Advanced Algorithms (Fall 2026)

Separation and Applications of LP Duality

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Outline

1. Separation Oracles
2. The s–t Cut Polytope
3. Maximum Flow and Minimum Cut
4. Zero-Sum Games and the Minimax Theorem
5. Optional Material
   - Spanning Trees and Matching: Separation
   - Ellipsoid Method: Running Time

Def. A separation oracle for a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is an algorithm that, given some $x^* \in \mathbb{R}^n$,

- either correctly claims that $x^* \in \mathcal{P}$,
- or outputs a linear constraint $a^{\mathrm{T}}x \leq b$ that separates x^* from $\mathcal{P}$: every $x \in \mathcal{P}$ satisfies $a^{\mathrm{T}}x \leq b$, but $a^{\mathrm{T}}x^* > b$. We say $a^{\mathrm{T}}x \leq b$ is a separation plane for x^*.

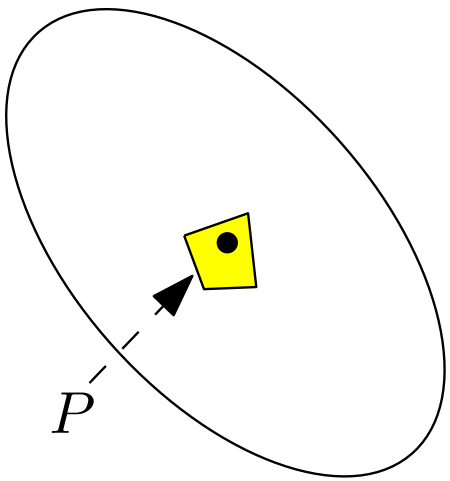
The separation oracle is efficient if its running time is polynomial in the size of the instance plus the size of x^*

- Clearly, if $\mathcal{P} \subseteq \mathbb{R}^n$ can be described using a polynomial-size LP, then it has an efficient separation oracle.
- However, there are cases where $\mathcal{P} \subseteq \mathbb{R}^n$ has exponential number of facets, but still admits an efficient separation oracle.

- We can use ellipsoid method to solve the LP $\min / \max w^{\mathrm{T}} x, x \in \mathcal{P}$, when $\mathcal{P}$ has an efficient separation oracle, using the ellipsoid method.

Ellipsoid Method

- maintain an ellipsoid that contains the feasible region
- query a separation oracle if the center of ellipsoid is in the feasible region:
  - yes: then the feasible region is not empty
  - no: cut the ellipsoid in half, find smaller ellipsoid to enclose the half-ellipsoid, and repeat



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s-t Cut Polytope

Def. Given a digraph $G = (V, E)$, C is a s-t cut in G, if there is no directed path from s to t in $(V, E \setminus C)$.

- $\mathcal{P}_{\min-\text{cut}} := \text{conv}\big(\{\chi^C : C \text{ is a } s\text{-}t \text{ cut in } G\}\big)$

Theorem $\mathcal{P}_{\min-\text{cut}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in P} x_e \geq 1 \qquad \forall \text{ simple } s\text{-}t \text{ path } P \qquad (*)$$

$$x_e \in [0, 1] \qquad \forall e \in E$$

Q: Given $x \in [0, 1]^E$, how can we check if x satisfies all constraints in (*)?

A: Use shortest path algorithm with weights $(x_e)_{e \in E}$.

Theorem $\mathcal{P}_{\min-\text{cut}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in P} x_e \geq 1 \qquad \forall \text{ simple } s\text{-}t \text{ path } P \qquad (*)$$

$$x_e \in [0,1] \qquad \forall e \in E$$

Proof of Lemma.

- Given $x \in [0,1]^E$ satisfying (*)
- $d_x(v), v \in V$: the minimum of 1 and the shortest-path distance from s to v with edge lengths x; so $d_x(s) = 0$ and $d_x(t) = 1$
- randomly choose a real $\theta \in (0,1)$
- $S := \{v \in V : d_x(v) \leq \theta\}, T := V \setminus S = \{v \in V : d_x(v) > \theta\}$
- $C := E(S,T)$

Claim For an edge $(u, v) \in E$, we have

$$\Pr[(u, v) \in C] \leq \max\{d_x(v) - d_x(u), 0\}.$$

Proof.

