Advanced Algorithms / Fall 2026

Linear Programming and Optimization

Integrality and Exact Algorithms

Matching polytopes and total unimodularity

Lecture 2: 100 minutes

These notes cover only the linear-programming strand taught by Shi Li. This module comprises nine 100-minute lectures, numbered consistently with the slides. Parts I and II are reading divisions within this lecture; optional appendices are for further study.

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Prepared from the instructor's course slides, with AI-assisted expansion of explanations, proofs, and independent-study exercises. The English and Chinese editions have matching mathematical content.

Learning goals and reading route

An LP relaxation is an exact algorithm only when we can recover an optimal discrete object. This unit proves integrality in two complementary ways: a geometric perturbation argument for bipartite matching and an algebraic determinant argument for totally unimodular matrices.

1. Prove that every fractional point of the bipartite matching relaxation fails to be a vertex.
2. Explain why odd-set inequalities are needed for general matching.
3. Prove the total-unimodularity integrality theorem and apply it to flow and interval scheduling.

Parts I and II form the required reading. The general perfect-matching proof and the matroid material are optional, matching their placement in the slide appendix. They supply further examples of the principle that a fractional vertex cannot have enough independent tight constraints. Write $x(F) = \sum_{e \in F} x_e$ for any set of edges or elements F.

Contents

1 Part I: Matching polytopes **1**
1.1 The bipartite matching theorem . . . 1
1.2 Why general graphs need more inequalities . . . 1

2 Part II: Total unimodularity **2**
2.1 A determinant certificate for integrality . . . 2
2.2 Useful closure properties and a sparse-matrix lemma . . . 2
2.3 Consecutive ones and interval scheduling . . . 3

3 Exercises for independent study **3**

A Optional: general perfect matching **4**

B Optional: matroids and their polytopes **4**
B.1 Independence, rank, and operations . . . 4
B.2 The rank-inequality polytope . . . 5
B.3 Bases, spanning trees, and intersection . . . 6

1 Part I: Matching polytopes

1.1 The bipartite matching theorem

Let $G = (L \uplus R, E)$ be a finite bipartite graph. A matching is a set of edges with no common endpoints. The natural relaxation of maximum-weight matching is

$$\max \sum_{e \in E} w_e x_e \quad \text{s.t.} \quad x(\delta(v)) \le 1 \ (v \in L \cup R), \qquad x_e \ge 0 \ (e \in E).$$

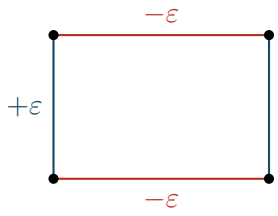
Each edge has an endpoint, so the degree constraints imply $x_e \le 1$. The region is therefore a polytope. Its 0–1 points are exactly matching indicators. The remaining task is to exclude fractional vertices.

Theorem 1.1. *The bipartite matching polytope is*

$$\operatorname{conv}\{\chi^M : M \text{ is a matching}\} = \{x \ge 0 : x(\delta(v)) \le 1 \text{ for all } v\}.$$

Proof. Take a fractional feasible vector x, and let H be the subgraph of edges with $0 < x_e < 1$. If H contains a cycle, bipartiteness makes that cycle even. Give its edges alternating signs $+1, -1, +1, -1, \ldots$ and put zero on every other edge, obtaining a nonzero direction d. At every cycle vertex, the two changes cancel. For sufficiently small $\varepsilon > 0$, both $x + \varepsilon d$ and $x - \varepsilon d$ remain nonnegative and preserve every degree sum. Thus x is not a vertex.

If H is a forest, take a path between two leaves of a component containing an edge, and again alternate signs. Internal degree sums do not change. At a leaf, exactly one incident edge is fractional. No incident edge can have value 1, because its sum with that positive fractional edge would exceed 1. All other incident edges therefore have value 0, and the degree sum is strictly less than 1. Choose ε smaller than the endpoint slacks and the positive values needed to preserve nonnegativity along the path. Both perturbations are feasible. In either case x is the midpoint of two distinct feasible points. Hence every vertex is integral, and the integrality criterion of Unit 1 proves the equality. □



Changes cancel at every vertex.
Reverse all signs for the second point.

