Advanced Algorithms (Fall 2026)

Linear Programming Duality

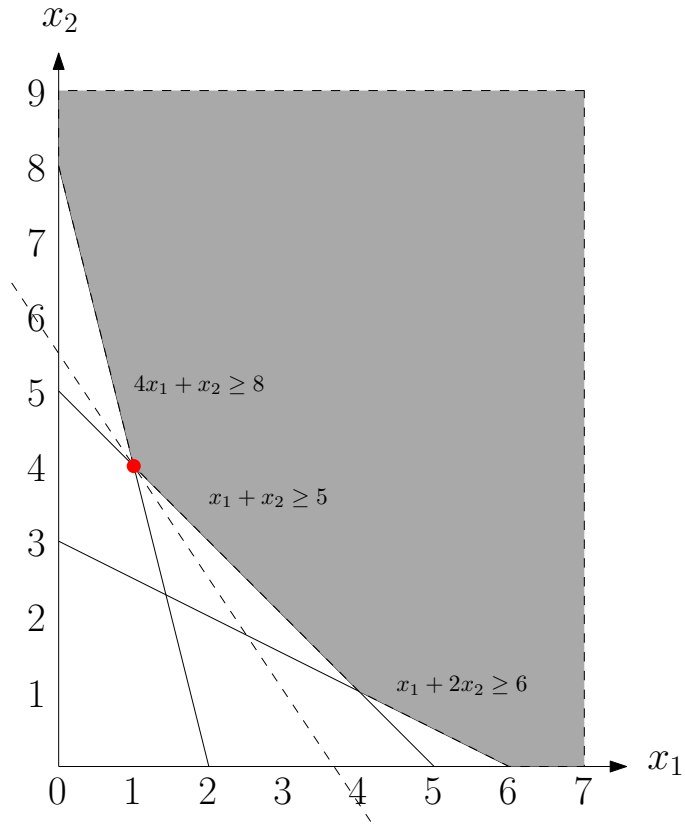
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Outline

1. Constructing Dual Programs and Weak Duality
2. Farkas's Lemma and Strong Duality
3. Complementary Slackness

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Q: How can we prove a lower bound for the value?

- $7x_1 + 4x_2 \geq 2(x_1 + x_2) + (x_1 + 2x_2) \geq 2 \times 5 + 6 = 16$
- $7x_1 + 4x_2 \geq (x_1 + x_2) + (x_1 + 2x_2) + (4x_1 + x_2) \geq 5 + 6 + 8 = 19$
- $7x_1 + 4x_2 \geq 4(x_1 + x_2) \geq 4 \times 5 = 20$
- $7x_1 + 4x_2 \geq 3(x_1 + x_2) + (4x_1 + x_2) \geq 3 \times 5 + 8 = 23$

Primal LP

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & 5y_1 + 6y_2 + 8y_3 \\ & y_1 + y_2 + 4y_3 \leq 7 \\ & y_1 + 2y_2 + y_3 \leq 4 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

A way to prove lower bound on the value of primal LP

$$\begin{aligned} & 7x_1 + 4x_2 \qquad (\text{if } 7 \geq y_1 + y_2 + 4y_3 \text{ and } 4 \geq y_1 + 2y_2 + y_3) \\ & \geq y_1(x_1 + x_2) + y_2(x_1 + 2x_2) + y_3(4x_1 + x_2) \quad (\text{if } y_1, y_2, y_3 \geq 0) \\ & \geq 5y_1 + 6y_2 + 8y_3. \end{aligned}$$

- Goal: need to maximize $5y_1 + 6y_2 + 8y_3$

Primal LP

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & 5y_1 + 6y_2 + 8y_3 \\ & y_1 + y_2 + 4y_3 \leq 7 \\ & y_1 + 2y_2 + y_3 \leq 4 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 4 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} \quad c = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$\begin{aligned} \min \quad & c^T x \qquad \text{s.t.} \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & b^T y \qquad \text{s.t.} \\ & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

Primal LP

$$\min \quad c^T x \qquad \text{s.t.}$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^T y \qquad \text{s.t.}$$
$$A^T y \leq c$$
$$y \geq 0$$

- $P =$ value of primal LP
- $D =$ value of dual LP

Theorem (weak duality theorem) $D \leq P$.

Theorem (strong duality theorem) If the primal LP has a finite optimum, then the dual has an optimum and $D = P$.

- An optimal dual solution certifies the value of any finite primal optimum.

Primal LP

$$\min \quad c^T x \qquad \text{s.t.}$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^T y \qquad \text{s.t.}$$
$$A^T y \leq c$$
$$y \geq 0$$

- $P =$ value of primal LP
- $D =$ value of dual LP

Theorem (weak duality theorem) $D \leq P$.

Proof.

- x^*: any feasible primal solution
- y^*: any feasible dual solution

$$b^{\mathrm{T}} y^* \leq (Ax^*)^{\mathrm{T}} y^* = (x^*)^{\mathrm{T}} A^{\mathrm{T}} y^* \leq (x^*)^{\mathrm{T}} c = c^{\mathrm{T}} x^*. \qquad \square$$

Primal LP

$$\begin{aligned} \min \quad & c^{\mathrm{T}} x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & b^{\mathrm{T}} y \\ & A^{\mathrm{T}} y \leq c \\ & y \geq 0 \end{aligned}$$

Relationships

Primal LP	dual LP
variables	constraints
constraints	variables
obj. coefficients	RHS constants
RHS constants	obj. coefficients

More Relationships

Primal LP	Dual LP
variable in $\mathbb{R}$	equalities
equalities	variable in $\mathbb{R}$

Outline

1 Constructing Dual Programs and Weak Duality

2 Farkas's Lemma and Strong Duality

3 Complementary Slackness

Fact If a point x does not belong to a nonempty closed convex polyhedron $\mathcal{P}$, then there is a hyperplane separating x and $\mathcal{P}$.

