Advanced Algorithms (Fall 2026)
Extension Complexity（扩展复杂度） of Polytopes（多胞体）

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Outline

1. Extended Formulations（扩展表述）
2. The Permutation Polytope（置换多胞体）
3. Nonnegative Rank（非负秩） and Slack Matrices（松弛矩阵）
4. Lower Bounds on Extension Complexity
5. Optional Material
   - The Spanning-Tree Polytope（生成树多胞体）

Motivation

Typical Combinatorial Optimization（组合优化） Problem

Input: $[n]$: ground set（基集）
$\mathcal{S}$: feasible sets（可行集）: a family of subsets of $[n]$, often implicitly given
$w_i, i \in [n]$: values/costs of elements

Output: the set $S \in \mathcal{S}$ with the minimum/maximum $w(S) := \sum_{i \in S} w_i$

$\mathcal{P} := \text{conv}(\{\chi^S : S \in \mathcal{S}\})$: convex hull（凸包） of all valid solutions

LP（线性规划） to Solve Problem Exactly

$$\min / \max \quad \sum_{i=1}^{n} w_i x_i \quad \text{s.t.} \qquad x \in \mathcal{P}$$

- within its affine hull（仿射包）, the minimum number of inequalities describing $\mathcal{P}$ is facets($\mathcal{P}$) := the number of facets（刻面） of $\mathcal{P}$

Q: Can we do better?

A: Yes in some cases, by introducing new variables that we call auxiliary variables（辅助变量）.

Def. An extension of a polytope（多胞体） $\mathcal{P} \subseteq \mathbb{R}^n$ is a polyhedron（多面体） $\mathcal{Q} \subseteq \mathbb{R}^{n+r}$ for some $r \geq 0$, such that $\mathcal{P}$ is the projection（投影） of $\mathcal{Q}$ to $\mathbb{R}^n$:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^r, (x, y) \in \mathcal{Q} \right\}$$

LP to Solve Problem Exactly with Auxiliary Variables

$$\min / \max \quad \sum_{i=1}^{n} c_i x_i \quad \text{s.t.} \qquad (x, y) \in \mathcal{Q},$$

where $\mathcal{Q}$ is an extension of $\mathcal{P}$.

- To require $(x, y) \in \mathcal{Q}$, the number of inequalities we need is facets($\mathcal{Q}$)
- It may be possible that facets($\mathcal{Q}$) $\ll$ facets($\mathcal{P}$)

Def. The extension complexity（扩展复杂度） of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denoted as xc($\mathcal{P}$), is defined as follows:

$$\text{xc}(\mathcal{P}) := \min\{\text{facets}(\mathcal{Q}) : \mathcal{Q} \text{ is an extension of } \mathcal{P}\}.$$

Def. An extended formulation（扩展表述） of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is a set of linear constraints（线性约束）:

$$(E, F)\begin{pmatrix} x \\ y \end{pmatrix} = g$$

$$y \geq 0$$

where $E \in \mathbb{R}^{N \times n}, F \in \mathbb{R}^{N \times r}, g \in \mathbb{R}^N$ are given, and $x \in \mathbb{R}^n$ is the vector of main variables, $y \in \mathbb{R}^r$ is the vector of auxiliary variables. The following property needs to be satisfied:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \exists y \geq 0, (E, F)\begin{pmatrix} x \\ y \end{pmatrix} = g \right\}.$$

The complexity of the extended formulation is defined as r.

Def. (An alternative definition) The extension complexity of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denoted as $\mathrm{xc}(\mathcal{P})$, is defined as the minimum complexity of an extended formulation of $\mathcal{P}$.

The Equivalence Between the Two Definitions

- $\mathrm{xc}_1(\mathcal{P}), \mathrm{xc}_2(\mathcal{P})$: $\mathrm{xc}(\mathcal{P})$ according to the first/second definition

$\mathrm{xc}_2(\mathcal{P}) \leq \mathrm{xc}_1(\mathcal{P})$

- Given an extension $\mathcal{Q}$ of $\mathcal{P}$, we can use facets($\mathcal{Q}$) inequalities (and some equalities, if the dimension（维数） of $\mathcal{Q}$ is smaller than the dimension of its host space) to describe $\mathcal{Q}$, one for each facet.
- For the i-th inequality $a_i x \geq b_i$, we introduce a variable y_i, and replace the inequality by $y_i = a_i x - b_i, y_i \geq 0$.
- This gives an extended formulation of $\mathcal{P}$ with facets($\mathcal{Q}$) y-variables.
- Remark: there might be some auxiliary variables with no non-negativity constraints（非负约束）; but they can be removed.

