Combinatorics

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The Twelvefold Way

Gian-Carlo Rota
(1932-1999)

The twelvefold way

$$f : N \to M \qquad |N| = n, \ \ |M| = m$$

elements of N	elements of M	any f	1-1	on-to
distinct	*distinct*			
identical	*distinct*			
distinct	*identical*			
identical	*identical*			

Knuth's version (in *TAOCP* vol.4A)

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n		
n identical balls, m distinct bins			
n distinct balls, m identical bins			
n identical balls, m identical bins			

Tuples

$$\{1, 2, \ldots, m\}$$

$$[m] = \cancel{\{0, 1, \ldots, m-1\}}$$

$$[m]^n = \underbrace{[m] \times \cdots \times [m]}_{n}$$

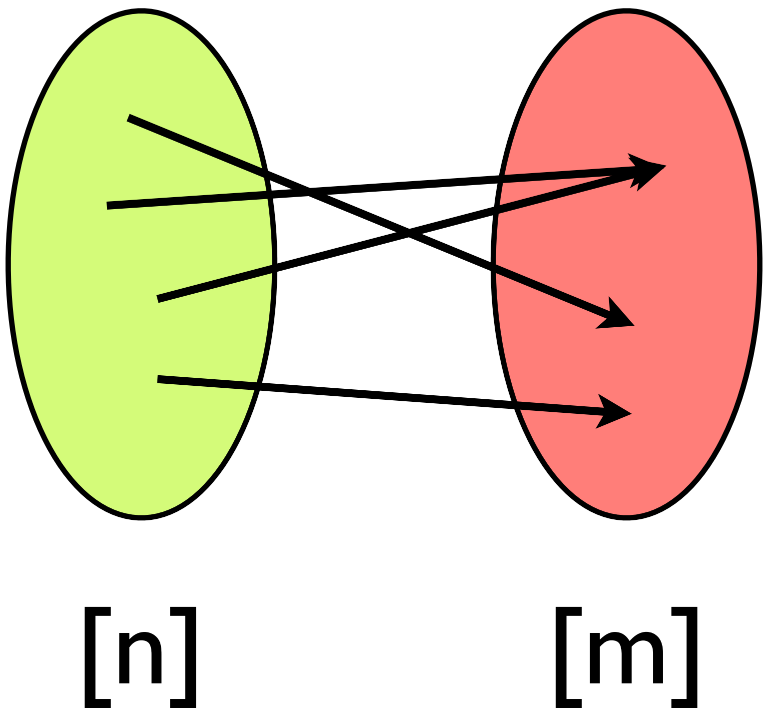
$$|[m]^n| = m^n$$

Product rule:

finite sets S and T

$$|S \times T| = |S| \cdot |T|$$

Functions



count the # of functions

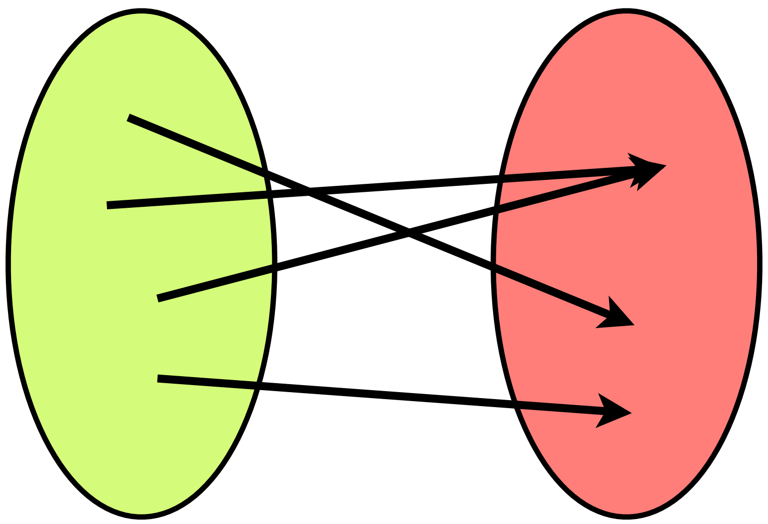
$$f : [n] \to [m]$$

$$(f(1), f(2), \ldots, f(n)) \in [m]^n$$

one-one correspondence

$$[n] \to [m] \Leftrightarrow [m]^n$$

Functions



[n] [m]