- $(u, v) \in C$ happens only if $d_x(u) \leq \theta < d_x(v)$.
- This happens with probability at most $\max\{d_x(v) - d_x(u), 0\} \leq x_{(u,v)}$. □

Proof of Lemma, Continued

- $\mathbb{E}_\theta[\chi^C] \leq x$
- We can define a random set C' so that $C' \supseteq C$ happens with probability 1, and $\mathbb{E}[\chi^{C'}] = x$.
- So $x \in \text{conv}(\{\chi^{C'} : C' \text{ is a } s\text{-}t \text{ cut in } G\})$ □

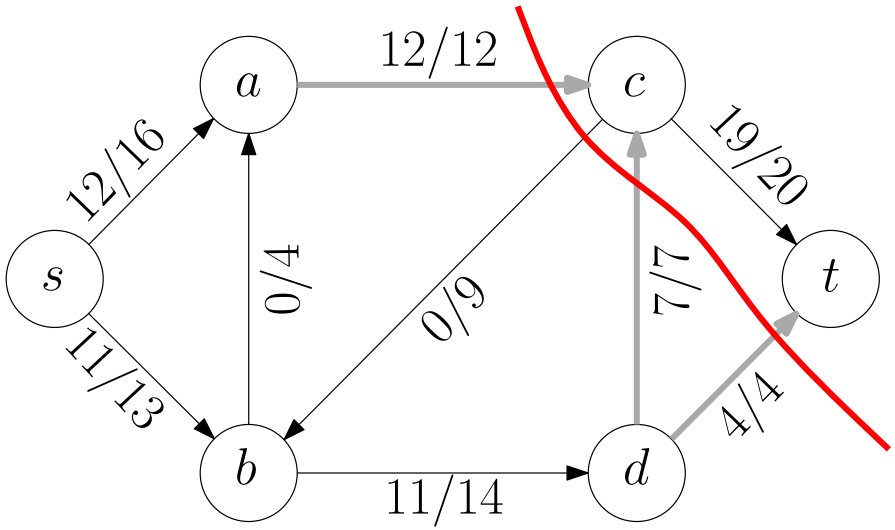
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Maximum Flow Problem

Input: flow network $(G = (V, E), c, s, t)$

Output: maximum value of a s-t flow f



LP for Maximum Flow

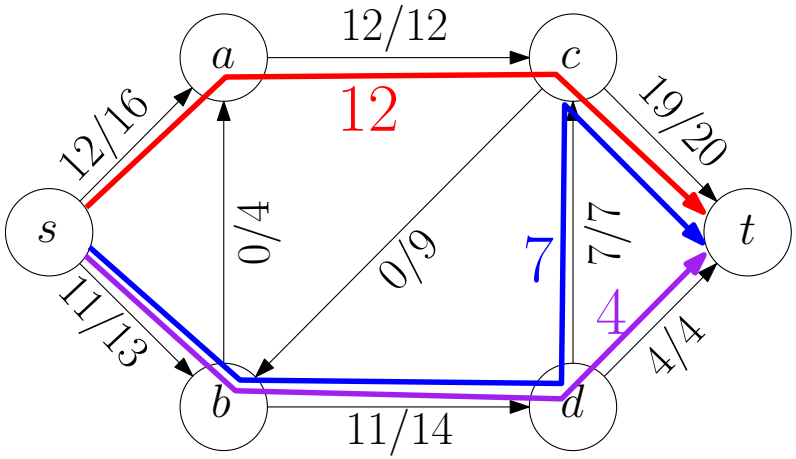
$$\max \sum_{e \in \delta^{\text{in}}(t)} x_e - \sum_{e \in \delta^{\text{out}}(t)} x_e$$

$$x_e \leq c_e \qquad \forall e \in E$$

$$\sum_{e \in \delta^{\text{out}}(v)} x_e - \sum_{e \in \delta^{\text{in}}(v)} x_e = 0 \qquad \forall v \in V \setminus \{s, t\}$$

$$x_e \geq 0 \qquad \forall e \in E$$

An Equivalent Packing LP



- $\mathcal{P}$: the set of all simple paths from s to t
- $f_P, P \in \mathcal{P}$: the flow on P

$$\max \quad \sum_{P \in \mathcal{P}} f_P$$

$$\sum_{P \in \mathcal{P}: e \in P} f_P \leq c_e \qquad \forall e \in E$$

$$f_P \geq 0 \qquad \forall P \in \mathcal{P}$$

$$\min \quad \sum_{e \in E} c_e y_e$$

$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$

$$y_e \geq 0 \qquad \forall e \in E$$

- dual constraints: the shortest s-t path w.r.t weights y has length ≥ 1

Dual LP

$$\min \quad \sum_{e \in E} c_e y_e$$

$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$

$$y_e \geq 0 \qquad \forall e \in E$$

Theorem The optimum value can be attained at an integral point y.

