Combinatorics
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Extremal Combinatorics

“How large or how small a collection of finite objects can be, if it has to satisfy certain restrictions.”

set system $\mathcal{F} \subseteq 2^{[n]}$ with ground set $[n]$
Sunflowers

$\mathcal{F}$ a sunflower of size $r$ with center $C$:

$$|\mathcal{F}| = r \quad \forall S, T \in \mathcal{F}, \quad S \cap T = C$$

a sunflower of size 6 with core $C$
Sunflowers

A sunflower of size \( r \) with center \( C \):

\[
|\mathcal{F}| = r \quad \forall S, T \in \mathcal{F}, \quad S \cap T = C
\]

A sunflower of size 6 with core \( \emptyset \)
Sunflower Lemma (Erdős-Rado 1960)

\[ F \subseteq \left( \begin{bmatrix} n \\ k \end{bmatrix} \right). \quad |F| > k!(r - 1)^k \]

\[ \exists \text{ a sunflower } G \subseteq F, \text{ such that } |G| = r \]

Induction on \( k \). when \( k=1 \)

\[ F \subseteq \left( \begin{bmatrix} n \\ 1 \end{bmatrix} \right), \quad |F| > r - 1 \]

\[ \exists r \text{ singletons} \]
Sunflower Lemma (Erdős-Rado 1960)

\[ \mathcal{F} \subseteq \binom{[n]}{k}. \quad |\mathcal{F}| > k!(r - 1)^k \]

\[ \exists \text{ a sunflower } \mathcal{G} \subseteq \mathcal{F}, \text{ such that } |\mathcal{G}| = r \]

For \( k \geq 2 \),

- take largest \( \mathcal{G} \subseteq \mathcal{F} \) with disjoint members \( \forall S, T \in \mathcal{G} \text{ that } S \neq T, S \cap T = \emptyset \)

- case 1: \( |\mathcal{G}| \geq r \), \( \mathcal{G} \) is a sunflower of size \( r \)
- case 2: \( |\mathcal{G}| \leq r - 1 \),

Goal: find a popular \( x \in [n] \)
**Goal:** find a popular $x \in [n]$

Consider

$$\mathcal{F} \subseteq \binom{[n]}{k}.$$  

$|\mathcal{F}| > k!(r - 1)^k$

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

$|\mathcal{G}| \leq r - 1,$  

**Goal:** find a popular $x \in [n]$

Consider

$$\{ S \in \mathcal{F} \mid x \in S \}$$

Remove $x$

$$\mathcal{H} = \{ S \setminus \{x\} \mid S \in \mathcal{F} \land x \in S \}$$

$\mathcal{H} \subseteq \binom{[n]}{k-1}$ if $|\mathcal{H}| > (k - 1)!(r - 1)^{k-1}$ I.H.
\[ \mathcal{F} \subseteq \binom{[n]}{k}. \quad |\mathcal{F}| > k!(r - 1)^k \]

**take largest** \( \mathcal{G} \subseteq \mathcal{F} \) **with disjoint members**

\[ |\mathcal{G}| \leq r - 1, \quad \text{let} \quad Y = \bigcup_{S \in \mathcal{G}} S \quad |Y| \leq k(r - 1) \]

**claim:** \( Y \) intersects all \( S \in \mathcal{F} \)

**if otherwise:** \( \exists T \in \mathcal{F}, \quad T \cap Y = \emptyset \)

\( T \) is disjoint with all \( S \in \mathcal{G} \)

**contradiction!**
\[ \mathcal{F} \subseteq \binom{[n]}{k}. \quad |\mathcal{F}| > k!(r - 1)^k \]

take maximal \( \mathcal{G} \subseteq \mathcal{F} \) with disjoint members

\[ |\mathcal{G}| \leq r - 1, \quad \text{let } Y = \bigcup_{S \in \mathcal{G}} S \quad |Y| \leq k(r - 1) \]

\( Y \) intersects all \( S \in \mathcal{F} \)

pigeonhole: \( \exists x \in Y, \quad \# \text{ of } S \in \mathcal{F} \text{ contain } x \)

\[ |\{S \in \mathcal{F} \mid x \in S\}| \geq \frac{|\mathcal{F}|}{|Y|} \geq \frac{k!(r - 1)^k}{k(r - 1)} \]

\[ = (k - 1)!(r - 1)^{k-1} \]

\[ \mathcal{H} = \{S \setminus \{x\} \mid S \in \mathcal{F} \land x \in S\} \]

\[ \mathcal{H} \subseteq \binom{[n]}{k - 1} \quad |\mathcal{H}| > (k - 1)!(r - 1)^{k-1} \]
Sunflower Lemma (Erdős-Rado 1960)

\[ F \subseteq \binom{[n]}{k}. \quad |F| > k!(r - 1)^k \]

\[ \exists \text{ a sunflower } G \subseteq F, \text{ such that } |G| = r \]

\[ \exists x \in Y, \quad \text{let } H = \{ S \setminus \{x\} \mid S \in F \land x \in S \} \]

\[ H \subseteq \binom{[n]}{k - 1} \]

\[ |H| > (k - 1)!(r - 1)^{k-1} \]