count the # of functions

$$f : [n] \to [m]$$

one-one correspondence

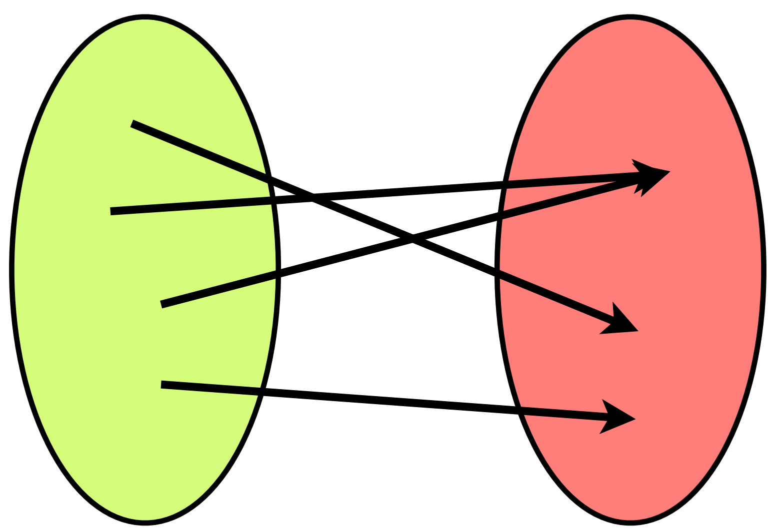
$$[n] \to [m] \Leftrightarrow [m]^n$$

Bijection rule:

finite sets S and T

$$\exists \phi : S \xrightarrow[\text{on-to}]{1-1} T \implies |S| = |T|$$

Functions



[n] [m]

count the # of functions

$$f : [n] \rightarrow [m]$$

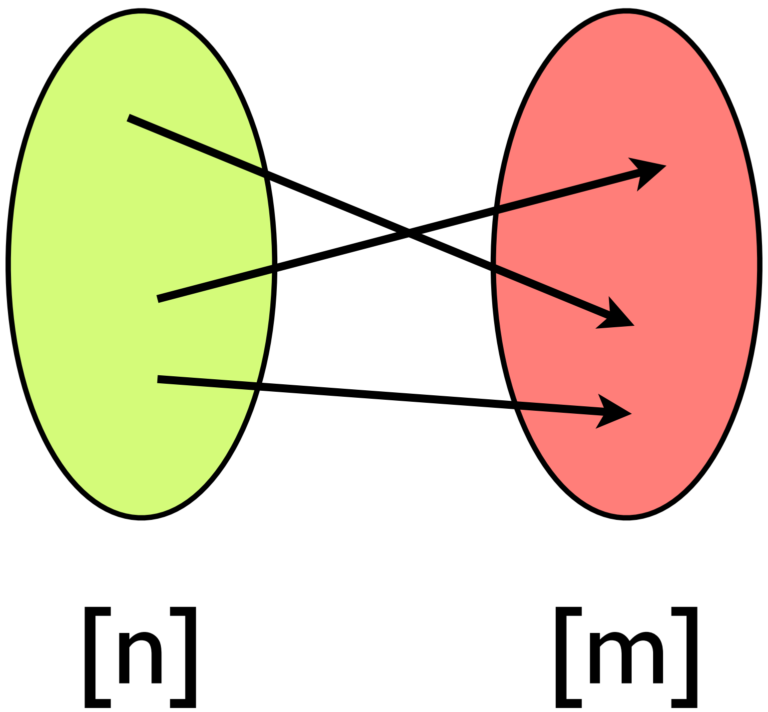
one-one correspondence

$$[n] \rightarrow [m] \Leftrightarrow [m]^n$$

$$|[n] \rightarrow [m]| = |[m]^n| = m^n$$

"Combinatorial proof."

