Advanced Algorithms (Fall 2026)

Semi-Definite Programming

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Outline

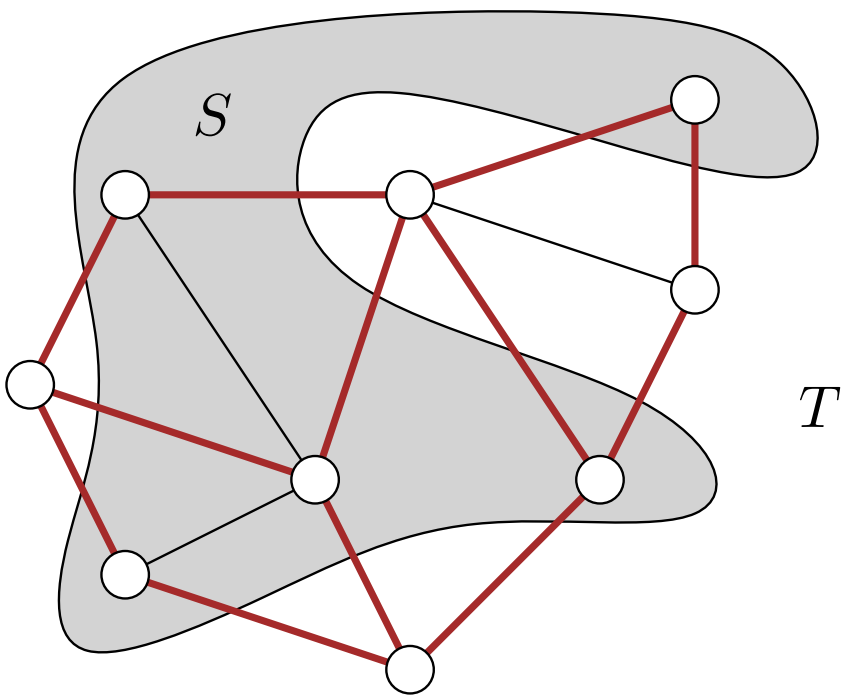
1. Max-Cut and Relaxations
2. Positive Semidefinite Matrices and SDP
3. Goemans–Williamson Rounding
4. Optional Material
   - Semidefinite Programming Duality

Maximum Cut Problem

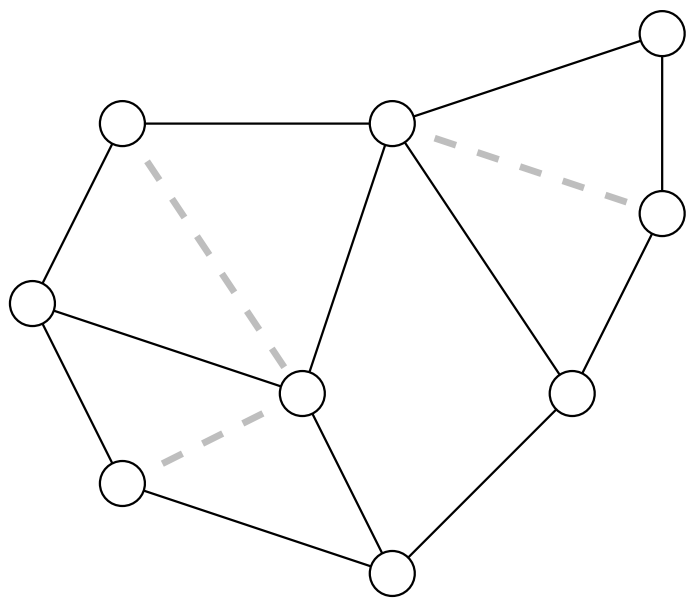
Input: $G = (V, E)$,

Output: a partition $(S \subseteq V, T := V \setminus S)$ of V so as to maximize $|E(S,T)|$,

where $E(S,T) = \{uv \in E : |\{u,v\} \cap S| = 1\}$



- Min-Uncut: remove minimum number of edges to make graph bipartite



- Max-Cut = Min-Uncut for exact algorithms, but not the same for approximation algorithms

- Recap: $1/2$-approximation algorithms for Max-Cut:

Randomized Algorithm

1: $S \leftarrow \emptyset$
2: **for** every $u \in V$ **do**
3: with probability $1/2$, add u to S
4: **return** $(S, V \setminus S)$