I.H.: \( H \) contains a sunflower of size \( r \)

adding \( x \) back, it is a sunflower in \( F \)
**Sunflower Conjecture**

\[
\mathcal{F} \subseteq \binom{[n]}{k}. \quad |\mathcal{F}| > c(r)^k
\]

\(\exists\) a **sunflower** \(\mathcal{G} \subseteq \mathcal{F}\), such that \(|\mathcal{G}| = r\)

\(c(r)\) : constant depending only on \(r\)

**Alon-Shpilka-Umans 2012:**

if sunflower conjecture is true
then matrix multiplication is slow
Erdős-Ko-Rado Theorem

Intersecting Families

\[ \mathcal{F} \subseteq \binom{[n]}{k} \]

intersecting:
\[ \forall S, T \in \mathcal{F}, \quad S \cap T \neq \emptyset \]

trivial case: \( n < 2k \)

nontrivial examples:

“How large can a nontrivial intersecting family be?”
Erdős-Ko-Rado Theorem

\[ \mathcal{F} \subseteq \binom{[n]}{k}, n \geq 2k. \]

\[ \forall S \text{ intersecting } \emptyset \implies |\mathcal{F}| \leq \binom{n-1}{k-1} \]

proved in 1938; published in 1961;

all \( S \ni x \)
Isoperimetric problem:

With fixed perimeter, what plane figure has the largest area?

Steiner symmetrization
Shifting

Isoperimetric problem:

With fixed area, what plane figure has the smallest perimeter?

Steiner symmetrization
Erdős-Ko-Rado Theorem

Let \( \mathcal{F} \subseteq \binom{[n]}{k} \), \( n \geq 2k \).

\[ \forall S, T \in \mathcal{F}, \ S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1} \]

**induction on** \( n \) and \( k \)

\[ \mathcal{F}_0 = \{ S \in \mathcal{F} \mid n \notin S \} \]

\[ \mathcal{F}_0 \subseteq \binom{[n-1]}{k} \]

**I.H.**

\[ |\mathcal{F}_0| \leq \binom{n-2}{k-1} \]

\[ \mathcal{F}_1 = \{ S \in \mathcal{F} \mid n \in S \} \]

\[ \mathcal{F}_1' = \{ S \setminus \{n\} \mid S \in \mathcal{F}_1 \} \]

\[ \mathcal{F}_1 \subseteq \binom{[n-1]}{k-1} \]

**I.H.**

\[ |\mathcal{F}_1'| \leq \binom{n-2}{k-2} \]

\[ |\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{F}_0| + |\mathcal{F}_1'| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1} \]
Shifting (compression)

special \( \mathcal{F} \subseteq \binom{[n]}{k} \)

\( \mathcal{F} \) remains intersecting after deleting \( n \)
Shifting (compression)

\[ \mathcal{F} \subseteq 2^{[n]} \quad \text{for} \quad 1 \leq i < j \leq n \]

\[ \forall T \in \mathcal{F}, \quad \text{write} \quad T_{ij} = (T \setminus \{j\}) \cup \{i\} \]

\((i,j)\)-shift: \(S_{ij}(\cdot)\)

\[ \forall T \in \mathcal{F}, \]

\[ S_{ij}(T) = \begin{cases} 
  T_{ij} & \text{if } j \in T, i \not\in T, \text{ and } T_{ij} \not\in \mathcal{F}, \\
  T & \text{otherwise}. 
\end{cases} \]

\[ S_{ij}(\mathcal{F}) = \{S_{ij}(T) \mid T \in \mathcal{F}\} \]
1 \leq i < j \leq n \quad \forall T \in \mathcal{F}, \text{ write } T_{i,j} = (T \setminus \{j\}) \cup \{i\}

S_{i,j}(T) = \begin{cases} T_{i,j} & \text{if } j \in T, i \not\in T, \text{ and } T_{i,j} \not\in \mathcal{F}, \\ T & \text{otherwise.} \end{cases}

S_{i,j}(\mathcal{F}) = \{ S_{i,j}(T) \mid T \in \mathcal{F} \}

1. \quad |S_{i,j}(T)| = |T| \quad \text{and} \quad |S_{i,j}(\mathcal{F})| = |\mathcal{F}|

2. \quad \mathcal{F} \text{ intersecting } \Rightarrow \quad S_{i,j}(\mathcal{F}) \text{ intersecting}