The proof works for arbitrary edge weights, not merely positive ones. Positivity in the problem statement is convenient but irrelevant to integrality. Algorithmically, solve the LP and select an optimal vertex; the chosen edges are those with coordinate 1.

1.2 Why general graphs need more inequalities

On a triangle, assigning $1/2$ to every edge satisfies every degree constraint and has total value $3/2$ under unit weights, while every matching has size at most 1. The alternating-cycle perturbation cannot close consistently around an odd cycle. For any odd vertex set S, a matching uses at most $(|S| - 1)/2$ edges entirely inside S. Write $E[S]$ for these internal edges. The following theorem, stated in the main slides, says these necessary inequalities are also sufficient.

Theorem 1.2 (General matching polytope)**.**

$$\operatorname{conv}\{\chi^M : M \text{ is a matching in } G\} = \left\{ x \ge 0 : \begin{array}{ll} x(\delta(v)) \le 1 & (v \in V), \\ x(E[S]) \le (|S| - 1)/2 & (|S| \text{ odd}) \end{array} \right\}.$$

There are exponentially many odd sets, so the theorem alone is not an explicit polynomial-size LP. Description size, separation, and optimization are different issues; we revisit them in Units 4 and 8.

For perfect matchings, every vertex is matched. The degree inequalities become equalities, and the equivalent odd-set form is especially useful:

$$x_e \geq 0, \qquad x(\delta(v)) = 1 \ (v \in V), \qquad x(\delta(S)) \geq 1 \ (|S| \text{ odd}). \tag{1.1}$$

Indeed, summing degrees in S gives $|S| = 2x(E[S]) + x(\delta(S))$. The odd-cut inequality expresses a parity fact: an odd number of vertices cannot all be paired internally. The convex hull of perfect-matching indicators is exactly (1.1); an optional proof appears below.

2 Part II: Total unimodularity

2.1 A determinant certificate for integrality

Definition 2.1. A matrix is totally unimodular (TU, called TUM in the slides) if every square submatrix has determinant in $\{-1, 0, 1\}$. This includes 1×1 submatrices, so every entry lies in $\{-1, 0, 1\}$.

Theorem 2.2. *If A is TU and b is integral, every vertex of $\{x : Ax \geq b,\ x \geq 0\}$ is integral. In particular, a bounded such region is an integral polytope.*

Proof. Take a vertex and fix its zero coordinates. The remaining positive coordinates are uniquely determined by an independent set of tight rows. Select a nonsingular square submatrix B on those coordinates, obtaining $Bx' = b'$. Total unimodularity gives $\det B = \pm 1$. By Cramer's rule,

$$x'_j = \frac{\det B_j}{\det B} \in \mathbb{Z},$$

where B_j replaces column j by the integral vector b'. Its determinant is an integer; it need not be ± 1. Reinstating the zero coordinates proves the claim. □

The right-hand side condition matters: the TU matrix (1) with equation $x = 1/2$ has a fractional solution. TU is a sufficient structural condition, not a necessary condition for a particular polytope to be integral.

2.2 Useful closure properties and a sparse-matrix lemma

Transposition, permuting rows or columns, and negating any row or column preserve TU. Adjoining a unit row or column also preserves it, by determinant expansion along that row or column. Thus upper bounds, nonnegativity constraints, and equality constraints can be included without destroying a TU formulation.

Lemma 2.3. *If each row of a matrix in $\{0, \pm 1\}^{m \times n}$ contains at most one $+1$ and at most one -1, the matrix and its transpose are TU.*

Proof. Consider any square submatrix B and induct on its order. A zero row gives determinant 0. A row with one nonzero entry reduces the determinant, up to sign, to a smaller submatrix. Otherwise every row has one $+1$ and one -1, so $B\mathbf{1} = 0$ and B is singular. All cases give determinant 0 or ± 1. Transposition preserves determinants of square submatrices. □

The vertex–arc incidence matrix of a directed graph has one $+1$ and one -1 in each column, so it is TU. Together with integer capacity bounds, flow-conservation equations therefore define a polytope whose vertices are integral. This proves existence of an integral maximum flow from LP geometry; it does not require an augmenting-path algorithm.

