Advanced Algorithms (Fall 2026)

Linear Programming: Foundations

Lecturers: 尹一通，刘景铖，栗师

Nanjing University

Outline

1. Modeling and LP Formulations
2. Polyhedral Geometry
3. Algorithms for Linear Programming
4. Combinatorial Polytopes and Integrality

Typical Combinatorial Optimization Problem

Given a ground set $[n]$, feasible sets $\mathcal{S} \subseteq 2^{[n]}$ (often implicit), and weights w_i, find $S \in \mathcal{S}$ minimizing or maximizing $\sum_{i \in S} w_i$.

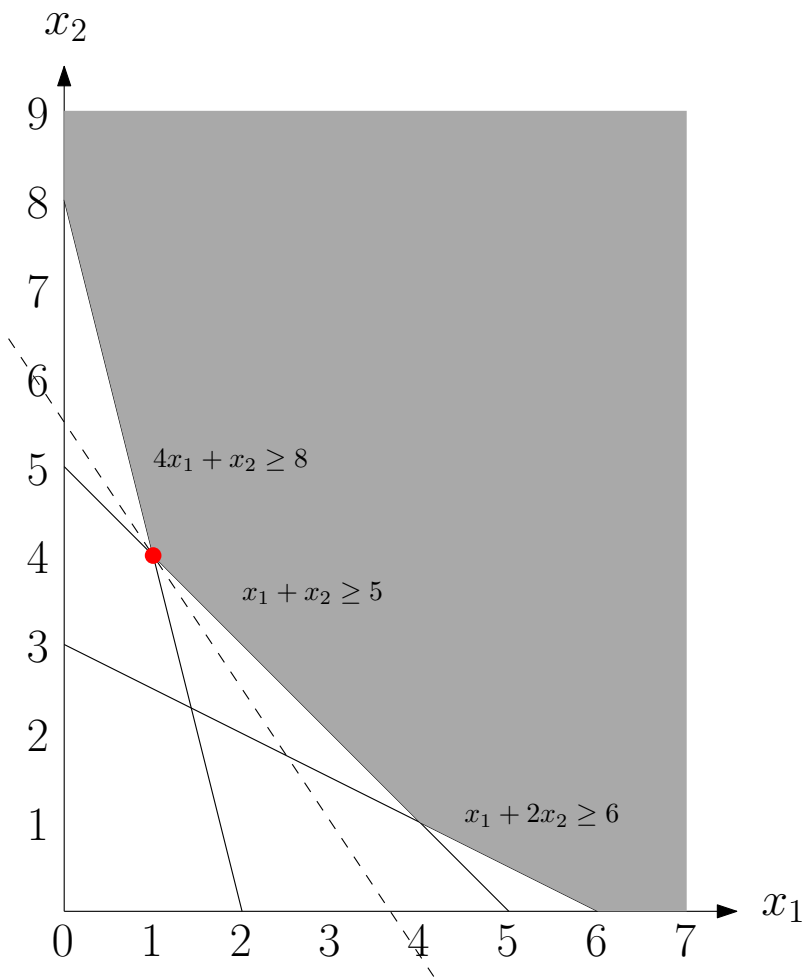
Examples

Shortest path, minimum spanning tree, maximum matching, maximum independent set, knapsack.

- Combinatorial optimization $\Longleftrightarrow$ integer program (IP). Relax integrality to obtain a linear program (LP).
- In general, integer programming is NP-hard; linear programming is in P.
- If the LP is integral, it yields an exact algorithm.
- Otherwise, solve the LP and round its fractional solution to obtain an approximation algorithm.

Linear Programming (LP), Linear Program (LP)

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$



- optimum solution:
  $x_1 = 1, x_2 = 4$
- optimum value $=$
  $7 \times 1 + 4 \times 4 = 23$

Standard Form of Linear Programs

$$
\begin{aligned}
\min \quad & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
& a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n \geq b_1 \\
& a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,n} x_n \geq b_2 \\
& \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
& a_{m,1} x_1 + a_{m,2} x_2 + \cdots + a_{m,n} x_n \geq b_m \\
& x_1, x_2, \cdots, x_n \geq 0
\end{aligned}
$$

- n: number of variables $\qquad$ m: number of constraints
- Other considerations: $\leq$ constraints? equalities?
- variables can be negative? maximization problem?