$\mathrm{xc}_1(\mathcal{P}) \leq \mathrm{xc}_2(\mathcal{P})$

- An extended formulation with m y-variables defines a polyhedron with at most m facets.

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Example: Permutation Polytope（置换多胞体）

- $\mathcal{S} := \{x \in [n]^{[n]} : x \text{ is a permutation of } [n]\}$
- $\mathcal{P} := \text{conv}(\mathcal{S})$
- note: $\mathcal{P}$ has dimension $n-1$, as $\sum_{i\in[n]} x_i = \frac{n(n+1)}{2}$ is valid.

Lemma For any $S \subsetneq [n], S \neq \emptyset$, $\sum_{i\in S} x_i \geq \frac{|S|(|S|+1)}{2}$ is a facet of $\mathcal{P}$.

- so, $\text{facets}(\mathcal{P}) = 2^{\Omega(n)}$

Proof Sketch.

- The constraint（约束） $\sum_{i\in S} x_i \geq \frac{|S|(|S|+1)}{2}$ gives a face（面）
- To show it's a facet, need to prove its dimension is $n-2$
- We can find $x^0, x^1, \cdots, x^{n-2}$ on the face such that $x^1 - x^0, x^2 - x^0, \cdots, x^{n-2} - x^0$ are linearly independent（线性无关的）.

□

Representation using Permutation Matrices（置换矩阵）

- Represent a permutation（置换） $x \in [n]^{[n]}$ by the permutation matrix $M \in \{0,1\}^{n \times n}$ so that $M_{ij} = 1$ iff $x_i = j$.

$$\text{Example :} \qquad (3,1,2) \Longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- Crucial property: x is a linear function（线性函数） of entries in M
- $\mathcal{P}' := \text{conv}\big(\{M : M \text{ is a permutation matrix}\}\big)$

Lemma $\mathcal{P}' = \big\{y \in [0,1]^{n \times n} : \sum_i y_{i,j} = 1, \forall j; \sum_j y_{i,j} = 1, \forall i\big\}$.

Proof.

- permutation $\Longleftrightarrow$ perfect matching（完美匹配） in complete bipartite graph（完全二分图） over $2n$ vertices（顶点）
- permutation matrix polytope $\Longleftrightarrow$ perfect matching polytope（完美匹配多胞体）

□

Extended Formulation of $\mathcal{P}$

$$\sum_{i \in [n]} y_{i,j} = 1 \qquad \forall j \in [n]$$

$$\sum_{j \in [n]} y_{i,j} = 1 \qquad \forall i \in [n]$$

$$y_{ij} \geq 0 \qquad \forall i, j \in [n]$$

$$x_i = \sum_{j=1}^{n} j \cdot y_{ij} \qquad \forall i \in [n]$$

Lemma The permutation polytope $\mathcal{P}$ has extension complexity $O(n^2)$.