Greedy Algorithms

1: $S \leftarrow \emptyset, T \leftarrow \emptyset$
2: **for** every $u \in V$ **do**
3: **if** $|E(u, S)| > |E(u, T)|$ **then**
4: $T \leftarrow T \cup \{u\}$
5: **else**
6: $S \leftarrow S \cup \{u\}$
7: **return** (S, T)

- Local Search: while we can improve the solution by switching the side of one vertex, perform the operation, stop if no swapping can improve the solution

Linear Programming Relaxation

First Attempt

- $y_v, v \in V$: if $v \in S$
- $x_{uv}, uv \in E$: if uv is cut

$$\max \quad \sum_{uv \in E} x_{uv}$$

$$x_{uv} \leq |y_u - y_v| \qquad \forall uv \in E$$
$$y_v \in [0,1] \qquad \forall v \in V$$

- $x_{uv} \leq |y_u - y_v|$ is not linear
- feasible region is not convex:

y_u	y_v	x_{uv}	Y/N
1	0	0.5	Y
0	1	0.5	Y
0.5	0.5	0.5	N

- $x_{uv} \geq |y_u - y_v|$ can be replaced by $x_{uv} \geq y_u - y_v$ and $x_{uv} \geq y_v - y_u$

Second Attempt

- $x_{uv}, uv \in \binom{V}{2}$: whether uv is cut

$$\max \quad \sum_{uv \in E} x_{uv}$$

$$x_{uv} + x_{vw} + x_{uw} \leq 2 \qquad \forall \text{ distinct } u, v, w \in V$$

$$x_{uv} \in [0, 1] \qquad \forall \{u, v\} \in \binom{V}{2}$$

- The ratio lp/opt can approach 2: triangle-free graphs can have opt $\approx |E|/2$, while $x_{uv} = 1$ on edges and 0 elsewhere gives lp $= |E|$.

Quadratic Program

- $y_v = \begin{cases} 1 & \text{if } v \in S \\ -1 & \text{if } v \notin S \end{cases}$

$$\max \quad \frac{1}{2} \sum_{uv \in E} (1 - y_u y_v)$$

$$y_v \in \{\pm 1\} \quad \forall v \in V$$

Semi-Definite Program

- $y_v \in \mathbb{R}^n, \forall v \in V$

$$\max \quad \frac{1}{2} \sum_{uv \in E} (1 - \langle y_u, y_v \rangle)$$

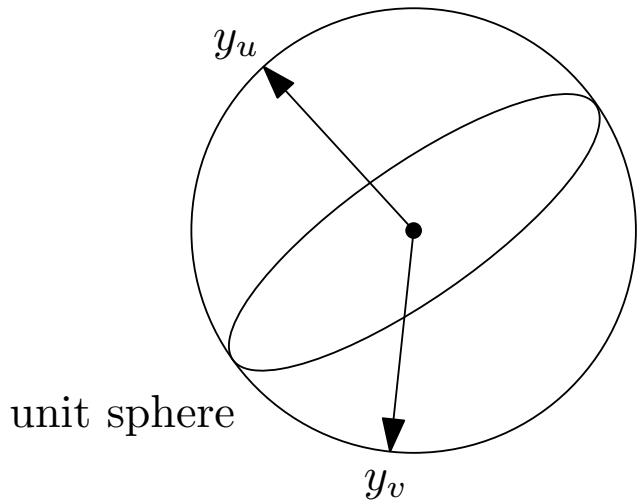
$$|y_v| = 1 \quad \forall v \in V$$

- $\langle y_u, y_v \rangle = y_u^{\mathrm{T}} y_v = \sum_{i=1}^{n} y_{u,i} \cdot y_{v,i}$: inner product of y_u and y_v
- requiring $y_v \in \mathbb{R}^n$ is the same as requiring $y_v \in \mathbb{R}^{n'}$ for any $n' \geq n$

SDP for Max-Cut

$$\max \quad \frac{1}{2}\sum_{uv\in E}(1-\langle y_u, y_v\rangle)$$

$$|y_v| = 1 \quad \forall v \in V$$



- SDP is a relaxation:

$$y_v = \begin{cases} (1,0,0,0,\cdots,0) & \text{if } v \in S \\ (-1,0,0,0,\cdots,0) & \text{if } v \in T \end{cases}$$

- sdp: the value of the SDP, $\quad$ sdp $\geq$ opt

Q: Can we solve the SDP?

A: Yes

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Def. A symmetric matrix $X \in \mathbb{R}^{n \times n}$ is Positive Semi-Definite (PSD) if $\forall y \in \mathbb{R}^n$, we have $y^{\mathrm{T}} X y \geq 0$. Use $X \succeq 0$ to denote X is PSD.