Standard Form of Linear Programs

$$x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \qquad c := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n,$$

$$A := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{m \times n}, \quad b := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m.$$

$$
\begin{aligned}
\min \quad & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
& a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n \geq b_1 \\
& a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,n} x_n \geq b_2 \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \\
& a_{m,1} x_1 + a_{m,2} x_2 + \cdots + a_{m,n} x_n \geq b_m \\
& x_1, x_2, \cdots, x_n \geq 0
\end{aligned}
$$

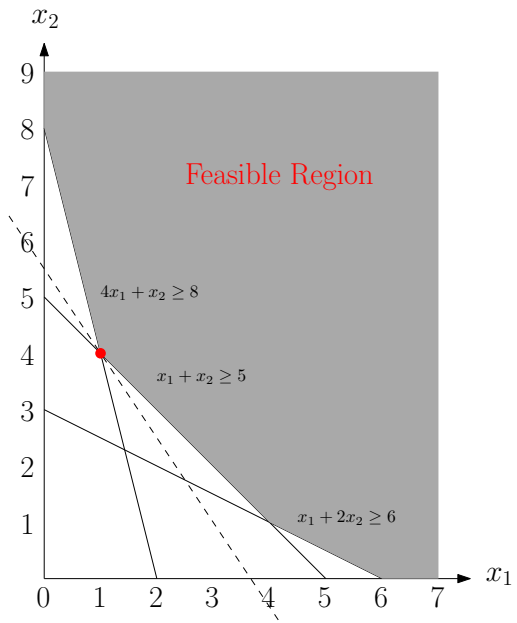
Standard Form of Linear Program

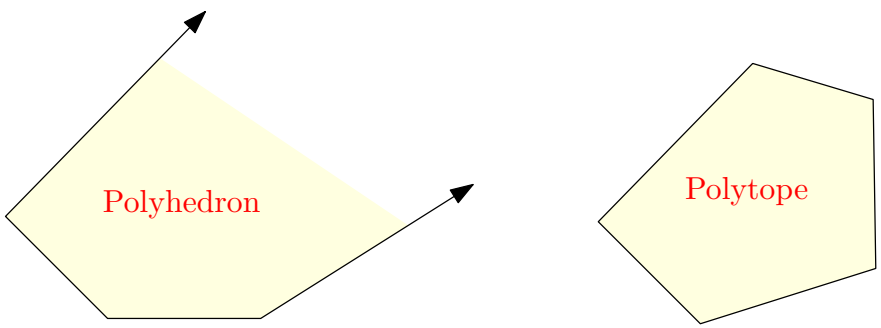
$$
\begin{aligned}
\min \quad & c^{\mathrm{T}} x \\
& Ax \geq b \\
& x \geq 0
\end{aligned}
$$

- $\geq$: coordinate-wise greater than or equal to

Preliminaries

- feasible region: the set of x's satisfying $Ax \geq b, x \geq 0$
- a polyhedron is the intersection of finite number of closed half-spaces
- so, feasible region is a polyhedron
- if every coordinate has an upper and lower bound in the polyhedron, then the polyhedron is a polytope





Applications of Linear Programming

- domain: computer science, mathematics, operations research, economics
- types of problems: transportation, scheduling, clustering, network routing, resource allocation, facility location

Research Directions

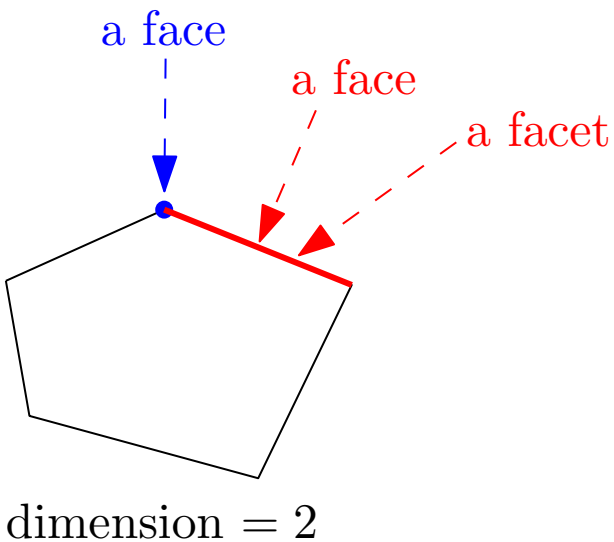
- polynomial time exact algorithm
- polynomial time approximation algorithm
- sub-routines for the branch-and-bound method for integer programming
- other algorithmic models: online algorithm, distributed algorithms, dynamic algorithms, fast algorithms