(2) the only bad case: \quad A, B \in \mathcal{F} \quad A \cap B = \{j\}\n
A_{i,j} = A \setminus \{j\} \cup \{i\} \in \mathcal{F} \quad B_{i,j} = B \setminus \{j\} \cup \{i\} \not\in \mathcal{F} \quad i \not\in B

\Rightarrow \quad A_{i,j} \cap B = \emptyset \quad \text{contradiction!}
1 \leq i < j \leq n \quad \forall T \in \mathcal{F}, \text{ write } T_{ij} = (T \setminus \{j\}) \cup \{i\}

S_{ij}(T) = \begin{cases} T_{ij} & \text{if } j \in T, i \notin T, \text{ and } T_{ij} \notin \mathcal{F}, \\ T & \text{otherwise}. \end{cases}

S_{ij}(\mathcal{F}) = \{S_{ij}(T) \mid T \in \mathcal{F}\}

1. \quad |S_{ij}(T)| = |T| \quad \text{and} \quad |S_{ij}(\mathcal{F})| = |\mathcal{F}|

2. \quad \mathcal{F} \text{ intersecting } \Rightarrow S_{ij}(\mathcal{F}) \text{ intersecting}

repeat applying \((i, j)\)-shifting \(S_{ij}(\mathcal{F})\) for \(1 \leq i < j \leq n\)

eventually, \(\mathcal{F}\) is unchanged by any \(S_{ij}(\mathcal{F})\)

called: \(\mathcal{F}\) is shifted
Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\forall S, T \in \mathcal{F}, S \cap T \neq \emptyset \quad \Rightarrow \quad |\mathcal{F}| \leq \binom{n-1}{k-1}$$

**Erdős-Ko-Rado’s proof:**

true for $k=1$;
when $n = 2k$,

$$\forall S \in \binom{[n]}{k} \quad \text{at most one of } S \text{ and } \bar{S} \text{ is in } \mathcal{F}$$

$$|\mathcal{F}| \leq \frac{1}{2} \binom{n}{k} = \frac{n!}{2 \cdot k!(n-k)!}$$

$$= \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1}$$
Let \( \mathcal{F} \subseteq \binom{[n]}{k} \), \( n \geq 2k \).

\[
\forall S, T \in \mathcal{F}, \ S \cap T \neq \emptyset \quad \Rightarrow \quad |\mathcal{F}| \leq \binom{n-1}{k-1}
\]

**arbitrary intersecting** \( \mathcal{F} \)  \( \Rightarrow \)  \( |\mathcal{F}| = |\mathcal{F}'| \)  \( \Rightarrow \)  \( \mathcal{F}' \) **shifted**

**keep intersecting**  \( |\mathcal{F}| \leq \binom{n-1}{k-1} \)  \( \Rightarrow \)  \( |\mathcal{F}'| \leq \binom{n-1}{k-1} \)
Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\forall S, T \in \mathcal{F}, S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$

when $n > 2k$, induction on $n$ \quad WLOG: $\mathcal{F}$ is shifted

$\mathcal{F}_1 = \{S \in \mathcal{F} \mid n \in S\}$ \quad $\mathcal{F}_1' = \{S \setminus \{n\} \mid S \in \mathcal{F}_1\}$

$\mathcal{F}_1'$ is intersecting

otherwise, \quad $\exists A, B \in \mathcal{F}$ \quad $A \cap B = \{n\}$

$|A \cup B| \leq 2k - 1 < n - 1 \Rightarrow \exists i < n, i \notin A \cup B$

$C = A \setminus \{n\} \cup \{i\} \in \mathcal{F}$ \quad $\mathcal{F}$ is shifted

$C \cap B = \emptyset$ \quad contradiction!
Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$. 

\[ \forall S, T \in \mathcal{F}, S \cap T \neq \emptyset \implies |\mathcal{F}| \leq \binom{n-1}{k-1} \]

when $n > 2k$, induction on $n$  

WLOG: $\mathcal{F}$ is shifted  

$\mathcal{F}_0 = \{ S \in \mathcal{F} \mid n \notin S \}$  

$\mathcal{F}_1 = \{ S \in \mathcal{F} \mid n \in S \}$  

$\mathcal{F}_0 \subseteq \binom{[n-1]}{k}$ and intersecting  

I.H.  

$|\mathcal{F}_0| \leq \binom{n-2}{k-1}$  

$\mathcal{F}_1' = \{ S \setminus \{n\} \mid S \in \mathcal{F}_1 \}$  

$\mathcal{F}_1' \subseteq \binom{[n-1]}{k-1}$ and intersecting  

I.H.  