Injections



count the # of 1-1 functions

$$f : [n] \xrightarrow{1\text{-}1} [m]$$

one-to-one correspondence

$$\pi = (f(1), f(2), \ldots, f(n))$$

n-permutation: $\pi \in [m]^n$ of distinct elements

$$(m)_n = \ m(m-1)\cdots(m-n+1) = \frac{m!}{(m-n)!}$$

"*m* lower factorial *n*"

Subsets

subsets of { 1, 2, 3 }:

$\varnothing$,
{1}, {2}, {3},
{1, 2}, {1, 3}, {2, 3},
{1, 2, 3}

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

Combinatorial proof:

A subset $S \subseteq [n]$ corresponds to a string of n bit, where bit i indicates whether $i \in S$.

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| = |\{0,1\}^n| = 2^n$$

Combinatorial proof:

$$S \subseteq [n] \Longleftrightarrow \chi_S \in \{0,1\}^n \quad \chi_S(i) = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$$

one-to-one correspondence

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

A not-so-combinatorial proof:

Let $f(n) = \left|2^{[n]}\right|$

$$\boxed{f(n) = 2f(n-1)}$$

$$f(n) = \left|2^{[n]}\right| \qquad \boxed{f(n) = 2f(n-1)}$$

$$2^{[n]} = \{S \subseteq [n] \mid n \notin S\} \cup \{S \subseteq [n] \mid n \in S\}$$

$$\left|2^{[n]}\right| = \left|2^{[n-1]}\right| + \left|2^{[n-1]}\right| = 2f(n-1)$$

Sum rule:

finite **disjoint** sets S and T

$$|S \cup T| = |S| + |T|$$

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| = 2^n$$

Let $f(n) = \left|2^{[n]}\right|$

$$\boxed{f(n) = 2f(n-1)}$$

$$f(0) = |2^{\emptyset}| = 1$$

Three rules

Sum rule:

finite **disjoint** sets S and T

$$|S \cup T| = |S| + |T|$$

Product rule:

finite sets S and T

$$|S \times T| = |S| \cdot |T|$$

Bijection rule:

finite sets S and T

$$\exists \phi : S \xrightarrow[\text{on}-\text{to}]{1-1} T \implies |S| = |T|$$

Subsets of fixed size

2-subsets of $\{1, 2, 3\}$: $\{1, 2\}, \{1, 3\}, \{2, 3\}$

k-uniform $$\binom{S}{k} = \{T \subseteq S \mid |T| = k\}$$

$$\binom{n}{k} = \left|\binom{[n]}{k}\right|$$

"*n* choose *k*"

Subsets of fixed size

$$\boxed{\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{k!(n-k)!}}$$

# of ordered k-subsets: $n(n-1)\cdots(n-k+1)$

# of permutations of a *k*-set: $k(k-1)\cdots 1$

Binomial coefficients

Binomial coefficient: $\binom{n}{k}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

1. $$\binom{n}{k} = \binom{n}{n-k}$$

choose a *k*-subset ⇔ choose its compliment

2. $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

0-subsets + 1-subsets + ... + n-subsets = all subsets

Binomial theorem

Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Proof:

$$(1+x)^n = \underbrace{(1+x)(1+x)\cdots(1+x)}_{n}$$

$$\# \text{ of } x^k: \quad \text{choose } k \text{ factors out of } n$$

Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

Let $x = 1$.

$$S = \{x_1, x_2, \ldots, x_n\}$$

$$\boxed{\begin{aligned} & \# \text{ of subsets of } S \text{ of odd sizes} \\ = \; & \# \text{ of subsets of } S \text{ of even sizes} \end{aligned}}$$

Let $x = -1$.

The twelvefold way

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins		$\binom{m}{n}$	
n distinct balls, m identical bins			
n identical balls, m identical bins			

Compositions of an integer

n beli $\qquad$ k pirates

How many ways to assign n beli to k pirates?

How many ways to assign n beli to k pirates,
so that each pirate receives at least 1 beli?

