Combinatorics

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Counting (labeled) trees

“How many different trees can be formed from $n$ distinct vertices?”
Cayley’s formula:

There are $n^{n-2}$ trees on $n$ distinct vertices.

Arthur Cayley
Prüfer Code

leaf: vertex of degree 1

removing a leaf from $T$, still a tree

$T_1 = T$;
for $i = 1$ to $n-1$

$u_i$: smallest leaf in $T_i$;
$(u_i, v_i)$: edge in $T_i$;
$T_{i+1} = $ delete $u_i$ from $T_i$;

Prüfer code:

$(v_1, v_2, \ldots, v_{n-2})$
edges of $T$: $(u_i, v_i), 1 \leq i \leq n-1$

$v_{n-1} = n$

$u_i$: smallest leaf in $T_i$

a tree has $\geq 2$ leaves} $\quad n$ is never deleted

$u_i \neq n$

$T$: 

$2, 4, 5, 6, 3, 1$

$4, 3, 1, 3, 1, 7$

$(v_1, v_2, \ldots, v_{n-2})$

Only need to recover every $u_i$ from $(v_1, v_1, \ldots, v_{n-2})$.

$u_i$ is the smallest number not in

$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$
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$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$

∀ vertex $v$ in $T$,

# occurrences of $v$ in $u_1, u_2, \ldots, u_{n-1}, v_{n-1}$: 1

# occurrences of $v$ in edges $(u_i,v_i)$, $1 \leq i \leq n-1$: $\text{deg}_T(v)$

$T$:

\begin{align*}
&T:  \\
&\begin{tikzpicture}
&\node[circle,fill, inner sep=1pt] (1) at (0,0) {1};
&\node[circle,fill, inner sep=1pt] (2) at (-1,1) {2};
&\node[circle,fill, inner sep=1pt] (3) at (0,1) {3};
&\node[circle,fill, inner sep=1pt] (4) at (-1,-1) {4};
&\node[circle,fill, inner sep=1pt] (5) at (1,0) {5};
&\node[circle,fill, inner sep=1pt] (6) at (0,-1) {6};
&\node[circle,fill, inner sep=1pt] (7) at (1,-1) {7};
&\draw (1) -- (2);
&\draw (1) -- (3);
&\draw (1) -- (4);
&\draw (3) -- (5);
&\draw (3) -- (6);
&\draw (3) -- (7);
&\end{tikzpicture}
\end{align*}

# occurrences of $v$ in Prüfer code: $(v_1, v_2, \ldots, v_{n-2})$

$u_i$: 2, 4, 5, 6, 3, 1

$v_i$: 4, 3, 1, 3, 1, 7

$(v_1, v_2, \ldots, v_{n-2})$
\( u_i \) is the smallest number not in 
\[ \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \]

\[ \forall \text{ vertex } v \text{ in } T_i, \]

\[ \# \text{ occurrences of } v \text{ in } u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1} : \quad 1 \]

\[ \# \text{ occurrences of } v \text{ in edges } (u_j,v_j), i \leq j \leq n-1: \quad \deg_{T_i}(v) \]

\[ T_3 : \]

\[ \begin{align*}
2 & \quad \text{blue} \\
4 & \quad \text{blue} \\
3 & \quad \text{blue} \\
6 & \quad \text{red} \\
7 & \quad \text{red} \\
5 & \quad \text{green} \\
\end{align*} \]

\[ \begin{align*}
u_i: & \quad 2, 4, 5, 6, 3, 1 \\
v_i: & \quad 4, 3, 1, 3, 1, 7 \\
(v_1, v_2, \ldots, v_{n-2}) & \quad \text{not in } \{v_i, v_{i+1}, \ldots, v_{n-2}\} \\
u_i: & \quad \text{smallest leaf in } T_i \]
$u_i$ is the smallest number not in 
$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$

$T$:

\begin{align*}
T &= \text{empty graph;} \\
v_{n-1} &= n; \\
\text{for } i = 1 \text{ to } n-1 \\
u_i &: \text{smallest number not in } \\
&\quad \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \\
\text{add edge } (u_i, v_i) \text{ to } T; \\
\end{align*}

