Advanced Algorithms (Fall 2026)

Integrality and Exact Algorithms

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Outline

1. Matching Polytopes
2. Total Unimodularity
3. Consecutive-Ones Matrices and Interval Scheduling
4. Total Unimodularity of Bipartite Incidence Matrices
5. Optional Material
   - Integrality of the General Perfect Matching Polytope
   - Matroids and Matroid Polytopes

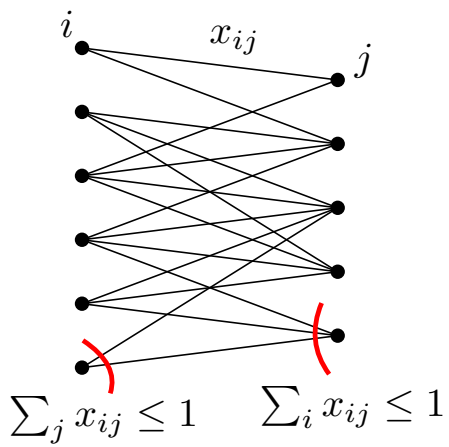
Bipartite Matching Polytope

Maximum Weight Bipartite Matching

Input: bipartite graph $G = (L \uplus R, E)$

edge weights $w \in \mathbb{Z}^E_{>0}$

Output: a matching $M \subseteq E$ so as to maximize $\sum_{e \in M} w_e$



Bipartite Matching Polytope

- Given bipartite graph $G = (L \cup R, E)$
- $\mathcal{P}_{\mathrm{BM}} := \mathrm{conv}\big(\{\chi^M : M \text{ is a matching in } G\}\big)$

Theorem $\mathcal{P}_{\mathrm{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

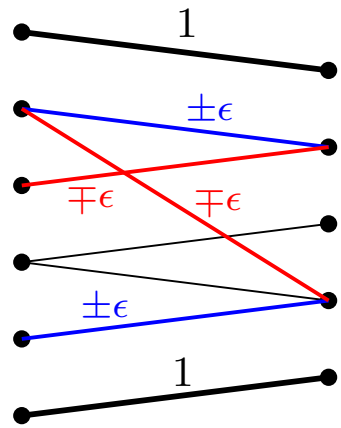
$$\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in L \cup R; \qquad x_e \geq 0, \forall e \in E.$$

Theorem $\mathcal{P}_{\mathrm{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

$$\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in L \cup R; \qquad x_e \geq 0, \forall e \in E.$$

Proof.

- take any x that satisfies the constraints
- prove: x non integral $\Longrightarrow$ x non-vertex
- find $x', x'' \in \mathcal{P}$: $x' \neq x'', x = \frac{1}{2}(x' + x'')$
- case 1: fractional edges contain a cycle
  - color edges in cycle blue and red
  - x': $+\epsilon$ for blue edges, $-\epsilon$ for red edges
  - x'': $-\epsilon$ for blue edges, $+\epsilon$ for red edges
- case 2: fractional edges form a forest
  - color edges in leaf-leaf path blue and red
  - x': $+\epsilon$ for blue edges, $-\epsilon$ for red edges
  - x'': $-\epsilon$ for blue edges, $+\epsilon$ for red edges □



General Graph Perfect Matching Polytope

General Perfect Matching Polytope

- Given a graph $G = (V, E)$, where $|V|$ is even
- $\mathcal{P}_{\mathrm{GPM}} := \mathrm{conv}\left(\left\{\chi^M : M \subseteq E \text{ is a perfect matching in } G\right\}\right)$

Theorem (General Perfect Matching Polytope Theorem)
$\mathcal{P}_{\mathrm{GPM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \forall v \in V$$

$$\sum_{e \in E(S, V \setminus S)} x_e \geq 1 \qquad \forall S \subseteq V, |S| \text{ is odd} \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

General Matching Polytope

- Given a graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{GM}} := \mathrm{conv}\left(\left\{\chi^M : M \subseteq E \text{ is a matching in } G\right\}\right)$

Theorem (General Matching Polytope Theorem) $\mathcal{P}_{\mathrm{GM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V$$

$$\sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2} \qquad \forall S \subseteq V, |S| \text{ is odd} \qquad (1)$$

$$x_e \geq 0 \qquad \forall e \in E$$

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Def. A matrix $A \in \mathbb{R}^{m \times n}$ is said to be totally unimodular (TUM), if every square submatrix of A has determinant in $\{-1, 0, 1\}$.

