# Advanced Algorithms (Fall 2026)
# Semi-Definite Programming（半正定规划）

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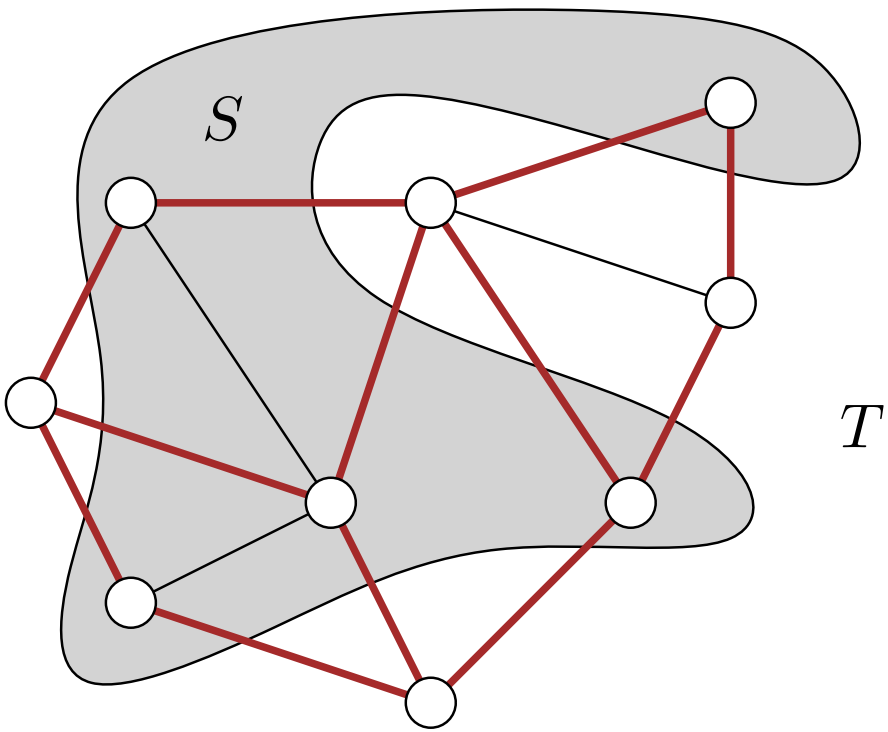
# Outline


1. Max-Cut（最大割） and Relaxations（松弛）
2. Positive Semidefinite Matrices（半正定矩阵） and SDP
3. Goemans–Williamson Rounding（Goemans–Williamson舍入）
4. Optional Material
   - Semidefinite Programming Duality（半正定规划对偶）

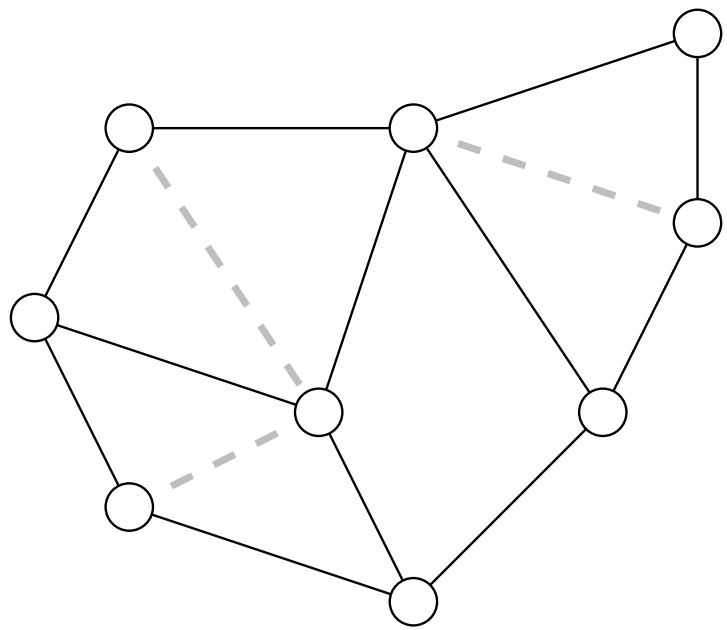
## Maximum Cut（最大割） Problem

**Input:** $G = (V, E)$,

**Output:** a partition $(S \subseteq V, T := V \setminus S)$ of $V$ so as to maximize $|E(S,T)|$, where $E(S,T) = \{uv \in E : |\{u,v\} \cap S| = 1\}$