Lemma (Farkas Lemma) $Ax = b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0$ is feasible.

Proof.

- $\{Ax : x \geq 0\}$ is a closed polyhedral cone. If b is outside it, $\exists$ some hyperplane separating b and $\{Ax : x \geq 0\}$.
- $y^{\mathrm{T}}b < g$ and $y^{\mathrm{T}}Ax > g$ for every $x \geq 0$
- $g < 0$ and $y^{\mathrm{T}}A \geq 0$
- Conversely, $Ax = b$, $x \geq 0$ would give $y^{\mathrm{T}}b = y^{\mathrm{T}}Ax \geq 0$, a contradiction. □

Lemma (Farkas Lemma) $Ax = b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0$ is feasible.

Lemma (Variant of Farkas Lemma) $Ax \leq b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0, y \geq 0$ is feasible.

Proof.

- system equivalent to $Ax + x' = b, x, x' \geq 0$

$$(A, I)\begin{pmatrix} x \\ x' \end{pmatrix} = b, \qquad \begin{pmatrix} x \\ x' \end{pmatrix} \geq 0$$

- By Farkas Lemma, $\exists y$ such that $y^{\mathrm{T}}(A, I) \geq 0, y^{\mathrm{T}}b < 0$
- $\iff y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}} \geq 0, y^{\mathrm{T}}b < 0$ □

Primal LP

$$\min \quad c^T x \qquad \text{s.t.}$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^T y \qquad \text{s.t.}$$
$$A^T y \leq c$$
$$y \geq 0$$

Lemma (Variant of Farkas Lemma) $Ax \leq b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}} A \geq 0, y^{\mathrm{T}} b < 0, y \geq 0$ is feasible.

Proof of Strong Duality Theorem

- $\forall \epsilon > 0, \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} x \leq \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix}, x \geq 0$ is infeasible
- There exists $y \in \mathbb{R}^m_{\geq 0}, \alpha \geq 0$, such that $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} \geq 0$,

  $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix} < 0$
- we can prove $\alpha > 0$, since the primal LP is feasible.

Proof of Strong Duality Theorem

- There exists $y \in \mathbb{R}^m_{\geq 0}, \alpha \geq 0$, such that $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} \geq 0$, $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix} < 0$
- assume $\alpha = 1$
- $-y^{\mathrm{T}}A + c^{\mathrm{T}} \geq 0, -y^{\mathrm{T}}b + P - \epsilon < 0 \iff A^{\mathrm{T}}y \leq c, b^{\mathrm{T}}y > P - \epsilon$
- $\forall \epsilon > 0, D > P - \epsilon \implies D = P$ (since $D \leq P$) $\square$

Strong Duality: Attainment

Fact A linear objective bounded below on a nonempty polyhedron attains its minimum. The analogous statement holds for a bounded-above maximum.

- The preceding argument gives feasible dual solutions with values arbitrarily close to P.
- Weak duality bounds every dual value above by the finite value P.
- Thus the dual is nonempty and its objective is bounded above. The fact gives an optimal dual solution, with value $D = P$.

- duality is mutual: the dual of the dual of an LP is the LP itself.

Primal LP

$$\begin{aligned} \max \quad & b^{\mathrm{T}} y \\ & A^{\mathrm{T}} y \leq c \\ & y \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & c^{\mathrm{T}} x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

- If one LP is unbounded in its optimization direction, the other is infeasible.
- If one LP is infeasible, the other may be unbounded or infeasible.
- Both can be infeasible: $A = (0)$, $b = (1)$, $c = (-1)$ give primal $0x \geq 1$ and dual $0y \leq -1$.

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Complementary Slackness

Primal LP

$$\begin{aligned} \min \quad & c^{\mathrm{T}} x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & b^{\mathrm{T}} y \\ & A^{\mathrm{T}} y \leq c \\ & y \geq 0 \end{aligned}$$

- x^* and y^*: optimum primal and dual solutions
- $D = b^{\mathrm{T}} y^* \leq (Ax^*)^{\mathrm{T}} y^* = (x^*)^{\mathrm{T}} A^{\mathrm{T}} y^* \leq (x^*)^{\mathrm{T}} c = c^{\mathrm{T}} x^* = P.$
- $P = D$: all the inequalities hold with equalities.

Complementary Slackness

- $y_i^* > 0 \implies \sum_j a_{ij} x_j^* = b_i.$
- $x_j^* > 0 \implies \sum_i a_{ij} y_i^* = c_j.$

Complementary Slackness: Certifying Optimality

For feasible x and y, the duality gap is

$$c^{\mathrm{T}}x - b^{\mathrm{T}}y = x^{\mathrm{T}}(c - A^{\mathrm{T}}y) + y^{\mathrm{T}}(Ax - b).$$

Every summand is nonnegative. Hence complementary slackness is also sufficient for x and y to be optimal.

The running example

Take $x = (1, 4)$ and $y = (3, 0, 1)$. Both are feasible, and

$$Ax - b = (0, 3, 0), \qquad c - A^{\mathrm{T}}y = (0, 0).$$

Only the second primal constraint is slack, and $y_2 = 0$. Both objective values are 23, certifying optimality.