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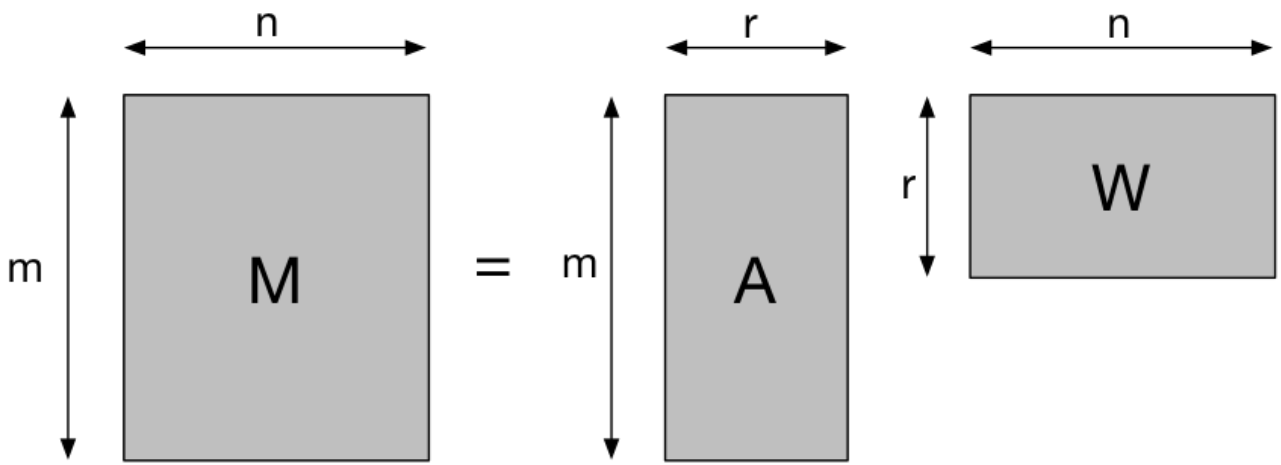
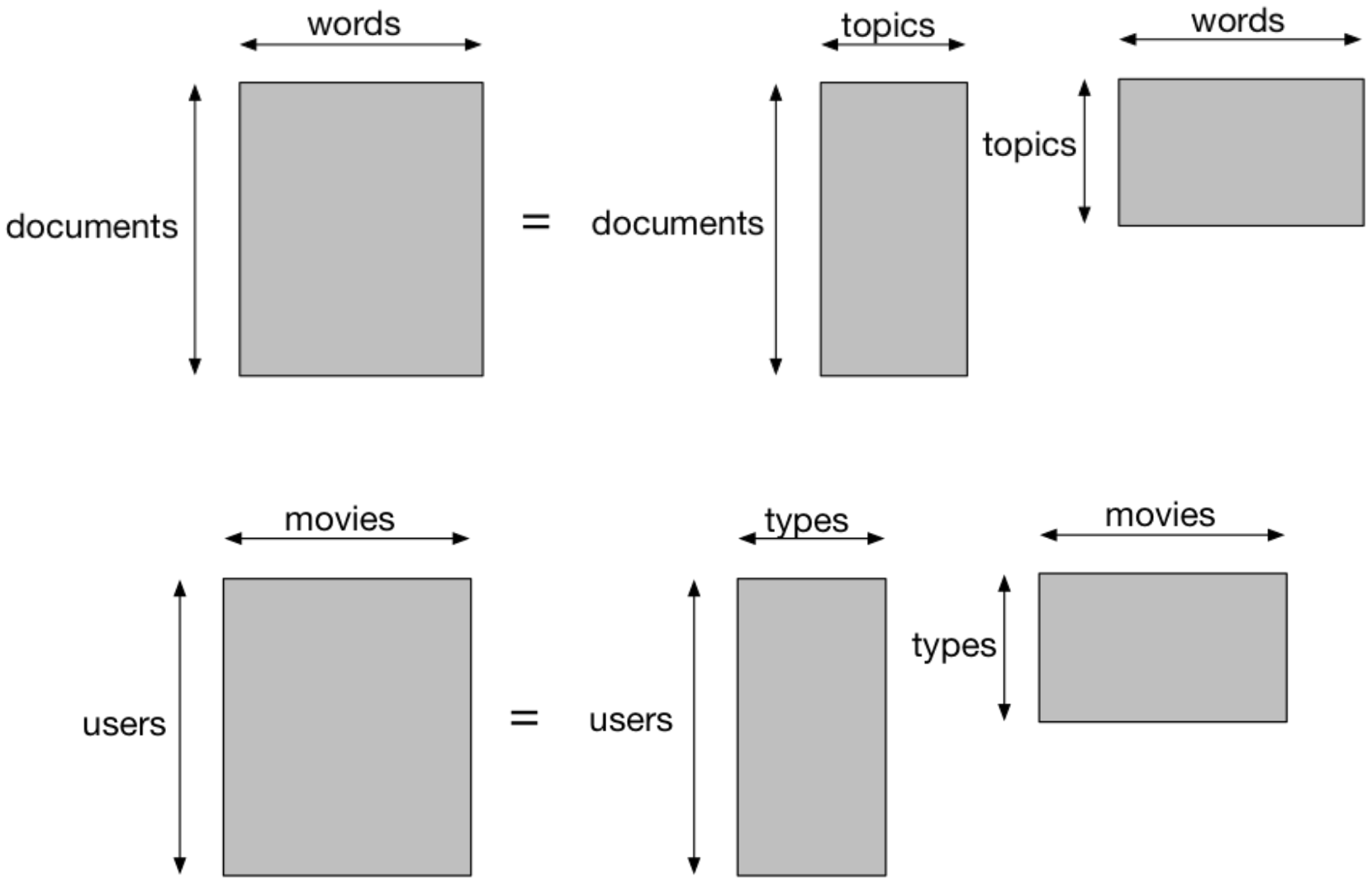
Def. The non-negative rank（非负秩） of a matrix（矩阵） $M \in \mathbb{R}_{\geq 0}^{m \times n}$ is the minimum $r \geq 0$ such that there are matrices $L \in \mathbb{R}_{\geq 0}^{m \times r}$ and $R \in \mathbb{R}_{\geq 0}^{r \times n}$ such that $M = LR$. We use $\text{rank}_+(M)$ to denote the non-negative rank of M.