Theorem If a polytope $\mathcal{P}$ is defined by $Ax \geq b, x \geq 0$ with a totally unimodular matrix A and integral b, then $\mathcal{P}$ is integral.

Proof.

- After substituting zero coordinates and reordering, a vertex x uniquely satisfies $\begin{pmatrix} A' & 0 \\ 0 & I \end{pmatrix} x = \begin{pmatrix} b' \\ 0 \end{pmatrix}$, where
  - A' is a nonsingular square submatrix of A, so $\det(A') = \pm 1$;
  - b' contains the corresponding entries of b.
- Let $x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$, so that $A'x^1 = b'$ and $x^2 = 0$.
- Cramer's rule: $x_i^1 = \frac{\det(A_i'|b')}{\det(A')}$ for every $i \implies x_i^1$ is integer

  $A_i'|b'$: the matrix of A' with the i-th column replaced by b' □

Example for the Proof

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The following equation system may give a vertex:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example for the Proof

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equivalently, the vertex satisfies

$$\begin{pmatrix} a_{1,2} & a_{1,3} & 0 & 0 & 0 \\ a_{3,2} & a_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Lemma Let $A' \in \{0, \pm 1\}^{n \times n}$ such that every row of A' contains at most one 1 and one -1. Then $\det(A') \in \{0, \pm 1\}$.

Proof.

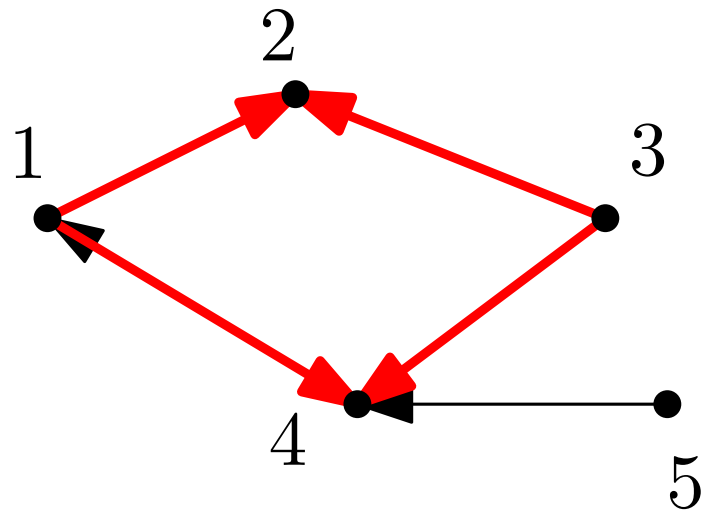
- wlog assume every row of A' contains one 1 and one -1
  - otherwise, we can reduce the matrix
- treat A' as a directed graph: columns $\equiv$ vertices, rows $\equiv$ arcs
- $\#\text{edges} = \#\text{vertices} \implies$ underlying undirected graph contains a cycle $\implies \det(A') = 0$ □

Lemma Let $A \in \{0, \pm 1\}^{m \times n}$ such that every row of A contains at most one 1 and one -1. Then A and A^{T} are TUM.

Coro. In the LP for s-t network flow problem with integer capacities, every vertex solution to the LP is integral.

Example for the Proof

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}$$



$$\begin{array}{rccccc} + & (1 & -1 & 0 & 0 & 0) \\ - & (0 & -1 & 1 & 0 & 0) \\ + & (0 & 0 & 1 & -1 & 0) \\ - & (1 & 0 & 0 & -1 & 0) \\ \\ = & (0 & 0 & 0 & 0 & 0) \end{array}$$

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Lemma A matrix $A \in \{0,1\}^{m \times n}$ where the 1's on every row form an interval is TUM.

Proof.