- $X \succeq X'$ means $X - X' \succeq 0$.

Lemma The following statements are equivalent for a symmetric matrix $X \in \mathbb{R}^{n \times n}$:

- $X \succeq 0$
- All the n eigenvalues of X are non-negative
- $X = V^{\mathrm{T}} V$ for some $V \in \mathbb{R}^{m \times n}, m \leq n$
- $X = \sum_{u=1}^{n} \lambda_u w_u w_u^{\mathrm{T}}$ for some reals $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ and orthonormal basis $\{w_u\}_{u \in [n]}$

Semi-definite Programming (SDP)

- matrices of size $n \times n \equiv$ flattened vectors of length n^2:
  - Use $\cdot$ as multiplication for flattened matrices,
  - $X \succeq 0$: view X as a matrix.
- $A \in \mathbb{R}^{m \times n^2}, b \in \mathbb{R}^m, c \in \mathbb{R}^{n^2}$
- Assume A_k's and c are symmetric matrices of size $n \times n$

Semi-Definite Program

$$\min \quad c^{\mathrm{T}} \cdot X$$

$$A \cdot X \geq b$$

$$X \succeq 0$$

An equivalent formulation

$$\min \quad \sum_{u,v \in [n]} c_{u,v} \cdot \langle y_u, y_v \rangle$$

$$\sum_{u,v} a_{k,u,v} \langle y_u, y_v \rangle \geq b_k \quad \forall k \in [m]$$

$$y_v \in \mathbb{R}^n \quad \forall v \in [n]$$

- requiring $y_v \in \mathbb{R}^n$ is the same as requiring $y_v \in \mathbb{R}^{n'}$ for any $n' \geq n$

Semi-Definite Program

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ & A \cdot X \geq b \\ & X \succeq 0 \end{aligned}$$

Example

$$\begin{aligned} \min \quad & 5y_1 + 6y_2 + 7y_3 \\ & y_1 + 3y_2 + 4y_3 \geq 5 \\ & 2y_1 + 3y_2 + y_3 \geq 10 \\ & 3y_1 + 2y_2 + 2y_3 \geq 7 \\ & \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} \succeq 0 \end{aligned}$$

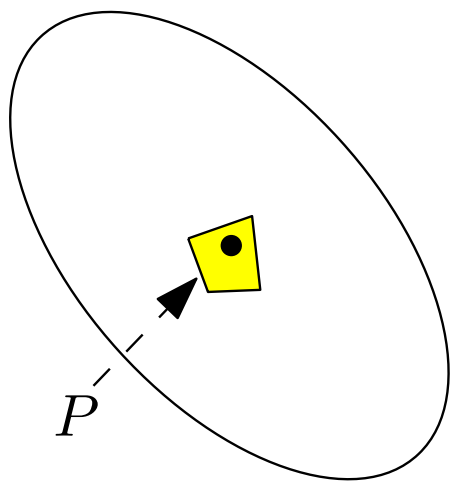
- $X \succeq 0 \iff X_{u,v} = X_{v,u}, \forall u, v \in [n]; (yy^{\mathrm{T}}) \cdot X \geq 0, \forall y \in \mathbb{R}^n$.
- SDP $\equiv$ LP with infinite number of linear constraints

Separation Oracle $\mathcal{O}$

- Given a symmetric $X \in \mathbb{R}^{n \times n}$, we need to either claim $X \succeq 0$, or return a $y \in \mathbb{R}^n$ such that $y^{\mathrm{T}} X y < 0$.
- An eigensolver (e.g., the QR algorithm) finds an eigenvector with negative eigenvalue, if one exists.

Recall: Ellipsoid Method

- maintain an ellipsoid that contains the feasible region
- query $\mathcal{O}$ if the center of ellipsoid is in the feasible region:
  - yes: then the feasible region is not empty
  - no: cut the ellipsoid in half, find smaller ellipsoid to enclose the half-ellipsoid, and repeat