- if we allow $L \in \mathbb{R}^{m \times r}$ and $R \in \mathbb{R}^{r \times n}$, then the non-negative rank becomes the rank（秩）
- the rank of a matrix can be computed efficiently
- it is NP-hard（NP困难的） to compute the non-negative rank of a matrix

Application of Non-Negative Rank



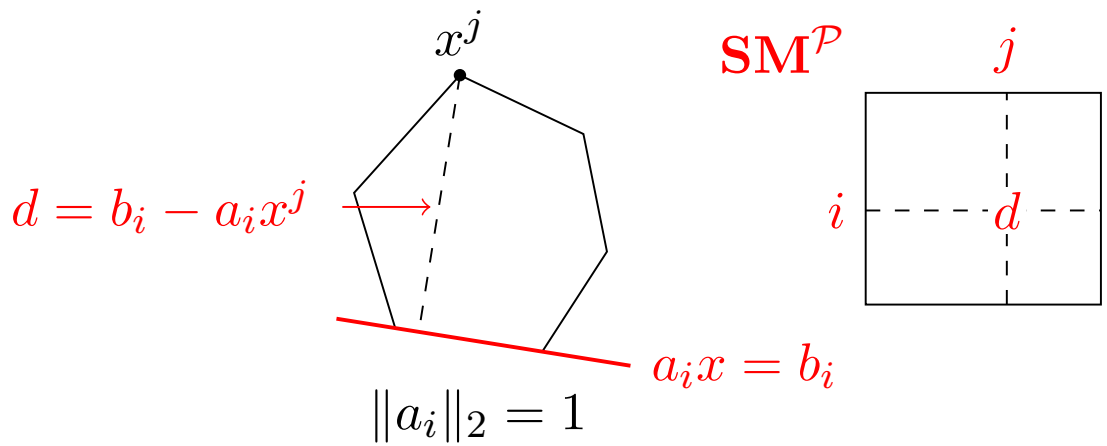
Def. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be defined as

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b; Ex = f\},$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $E \in \mathbb{R}^{m' \times n}$, $f \in \mathbb{R}^{m'}$. Assume the equations $Ex = f$ are linearly independent, and there is a 1-1 correspondence between inequalities in $Ax \leq b$ and facets of $\mathcal{P}$.
Let $x^1, x^2, \cdots, x^v$ be all the vertices of $\mathcal{P}$. The slack matrix（松弛矩阵） $\mathbf{SM}^{\mathcal{P}}$ of $\mathcal{P}$ w.r.t this description is a matrix in $\mathbb{R}^{m \times v}_{\geq 0}$ such that

$$\mathbf{SM}^{\mathcal{P}}_{i,j} = b_i - a_i x^j.$$

Here a_i is the i-th row vector（行向量） of A.



Slack Matrix Theorem（松弛矩阵定理）

Theorem [Yannakakis 91] For any polytope $\mathcal{P}$, we have $\mathrm{xc}(\mathcal{P}) = \mathrm{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

Notes

- Considering non-vertex points in $\mathcal{P}$ for the columns of $\mathbf{SM}^{\mathcal{P}}$ does not increase its non-negative rank
- Considering non-facet faces of $\mathcal{P}$ for rows of $\mathbf{SM}^{\mathcal{P}}$ does not increase its non-negative rank

Theorem [Yannakakis 91] For any polytope $\mathcal{P}$, we have $\mathsf{xc}(\mathcal{P}) = \mathsf{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

Proof of $\mathsf{xc}(\mathcal{P}) \leq \mathsf{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