$u_i: 2, 4, 5, 6, 3, 1$

$v_i: 4, 3, 1, 3, 1, 7$

$(v_1, v_2, \ldots, v_{n-2})$
Prüfer code is reversible \( \Rightarrow \) 1-1

every \( (v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2} \)

is decodable to a tree \( \Rightarrow \) onto

\[ T : \]

\[ T = \text{empty graph; } \]
\[ v_{n-1} = n ; \]
for \( i = 1 \) to \( n-1 \)

\( u_i : \) smallest number not in \( \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \)

add edge \( (u_i, v_i) \) to \( T ; \)

\( u_i : 2, 4, 5, 6, 3, 1 \)

\( v_i : 4, 3, 1, 3, 1, 7 \)

(\( v_1, v_2, \ldots, v_{n-2} \))
Prüfer code is reversible \[1-1\]

every \((v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2}\) is decodable to a tree \[\text{onto}\]

Cayley’s formula:

There are \(n^{n-2}\) trees on \(n\) distinct vertices.
# of sequences of adding directed edges to an empty graph to form a rooted tree
$T_n$: # of trees on $n$ distinct vertices.

# of sequences of adding directed edges to an empty graph to form a rooted tree

From a tree:
- pick a root;
- pick an order of edges.

$$T_n n(n - 1)!$$

$$= n!T_n$$
$T_n$: # of trees on $n$ distinct vertices.

# of sequences of adding directed edges to an empty graph to form a rooted tree

From an empty graph:
- add edges one by one
# of sequences of adding directed edges to an empty graph to form a rooted tree

From an empty graph:  • add edges one by one

Start from $n$ isolated vertices

rooted trees

Each step joins 2 trees.
Let us now start at the other end. To produce from adding directed edges, from any vertex the roots on top. We now regard each component tree. Let \( F \) choices. Continuing this way, we arrive at.

Start from \( n \) rooted trees. After adding \( k \) edges \( n-k \) rooted trees

add an edge

\[ \text{any vertex} \quad \rightarrow \quad \text{root of another tree} \]

\( n \quad \rightarrow \quad n-k-1 \)
# of sequences of adding directed edges to an empty graph to form a rooted tree

From an empty graph: • add edges one by one

\[
\prod_{k=0}^{n-2} n(n-k-1)
\]

\[
= n^{n-1} \prod_{k=1}^{n-1} k
\]

\[
= n^{n-2} n!
\]

Start from \(n\) rooted trees.

After adding \(k\) edges

\(n-k\) rooted trees

add an edge

any vertex \(\rightarrow\) root of another tree

\(n\) \hspace{1cm} \(n-k-1\)
# of sequences of adding directed edges to an empty graph to form a rooted tree

From a tree:
- pick a root;
- pick an order of edges.

\[ T_n n(n-1)! = n!T_n \]

From an empty graph:
- add edges one by one

\[ \prod_{k=2}^{n} n(k-1) = n^{n-2}n! \]

\[ T_n = n^{n-2} \]
Cayley’s formula:

There are $n^{n-2}$ trees on $n$ distinct vertices.
Graph Laplacian

Graph \( G(V,E) \)

adjacency matrix \( A \)

\[
A(i, j) = \begin{cases} 
1 & \{i, j\} \in E \\
0 & \{i, j\} \notin E 
\end{cases}
\]

diagonal matrix \( D \)

\[
D(i, j) = \begin{cases} 
\deg(i) & i = j \\
0 & i \neq j 
\end{cases}
\]

graph Laplacian \( L \)

\[
L = D - A
\]
**Graph Laplacian**

**graph Laplacian** $L$

$$L(i, j) = \begin{cases} 
\deg(i) & i = j \\
-1 & i \neq j, \{i, j\} \in E \\
0 & \text{otherwise}
\end{cases}$$

**quadratic form:**

$$x L x^T = \sum_{i} d_i x_i^2 - \sum_{ij \in E} x_i x_j = \frac{1}{2} \sum_{ij \in E} (x_i - x_j)^2$$

**incidence matrix** $B : n \times m$

$i \in V, e \in E$

$$B(i, e) = \begin{cases} 
1 & e = \{i, j\}, i < j \\
-1 & e = \{i, j\}, i > j \\
0 & \text{otherwise}
\end{cases}$$

$$L = B B^T$$
Kirchhoff’s matrix-tree theorem

$L_{i,i}$ : submatrix of $L$ by removing $i$th row and $i$th column

t($G$) : number of spanning trees in $G$
Kirchhoff’s matrix-tree theorem

\[ L_{i,i} : \text{submatrix of } L \text{ by removing } \]

\[ \text{ith row and ith column} \]

\[ t(G) : \text{number of spanning trees in } G \]