Compositions of an integer

$n \in \mathbb{Z}^+$

k-composition of *n*:

an ordered sum of k positive integers

Compositions of an integer

$n \in \mathbb{Z}^+$

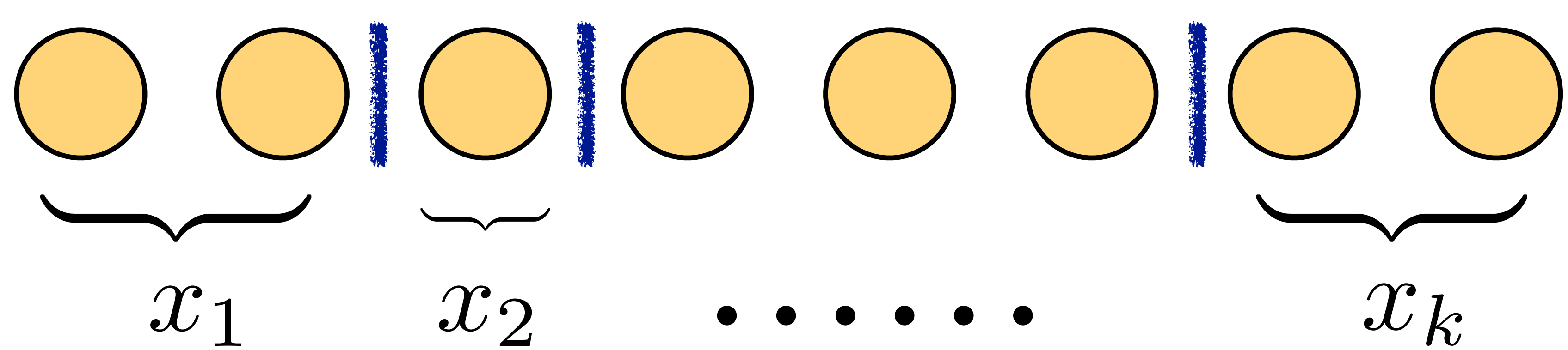
k-composition of *n*:

a *k*-tuple $(x_1, x_2, \cdots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

\# of *k*-compositions of *n*? $\binom{n-1}{k-1}$

n identical balls



Compositions of an integer

a *k*-tuple $(x_1, x_2, \cdots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ **and** $x_i \in \mathbb{Z}^+$

\# of *k*-compositions of *n*? $\binom{n-1}{k-1}$

$$\begin{aligned}\phi((x_1, x_2, \ldots, x_k)) \quad = \quad & \{x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3, \\ & \ldots,\ x_1 + x_2 + \cdots + x_{k-1}\}\end{aligned}$$

ϕ is a 1-1 correspondence between
$\{k\text{-compositions of } n\}$ and $\binom{\{1,2,\ldots,n-1\}}{k-1}$

Compositions of an integer

weak k-composition of n:

an ordered sum of k nonnegative integers

Compositions of an integer

weak *k*-composition of *n*:

a *k*-tuple $(x_1, x_2, \cdots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{N}$

# of weak *k*-compositions of *n*? $\binom{n+k-1}{k-1}$

$$(x_1 + 1) + (x_2 + 1) + \cdots + (x_k + 1) = n + k$$

a *k*-composition of *n*+*k*

1-1 correspondence

Multisets

k-subset of *S* "*k*-combination of *S* without repetition"

3-combinations of { 1, 2, 3, 4 }

without repetition:

{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}

with repetition:

{1,1,1}, {1,1,2}, {1,1,3}, {1,1,4}, {1,2,2}, {1,3,3},
{1,4,4}, {2,2,2}, {2,2,3}, {2,2,4}, {2,3,3}, {2,4,4},
{3,3,3}, {3,3,4}, {3,4,4}, {4,4,4}

Multisets

multiset *M* on set *S*:

$$m : S \to \mathbb{N}$$

multiplicity of $x \in S$

$m(x)$: # of repetitions of x in M

cardinality $|M| = \sum_{x \in S} m(x)$

"*k*-combination of *S* with repetition" $\Longleftrightarrow$ *k*-multiset on *S*

$\left(\!\binom{n}{k}\!\right)$: # of *k*-multisets on an *n*-set

Multisets

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

k-multiset on $S = \{x_1, x_2, \ldots, x_n\}$

$$m(x_1) + m(x_2) + \cdots + m(x_n) = k$$

$$m(x_i) \geq 0$$

a weak *n*-composition of *k*

Multinomial coefficients

permutations of a multiset
of size *n* with multiplicities $m_1, m_2 \ldots, m_k$