- take any square submatrix A' of A,
- the 1's on every row of A' form an interval.
- $A'M$ is a matrix satisfying condition of first lemma, where

$$M = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \ \det(M) = 1.$$

- $\det(A'M) \in \{0, \pm 1\} \implies \det(A') \in \{0, \pm 1\}$. □

Example for the Proof

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

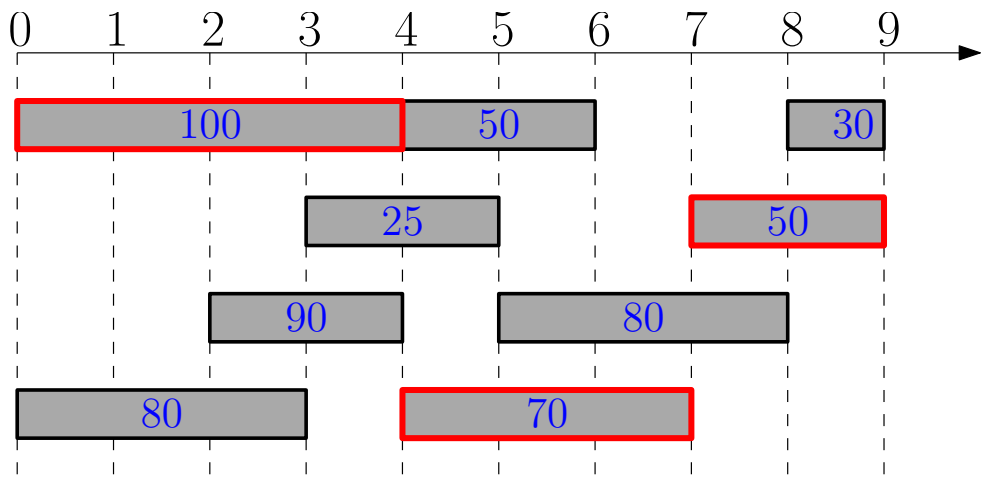
- $(\text{col } 1, \text{col } 2 - \text{col } 1, \text{col } 3 - \text{col } 2, \text{col } 4 - \text{col } 3, \text{col } 5 - \text{col } 4)$
- every row has at most one 1, at most one -1

Weighted Interval Scheduling Problem

Input: n activities, activity i starts at time s_i, finishes at time $f_i > s_i$, and has weight $w_i > 0$

i and j can be scheduled together iff $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: maximum weight subset of jobs that can be scheduled



- optimum value$= 220$
- Classic Problem for Dynamic Programming

Weighted Interval Scheduling Problem

Let $\mathcal{T} = \{s_1, \ldots, s_n\}$, ordered by time.

Linear Program

$$\max \quad \sum_{j=1}^{n} w_j x_j$$

$$\sum_{j:t\in[s_j,f_j)} x_j \leq 1 \quad \forall t \in \mathcal{T}$$

$$x_j \geq 0 \quad \forall j \in [n].$$

- Checking the distinct start times suffices.
- The 1's in each column are consecutive. The matrix is TUM, so the polytope is integral.

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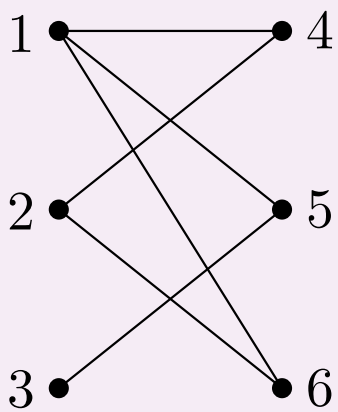
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Lemma The edge-vertex incidence matrix A of a bipartite graph is totally-unimodular.

Proof.

- $G = (L \uplus R, E)$: the bipartite graph
- A': obtained from A by negating columns correspondent to R
- each row of A' has exactly one $+1$, and exactly one -1
- $\Longrightarrow$ A' is TUM $\Longleftrightarrow$ A is TUM $\square$

Example



$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

A different proof for the theorem we proved:

Theorem $\mathcal{P}_{\mathrm{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

$$\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in L \cup R; \qquad x_e \geq 0, \forall e \in E.$$

Proof.

The coefficient matrix for the constraints $\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in L \cup R$ is the vertex-edge incidence matrix of the graph G. Therefore, the polytope is integral. □

- remark: bipartiteness is needed. The edge-vertex incidence matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ of a triangle has determinant 2.