For a bipartite graph, start with the unsigned vertex–edge incidence matrix and negate rows belonging to R. Every edge-column now has opposite signs, so the matrix is TU. Theorem 2.2 gives a second proof of Theorem 1.1. The triangle's unsigned incidence matrix has determinant of absolute value 2, displaying exactly where this argument fails outside bipartite graphs.

2.3 Consecutive ones and interval scheduling

Lemma 2.4. *A 0–1 matrix with consecutive ones in each row is TU. The same holds when each column has consecutive ones.*

Proof. Every square submatrix B inherits consecutive ones in each row. Replace its columns by $B_1, B_2 - B_1, \ldots, B_k - B_{k-1}$, where differences use the original columns. This multiplies B by an upper-bidiagonal matrix of determinant 1. Each interval of ones becomes one $+1$ and at most one -1, so Lemma 2.3 implies $\det B \in \{0, \pm 1\}$. Apply transposition for the column version. □

In weighted interval scheduling, activity j occupies $[s_j, f_j)$ with $f_j > s_j$ and weight $w_j > 0$. Let $\mathcal{T}$ be the sorted distinct start times. The LP is

$$\max \sum_j w_j x_j \quad \text{s.t.} \quad \sum_{j : t \in [s_j, f_j)} x_j \le 1 \ (t \in \mathcal{T}), \qquad x_j \ge 0.$$

It suffices to check start times. At any time t with active intervals, take the latest start time $s \le t$ among them. All intervals active at t were also active at s, so nonnegativity bounds their total by the constraint at s. Each column is a consecutive block of ones. The right-hand sides are integers and each activity appears in its own start-time row, so the feasible region is bounded and integral. An optimal vertex is an optimal schedule. The half-open convention matters: an activity finishing at time t can coexist with one starting at time t. This LP proof complements the familiar dynamic program; it also shows how a temporal ordering becomes a matrix integrality property.

3 Exercises for independent study

1. In the forest case of the matching proof, give an explicit positive upper bound on ε using the path-edge values and endpoint slacks.
2. Compute the determinant of the unsigned incidence matrix of a triangle. Write its missing odd-set inequality and show that it excludes the half-integral vector.
3. Prove carefully that adjoining unit rows preserves total unimodularity. Use this to handle integer lower and upper bounds on network-flow arcs.
4. Form the interval-scheduling LP for $[0, 2), [1, 4), [2, 4)$ with weights $4, 7, 5$. Find an optimal schedule and a corresponding optimal vertex.
5. Explain why "every row has at most two nonzero entries" does not imply TU without the sign condition.
6. Optional: fill in the parity check for odd cuts after contracting a tight odd set in the proof below.

A Optional: general perfect matching

We expand the contraction proof from the slide appendix. Parallel edges are permitted, and self-loops are discarded on contraction. Let $Q(G)$ be the polytope in (1.1). If $|V|$ is odd, the constraint for $S = V$ makes it empty, as it should. Assume $|V|$ is even.

Choose a counterexample with minimum $|V| + |E|$ and a fractional vertex x. A zero edge can be deleted. If $x_e = 1$, its two endpoints have all other incident coordinates zero; remove the matched pair, decompose the remaining vector by minimality, and add e back. Each operation would express x as a convex combination of perfect matchings, a contradiction. Thus we may assume $0 < x_e < 1$ for every edge.

Every vertex then has degree at least 2 in the support graph. If all degrees are 2, the graph consists of cycles. An odd component violates its own odd-cut constraint. Every remaining component is even, and the fractional solution on each even cycle is directly a convex combination of its two alternating perfect matchings. Taking the product distribution across components decomposes the entire vector. Hence x is not a fractional vertex.