- Min-Uncut（最小不割边问题）: remove minimum number of edges（边） to make graph（图） bipartite（二分图）



- Max-Cut = Min-Uncut for exact algorithms（精确算法）, but not the same for approximation algorithms（近似算法）
- Recap: $1/2$-approximation algorithms for Max-Cut:

**Randomized Algorithm（随机算法）**

$S \leftarrow \emptyset$
**for** every $u \in V$ **do**
    with probability（概率）$1/2$, add $u$ to $S$
**return** $(S, V \setminus S)$

**Greedy Algorithms（贪心算法）**

$S \leftarrow \emptyset, T \leftarrow \emptyset$
**for** every $u \in V$ **do**
    **if** $|E(u, S)| > |E(u, T)|$ **then**
        $T \leftarrow T \cup \{u\}$
    **else**
        $S \leftarrow S \cup \{u\}$
**return** $(S, T)$

- Local Search（局部搜索）: while we can improve the solution by switching the side of one vertex（顶点）, perform the operation, stop if no swapping can improve the solution

# Linear Programming Relaxation（线性规划松弛）

## First Attempt

- $y_v, v \in V$: if $v \in S$
- $x_{uv}, uv \in E$: if $uv$ is cut（割）

$$\max \quad \sum_{uv \in E} x_{uv}$$

$$x_{uv} \leq |y_u - y_v| \qquad \forall uv \in E$$

$$y_v \in [0, 1] \qquad \forall v \in V$$

- $x_{uv} \leq |y_u - y_v|$ is not linear
- feasible region（可行域） is not convex（凸的）:

| $y_u$ | $y_v$ | $x_{uv}$ | Y/N |
|---|---|---|---|
| 1 | 0 | 0.5 | Y |
| 0 | 1 | 0.5 | Y |
| 0.5 | 0.5 | 0.5 | N |

- $x_{uv} \geq |y_u - y_v|$ can be replaced by $x_{uv} \geq y_u - y_v$ and $x_{uv} \geq y_v - y_u$

## Second Attempt

- $x_{uv}, uv \in \binom{V}{2}$: whether $uv$ is cut

$$\max \quad \sum_{uv \in E} x_{uv}$$

$$x_{uv} + x_{vw} + x_{uw} \leq 2 \qquad \forall \text{ distinct } u, v, w \in V$$

$$x_{uv} \in [0, 1] \qquad \forall \{u, v\} \in \binom{V}{2}$$

- The ratio lp/opt can approach 2: triangle-free graphs（无三角形图） can have opt $\approx |E|/2$, while $x_{uv} = 1$ on edges and 0 elsewhere gives lp $= |E|$.

## Quadratic Program（二次规划）

- $y_v = \begin{cases} 1 & \text{if } v \in S \\ -1 & \text{if } v \notin S \end{cases}$

$$\max \quad \frac{1}{2} \sum_{uv \in E} (1 - y_u y_v)$$

$$y_v \in \{\pm 1\} \quad \forall v \in V$$

## Semi-Definite Program（半正定规划）

- $y_v \in \mathbb{R}^n, \forall v \in V$

$$\max \quad \frac{1}{2} \sum_{uv \in E} (1 - \langle y_u, y_v \rangle)$$

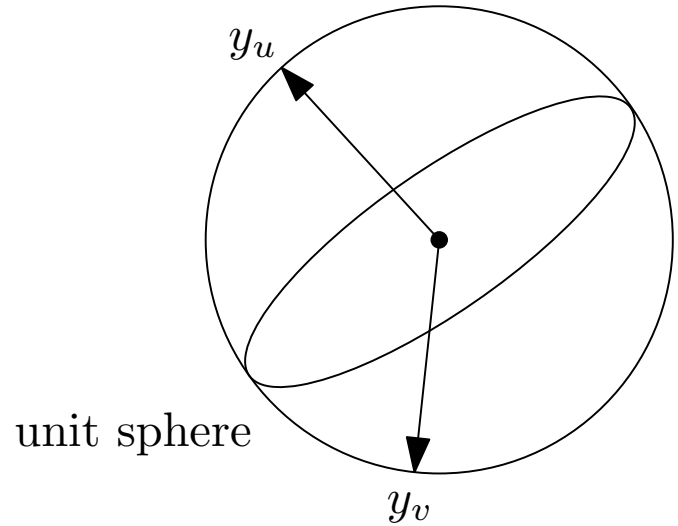
$$|y_v| = 1 \quad \forall v \in V$$

- $\langle y_u, y_v \rangle = y_u^{\mathrm{T}} y_v = \sum_{i=1}^{n} y_{u,i} \cdot y_{v,i}$: inner product（内积） of $y_u$ and $y_v$
- requiring $y_v \in \mathbb{R}^n$ is the same as requiring $y_v \in \mathbb{R}^{n'}$ for any $n' \geq n$