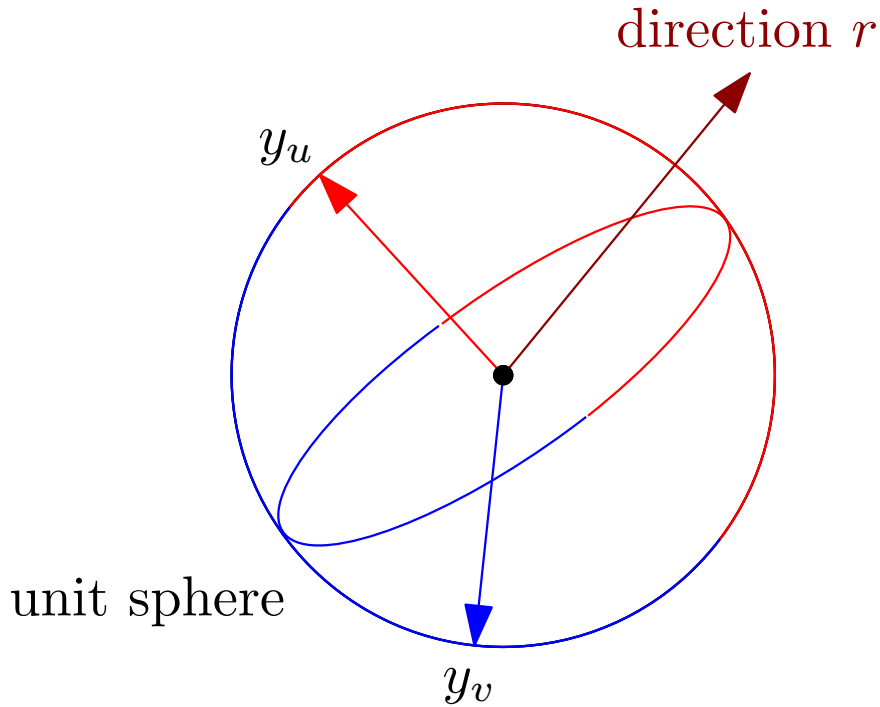
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SDP for Max-Cut

$$\max \quad \frac{1}{2}\sum_{uv\in E}(1-\langle y_u, y_v\rangle)$$

$$|y_v| = 1 \quad \forall v \in V$$



- let $(y_v)_{v\in V}$ be the vectors obtained from solving SDP
- sdp $= \frac{1}{2}\sum_{uv\in E}(1-y_u^{\mathrm{T}}y_v) \geq$ opt

[Goemans-Williamson'95] Rounding Algorithm

1: randomly choose a direction $r \in \mathbb{R}^n$:
   - choose each $r_u \sim N(0,1)$ i.i.d

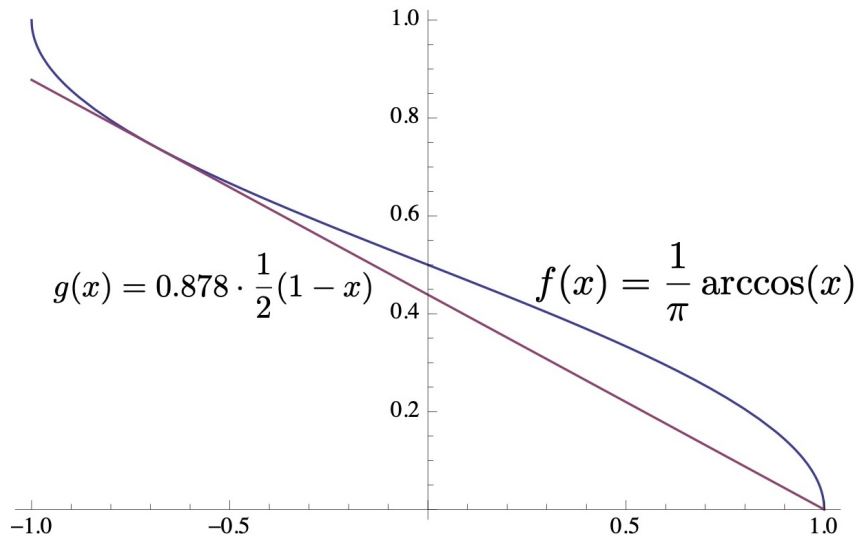
     $N(0,1)$: standard normal distribution

2: $\bar{y}_v = \mathrm{sgn}(\langle y_v, r\rangle)$, $S = \{v \in V : \bar{y}_v > 0\}$, **return** $(S, V \setminus S)$

$$\Pr[uv \text{ is cut}] = \frac{\text{radian angle between } y_u \text{ and } y_v}{\pi} = \frac{\arccos\langle y_u, y_v\rangle}{\pi}$$

$$x := \langle y_u, y_v\rangle \in [-1, 1),$$

$$\frac{\Pr[uv \text{ is cut}]}{\frac{1}{2}(1-x)} = \frac{2\arccos(x)}{\pi(1-x)}.$$