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$$\sum_{e \in E(S, V \setminus S)} x_e \geq 1 \qquad \forall S \subseteq V, |S| \text{ is odd} \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

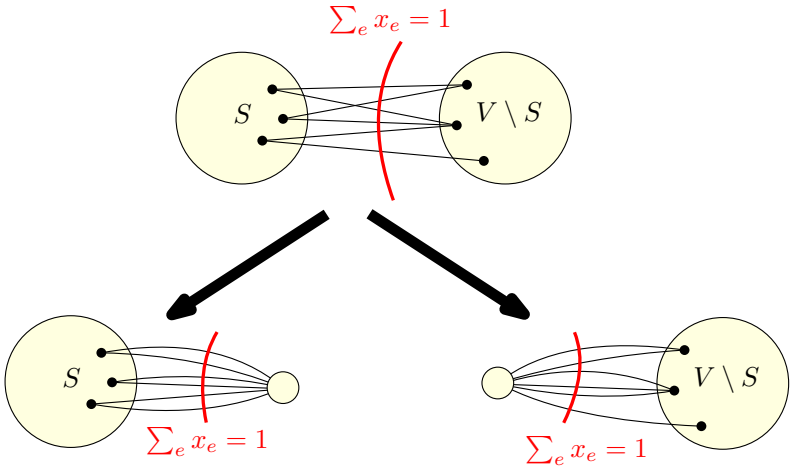
Proof of General Perfect Matching Polytope Theorem

- Clearly, every $x \in \mathcal{P}_{\mathsf{GPM}}$ satisfies all the LP constraints
- We prove the LP polytope is integral; this implies lemma
- We choose the counter-example G with the smallest $|V| + |E|$, and focus on a non-integral vertex x of the LP polytope

Proof of General Perfect Matching Polytope Theorem

- $x_e = 0$ for some $e \in E$: e could be removed.
- $x_e = 1$ for some $e \in E$: e and its 2 end vertices could be removed.
- So $x_e \in (0, 1)$ for every $e \in E$.
- Every $v \in V$ has degree at least 2.
- If all degrees are 2, G is a union of cycles. Odd cycles violate odd-cut constraints; alternating perturbations on even cycles show x is not a vertex.
- Assume some $v \in V$ has degree at least 3; $|E| \geq |V| + 1$.
- x is the unique solution to a system of $|E|$ independent tight linear equations from the LP.
- So, some linear equation is

$$\sum_{e \in E(S, V \setminus S)} x_e = 1, \quad |S| \text{ odd}, \quad 3 \leq |S| \leq |V| - 3.$$



Proof of General Perfect Matching Polytope Theorem

- Consider two instances: $(G/S, x'), (G/(V \setminus S), x'')$
- Both x' and x'' satisfy the LP constraints for their respective graphs.

- By the minimality assumption:

$$x' \in \text{conv}(\{\chi^M : M \text{ is a perfect matching in } G/S\})$$
$$x'' \in \text{conv}(\{\chi^M : M \text{ is a perfect matching in } G/(V \setminus S)\})$$

- Decompose x' and x'' into convex combinations of perfect matchings
- Each cut edge e has the same total weight in both combinations
- Pair matchings using the same cut edge to obtain a convex combination of perfect matchings in G. Thus x is not a vertex. □

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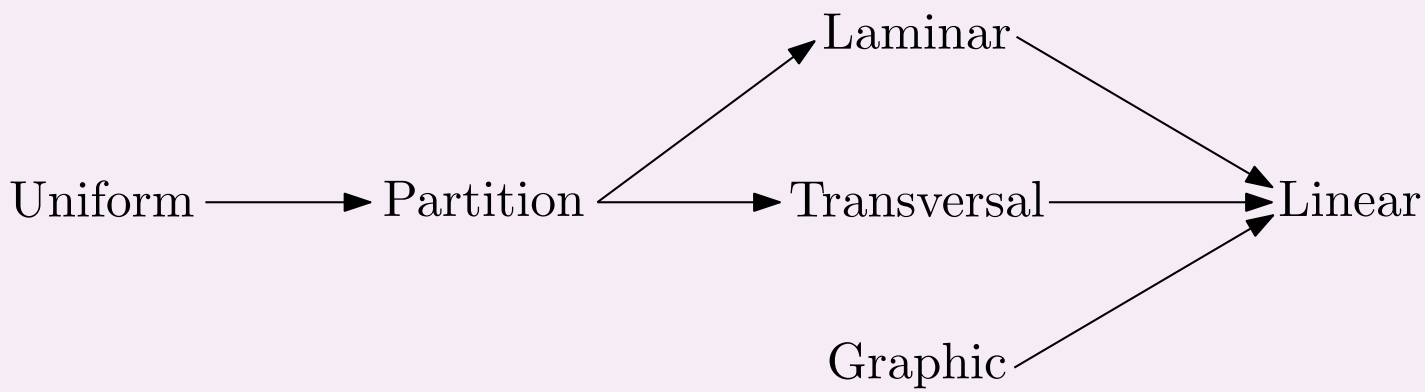
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Recall Definition and Examples of Matroid

Def. A (finite) matroid $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where E is a finite set (called the ground set) and $\mathcal{I}$ is a family of subsets of E (called independent sets) with the following properties:

1. $\emptyset \in \mathcal{I}$.
2. (downward-closed property) If $B \subsetneq A \in \mathcal{I}$, then $B \in \mathcal{I}$.
3. (augmentation/exchange property) If $A, B \in \mathcal{I}$ and $|B| < |A|$, then there exists $e \in A \setminus B$ such that $B \cup \{e\} \in \mathcal{I}$.