**Kirchhoff’s Matrix-Tree Theorem:**

\[ \forall i, \quad t(G) = \det(L_{i,i}) \]
Kirchhoff’s Matrix-Tree Theorem:

\[ \forall i, \quad t(G) = \det(L_{i,i}) \]

\( B_i: (n - 1) \times m \)

incidence matrix \( B \) removing \( i \)th row

\[ L = BB^T \]

\[ L_{i,i} = B_iB_i^T \quad \det(L_{i,i}) = \det(B_iB_i^T) = ? \]
Cauchy-Binet Theorem:

\[
\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],S}) \det(B_{S,[n]})
\]

\[A : n \times m\]

\[B : m \times n\]
Cauchy-Binet Theorem:

\[
\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n]}, S) \det(B_{S,[n]})
\]

\[
\det(L_{i,i}) = \det(B_i B_i^T)
\]

\[
= \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\{i\},S}) \det(B_{S,[n]\{i\}}^T)
\]

\[
= \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\{i\},S})^2
\]
\[
\det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\{i\},S})^2
\]

\(j \in [n] \setminus \{i\}, e \in S\)

\[
B_{[n]\{i\},S}(j, e) = \begin{cases} 
1 & e = \{j, k\}, j < k \\
-1 & e = \{j, k\}, j > k \\
0 & \text{otherwise}
\end{cases}
\]

\[
\det(B_{[n]\{i\},S}) = \begin{cases} 
\pm 1 & S \text{ is a spanning tree of } G \\
0 & \text{otherwise}
\end{cases}
\]
\[
\det(B_{[n]\setminus \{i\},S}) = \begin{cases} 
\pm 1 & \text{if } S \text{ is a spanning tree of } G \\
0 & \text{otherwise}
\end{cases}
\]

\[B' = B_{[n]\setminus \{i\},S}\]

\((n-1) \times (n-1)\) matrix:

- every column contains at most one 1 and at most one -1
- and all other entries are 0

\[
\det(B') \in \{-1, 0, 1\}
\]

\[\det(B') \neq 0 \text{ iff } S \text{ is a spanning tree}\]
\[ \text{det}(B') \neq 0 \iff S \text{ is a spanning tree} \]

\[ S \text{ is not a spanning tree:} \]
\[ \exists \text{ a connected component } R \text{ s.t. } i \notin R \]
\[ \Rightarrow \text{ det}(B') = 0 \]

\[ S \text{ is a spanning tree:} \]
\[ \exists \text{ a leaf } j_1 \neq i \text{ with incident edge } e_1, \text{ delete } e_1 \]
\[ \exists \text{ a leaf } j_2 \neq i \text{ with incident edge } e_2, \text{ delete } e_2 \]
\[ \vdots \]
\[ \text{ vertices: } j_1, j_2, \ldots, j_{n-1} \]
\[ \text{ edges: } e_1, e_2, \ldots, e_{n-1} \]

\[ \text{ det}(B') = \pm 1 \]
**Cauchy-Binet**

\[
det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} det(B_{[n] \setminus \{i\},S})^2
\]

\[
j \in [n] \setminus \{i\}, e \in S
\]

\[
B_{[n] \setminus \{i\},S}(j, e) = \begin{cases}
1 & e = \{j, k\}, j < k \\
-1 & e = \{j, k\}, j > k \\
0 & \text{otherwise}
\end{cases}
\]

\[
det(B_{[n] \setminus \{i\},S}) = \begin{cases}
\pm 1 & S \text{ is a spanning tree of } G \\
0 & \text{otherwise}
\end{cases}
\]
Kirchhoff’s Matrix-Tree Theorem:
\[ \forall i, \quad t(G) = \det(L_{i,i}) \]

all \( n \)-vertex trees: spanning trees of \( K_n \)

\[ L_{i,i} = \begin{bmatrix}
  n - 1 & -1 & \cdots & -1 \\
  -1 & n - 1 & \cdots & -1 \\
  \vdots & \vdots & \ddots & \vdots \\
  -1 & -1 & \cdots & n - 1
\end{bmatrix} \]

Cayley formula:
\[ T_n = t(K_n) = \det(L_{i,i}) = n^{n-2} \]