## SDP for Max-Cut

$$\max \quad \frac{1}{2}\sum_{uv\in E}(1-\langle y_u, y_v\rangle)$$

$$|y_v| = 1 \quad \forall v \in V$$




- SDP is a relaxation（松弛）:

$$y_v = \begin{cases} (1,0,0,0,\cdots,0) & \text{if } v \in S \\ (-1,0,0,0,\cdots,0) & \text{if } v \in T \end{cases}$$

- sdp: the value of the SDP, $\quad$ sdp $\geq$ opt

**Q:** Can we solve the SDP?

**A:** Yes

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**Def.** A symmetric matrix（对称矩阵） $X \in \mathbb{R}^{n \times n}$ is Positive Semi-Definite（半正定的） (PSD) if $\forall y \in \mathbb{R}^n$, we have $y^{\mathrm{T}} X y \geq 0$. Use $X \succeq 0$ to denote $X$ is PSD.

- $X \succeq X'$ means $X - X' \succeq 0$.

**Lemma** The following statements are equivalent for a symmetric matrix $X \in \mathbb{R}^{n \times n}$:

- $X \succeq 0$
- All the $n$ eigenvalues（特征值） of $X$ are non-negative
- $X = V^{\mathrm{T}} V$ for some $V \in \mathbb{R}^{m \times n}, m \leq n$
- $X = \sum_{u=1}^{n} \lambda_u w_u w_u^{\mathrm{T}}$ for some reals $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ and orthonormal（标准正交的） basis（基） $\{w_u\}_{u \in [n]}$

# Semi-definite Programming (SDP)

- matrices（矩阵） of size $n \times n \equiv$ flattened vectors of length $n^2$:
  - Use $\cdot$ as multiplication for flattened matrices,
  - $X \succeq 0$: view $X$ as a matrix.
- $A \in \mathbb{R}^{m \times n^2}, b \in \mathbb{R}^m, c \in \mathbb{R}^{n^2}$
- Assume $A_k$'s and $c$ are symmetric matrices of size $n \times n$

## Semi-Definite Program

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ A \cdot X &\geq b \\ X &\succeq 0 \end{aligned}$$

## An equivalent formulation

$$\begin{aligned} \min \quad & \sum_{u,v \in [n]} c_{u,v} \cdot \langle y_u, y_v \rangle \\ \sum_{u,v} a_{k,u,v} \langle y_u, y_v \rangle &\geq b_k \quad \forall k \in [m] \\ y_v &\in \mathbb{R}^n \quad \forall v \in [n] \end{aligned}$$

- requiring $y_v \in \mathbb{R}^n$ is the same as requiring $y_v \in \mathbb{R}^{n'}$ for any $n' \geq n$

## Semi-Definite Program

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ & A \cdot X \geq b \\ & X \succeq 0 \end{aligned}$$

## Example

$$\begin{aligned} \min \quad & 5y_1 + 6y_2 + 7y_3 \\ & y_1 + 3y_2 + 4y_3 \geq 5 \\ & 2y_1 + 3y_2 + y_3 \geq 10 \\ & 3y_1 + 2y_2 + 2y_3 \geq 7 \\ & \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \succeq 0 \end{aligned}$$

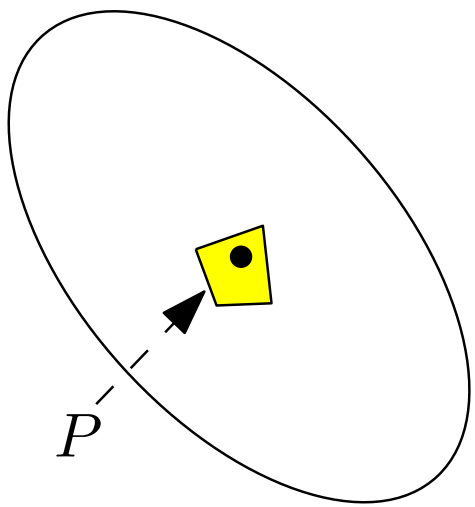
- $X \succeq 0 \Longleftrightarrow X_{u,v} = X_{v,u}, \forall u, v \in [n]; (yy^{\mathrm{T}}) \cdot X \geq 0, \forall y \in \mathbb{R}^n$.
- SDP $\equiv$ LP （线性规划） with infinite number of linear constraints （线性约束）

