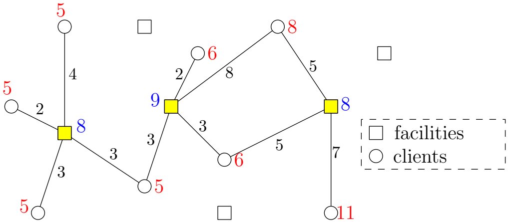
$$\begin{aligned} \min \quad & \sum_{(i,J)} \text{cost}(i,J) \cdot x_{i,J} \\ \sum_{(i,J): j \in J} & x_{i,J} \geq 1 \qquad \forall j \in C \\ & x_{i,J} \geq 0 \qquad \forall (i,J) \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{j \in C} \alpha_j \\ \sum_{j \in J} \alpha_j \leq \text{cost}(j,J) & \qquad \forall (i,J) \\ \alpha_j \geq 0 & \qquad \forall j \in C \end{aligned}$$

- α_j: budget of j
- dual constraints: total budget in any star is $\leq$ its cost
- $\implies$ opt $\geq$ total budget $=$ dual value

Construction of Dual Solution α

- α_j's can only increase
- α is always feasible
- if a dual constraint becomes tight, freeze all clients in star
- unfrozen clients are called active clients

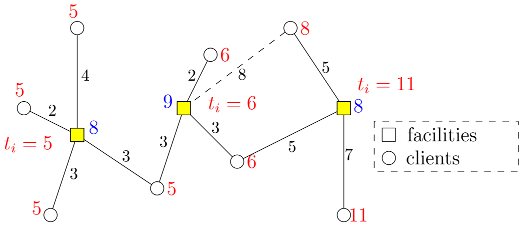


Construction of Dual Solution α

1: $\alpha_j \leftarrow 0, \forall j \in C$
2: **while** exists at least one active client **do**
3: increase the budgets α_j for all active clients j at uniform rate, until (at least) one new client is frozen

Construction of Dual Solution α

- ■: tight facilities; they are temporarily open
- □: pemanently closed
- t_i: time when facility i becomes tight
- construct a bipartite graph: (i, j) exists $\iff \alpha_j > d(i, j)$,



$\alpha_j > d(i, j)$: j contributes to i, (solid lines)

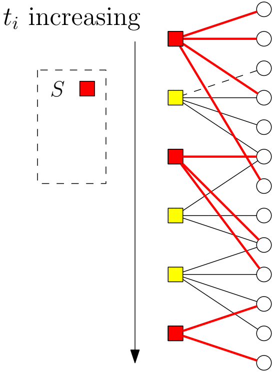
$\alpha_j = d(i, j)$: j does not contribute to i, but its budget is just enough for it to connect to i (dashed lines)

$\alpha_j < d(i, j)$: budget of j is not enough to connect to i

Construction of Integral Primal Solution

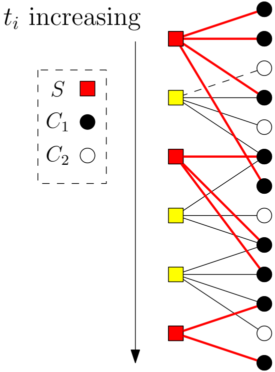
Construction of Integral Primal Solution

1: $S \leftarrow \emptyset$, all clients are unowned
2: **for** every temporarily open facility i, in increasing order of t_i **do**
3: **if** all (solid-line) neighbors of i are unowned **then**
4: $S \leftarrow S \cup \{i\}$, open facility i
5: connect to all its neighbors to i
6: let i own them
7: connect unconnected clients to their nearest facilities in S



- S: set of open facilities
- C_1: clients that make contributions
- C_2: clients that do not make contributions

- f: total facillity cost
- c_j: connection cost of client j
- $c = \sum_{j \in C} c_j$: total connection cost
- $D = \sum_{j \in C} \alpha_j$: value of α



Lemma

- $f + \sum_{j \in C_1} c_j \leq \sum_{j \in C_1} \alpha_j$
- for any client $j \in C_2$, we have $c_j \leq 3\alpha_j$

Lemma

- $f + \sum_{j \in C_1} c_j \leq \sum_{j \in C_1} \alpha_j$
- for any client $j \in C_2$, we have $c_j \leq 3\alpha_j$

- So, $f + c = f + \sum_{j \in C} c_j \leq 3 \sum_{j \in C} \alpha_j = 3D \leq 3 \cdot \mathrm{opt}.$

- stronger statement:

$$3f + c = 3f + \sum_{j \in C} c_j \leq 3 \sum_{j \in C} \alpha_j = 3D \leq 3 \cdot \mathrm{opt}.$$

Proof of $\forall j \in C_2, c_j \le 3\alpha_j$

- at time α_j, j is frozen.
- let i be the temporarily open facility it connects to
- $i \in S$: then $c_j \le \alpha_j$. assume $i \notin S$.
- there exists a client j', which made contribution to i, and owned by another facility $i' \in S$
- $d(j,i) \le \alpha_j$
- $d(j',i) < \alpha_{j'}, d(j',i') < \alpha_{j'}$
- $\alpha_{j'} = t_i' \le t_i \le \alpha_j$
- $d(j,i') \le d(j,i) + d(i,j') + d(j',i') \le \alpha_j + \alpha_j + \alpha_j = 3\alpha_j$