Relationship between matroids



Other Terminologies Related To a Matroid $\mathcal{M} = (E, \mathcal{I})$

- A subset of E that is not independent is dependent.
- A maximal independent set is called a basis (plural: bases)
- A minimal dependent set is called a circuit

- Graphic matroid for a connected graph $G = (V, E)$:
  basis $\Longleftrightarrow$ spanning tree circuit $\Longleftrightarrow$ cycle

Lemma All bases of a matroid have the same size.

Proof.

- Assume two A and A' are both bases of $\mathcal{M}$ and $|A| > |A'|$
- By exchange property: $\exists i \in A \setminus A', A' \cup \{i\} \in \mathcal{I}$
- contradiction with that A' is a basis $\square$

- Recall: Matroid Rank Function:

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the rank of any $A \subseteq E$ is defined as

$$r_{\mathcal{M}}(A) = \max\left\{|A'| : A' \subseteq A, A' \in \mathcal{I}\right\}.$$

The function $r_{\mathcal{M}} : 2^E \to \mathbb{Z}_{\geq 0}$ is called the rank function of $\mathcal{M}$.

- $r_{\mathcal{M}}(A)$ is size of maximum independent subset of A

Trivial properties of $r_{\mathcal{M}}$

- $r_{\mathcal{M}}(\emptyset) = 0$
- $r_{\mathcal{M}}(A \cup \{i\}) - r_{\mathcal{M}}(A) \in \{0, 1\}$ for every $A \subseteq E, i \in E \setminus A$

Theorem The rank function $r_{\mathcal{M}}$ of a matroid $\mathcal{M} = (E, \mathcal{I})$ is submodular.

Greedy algorithm finds max ind. subset of any given $X \subseteq E$:

1: $S \leftarrow \emptyset$
2: **while** $\exists e \in X \setminus S$ s.t. $S \cup \{e\} \in \mathcal{I}$ **do**
3: let e be an arbitrary element satisfying the condition
4: $S \leftarrow S \cup \{e\}$
5: **return** S

Proof of Submodularity of $r_{\mathcal{M}}$.

- Take $A \subsetneq E, i, j \in E \setminus A, i \neq j$, need to prove:
  $r_{\mathcal{M}}(A \cup \{i,j\}) - r_{\mathcal{M}}(A \cup \{i\}) \leq r_{\mathcal{M}}(A \cup \{j\}) - r_{\mathcal{M}}(A)$
- if not, then LHS $= 1$, RHS $= 0$
- S: max ind. subset of A, S': max ind. subset of $A \cup \{i\}$
- $|S| = r_{\mathcal{M}}(A), |S'| = r_{\mathcal{M}}(A \cup \{i\}), \quad S' = S$ or $S' = S \cup \{i\}$
- RHS $= 0 \implies S \cup \{j\} \notin \mathcal{I}$, LHS $= 1 \implies S' \cup \{j\} \in \mathcal{I}$
- contradiction □

Lemma A function $r : 2^E \to \mathbb{R}$ is the rank function of a matroid if and only if

1. $r(\emptyset) = 0$
2. $r(A \cup \{i\}) - r(A) \in \{0, 1\}$ for all $A \subseteq E, i \in E \setminus A$
3. r is submodular.

Proof.