# of reordering of "*multinomial*"

permutations of $\{a, i, i, l, l, m, m, n, o, t, u\}$

assign *n* distinct balls to *k* distinct bins
with the *i-th* bin receiving m_i balls

multinomial coefficient $\binom{n}{m_1, \ldots, m_k}$

$$m_1 + m_2 + \cdots + m_k = n$$

Multinomial coefficients

permutations of a multiset
of size *n* with multiplicities $m_1, m_2, \ldots, m_k$

$$=$$

assign *n* distinct balls to *k* distinct bins
with the *i-th* bin receiving m_i balls

$$\binom{n}{m_1, \ldots, m_k} = \frac{n!}{m_1! m_2! \cdots m_k!}$$

$$\binom{n}{m, n-m} = \binom{n}{m}$$

Multinomial theorem

Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_k)^n$$
$$= \sum_{m_1+\cdots+m_k=n} \binom{n}{m_1, \ldots, m_k} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$$

Proof: $(x_1 + x_2 + \cdots + x_k)^n$

$$= \underbrace{(x_1 + x_2 + \cdots + x_k) \cdots\cdots (x_1 + x_2 + \cdots + x_k)}_{n}$$

$\#$ of $x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$:

assign n factors to k groups of sizes $m_1, m_2, \ldots, m_k$

The twelvefold way

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins			
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The twelvefold way

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balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins			
n identical balls, m identical bins			

Partitions of a set

n pirates *k* boats

$P = \{A_1, A_2, \ldots, A_k\}$ is a partition of S:

$$A_i \neq \emptyset$$

$$A_i \cap A_j = \emptyset$$

$$A_1 \cup A_2 \cup \cdots \cup A_k = S$$

Partitions of a set

$P = \{A_1, A_2, \ldots, A_k\}$ is a partition of S:

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$$A_i \cap A_j = \emptyset$$

$$A_1 \cup A_2 \cup \cdots \cup A_k = S$$

$\left\{ n \atop k \right\}$ # of *k*-partitions of an *n*-set

"Stirling number of the second kind"

$$B_n = \sum_{k=1}^{n} \left\{ n \atop k \right\}$$ # of partitions of an *n*-set

"Bell number"

Stirling number of the 2nd kind

$\left\{ {n \atop k} \right\}$ # of k-partitions of an n-set

$$\boxed{\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}}$$

Case.1 {n} is not a partition block

n is in one of the k blocks in a k-partition of $[n\text{-}1]$

Case.2 {n} is a partition block

the remaining k-1 blocks forms a (k-1)-partition of $[n\text{-}1]$

The twelvefold way

$f : N \to M$ *n* balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins	$\left(\binom{m}{n}\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins	$\sum_{k=1}^{m} \left\{ n \atop k \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ n \atop m \right\}$
n identical balls, m identical bins			

Surjections

$$f : [n] \xrightarrow{\text{on-to}} [m]$$

$\forall i \in [m]$

$f^{-1}(i) \neq \emptyset$

$$\Updownarrow$$

$$(f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(m))$$

ordered m-partition of $[n]$

$$m! \left\{ {n \atop m} \right\}$$

The twelvefold way

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	$m! \left\{ {n \atop m} \right\}$
n identical balls, m distinct bins	$\left(\!\!\binom{m}{n}\!\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins			

Partitions of a number

n beli

k boxes

a partition of *n* into *k* parts:

an unordered sum of *k* positive integers

Partitions of a number

a partition of *n* into *k* parts: “positive”

n=7 {7} “unordered”