- $\alpha_{\mathsf{GW}} := \inf_{x\in[-1,1)} \frac{2}{\pi} \cdot \frac{\arccos(x)}{(1-x)} \geq 0.878$

$$\mathbb{E}[|E(S,T)|] = \sum_{uv \in E} \Pr[uv \text{ is cut}] \geq \alpha_{\mathsf{GW}} \sum_{uv \in E} \frac{1}{2}(1 - \langle y_u, y_v \rangle)$$
$$= \alpha_{\mathsf{GW}} \cdot \mathsf{sdp} \geq \alpha_{\mathsf{GW}} \cdot \mathsf{opt} \geq 0.878 \cdot \mathsf{opt}.$$

- Assuming Unique Games Conjecture (UGC), no polynomial-time algorithm can give an approximation ratio of $\alpha_{\mathsf{GW}} + \epsilon$ for any constant $\epsilon > 0$.

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Duality for Semi-Definite Programming

Semi-Definite Program

$$\min \quad c^{\mathrm{T}} \cdot X$$

$$A \cdot X \geq b$$

$$X \succeq 0$$

Semi-Definite Program

$$\min \quad c^{\mathrm{T}} \cdot X$$

$$\sum_{u,v \in [n]} a_{k,u,v} X_{u,v} \geq b_k \qquad \forall k \in [m]$$

$$\sum_{u,v \in [n]} r_u r_v X_{u,v} \geq 0 \qquad \forall r \in \mathbb{R}^n$$

- replace $X \succeq 0$ with infinite number of linear constraints: $(rr^{\mathrm{T}}) \cdot X \geq 0, \forall r \in \mathbb{R}^n$.
- Symmetry can be imposed without loss: replace X by $(X + X^{\mathrm{T}})/2$, since A_k and c are symmetric.

Duality for Semi-Definite Programming

Semi-Definite Program

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ A \cdot X &\geq b \\ X &\succeq 0 \end{aligned}$$

Semi-Definite Program

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ \sum_{u,v \in [n]} a_{k,u,v} X_{u,v} &\geq b_k && \forall k \in [m] \\ \sum_{u,v \in [n]} r_u r_v X_{u,v} &\geq 0 && \forall r \in \mathbb{R}^n \end{aligned}$$

Dual : $\max \quad \sum_{k=1}^{m} b_k \cdot y_k$

$$\begin{aligned} \sum_{k=1}^{m} a_{k,u,v} \cdot y_k + \sum_{r \in \mathbb{R}^n} r_u r_v \cdot z_r &= c_{u,v} && \forall u, v \in [n] \\ y_k &\geq 0 && \forall k \in [m] \\ z_r &\geq 0 && \forall r \in \mathbb{R}^n \end{aligned}$$

Duality for Semi-Definite Programming

Dual : $\max \quad \sum_{k=1}^{m} b_k \cdot y_k$

$$\sum_{k=1}^{m} a_{k,u,v} \cdot y_k + \sum_{r \in \mathbb{R}^n} r_u r_v \cdot z_r = c_{u,v} \qquad \forall u, v \in [n]$$

$$y_k \geq 0 \qquad \forall k \in [m]$$

$$z_r \geq 0 \qquad \forall r \in \mathbb{R}^n$$

- Take z with finite support; each PSD matrix needs at most n terms in this sum.
- first red constraint $\Leftrightarrow A^{\mathrm{T}} y + \sum_{r \in \mathbb{R}^n} z_r \cdot r r^{\mathrm{T}} = c$
- $\sum_{r \in \mathbb{R}^n} z_r \cdot r r^{\mathrm{T}}$ is PSD
- moreover, any PSD matrix can be written in this form
- $\Longrightarrow$ red constraints can be replaced by $A^{\mathrm{T}} y \preceq c$

Duality for Semi-Definite Programming

Semi-Definite Program

$$\begin{aligned} \min \quad & c^{\mathrm{T}} \cdot X \\ A \cdot X & \geq b \\ X & \succeq 0 \end{aligned}$$

Dual for SDP

$$\begin{aligned} \max \quad & b^{\mathrm{T}} y \\ A^{\mathrm{T}} y & \preceq c \\ y & \geq 0 \end{aligned}$$

- Linear Program: $X \geq 0$
- In Dual of LP: $A^{\mathrm{T}} y \leq c$