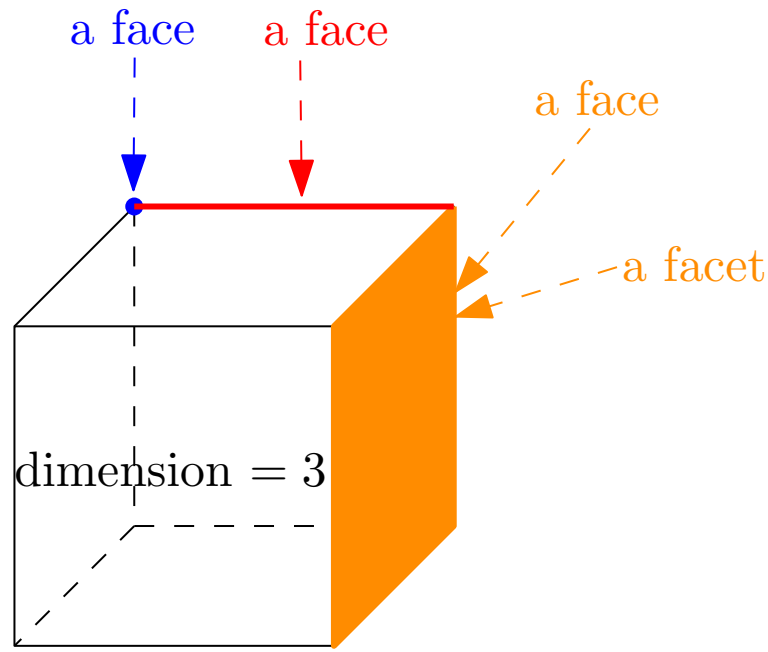
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4. Combinatorial Polytopes and Integrality

Given a polytope $\mathcal{P} \subseteq \mathbb{R}^n$:

- The dimension of $\mathcal{P}$ is n minus the maximum number of linearly-independent equalities satisfied by all points in $\mathcal{P}$.
- Assume the linear inequality $a^{\mathrm{T}}x \leq b$ holds for every $x \in \mathcal{P}$, and some $x \in \mathcal{P}$ satisfies $a^{\mathrm{T}}x = b$. Then $\{x \in \mathcal{P} : a^{\mathrm{T}}x = b\}$ is said to be a face of $\mathcal{P}$.
- A face of $\mathcal{P}$ is also a polytope.
- Assume the dimension of $\mathcal{P}$ is d. Then a face of $\mathcal{P}$ of dimension $d-1$ is said to be a facet of $\mathcal{P}$.