{1,6}, {2,5}, {3,4}

{1,1,5}, {1,2,4}, {1,3,3}, {2,2,3}

{1,1,1,4}, {1,1,2,3}, {1,2,2,2}

{1,1,1,1,3}, {1,1,1,2,2}

{1,1,1,1,1,2}

{1,1,1,1,1,1,1}

$p_k(n)$ # of partitions of *n* into *k* parts

$p_k(n)$ # of partitions of *n* into *k* parts

integral solutions to
$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

$$p_k(n) = ?$$

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

$$\boxed{p_k(n) = p_{k-1}(n-1) + p_k(n-k)}$$

Case.1 $x_k = 1$

$(x_1, \ldots, x_{k-1})$ is a $(k-1)$-partition of $n-1$

Case.2 $x_k > 1$

$(x_1 - 1, \ldots, x_k - 1)$ is a k-partition of $n-k$

partition
$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

composition
$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_i \geq 1 \end{cases}$$

$$\begin{array}{ccc} \text{partition} & & \text{composition} \\ \{x_1, \cdots, x_k\} & \xrightarrow{\pi} & (x_1, \cdots, x_k) \\ & \text{permutation} & \end{array}$$

“on-to”

$$k!p_k(n) \geq \binom{n-1}{k-1}$$

partition $\{x_1, \cdots, x_k\}$ $\quad$ $y_i = x_i + k - i$

$$x_1 \geq x_2 \geq \cdots \geq x_{k-2} \geq x_{k-1} \geq x_k \geq 1$$

$+k-1 \quad +k-2 \quad\quad +2 \quad +1$

$$y_1 > y_2 > \cdots > y_{k-2} > y_{k-1} > y_k > 1$$

$\xrightarrow[\text{permutation}]{\pi}$ composition of $n + \frac{k(k-1)}{2}$

$$(y_1, y_2, \ldots, y_k)$$

"1-1"

$$k! p_k(n) \leq \binom{n + \frac{k(k-1)}{2} - 1}{k-1}$$

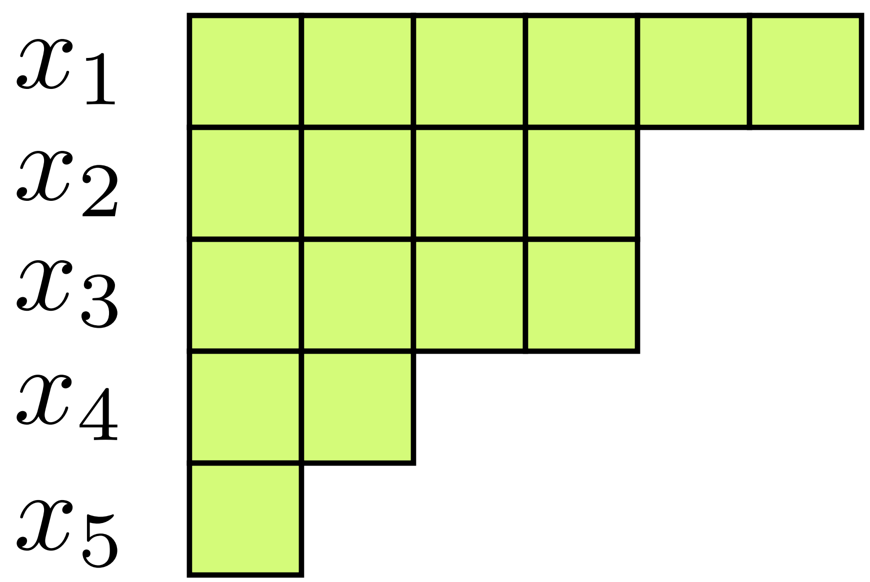
$$\boxed{\frac{\binom{n-1}{k-1}}{k!} \leq p_k(n) \leq \frac{\binom{n+\frac{k(k-1)}{2}-1}{k-1}}{k!}}$$