Otherwise $|E| > |V|$. A vertex needs $|E|$ independent tight equations, while degree equations contribute at most $|V|$. Because no nonnegativity constraint is tight, there must be a tight nontrivial odd cut S with $3 \le |S| \le |V| - 3$ and $x(\delta(S)) = 1$. Contract S in one copy of the graph and its complement in another. Original degrees remain 1, and the new contracted vertex has degree-sum 1. An odd set in a contracted graph either avoids the supervertex or lifts by replacing it with the odd set contracted into it; its lift still has odd cardinality. Thus both induced vectors satisfy their respective systems.

By minimality, each vector is a convex combination of perfect matchings. Every such matching uses exactly one cut edge. Let the weights in the two decompositions be λ_a and μ_b. For a fixed cut edge e, both decompositions place total mass x_e on matchings using e. Pair any such two matchings with weight $\lambda_a \mu_b / x_e$. Gluing the pair along e produces a perfect matching of the original graph. The paired weights have the correct marginals on each side, so all original edge coordinates are recovered, and their total is $\sum_{e \in \delta(S)} x_e = 1$. This is a convex decomposition of x, contradicting fractional extremality.

Why the coupling is necessary

Independently choosing one matching from each contracted graph need not choose the same cut edge, so the two pieces may not glue. Conditioning on a common cut edge preserves both marginal distributions and makes every glued object feasible.

B Optional: matroids and their polytopes

B.1 Independence, rank, and operations

Definition B.1. A matroid is a pair $\mathcal{M} = (E, \mathcal{I})$ with $\varnothing \in \mathcal{I}$, such that $\mathcal{I}$ is closed under subsets and satisfies augmentation: if $I, J \in \mathcal{I}$ and $|I| < |J|$, some $e \in J \setminus I$ has $I \cup \{e\} \in \mathcal{I}$.

Members of $\mathcal{I}$ are independent sets. Maximal independent sets are bases; minimal dependent sets are circuits. Augmentation implies all bases have the same size. In a graphic matroid, independent sets are forests, bases of a connected graph are spanning trees, and circuits are cycles. Linear independence of columns gives a linear matroid. A partition matroid permits at most a specified number of elements from each part of a partition.

Define $r(S) = \max\{|I| : I \subseteq S,\ I \in \mathcal{I}\}$. Greedily adding any element that preserves independence finds a set of size $r(S)$: if a maximal set were smaller, augmentation would extend it. The rank is

normalized, increases by 0 or 1 when one element is added, and is submodular:

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B). \tag{B.1}$$

Here is a direct proof. Choose a basis of $A \cap B$, extend it to a basis of A, then extend to a basis of $A \cup B$. All newly added elements in the last step lie in $B \setminus A$. Together with the original basis of $A \cap B$, they form an independent subset of B. Hence $r(A \cap B) + r(A \cup B) - r(A) \leq r(B)$, which is (B.1).

Conversely, a normalized set function with 0–1 increments and submodularity is a matroid rank function. Define independent sets by $r(I) = |I|$. Removing elements preserves this equality because each removal lowers rank by at most one. If $|I| < |J|$ and both are independent, then $r(I \cup J) > r(I)$. If no single element of $J \setminus I$ increased rank, diminishing returns would prevent any increase from adding them all, a contradiction. This proves augmentation. Extending a maximal independent subset within S similarly shows that its size equals $r(S)$.

Operation	Meaning
$\mathcal{M} \setminus e$	Delete e and keep independent sets avoiding it.
$\mathcal{M}/e$	For a nonloop e, I is independent iff $I \cup \{e\}$ was independent. Its rank is $r_{/e}(S) = r(S \cup \{e\}) - 1$.
$\mathcal{M}[S]$	Restrict the ground set to S.
$\mathcal{M}^*$	The dual matroid has complements of original bases as its bases. This is matroid duality, a different notion from LP duality.

B.2 The rank-inequality polytope

Theorem B.2 (Matroid independence polytope)**.**

$$\operatorname{conv}\{\chi^I : I \in \mathcal{I}\} = Q := \{x \geq 0 : x(S) \leq r(S) \text{ for every } S \subseteq E\}.$$

Singleton constraints imply $x_e \leq 1$, so Q is bounded. Its 0–1 points are precisely the independent sets. We give the tight-chain proof in the slides, making the rank argument explicit.