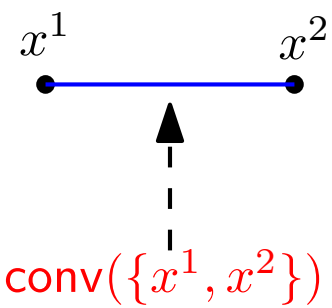


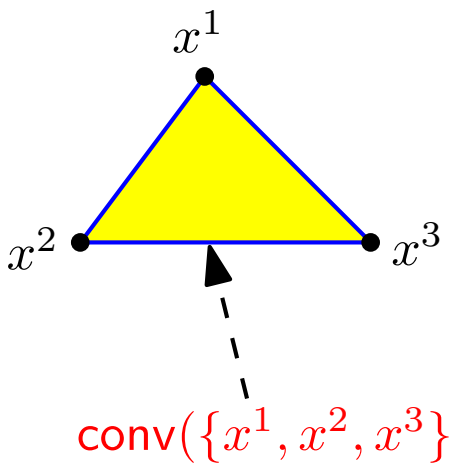
Preliminaries

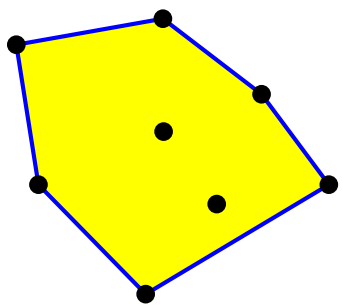
- x is a convex combination of $\{x^{(1)}, x^{(2)}, \cdots, x^{(t)}\}$ if the following condition holds: there exist $\lambda_1, \lambda_2, \cdots, \lambda_t \in [0, 1]$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_t = 1, \qquad \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_t x^{(t)} = x$$

- the convex hull of a set S of points in $\mathbb{R}^n$, denoted as conv(S), is the set of convex combinations of S



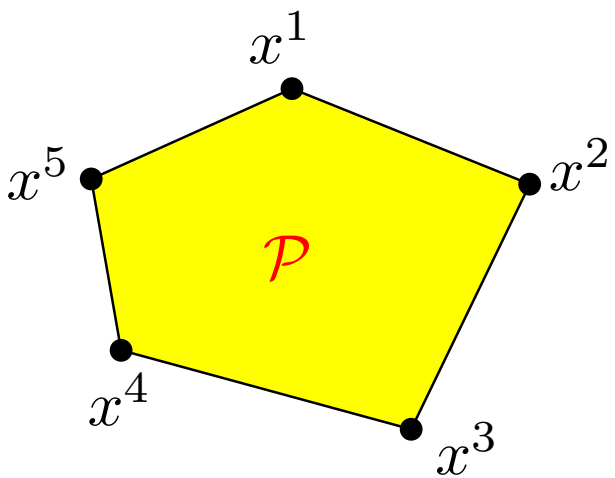




Terminology and Preliminaries

- let $\mathcal{P}$ be polytope, $x \in \mathcal{P}$. If there are no other points $x', x'' \in \mathcal{P}$ such that x is a convex combination of x' and x'', then x is called a vertex/extreme point of $\mathcal{P}$

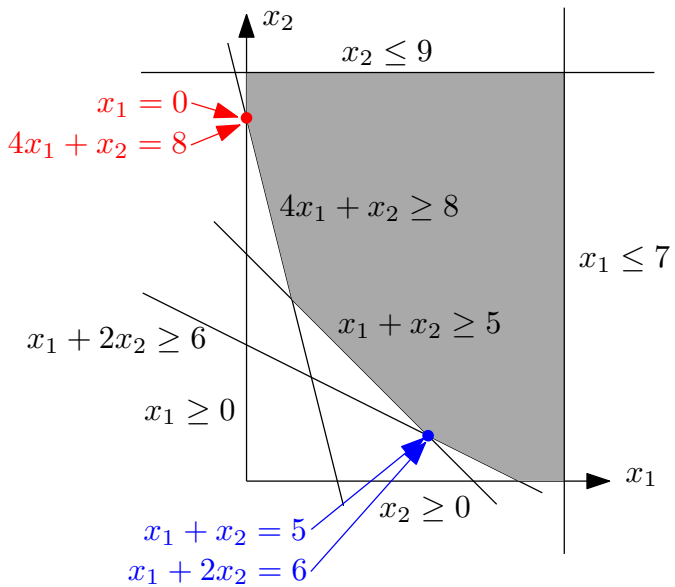
Lemma A polytope has finite number of vertices, and it is the convex hull of the vertices.



$$\mathcal{P} = \text{conv}(\{x^1, x^2, x^3, x^4, x^5\})$$

Terminology and Preliminaries

Lemma Let $x \in \mathbb{R}^n$ be a vertex of a polytope. Then, there are n constraints in the definition of the polytope, such that x is the unique solution to the linear system obtained from the n constraints by replacing inequalities to equalities.



Lemma If the feasible region of a linear program is a nonempty polytope, then the optimum value can be attained at some vertex of the polytope.