Maximum Flow Minimum Cut Theorem The value of the maximum flow equals the value of the minimum cut.

Proof of Theorem.

- Given any optimum y, let d_v be the minimum of 1 and the shortest-path distance from s to v using lengths y, for every $v \in V$. $d_s = 0, d_t = 1$
- Randomly choose $\theta \in (0, 1)$, and output cut $(S := \{v : d_v \leq \theta\}, T := \{v : d_v > \theta\})$
- Lemma: $\mathbb{E}[\text{cut value of}(S, T)] \leq \sum_{e \in E} c_e y_e$
- Any cut (S, T) in the support is optimum □

$$\max \quad \sum_{P \in \mathcal{P}} f_P$$

$$\sum_{P \in \mathcal{P}: e \in P} f_P \leq c_e \qquad \forall e \in E$$

$$f_P \geq 0 \qquad \forall P \in \mathcal{P}$$

$$\min \quad \sum_{e \in E} c_e y_e$$

$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$

$$y_e \geq 0 \qquad \forall e \in E$$

- pros of new LP: it is a packing LP, dual is a covering LP, easier to understand and analyze
- cons of new LP: exponential size, can not be solved directly
  - when we only need to do non-algorithmic analysis
  - ellipsoid method with separation oracle can solve some exponential size LP

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0-Sum Game

Input: a payoff matrix $M \in \mathbb{R}^{m \times n}, m, n \geq 1$,

two players: row player R, column player C

Output: R plays a row $i \in [m]$, C plays a column $j \in [n]$

payoff of game is M_{ij}

R wants to minimize M_{ij}, C wants to maximize M_{ij}

Rock-Scissor-Paper Game

payoff	R	S	P
R	0	-1	1
S	1	0	-1
P	-1	1	0

- game depends on who plays first

By allowing mixed strategies, each player has a best strategy, regardless of who plays first

	row player R	column player C
pure strategy	row $i \in [m]$	column $j \in [n]$
mixed strategy	distribution x over $[m]$ $x \in [0,1]^m, \sum_{i=1}^m x_i = 1$	distribution y over $[n]$ $y \in [0,1]^n, \sum_{j=1}^n y_j = 1$

$$M(x,y) := \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j M_{ij}$$

$$M(x,j) := \sum_{i=1}^{m} x_i M_{ij}, \qquad M(i,y) := \sum_{j=1}^{n} y_j M_{ij}$$

- If R plays a mixed strategy x first, then it is the best for C to play a pure strategy j. Value of game is $\inf_x \max_{j\in[n]} M(x,j)$.
- If C plays a mixed strategy y first, then it is the best for R to play a pure strategy i. Value of game is $\sup_y \min_{i\in[m]} M(i,y)$.

Theorem (von Neumann (1928), Minimax Theorem)

$$\inf_x \max_{j\in[n]} M(x,j) = \sup_y \min_{i\in[m]} M(i,y).$$

Coro. $\inf_x \sup_y M(x,y) = \sup_y \inf_x M(x,y)$.

Coro. There are mixed strategies x^* and y^* satisfying $M(x,y^*) \geq M(x^*,y^*), \forall x$ and $M(x^*,y) \leq M(x^*,y^*), \forall y$.

Proof.

- $V := \inf_x \sup_y M(x,y) = \sup_y \inf_x M(x,y)$
- x^*: the strategy x that minimizes $\sup_y M(x,y)$
- y^*: the strategy y that maximizes $\inf_x M(x,y)$
- $M(x^*,y^*) \leq V, M(x^*,y^*) \geq V \implies M(x^*,y^*) = V$
- $M(x^*,y) \leq V, \forall y$ and $M(x,y^*) \geq V, \forall x$. □

- As long as the first player can play a mixed strategy, then he will not be at a disadvantage.
- If both players can play mixed strategies, then they do not need to know the strategy of the other player.

Def. $\inf_x \sup_y M(x,y) = \sup_y \inf_x M(x,y)$ is called the value of the game. The two strategies x^* and y^* in the corollary are called the optimum strategies for R and C respectively.

Theorem (von Neumann (1928), Minimax Theorem)

$$\inf_x \max_{j\in[n]} M(x,j) = \sup_y \min_{i\in[m]} M(i,y).$$

- Can be proved by LP duality.