- Given non-negative decomposition（非负分解） $\mathbf{SM}^{\mathcal{P}} = FV$ with $F \in \mathbb{R}^{m \times r}_{\geq 0}$ and $V \in \mathbb{R}^{r \times v}_{\geq 0}$
- we show the following is an extended formulation of $\mathcal{P}$ with complexity r:

$$\mathcal{P}' = \{x : Ex = f,\ \exists y \geq 0,\ Ax + Fy = b\}$$

- if $x \in \mathcal{P}'$, then $Ax \leq b$ and $Ex = f$ $\qquad \mathcal{P}' \subseteq \mathcal{P}$
- fix vertex x^j: $b - Ax^j$ is the j-th column of $\mathbf{SM}^{\mathcal{P}}$
- it is a non-negative combination（非负组合） of columns of F
- so, $\exists y \geq 0$ with $b - Ax^j = Fy$ $\qquad \mathcal{P} \subseteq \mathcal{P}'$ $\square$

Slack Matrix Theorem

Proof of $\mathrm{xc}(\mathcal{P}) \geq \mathrm{rank}_+(\mathbf{SM}^{\mathcal{P}})$

- Write the extension as $Ex + Fy = g$, $y \geq 0$, with N equations and r auxiliary variables; x is free.
- For facet i, maximize $a_i x$ over this extension. Its optimum is b_i, because the facet is tight（紧的） on $\mathcal{P}$.
- Strong duality（强对偶性） gives $\mu^i \in \mathbb{R}^N$ with

$$\mu^i E = a_i, \qquad \mu^i F \geq 0, \qquad \mu^i g = b_i.$$

- Put $\nu^i = \mu^i F$. For a feasible lift（可行提升） y^j of vertex x^j,

$$b_i - a_i x^j = \mu^i (g - E x^j) = \nu^i y^j.$$

Slack Matrix Theorem

Theorem [Yannakakis 91] For any polytope $\mathcal{P}$, we have $\mathrm{xc}(\mathcal{P}) = \mathrm{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

Proof of $\mathrm{xc}(\mathcal{P}) \geq \mathrm{rank}_+(\mathbf{SM}^{\mathcal{P}})$.

- $b_i - a_i x^j = \nu^i y^j$
- Then,

$$\mathbf{SM}^{\mathcal{P}} = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \vdots \\ \nu^m \end{pmatrix} (y^1, y^2, \cdots, y^v)$$

- This is a nonnegative factorization of inner dimension r, so $\mathrm{rank}_+(\mathbf{SM}^{\mathcal{P}}) \leq r$. □

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Travelling Salesman Problem（旅行商问题） (TSP) Polytope

- Given the complete graph（完全图） $G = (V, \binom{V}{2})$
- $\mathcal{P}_{\text{TSP}} := \text{conv}(\{\chi^S : S \subseteq \binom{V}{2} \text{ is a TSP tour of } V\})$

Cut Polytope（割多胞体）

- $G = (V, E) = K_n$: the complete graph
- $\mathcal{P}_{\text{cut}} := \text{conv}\left(\left\{\chi^{E(S, V\setminus S)} : S \subseteq V\right\}\right)$

Correlation Polytope（相关多胞体）

- $\mathcal{P}_{\text{corr}} = \text{conv}\left(\left\{bb^{\text{T}} : b \in \{0,1\}^n\right\}\right)$.

- [Samuel Fiorini, Serge Massar, Sebastian Pokutta, Hans Raj Tiwary and Ronald de Wolf]: "Exponential Lower Bounds for Polytopes in Combinatorial Optimization": The TSP, cut（割）, and correlation polytope families have exponential extension complexity.
- 2023 Godel Prize Winner Paper

General Matching Polytope（一般图匹配多胞体）

- Given a graph（图） $G = (V, E)$
- $\mathcal{P}_{\mathrm{GM}} := \mathrm{conv}\left(\left\{\chi^M : M \subseteq E \text{ is a matching in } G\right\}\right)$

Theorem (General Matching Polytope Theorem) $\mathcal{P}_{\mathrm{GM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\begin{aligned} \sum_{e \in \delta(v)} x_e &\leq 1 && \forall v \in V \\ \sum_{e \in E(S)} x_e &\leq \frac{|S| - 1}{2} && \forall S \subseteq V, |S| \text{ is odd} \\ x_e &\geq 0 && \forall e \in E \end{aligned} \tag{1}$$

- [Rothvoss 2017]: "The Matching Polytope（匹配多胞体） has Exponential Extension Complexity." 2023 Godel Prize Winner Paper

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Spanning Tree Polytope（生成树多胞体）