- Define $\mathcal{I} = \{A \subseteq E : r(A) = |A|\}$.
- Claim: $(E, \mathcal{I})$ is a matroid and r is its rank function.
- ①, ② $\implies$ $\mathcal{I}$ is closed under taking subsets
- $A, A' : r(A) = |A|, r(A') = |A'|, |A| < |A'|$
- $U := A \cup A' : r(U) \geq r(A') > r(A), \qquad A \subsetneq U$
- ③ $\implies$ $\exists i \in U \setminus A = A' \setminus A : r(A \cup \{i\}) > r(A)$
- $i \in A' \setminus A$ and $r(A \cup \{i\}) = r(A) + 1 = |A \cup \{i\}|$
- so, $A \cup \{i\} \in \mathcal{I}$ $\implies$ exchange property □

Derivatives of Matroids

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and an element $e \in E$, the matroid obtained from $\mathcal{M}$ by removing e, denoted as $\mathcal{M} \setminus e$, is defined as follows:

$$\mathcal{M} \setminus e = (E \setminus \{e\}, \{A \subseteq E \setminus \{e\} : A \in \mathcal{I}\}).$$

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and a non-loop $e \in E$ (i.e., $\{e\} \in \mathcal{I}$), the matroid obtained by contracting e, denoted as $\mathcal{M}/e$, is defined as follows:

$$\mathcal{M}/e = (E \setminus \{e\}, \{A \subseteq E \setminus \{e\} : A \cup \{e\} \in \mathcal{I}\}).$$

Derivatives of Matroids

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and a subset $E' \subseteq E$, the matroid of $\mathcal{M}$ restricted to E', denoted as $\mathcal{M}[E']$, is defined as follows:

$$\mathcal{M}[E'] = (E', \{A \subseteq E' : A \in \mathcal{I}\}).$$

Def. For a matroid $\mathcal{M} = (E, \mathcal{I})$, the dual matroid $\mathcal{M}^* = (E, \mathcal{I}^*)$ is defined so that the bases in $\mathcal{M}^*$ are exactly the complements of the bases of $\mathcal{M}$.

Theorem $\mathcal{M}^*$ is a matroid.

Matroid Polytope

- Given a matroid $\mathcal{M} = (E, \mathcal{I})$
- The matroid polytope for $\mathcal{M}$ is defined as

$$\mathcal{P}_{\mathcal{M}} := \text{conv}(\{\chi^A : A \in \mathcal{I}\}).$$

- Recall: $\chi^A \in \{0,1\}^E, \quad \chi^A_i = \begin{cases} 1 & i \in A \\ 0 & i \notin A \end{cases}$

Theorem (Matroid Polytope Theorem) For a matroid $\mathcal{M} = (E, \mathcal{I})$, we have

$$\mathcal{P}_{\mathcal{M}} = \left\{ x \in [0,1]^E : x(S) \leq r_{\mathcal{M}}(S), \forall S \subseteq E \right\},$$

where $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq E$.

Proof of Matroid Polytope Theorem

- $\mathcal{Q} := \left\{ x \in [0,1]^E : \sum_{i \in A} x_i \leq r_{\mathcal{M}}(A), \forall A \subseteq E \right\}$
- $\mathcal{Q} \cap \{0,1\}^E = \{\chi^A : A \in \mathcal{I}\}$; it suffices to prove $\mathcal{Q}$ is integral
- Focus on the counter example with the smallest $|E|$
- assume some vertex x of $\mathcal{Q}$ is non-integral
- If $x_e = 0$ for some $e \in E$, removing e gives a smaller counterexample
- If $x_e = 1$ for some $e \in E$, contracting e gives a smaller counterexample
- So, $x_e \in (0,1)$ for every $e \in E$.

Proof of Matroid Polytope Theorem

Def. We say a set $A \subseteq E$ is tight if $x(A) = r_{\mathcal{M}}(A)$. Let $\mathcal{T}$ be the family of all tight subsets of E.

Lemma If $A, B \in \mathcal{T}$, then both $A \cup B$ and $A \cap B$ are in $\mathcal{T}$.

Proof.

$$x(A) + x(B) = r_{\mathcal{M}}(A) + r_{\mathcal{M}}(B)$$
$$\geq r_{\mathcal{M}}(A \cup B) + r_{\mathcal{M}}(A \cap B) \geq x(A \cup B) + x(A \cap B).$$

- equality: A and B are tight
- first inequality: $r_{\mathcal{M}}$ is submodular
- second inequality: $x(S) \leq r_{\mathcal{M}}(S)$ for every $S \subseteq E$

But $x(A) + x(B) = x(A \cup B) + x(A \cap B)$. So, both inequalities hold with equality. □

Proof of Matroid Polytope Theorem

Def. A chain is a sequence of subsets $S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_t$ of E.