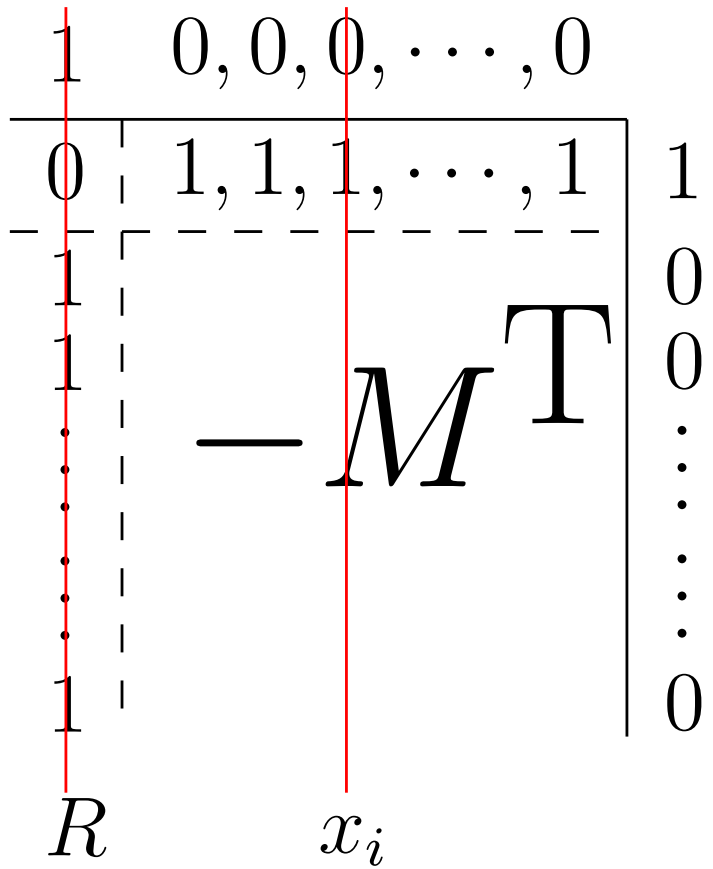
LP for Row Player

$$\begin{aligned} \min \quad & R \\ \sum_{i=1}^{m} x_i &= 1 \\ R - \sum_{i=1}^{m} M_{ij} x_i &\geq 0 \quad \forall j \in [n] \\ x_i &\geq 0 \quad \forall i \in [m] \end{aligned}$$

LP for Column Player

$$\begin{aligned} \max \quad & C \\ \sum_{j=1}^{n} y_j &= 1 \\ C - \sum_{j=1}^{n} M_{ij} y_j &\leq 0 \quad \forall i \in [m] \\ y_j &\geq 0 \quad \forall j \in [n] \end{aligned}$$

- The two LPs are dual to each other.



LP for Row Player

$$\begin{aligned} \min \quad & R \\ \sum_{i=1}^{m} x_i &= 1 \\ R - \sum_{i=1}^{m} M_{ij} x_i &\geq 0 \quad \forall j \in [n] \\ x_i &\geq 0 \quad \forall i \in [m] \end{aligned}$$

LP for Column Player

$$\begin{aligned} \max \quad & C \\ \sum_{j=1}^{n} y_j &= 1 \\ C - \sum_{j=1}^{n} M_{ij} y_j &\leq 0 \quad \forall i \in [m] \\ y_j &\geq 0 \quad \forall j \in [n] \end{aligned}$$

- The two LPs are dual to each other.

$x_i, i \in [m]$	primal variable ($\in \mathbb{R}_{\geq 0}$)	dual constraint ($\leq$)
$y_j, j \in [n]$	dual variable ($\in \mathbb{R}_{\geq 0}$)	primal constraint ($\geq$)
R	primal variable ($\in \mathbb{R}$)	dual constraint ($=$)
C	dual variable ($\in \mathbb{R}$)	primal constraint ($=$)

- Let V be the value of the game, x^* and y^* be the two optimum strategies. Complementary slackness implies:
  - If $x_i^* > 0$, then $M(i, y^*) = V$.
  - If $y_j^* > 0$, then $M(x^*, j) = V$.
- The game is called 0-sum game as the payoff for R is the negative of the payoff for C.

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Spanning Tree Polytope

- Given a connected graph $G = (V, E)$, $n := |V|$
- $\mathcal{P}_{\mathrm{ST}} := \mathrm{conv}\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subseteq V, 2 \leq |S| \leq n - 1 \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

- Spanning trees correspond to bases of graphic matroid for G
- The optional material in Lecture 2 proves this via matroid polytopes; we use the theorem here without that proof

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in E} x_e = n - 1$$

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$$x_e \geq 0 \qquad \forall e \in E$$

Q: How can we check if all constraints in (*) are satisfied?