We use the following theorem (stated here without proof):

Spanning Tree Polytope

- Given a connected graph（连通图） $G = (V, E)$, $n := |V|$
- $\mathcal{P}_{\mathrm{ST}} := \mathrm{conv}\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subseteq V, 2 \leq |S| \leq n - 1 \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

- Choose a root $r \in V$ arbitrarily. We use a flow（流） formulation:
- For any spanning tree（生成树）, we direct the edges（边） from r to leaves（叶节点）: the tree becomes an out-arborescence（外向树形图） rooted at r
- $y_{u \to v}$: whether (u, v) is a directed edge in the arborescence（树形图）.

$$\sum_{(u,v) \in E} y_{u \to v} = 1 \qquad \forall v \in V \setminus \{r\}$$

$$y_{v \to r} = 0 \qquad \forall (v, r) \in E$$

$$y_{u \to v} \geq 0 \qquad \forall u, v \text{ with } (u, v) \in E$$

$$x_{\{u,v\}} = y_{u \to v} + y_{v \to u} \qquad \forall (u, v) \in E$$

y supports 1 unit flow from r to v $\qquad \forall v \in V \setminus \{r\} \qquad (\dagger)$

- $(\dagger)$ for every v can be captured using a maximum-flow（最大流） LP, with $O(|E|)$ variables and constraints.

Theorem The formulation is an extended formulation of $\mathcal{P}_{\mathrm{ST}}$.

Proof.

- For any ST T of G, χ^T (with extension) is a valid solution
- Remaining goal: prove that every valid (x, y) satisfies:

$$\sum_{e \in E} x_e = n - 1 \tag{2}$$

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subseteq V, 2 \leq |S| \leq n - 1 \tag{3}$$

- every $v \in V \setminus \{r\}$ has 1 fractional incoming edge
- $\implies$ total fractional number of edges is $n - 1$ $\implies$ (1)

Theorem The formulation is an extended formulation of $\mathcal{P}_{\mathrm{ST}}$.

Proof.

- Remaining goal: prove that every valid (x, y) satisfies:

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subseteq V, 2 \leq |S| \leq n-1 \tag{2}$$

- Focus on $S \ni r$: $|S| - 1$ fractional edges with head in S
- Focus on $S \not\ni r, |S| \geq 2$. Let $v \in S$ be arbitrary.
- y supports 1 unit $r \to v$ flow
  $\implies \geq 1$ fractional edge from $V \setminus S$ to S
  $\implies$ at most $|S| - 1$ fractional edges inside S □

- When G is complete graph, $\mathcal{P}_{\mathrm{ST}}$ has $O(n^3)$ extension complexity
- The lower bound is $\Omega(n^2)$
- Big open problem to close the gap.

Appendix: 英汉术语对照 (1/2)

English	中文	English	中文
affine hull	仿射包	arborescence	树形图
auxiliary variable	辅助变量	combinatorial optimization	组合优化
complete bipartite graph	完全二分图	complete graph	完全图
connected graph	连通图	constraint	约束
convex hull	凸包	correlation polytope	相关多胞体
cut	割	cut polytope	割多胞体
dimension	维数	edge	边
extended formulation	扩展表述	extension complexity	扩展复杂度
face	面	facet	刻面
feasible lift	可行提升	feasible set	可行集
flow	流	general matching polytope	一般图匹配多胞体
graph	图	ground set	基集
leaf	叶节点	linear constraint	线性约束
linear function	线性函数	linear programming	线性规划

Appendix: 英汉术语对照 (2/2)

English	中文	English	中文
linearly independent	线性无关的	matching polytope	匹配多胞体
matrix	矩阵	maximum flow	最大流
non-negative combination	非负组合	non-negativity constraint	非负约束
nonnegative factorization	非负分解	nonnegative rank	非负秩
NP-hard	NP困难的	out-arborescence	外向树形图
perfect matching	完美匹配	perfect matching polytope	完美匹配多胞体
permutation	置换	permutation matrix	置换矩阵
permutation polytope	置换多胞体	polyhedron	多面体
polytope	多胞体	projection	投影
rank	秩	row vector	行向量
slack matrix	松弛矩阵	slack matrix theorem	松弛矩阵定理
spanning tree	生成树	spanning tree polytope	生成树多胞体
strong duality	强对偶性	tight	紧的
travelling salesman problem	旅行商问题	vertex	顶点