Special cases (for minimization linear programs):

- if feasible region is empty, then its value is ∞
- if the feasible region is unbounded, then its value can be $-\infty$

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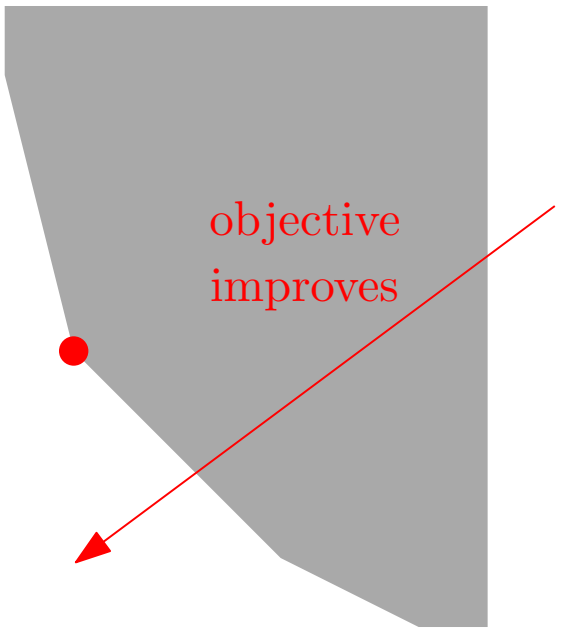
Algorithms for Linear Programming

algorithm	running time	practice
Simplex Method	exponential time	fast
Ellipsoid Method	polynomial time	slow
Interior Point Method	polynomial time	fast

Simplex Method

- [Dantzig, 1947]

- move from one vertex to another, so as to improve the objective
- repeat until we reach an optimum vertex



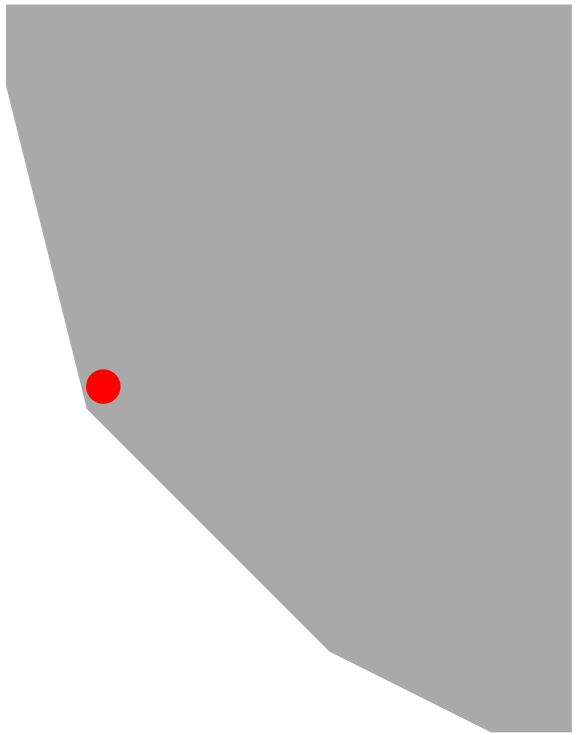
- the number of iterations might be exponentially large; but algorithm runs fast in practice
- [Spielman-Teng,2002]: smoothed analysis

Interior Point Method

- [Karmarkar, 1984]

- keep the solution inside the polytope
- design penalty function so that the solution is not too close to the boundary
- the final solution will be arbitrarily close to the optimum solution

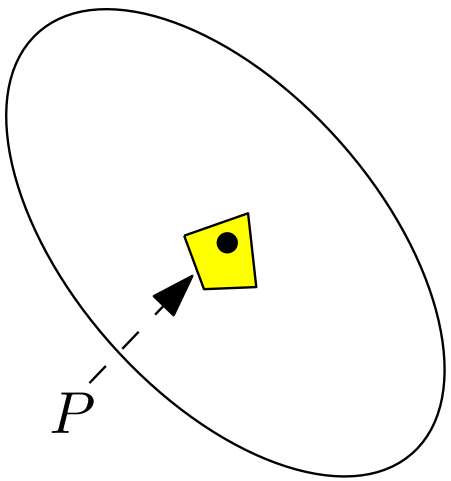
- polynomial time



Ellipsoid Method

- [Khachiyan, 1979]
- used to decide if the feasible region is empty or not

- maintain an ellipsoid that contains the feasible region
- query a separation oracle if the center of ellipsoid is in the feasible region:
  - yes: then the feasible region is not empty
  - no: cut the ellipsoid in half, find smaller ellipsoid to enclose the half-ellipsoid, and repeat



- polynomial time, but impractical

Q: The exact running time of these algorithms?