Proof. Assume a smallest-ground-set counterexample and a fractional vertex x. A coordinate $x_e = 0$ reduces to deletion. A coordinate $x_e = 1$ reduces to contraction of the nonloop e, since constraints on $S \cup \{e\}$ become $x(S) \leq r(S \cup \{e\}) - 1$. Conversely the contracted inequalities imply those not containing e, because $r(S \cup \{e\}) - 1 \leq r(S)$. The reduced polytopes are integral, so these cases cannot yield a fractional vertex. We may assume $0 < x_e < 1$ for every e.

Call S tight if $x(S) = r(S)$. If A, B are tight, then

$$x(A) + x(B) = r(A) + r(B) \geq r(A \cup B) + r(A \cap B) \geq x(A \cup B) + x(A \cap B).$$

The first and last expressions are equal by the inclusion–exclusion identity for sums. Every inequality is equality, so $A \cup B$ and $A \cap B$ are tight.

Choose a longest strict chain of nonempty tight sets $\mathcal{C}$. We claim that its indicator vectors span the indicators of all tight sets. Otherwise choose a tight set B outside this span minimizing the number of members of $\mathcal{C}$ incomparable with B. If this number is zero, B can be inserted in the chain, contradicting maximal length. Otherwise choose an incomparable $T \in \mathcal{C}$. Both $B \cap T$ and $B \cup T$ are tight and are incomparable with fewer chain members: a chain member below T is comparable with $B \cup T$, and one above T is comparable with $B \cap T$; neither operation creates a new conflict. By minimality their indicators lie in the chain span. But

$$\chi^B = \chi^{B \cup T} + \chi^{B \cap T} - \chi^T,$$

which contradicts the choice of B.

At a vertex with all coordinates strictly between 0 and 1, tight rank inequalities must span $\mathbb{R}^E$. The chain must therefore have $|E|$ nonempty members. A strict chain of that length adds exactly one element at each step and ends at E. Subtracting successive tight equalities makes each coordinate a difference of two integer ranks, so every coordinate is integral. This contradicts $0 < x_e < 1$ and proves the theorem. □

B.3 Bases, spanning trees, and intersection

Adding $x(E) = r(E)$ selects the face consisting of convex combinations of bases. It is a face, but need not be a facet. For a connected graph, the graphic rank is $r(F) = |V| - \mathrm{cc}(V, F)$, where isolated vertices count as components. Its base polytope becomes

$$x \geq 0, \quad x(E) = |V| - 1, \quad x(E[S]) \leq |S| - 1 \qquad (\varnothing \neq S \subseteq V).$$

To recover any rank inequality for an arbitrary edge set F, sum the vertex-set inequalities over the connected components of (V, F) and use $F \cap E[S] \subseteq E[S]$ with nonnegative coordinates. This explains why vertex subsets suffice for spanning trees.

Theorem B.3 (Matroid intersection; stated without proof)**.** *For two matroids on the same ground set,*

$$P_{\mathcal{M}_1} \cap P_{\mathcal{M}_2} = \mathrm{conv}\{\chi^I : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}.$$

Requiring a common independent set to be a basis of one matroid corresponds to intersecting with that matroid's base-polytope face.

This is a special theorem: intersections of arbitrary integral polytopes need not be integral. Bipartite matchings are common independent sets of two partition matroids, one for the edges incident to each left vertex and one for the right vertices. A directed arborescence rooted at r is a basis of the underlying graphic matroid and an independent set of the partition matroid imposing indegree at most one at $v \neq r$ and zero at r. A spanning tree with these indegree conditions has a directed path from r to every vertex. These applications show how different combinatorial restrictions can combine without losing integrality.

Source and reading notes

Primary source: the instructor's slide folder

`2.Integrality`

`Topics.tex` and `Optional.tex`.

Optional sections follow the optional slides where present. Explanatory examples and exercises are added to support independent reading; they do not change the lecture allocation.

Course website. The early-course notes linked there were used as examples of organization and level of detail, not as the source of this module's subject matter.