## Separation Oracle（分离预言机） $\mathcal{O}$

- Given a symmetric $X \in \mathbb{R}^{n \times n}$, we need to either claim $X \succeq 0$, or return a $y \in \mathbb{R}^n$ such that $y^{\mathrm{T}} X y < 0$.
- An eigensolver（特征值求解器） (e.g., the QR algorithm（QR算法）) finds an eigenvector（特征向量） with negative eigenvalue, if one exists.

## Recall: Ellipsoid Method（椭球法）

- maintain an ellipsoid（椭球） that contains the feasible region
- query $\mathcal{O}$ if the center of ellipsoid is in the feasible region:
  - yes: then the feasible region is not empty
  - no: cut the ellipsoid in half, find smaller ellipsoid to enclose the half-ellipsoid, and repeat



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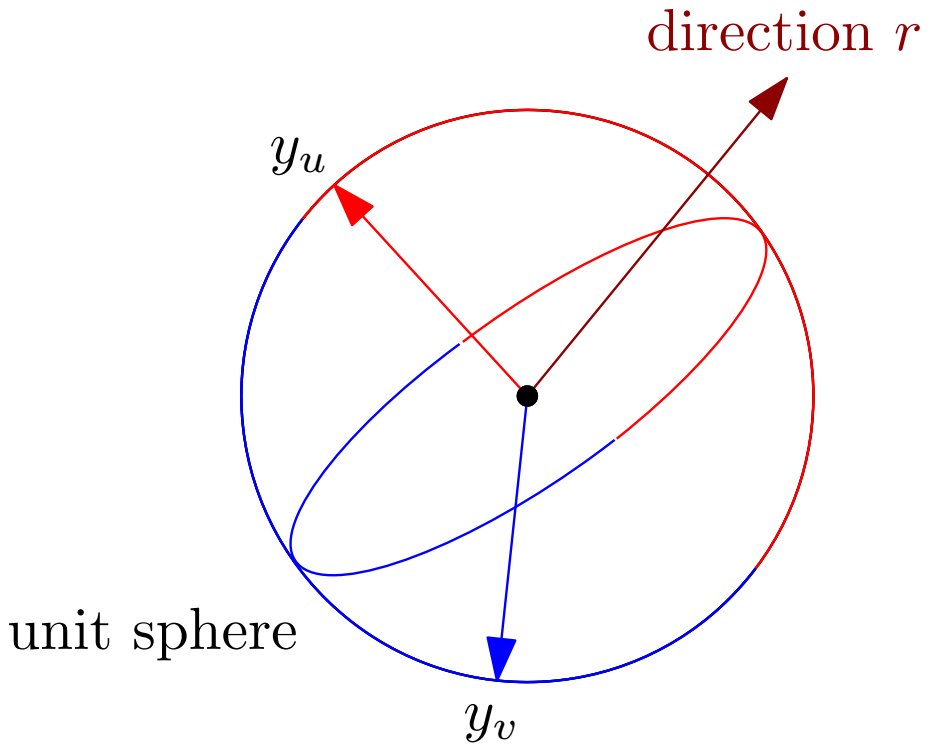
## SDP for Max-Cut

$$\max \quad \frac{1}{2}\sum_{uv\in E}(1-\langle y_u, y_v\rangle)$$

$$|y_v| = 1 \quad \forall v \in V$$




- let $(y_v)_{v\in V}$ be the vectors obtained from solving SDP
- sdp $= \frac{1}{2}\sum_{uv\in E}(1-y_u^{\mathrm{T}}y_v) \geq$ opt

## [Goemans-Williamson'95] Rounding Algorithm（舍入算法）

1: randomly choose a direction $r \in \mathbb{R}^n$:
   - choose each $r_u \sim N(0,1)$ i.i.d（独立同分布）