- it depends on many parameters: #variables, #constraints, #(non-zero coefficients), magnitude of integers
- precision issue

Open Problem

Can linear programming be solved in strongly polynomial time?

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Typical Combinatorial Optimization Problem

Input: $[n]$: ground set

$\mathcal{S}$: feasible sets: a family of subsets of $[n]$, often implicitly given

$w_i, i \in [n]$: values/costs of elements

Output: the set $S \in \mathcal{S}$ with the minimum/maximum $w(S) := \sum_{i \in S} w_i$

Def. For any $S \subseteq [n]$, we use $\chi^S \in \{0,1\}^{[n]}$ to denote the indicator vector for S:

$$\chi_i^S = \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S \end{cases}$$

polytope of interest: $\mathcal{P} = \text{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

Examples

Bipartite Matching Polytope

- Given bipartite graph $G = (L \cup R, E)$
- $\mathcal{P}_{\mathrm{BM}} := \mathrm{conv}\big(\{\chi^M : M \text{ is a matching in } G\}\big)$

General Matching Polytope

- Given a graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{GM}} := \mathrm{conv}\left(\left\{\chi^M : M \subseteq E \text{ is a matching in } G\right\}\right)$

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$, $n := |V|$
- $\mathcal{P}_{\mathrm{ST}} := \mathrm{conv}\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Examples

Travelling Salesman Problem (TSP) Polytope

- Given the complete graph $G = (V, \binom{V}{2})$
- $\mathcal{P}_{\mathrm{TSP}} := \mathrm{conv}(\{\chi^S : S \subseteq \binom{V}{2}$ is a TSP tour of $V\})$

polytope of interest: $\mathcal{P} = \text{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

- Mechanic description of $\mathcal{P}$:

$$\sum_{i\in[n]} w_i x_i \leq \max_{S\in\mathcal{S}} \sum_{i\in S} w_i \qquad \forall w \in \mathbb{R}^{[n]}$$

- However, the description is often useless; many constraints are redundant
- It is often interesting and important to find the facet-defining constraints; those are the constraints that can not be removed

1. In some cases, $\mathcal{P}$ has polynomial number of facets
2. In some cases, $\mathcal{P}$ has exponential number of facets, but has an efficient separation oracle.
3. In some cases, $\mathcal{P}$ does not have an efficient separation oracle, unless P = NP.

polytope of interest: $\mathcal{P} = \text{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

Def. A polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is integral if all its vertices are in $\mathbb{Z}^n$. For $\mathcal{P} \subseteq [0,1]^n$, this means all vertices are in $\{0,1\}^n$.

Lemma For a polytope $\mathcal{Q} \subseteq [0,1]^n$, if $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$ and $\mathcal{Q}$ is integral, then $\mathcal{Q} = \mathcal{P}$.

Proof.

- $\mathcal{P} \subseteq \mathcal{Q}$, as every vertex of $\mathcal{P}$ is χ^S for some $S \in \mathcal{S}$, and $\chi^S \in \mathcal{Q}$.
- $\mathcal{Q} \subseteq \mathcal{P}$: take some vertex x of $\mathcal{Q}$
- $\mathcal{Q}$ is integral $\implies$ x is integral $\implies$ $x = \chi^S$ for some $S \subseteq [n]$
- As $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$, $x = \chi^S$ for some $S \in \mathcal{S}$
- $x \in \mathcal{P}$ □

polytope of interest: $\mathcal{P} = \text{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

Lemma For a polytope $\mathcal{Q} \subseteq [0,1]^n$, if $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$ and $\mathcal{Q}$ is integral, then $\mathcal{Q} = \mathcal{P}$.

- Often, it is easy to guarantee $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$
- The linear program that defines such a $\mathcal{Q}$ is often called a LP relaxation for the problem.
- The harder part is often to prove that $\mathcal{Q}$ is integral.