- We use span$(\mathcal{S})$ for span$(\{\chi^S : S \in \mathcal{S}\})$, for any $\mathcal{S} \subseteq \mathcal{T}$.

Lemma (Key Lemma) Let $\mathcal{C}$ be a longest chain of tight subsets of E (i.e., subsets in $\mathcal{T}$). Then, we have span$(\mathcal{C}) =$ span$(\mathcal{T})$.

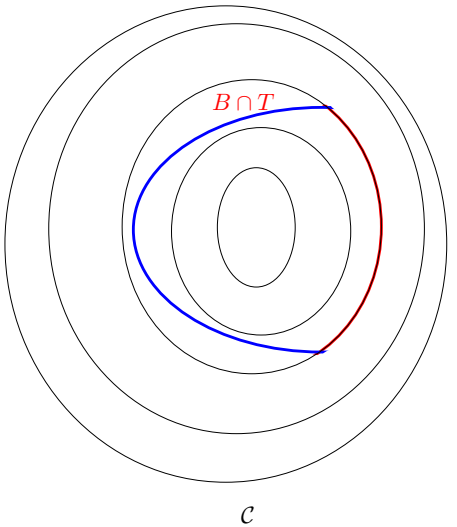
Proof of Key Lemma

- We say two sets B and T conflict with each other, if $B \not\subseteq T$ and $T \not\subseteq B$.
- Define $\tau(B) := \{T \in \mathcal{C} : B$ conflicts with $T\}, \forall B$
- Assume span$(\mathcal{C}) \subsetneq$ span$(\mathcal{T})$
- Let $B = \arg\min_{B \in \mathcal{T}, \chi^B \notin \text{span}(\mathcal{C})} |\tau(B)|$

Proof of Matroid Polytope Theorem

Proof of Key Lemma

- Let $T \in \mathcal{C}$ be a set conflicting with B;
- We prove $\tau(B \cup T), \tau(B \cap T) \subsetneq \tau(B)$.



- For $\tau(B \cup T) \subseteq \tau(B)$:
  - $S \subsetneq T$: S does not conflict with $B \cup T$, and may conflict with B.
  - $S \supsetneq T$: S not conflict with $B \implies S$ not conflict with $B \cup T$.
- For $\tau(B \cap T) \subseteq \tau(B)$:
  - $S \subsetneq T$: S not conflict with $B \implies S$ not conflict with $B \cap T$.
  - $S \supsetneq T$: S does not conflict with $B \cap T$, and may conflict with B.
- "$\neq$" : B conflicts with T, but $B \cup T$ and $B \cap T$ do not.

Proof of Matroid Polytope Theorem

Proof of Key Lemma

- By our choice of B, we have $\chi^{B\cup T}, \chi^{B\cap T} \in \text{span}(\mathcal{C})$.
- However, as $\chi^B = \chi^{B\cup T} + \chi^{B\cap T} - \chi^T$ and all three vectors are in $\text{span}(\mathcal{C})$, contradiction with $\chi^B \notin \text{span}(\mathcal{C})$. □

Recall the key lemma:

Lemma (Key Lemma) Let $\mathcal{C}$ be a longest chain of tight subsets of E (i.e., subsets in $\mathcal{T}$). Then, we have $\text{span}(\mathcal{C}) = \text{span}(\mathcal{T})$.

- Therefore, $x \in [0,1]^E$ is defined by the system of linear equations correspondent to $\mathcal{C}$.
- $\mathcal{C}$ has $|E|$ nonempty sets, so its successive differences are singletons.
- The system gives an integer solution x. Contradiction. □

What we proved:

Matroid Polytope

- Given a matroid $\mathcal{M} = (E, \mathcal{I})$
- The matroid polytope for $\mathcal{M}$ is defined as

$$\mathcal{P}_{\mathcal{M}} := \text{conv}(\{\chi^A : A \in \mathcal{I}\}).$$

Theorem (Matroid Polytope Theorem) For a matroid $\mathcal{M} = (E, \mathcal{I})$, we have

$$\mathcal{P}_{\mathcal{M}} = \left\{ x \in [0,1]^E : x(S) \leq r_{\mathcal{M}}(S), \forall S \subseteq E \right\},$$

where $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq E$.