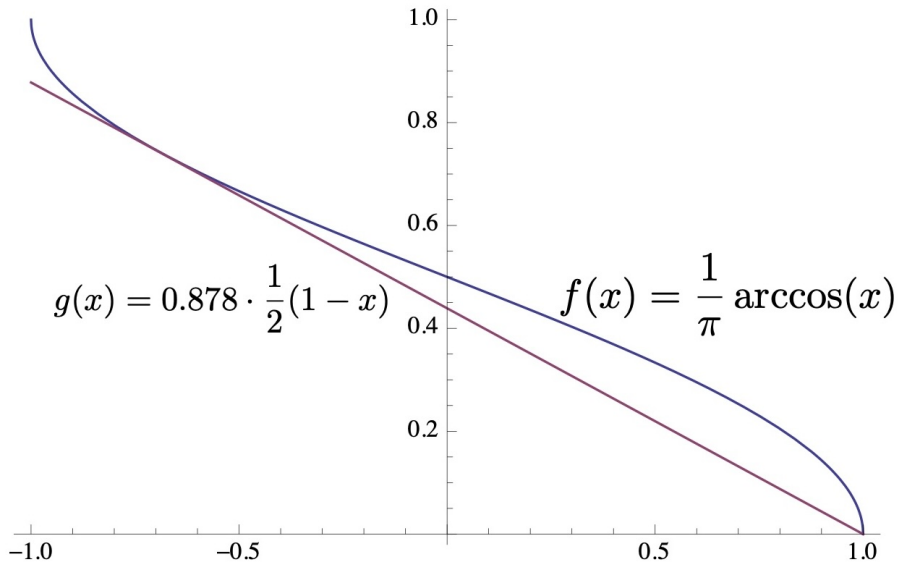
   $N(0,1)$: standard normal distribution（标准正态分布）

2: $\bar{y}_v = \mathrm{sgn}(\langle y_v, r\rangle)$, $S = \{v \in V : \bar{y}_v > 0\}$, **return** $(S, V \setminus S)$

$$\Pr[uv \text{ is cut}] = \frac{\text{radian angle between } y_u \text{ and } y_v}{\pi} = \frac{\arccos\langle y_u, y_v\rangle}{\pi}$$

$$x := \langle y_u, y_v\rangle \in [-1, 1),$$

$$\frac{\Pr[uv \text{ is cut}]}{\frac{1}{2}(1-x)} = \frac{2\arccos(x)}{\pi(1-x)}.$$




- $\alpha_{\mathsf{GW}} := \inf_{x\in[-1,1)} \frac{2}{\pi} \cdot \frac{\arccos(x)}{(1-x)} \geq 0.878$

$$\mathbb{E}[|E(S,T)|] = \sum_{uv \in E} \Pr[uv \text{ is cut}] \geq \alpha_{\mathsf{GW}} \sum_{uv \in E} \frac{1}{2}(1 - \langle y_u, y_v \rangle)$$
$$= \alpha_{\mathsf{GW}} \cdot \mathsf{sdp} \geq \alpha_{\mathsf{GW}} \cdot \mathsf{opt} \geq 0.878 \cdot \mathsf{opt}.$$

- Assuming Unique Games Conjecture（唯一博弈猜想） (UGC), no polynomial-time（多项式时间） algorithm can give an approximation ratio（近似比） of $\alpha_{\mathsf{GW}} + \epsilon$ for any constant $\epsilon > 0$.

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# Duality（対偶性） for Semi-Definite Programming

**Semi-Definite Program**

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ & A \cdot X \geq b \\ & X \succeq 0 \end{aligned}$$

**Semi-Definite Program**

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ & \sum_{u,v \in [n]} a_{k,u,v} X_{u,v} \geq b_k \qquad \forall k \in [m] \\ & \sum_{u,v \in [n]} r_u r_v X_{u,v} \geq 0 \qquad \forall r \in \mathbb{R}^n \end{aligned}$$

- replace $X \succeq 0$ with infinite number of linear constraints: $(rr^{\mathrm{T}}) \cdot X \geq 0, \forall r \in \mathbb{R}^n$.
- Symmetry can be imposed without loss: replace $X$ by $(X + X^{\mathrm{T}})/2$, since $A_k$ and $c$ are symmetric.