A: $\xrightarrow{\text{reduce}}$ densest sub-graph $\xrightarrow{\text{reduce}}$ maximum flow

Separation for the Spanning Tree Polytope

- We need to check if $\exists S \subseteq V,\ |S| \geq 2 : \quad \frac{\sum_{e \in E[S]} x_e}{|S|-1} > 1$:

- Guess a vertex $v \in S$; set $w_v = 0$ and $w_u = 1$ for every $u \in V \setminus \{v\}$
- The problem becomes to check if $\exists S \subseteq V,\ v \in S,\ |S| \geq 2 : \quad \sum_{e \in E[S]} x_e > \sum_{u \in S} w_u$
- This is a (weighted) densest subgraph problem
- Exercise: It can be solved using maximum flow

Theorem (General Perfect Matching Polytope Theorem)
$\mathcal{P}_{\mathrm{GPM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\begin{aligned} \sum_{e \in \delta(v)} x_e &= 1 && \forall v \in V \\ \sum_{e \in E(S, V \setminus S)} x_e &\geq 1 && \forall S \subseteq V, |S| \text{ is odd} \qquad (*) \\ x_e &\geq 0 && \forall e \in E \end{aligned}$$

Q: How can we check if all constraints in (*) are satisfied?

A: Use the Gomory-Hu Tree structure.

- inequality in (*) can be replaced by $\sum_{e \in E[S]} x_e \leq \frac{|S|-1}{2}$
- more convenient to obtain general matching polytope

Remark

- For all the polytopes, we identified a set of linear inequalities that are sufficient to define the polytope.
- However, not all the constraints are facet-defining.
- Within the affine hull, an irredundant description has one inequality per facet. Other inequalities are redundant. (We keep them for convenience.)

- Example: in spanning tree polytope, $\sum_{e \in E[S]} x_e \leq |S| - 1$ is not needed if $(S, E[S])$ is disconnected, or contains a bridge and $|S| \geq 3$: sum the constraints for the pieces, together with $x_e \leq 1$ for the bridge.

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Focus on $\mathbb{R}^n$, $n \geq 2$ (for $n = 1$, bisect an interval):

- axis-aligned ellipsoid centered at c with axis lengths $a \in \mathbb{R}^n_{>0}$, $\mathcal{Q}_{c,a}$: $\left\{x \in \mathbb{R}^n : \sum_{i\in[n]} \frac{(x_i - c_i)^2}{a_i^2} \leq 1\right\}$
- axis-aligned half-ellipsoid: $\mathcal{R}_{c,a,w} := \left\{x \in \mathcal{Q}_{c,a} : w^{\mathrm{T}}(x - c) \geq 0\right\}$, $w \in \mathbb{R}^n \setminus \{0\}$

Lemma For any ellipsoid $\mathcal{Q}$ in $\mathbb{R}^n$, $n \geq 2$, and any central half $\mathcal{R}$, we can efficiently find an ellipsoid $\mathcal{Q}'$ such that

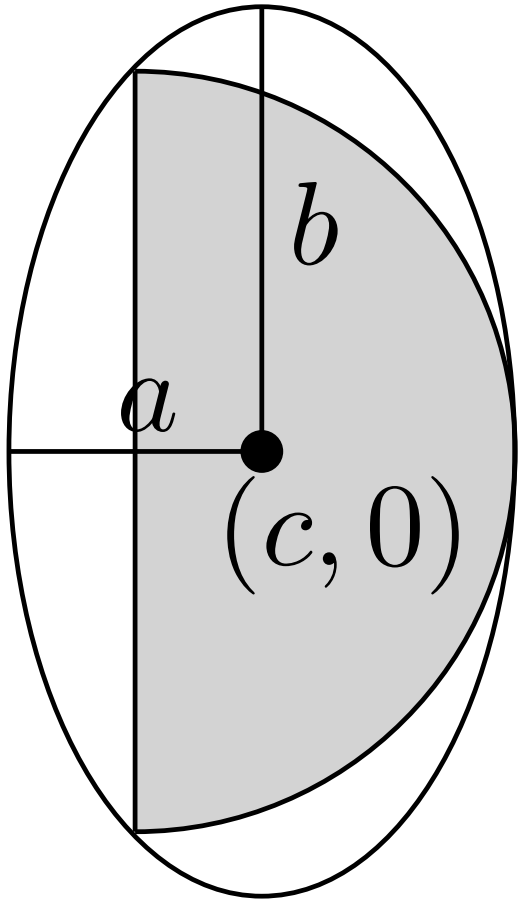
- $\mathcal{R} \subseteq \mathcal{Q}'$
- $\frac{\mathrm{vol}(\mathcal{Q}')}{\mathrm{vol}(\mathcal{Q})} \leq e^{-\frac{1}{2(n+1)}} = 1 - \Omega\left(\frac{1}{n}\right)$

Proof.