If *k* is fixed,

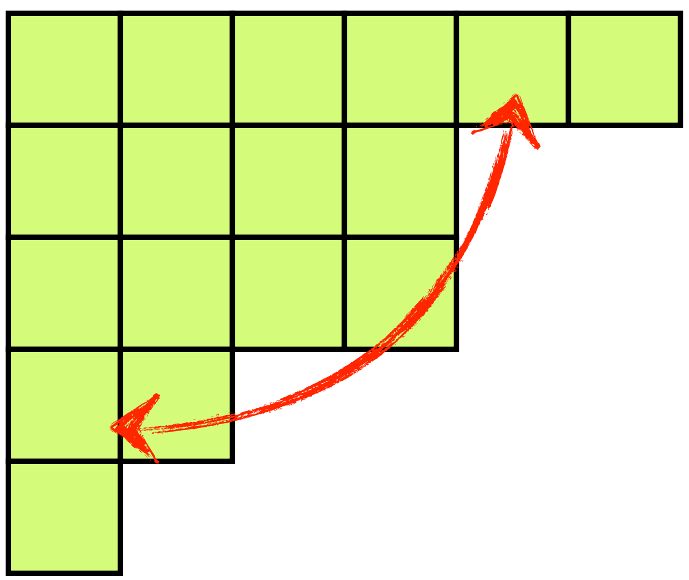
$$p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!} \qquad \text{as} \quad n \to \infty$$

Ferrers diagram

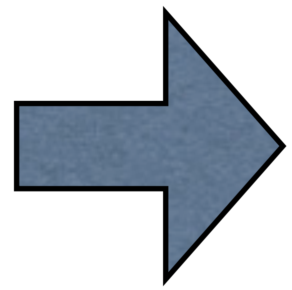
(Young diagram)



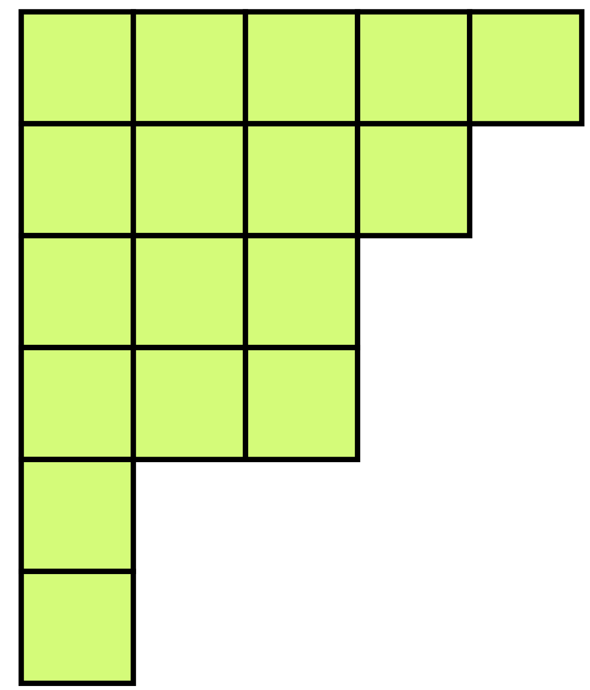
partition $\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$



(6,4,4,2,1)