Matroid Basis Polytope

- Given a matroid $\mathcal{M} = (E, \mathcal{I})$
- The matroid basis polytope for $\mathcal{M}$ is defined as

$$\mathcal{P}_{\mathcal{M}}^{\text{basis}} := \text{conv}(\{\chi^A : A \in \mathcal{I}, r_{\mathcal{M}}(A) = r_{\mathcal{M}}(E)\}).$$

Theorem (Matroid Basis Polytope Theorem) For a matroid $\mathcal{M} = (E, \mathcal{I})$, we have

$$\mathcal{P}_{\mathcal{M}}^{\text{basis}} = \left\{x \in [0,1]^E : x(S) \leq r_{\mathcal{M}}(S), \forall S \subseteq E; x(E) = r_{\mathcal{M}}(E)\right\},$$

where $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq E$.

Proof.

- $\mathcal{P}_{\mathcal{M}}^{\text{basis}}$ is a face (not necessarily a facet) of $\mathcal{P}_{\mathcal{M}}$.
- $\mathcal{P}_{\mathcal{M}}$ is integral $\implies$ $\mathcal{P}_{\mathcal{M}}^{\text{basis}}$ is integral $\square$

Recall: Spanning Tree Polytope

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$, $n := |V|$
- $\mathcal{P}_{\mathrm{ST}} := \mathrm{conv}\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subseteq V, 2 \leq |S| \leq n - 1 \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

- Graphic matroid:
  - independent sets $\leftrightarrow$ spanning forests
  - bases $\leftrightarrow$ spanning trees.

- So, $\mathcal{P}_{\mathrm{ST}}$ is the set of $x \in [0,1]^E$ satisfying

$$x(E') \leq n - \mathsf{CC}(E'), \forall E' \subseteq E; \quad x(E) = n - 1,$$

  where $\mathsf{CC}(E')$ is the number of connected components in (V, E').

- It suffices to consider the case where $E' = E[S]$ for some connected set $S \subseteq V$, in which case $n - \mathrm{CC}(E') = |S| - 1$.
- $\implies$ Spanning Tree Polytope Theorem.

Theorem (Matroid Intersection Polytope Theorem) Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids with the common ground set E. Then

$$\begin{aligned}&\text{conv}\big(\{\chi^A : A \in \mathcal{I}_1 \cap \mathcal{I}_2\}\big) = \mathcal{P}_{\mathcal{M}_1} \cap \mathcal{P}_{\mathcal{M}_2} \\ &= \Big\{x \in [0,1]^E : x(S) \le r_{\mathcal{M}_1}(S), x(S) \le r_{\mathcal{M}_2}(S), \forall S \subseteq E\Big\}.\end{aligned}$$

- We will not prove the theorem.
- A similar theorem works if we require A to be a basis for the matroid $\mathcal{M}_1$ or $\mathcal{M}_2$:

$$\begin{aligned}&\text{conv}\big(\{\chi^A : A \in \mathcal{I}_1 \cap \mathcal{I}_2, r_{\mathcal{M}_1}(A) = r_{\mathcal{M}_1}(E)\}\big) \\ &= \mathcal{P}^{\text{basis}}_{\mathcal{M}_1} \cap \mathcal{P}_{\mathcal{M}_2}\end{aligned}$$

Applications

Bipartite Matching Polytope

- Given bipartite graph $G = (L \cup R, E)$
- $\mathcal{P}_{\mathrm{BM}} := \mathrm{conv}\big(\{\chi^M : M \text{ is a matching in } G\}\big)$

Theorem $\mathcal{P}_{\mathrm{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

$$\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in L \cup R; \qquad x_e \geq 0, \forall e \in E.$$

- A matching is an independent set of two partition matroids, one for each side of the bipartite graph.
- Matching polytope is intersection of two partition matroid polytopes.

Applications

Arborescence Polytope

- Given a directed graph $G = (V, E)$, a root $r \in V$
- $\mathcal{P}_{\text{Arbo}} := \text{conv}(\{\chi^{E'} : E' \text{ is an arborescence of } G \text{ rooted at } r\})$

- We define two matroids:
  - Graphic Matroid: we ignore the directions of G, and require E' to be a spanning forest
  - Partition Matroid: in-degree is at most 1 at each $v \neq r$, and 0 at r
- E' is a rooted arborescence iff it is a basis of the graphic matroid and independent in the partition matroid.

Summary

- linear programming, simplex method, interior point method, ellipsoid method
- Polytopes with totally-unimodular coefficient matrix:
  - integral LP polytopes: bipartite matching polytope, s-t flow polytope, weighted interval scheduling polytope
- Matroid Polytope