- An invertible affine map and a rotation reduce any central half-ellipsoid to the unit half-ball; volume ratios are preserved.
- half-ellipsoid becomes half ball: $\{x \in \mathbb{R}^n : |x|_2 \leq 1, x_1 \geq 0\}$

Proof.

- center of new ellipsoid : $(c, 0, \cdots, 0)$, $c \in [0, 1]$
- axis lengths: $(a, b, b, \cdots, b), a < b$
- the ellipsoid: $\frac{(x_1-c)^2}{a^2} + \sum_{i=2}^{n} \frac{x_i^2}{b^2} \leq 1$
- $(1, 0, 0, \cdots, 0)$ in ellipsoid: $\frac{(1-c)^2}{a^2} \leq 1$
- $(0, 1, 0, \cdots, 0)$ in ellipsoid: $\frac{c^2}{a^2} + \frac{1}{b^2} \leq 1$
- set a, b and c so that both constraints are tight: $a = 1 - c, b = \sqrt{\frac{(1-c)^2}{(1-c)^2-c^2}} = \frac{1-c}{\sqrt{1-2c}}$



Proof.

- we won't prove that the ellipsoid contains all points in half ball.
- volume of ellipsoid is minimized when $c = \frac{1}{n+1}$
- $a = \frac{n}{n+1}, b = \frac{n/(n+1)}{\sqrt{(n-1)/(n+1)}}$

Proof.

- $a = \frac{n}{n+1}, b = \frac{n/(n+1)}{\sqrt{(n-1)/(n+1)}}$

$$\frac{\text{vol(ellipsoid)}}{\text{vol(unit ball)}} = ab^{n-1} = \left(\frac{n}{n+1}\right)^n \cdot \left(\frac{n+1}{n-1}\right)^{\frac{n-1}{2}}$$

$$= \left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} \cdot \frac{n}{n+1}$$

$$\ln \frac{\text{vol(ellipsoid)}}{\text{vol(unit ball)}} = \frac{n-1}{2} \ln \frac{n^2}{n^2-1} + \ln \frac{n}{n+1}$$

$$\leq \frac{n-1}{2} \cdot \frac{1}{n^2-1} - \frac{1}{n+1} = -\frac{1}{2(n+1)}$$

- we used $\ln(1+x) \leq x, \forall x > -1$
- $\frac{\text{vol(ellipsoid)}}{\text{vol(unit ball)}} \leq e^{-\frac{1}{2(n+1)}}$

□

Lemma For any ellipsoid $\mathcal{Q}$ in $\mathbb{R}^n$, $n \geq 2$, and any central half $\mathcal{R}$, we can efficiently find an ellipsoid $\mathcal{Q}'$ such that

- $\mathcal{R} \subseteq \mathcal{Q}'$
- $\frac{\mathsf{vol}(\mathcal{Q}')}{\mathsf{vol}(\mathcal{Q})} \leq e^{-\frac{1}{2(n+1)}} = 1 - \Omega\left(\frac{1}{n}\right)$

Assumption

- The initial polytope is contained in a ball of radius R, where $R \leq 2^{\mathsf{poly(input\ size)}}$.
- When the polytope is not empty, it contains a ball of radius at least r, where $r \geq 1/2^{\mathsf{poly(input\ size)}}$.

- The first assumption is easy to guarantee; the second assumption needs some twisting.
- If no feasible center has been found, volume below that of a radius-r ball certifies emptiness under these assumptions.

- $R \leq 2^{\text{poly(input size)}}, \qquad r \geq 1/2^{\text{poly(input size)}}$
- number of iterations for ellipsoid method is at most

$$\log_{e^{\frac{1}{2(n+1)}}} \left(\frac{R}{r}\right)^n = n \cdot \frac{\ln(R/r)}{1/(2(n+1))} = O(n^2) \cdot \ln \frac{R}{r}$$
$$\leq O(n^2) \cdot \text{poly(input size)}$$