Combinatorics

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PIE

(Principle of Inclusion-Exclusion)

$$A_1, A_2, \ldots, A_n \subseteq U$$

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \, |A_I|$$

$$A_I = \bigcap_{i \in I} A_i$$

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(Principle of Inclusion-Exclusion)

$$A_1, A_2, \ldots, A_n \subseteq U$$

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_n}\right| = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \left|A_I\right|$$

$$A_I = \bigcap_{i \in I} A_i \qquad\qquad A_\emptyset = U$$

PIE

(Principle of Inclusion-Exclusion)

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for $T \subseteq S$

$$\sum_{T \subseteq I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = T \\ 0 & \text{otherwise} \end{cases}$$

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$$A_1 = A_2 = \cdots = A_n = \{1\}$$

$$1 = \left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \, |A_I|$$

$$A_I = \bigcap_{i \in I} A_i \; = \{1\}$$

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1}$$

when $\{1, 2, \ldots, n\} \neq \emptyset$

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{n-|I|} = 0$$

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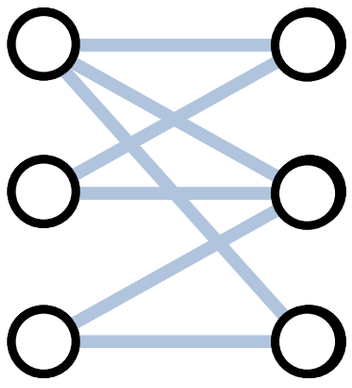
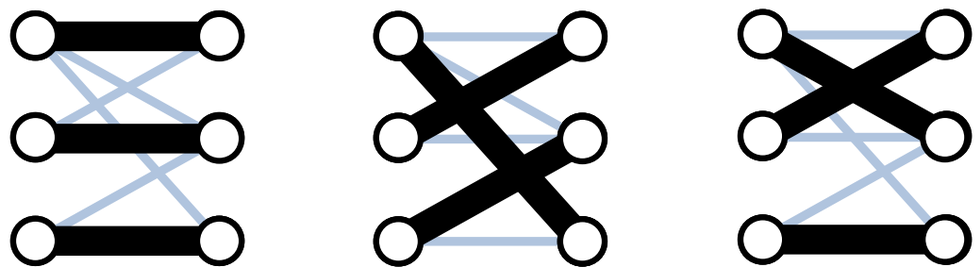
Bipartite Perfect Matching

bipartite graph



perfect matchings



$G([n],[n],E)$

permutation π of $[n]$ **s.t.** $(i, \pi(i)) \in E$

$n \times n$ matrix A :

$$A_{i,j} = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

# of P.M. in G

$$= \sum_{\pi \in S_n} \prod_{i \in [n]} A_{i,\pi(i)}$$

Permanent

$n \times n$ matrix A :

$$\mathrm{perm}(A) = \sum_{\pi \in S_n} \prod_{i \in [n]} A_{i,\pi(i)}$$

#P-hard

determinant:

$$\det(A) = \sum_{\pi \in S_n} (-1)^{r(\pi)} \prod_{i \in [n]} A_{i,\pi(i)}$$

poly-time by Gaussian elimination

Ryser's formula

$$\sum_{\pi \in S_n} \prod_{i \in [n]} A_{i,\pi(i)} = \sum_{I \subseteq [n]} (-1)^{n-|I|} \prod_{i \in [n]} \sum_{j \in I} A_{i,j}$$

term in : $\prod_{i \in [n]} A_{i,f(i)}$ for some $f : [n] \to [n]$

$$T = f([n]) \subseteq I$$

coefficient of $\prod_{i \in [n]} A_{i,f(i)}$ in :

$$\sum_{T \subseteq I \subseteq [n]} (-1)^{n-|I|} = \begin{cases} 1 & T = [n] \leftarrow f \text{ is a permutation} \\ 0 & o.w. \end{cases}$$