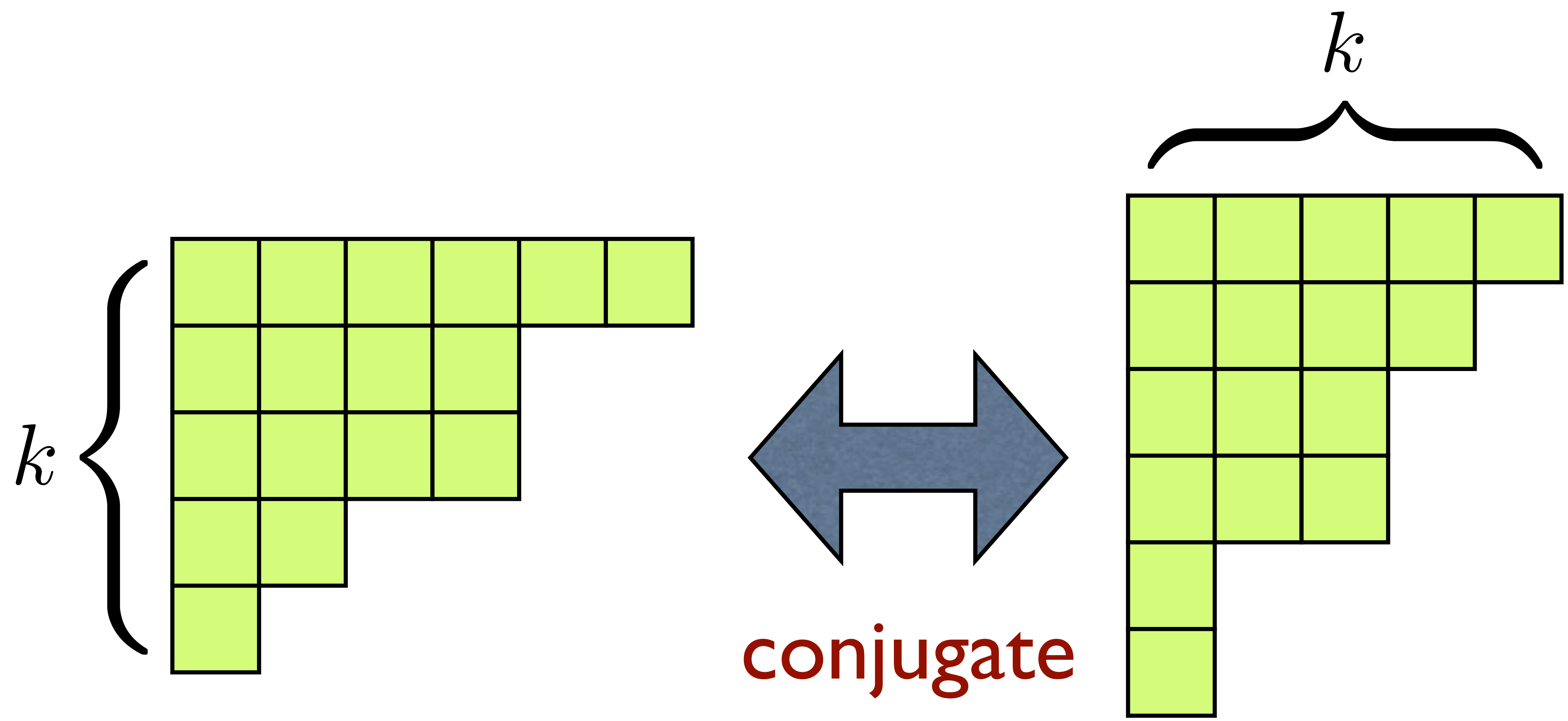


conjugate

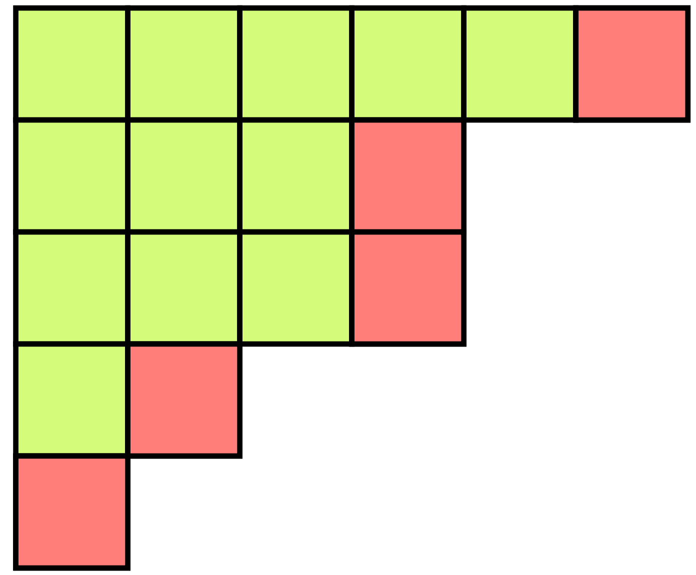


(5,4,3,3,1,1)

one-to-one
correspondence



# of partitions of *n* into *k* parts	=	# of partitions of *n* with largest part *k*



# of partitions of *n* into *k* parts = # of partitions of *n*-*k* into at most *k* parts

$$p_k(n) = \sum_{j=1}^{k} p_j(n-k)$$

The twelvefold way

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, *m distinct bins*	m^n	$(m)_n$	$m! \left\{ {n \atop m} \right\}$
n identical balls, *m distinct bins*	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, *m identical bins*	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, *m identical bins*	$\sum_{k=1}^{m} p_k(n)$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$p_m(n)$