# Duality for Semi-Definite Programming

**Semi-Definite Program**

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ A \cdot X &\geq b \\ X &\succeq 0 \end{aligned}$$

**Semi-Definite Program**

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ \sum_{u,v \in [n]} a_{k,u,v} X_{u,v} &\geq b_k && \forall k \in [m] \\ \sum_{u,v \in [n]} r_u r_v X_{u,v} &\geq 0 && \forall r \in \mathbb{R}^n \end{aligned}$$

Dual（对偶问题）：

$$\begin{aligned} \max \quad & \textstyle\sum_{k=1}^{m} b_k \cdot y_k \\ \sum_{k=1}^{m} a_{k,u,v} \cdot y_k + \sum_{r \in \mathbb{R}^n} r_u r_v \cdot z_r &= c_{u,v} && \forall u, v \in [n] \\ y_k &\geq 0 && \forall k \in [m] \\ z_r &\geq 0 && \forall r \in \mathbb{R}^n \end{aligned}$$

# Duality for Semi-Definite Programming

Dual : $\max \quad \sum_{k=1}^{m} b_k \cdot y_k$

$$\sum_{k=1}^{m} a_{k,u,v} \cdot y_k + \sum_{r \in \mathbb{R}^n} r_u r_v \cdot z_r = c_{u,v} \qquad \forall u, v \in [n]$$

$$y_k \geq 0 \qquad \forall k \in [m]$$

$$z_r \geq 0 \qquad \forall r \in \mathbb{R}^n$$

- Take $z$ with finite support（有限支撑）; each PSD matrix（半正定矩阵）needs at most $n$ terms in this sum.
- first red constraint（约束） $\Leftrightarrow A^{\mathrm{T}} y + \sum_{r \in \mathbb{R}^n} z_r \cdot r r^{\mathrm{T}} = c$
- $\sum_{r \in \mathbb{R}^n} z_r \cdot r r^{\mathrm{T}}$ is PSD
- moreover, any PSD matrix can be written in this form
- $\Longrightarrow$ red constraints can be replaced by $A^{\mathrm{T}} y \preceq c$

# Duality for Semi-Definite Programming

## Semi-Definite Program

$$\min \quad c^{\mathrm{T}} \cdot X$$

$$A \cdot X \geq b$$

$$X \succeq 0$$

## Dual for SDP

$$\max \quad b^{\mathrm{T}} y$$

$$A^{\mathrm{T}} y \preceq c$$

$$y \geq 0$$

- Linear Program: $X \geq 0$
- In Dual of LP: $A^{\mathrm{T}} y \leq c$

# Appendix: 英汉术语对照 (1/2)

| **English** | 中文 | **English** | 中文 |
|---|---|---|---|
| approximation algorithm | 近似算法 | approximation ratio | 近似比 |
| basis | 基 | bipartite graph | 二分图 |
| constraint | 约束 | convex | 凸的 |
| cut | 割 | dual | 对偶问题 |
| duality | 对偶性 | edge | 边 |
| eigensolver | 特征值求解器 | eigenvalue | 特征值 |
| eigenvector | 特征向量 | ellipsoid | 椭球 |
| ellipsoid method | 椭球法 | exact algorithm | 精确算法 |
| feasible region | 可行域 | finite support | 有限支撑 |
| Goemans–Williamson rounding | Goemans–Williamson舍入 | graph | 图 |
| greedy algorithm | 贪心算法 | i.i.d | 独立同分布 |
| inner product | 内积 | linear constraint | 线性约束 |

# Appendix: 英汉术语对照 (2/2)

| English | 中文 | English | 中文 |
|---|---|---|---|
| linear programming | 线性规划 | local search | 局部搜索 |
| LP relaxation | 线性规划松弛 | matrix | 矩阵 |
| Max-Cut | 最大割 | Min-Uncut | 最小不割边问题 |
| orthonormal | 标准正交的 | polynomial time | 多项式时间 |
| positive semidefinite | 半正定的 | positive semidefinite matrix | 半正定矩阵 |
| probability | 概率 | QR algorithm | QR算法 |
| quadratic program | 二次规划 | randomized algorithm | 随机算法 |
| relaxation | 松弛 | rounding algorithm | 舍入算法 |
| semidefinite programming | 半正定规划 | semidefinite programming duality | 半正定规划对偶 |
| separation oracle | 分离预言机 | standard normal distribution | 标准正态分布 |
| symmetric matrix | 对称矩阵 | triangle-free graph | 无三角形图 |
| Unique Games Conjecture | 唯一博弈猜想 | vertex | 顶点 |