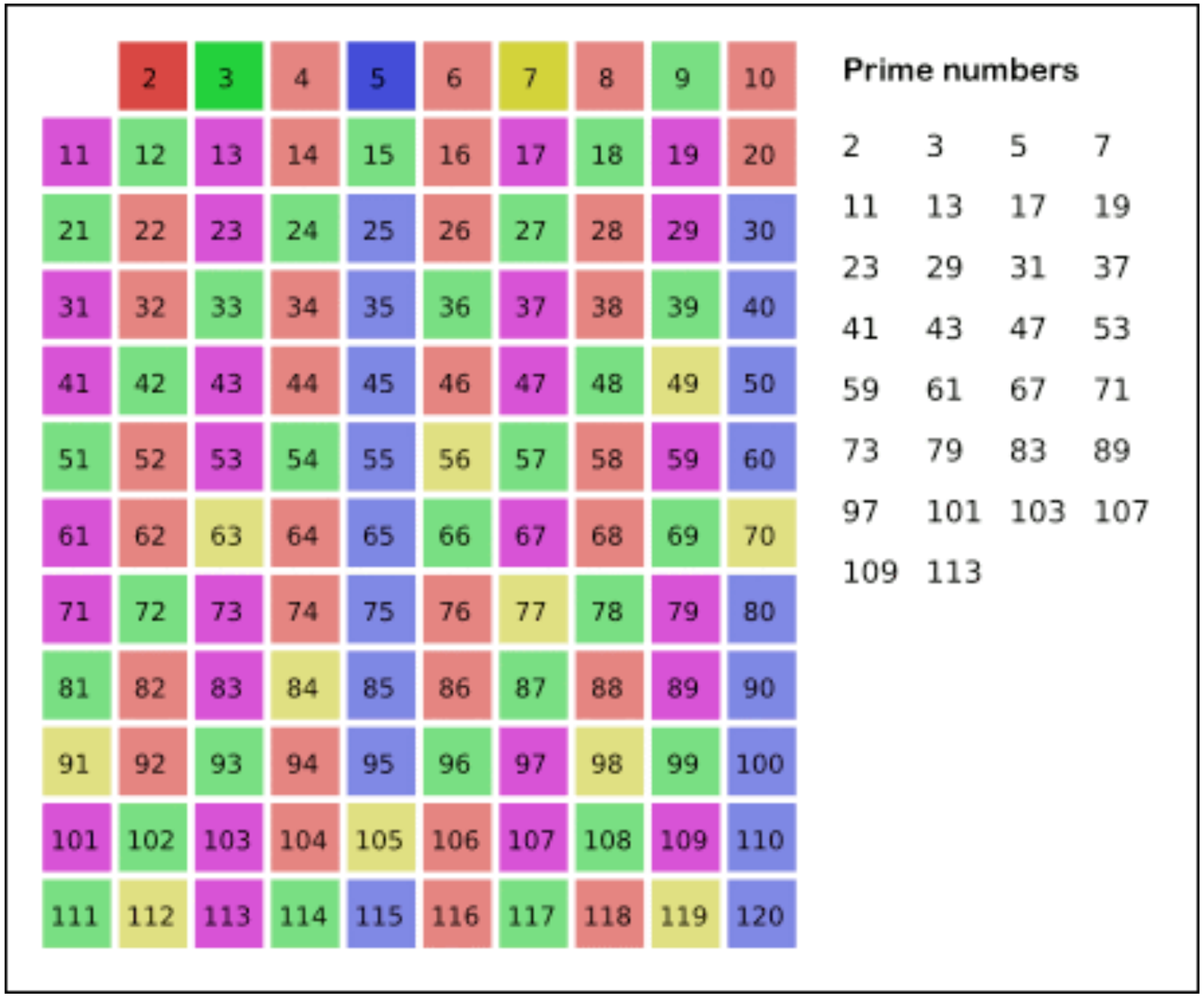
Ryser's formula

$$\sum_{\pi \in S_n} \prod_{i \in [n]} A_{i,\pi(i)} = \sum_{I \subseteq [n]} (-1)^{n-|I|} \prod_{i \in [n]} \sum_{j \in I} A_{i,j}$$

O($n!$) time

O($n2^n$) time

Sieve of Eratosthenes



Euler Totient Function

$\phi(n) = |\{1 \le a \le n \mid \gcd(a,n)=1\}|$

of $a \in \{1,2,\ldots,n\}$ relative prime to n

prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)$$

Euler Totient Function

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Universe: $U = \{1, 2, ..., n\}$

$$i = 1, 2, \ldots, r \qquad A_i = \{1 \le a \le n \mid p_i | a\}$$

$$I \subseteq \{1, 2, \ldots, r\} \qquad A_I = \{1 \le a \le n \mid \forall i \in I, p_i | a\}$$

$$|A_i| = \frac{n}{p_i} \qquad |A_I| = \frac{n}{\prod_{i \in I} p_i}$$

$$\phi(n) = \left| \bigcap_{i \in \{1,\ldots,r\}} \overline{A_i} \right| = \sum_{I \subseteq \{1,\ldots,r\}} (-1)^{|I|} |A_I|$$

Euler Totient Function

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prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$I \subseteq \{1, 2, \ldots, r\}$ $A_I = \{1 \le a \le n \mid \forall i \in I, p_i | a\}$

$$|A_I| = \frac{n}{\prod_{i \in I} p_i}$$

$$\phi(n) = \sum_{I \subseteq \{1,\ldots,r\}} (-1)^{|I|} |A_I|$$

$$= n \sum_{I \subseteq \{1,\ldots,r\}} \prod_{i \in I} \left(\frac{-1}{p_i} \right) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i} \right)$$

Euler Totient Function

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prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

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