$|\mathcal{F}_1'| \leq \binom{n-2}{k-2}$  

$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{F}_0| + |\mathcal{F}_1'| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$
Katona’s proof (1972)

$n$-cycle:

$k$-arc: length $k$ path on cycle

intersecting arcs: share edges

Lemma

$n \geq 2k$. Suppose $A_1, A_2, ..., A_t$ are distinct pairwise intersecting $k$-arcs. Then $t \leq k$.

every node can be endpoint of at most 1 arc

take $A_1$: $A_1$ has $k+1$ nodes

2 endpoints of itself
Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

\[ \forall S, T \in \mathcal{F}, S \cap T \neq \emptyset \quad \Rightarrow \quad |\mathcal{F}| \leq \binom{n-1}{k-1} \]

take an $n$-cycle $\pi$ of $[n]$

family of all $k$-arcs in $\pi$

\[ \mathcal{G}_\pi = \{ \{ \pi(i+j) \mod n \mid j \in [k] \} \mid i \in [n] \} \]

double counting: \[ X = \{ (S, \pi) \mid S \in \mathcal{F} \cap \mathcal{G}_\pi \} \]

each $n$-cycle $\pi$ an $n$-cycle has $\leq k$ intersecting $k$-arcs

\[ |\mathcal{F} \cap \mathcal{G}_\pi| \leq k \]

# of $n$-cycles: $(n-1)!$

\[ |X| = \sum_{n\text{-cycle } \pi} |\mathcal{F} \cap \mathcal{G}_\pi| \leq k(n-1)! \]
Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\forall S, T \in \mathcal{F}, S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$

take an $n$-cycle $\pi$ of $[n]$

family of all $k$-arcs in $\pi$

$\mathcal{G}_\pi = \left\{ \left\{ \pi(i+j) \mod n \mid j \in [k] \right\} \mid i \in [n] \right\}$

double counting: $X = \left\{ (S, \pi) \mid S \in \mathcal{F} \cap \mathcal{G}_\pi \right\}$

$|X| \leq k(n - 1)!$

each $S$ is a $k$-arc in $k!(n-k)!$ cycles

$|X| = \sum_{S \in \mathcal{F}} |\{\pi \mid S \in \mathcal{G}_\pi\}| = |\mathcal{F}|k!(n - k)!$
Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

For all $S, T \in \mathcal{F}$, $S \cap T \neq \emptyset$ implies $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Take an $n$-cycle $\pi$ of $[n]$ family of all $k$-arcs in $\pi$

$$\mathcal{G}_\pi = \left\{ \{\pi(i+j) \mod n \mid j \in [k]\} \mid i \in [n] \right\}$$

Double counting:

$$|X| \leq k(n-1)!$$

$$|X| = |\mathcal{F}|k!(n-k)!$$

$$|\mathcal{F}| \leq \frac{k(n-1)!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1}$$
\[ \mathcal{F} \subseteq 2^{[n]} \text{ is an antichain} \]
\[ \forall A, B \in \mathcal{F}, \ A \nsubseteq B \]
\[ \binom{[n]}{k} \text{ is antichain} \]

largest size: \( \binom{n}{\lfloor n/2 \rfloor} \)

“Is this the largest size for all antichains?”
Theorem (Sperner 1928) \( \mathcal{F} \subseteq 2^{[n]} \) is an antichain.

\[ |\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}. \]
Sperner’s proof
\[ \mathcal{F} \subseteq \binom{[n]}{k} \]

**shade:** \[ \nabla \mathcal{F} = \left\{ T \in \binom{[n]}{k+1} \mid \exists S \in \mathcal{F}, S \subset T \right\} \]

**shadow:** \[ \Delta \mathcal{F} = \left\{ T \in \binom{[n]}{k-1} \mid \exists S \in \mathcal{F}, T \subset S \right\} \]

\[ [n] = \{1,2,3,4,5\} \]

\[ \mathcal{F} = \{ \{1,2,3\}, \{1,3,4\}, \{2,3,5\} \} \]

\[ \nabla \mathcal{F} = \{ \{1,2,3,4\}, \{1,2,3,5\}, \{1,3,4,5\}, \{2,3,4,5\} \} \]

\[ \Delta \mathcal{F} = \{ \{1,2\}, \{2,3\}, \{1,3\}, \{3,4\}, \{1,4\}, \{2,5\}, \{3,5\} \} \]
Lemma (Sperner)
Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Then

$$|\nabla \mathcal{F}| \geq \frac{n-k}{k+1} |\mathcal{F}| \quad \text{ (for } k < n)$$

$$|\Delta \mathcal{F}| \geq \frac{k}{n-k+1} |\mathcal{F}| \quad \text{ (for } k > 0)$$

double counting

$$\mathcal{R} = \{(S, T) \mid S \in \mathcal{F}, T \in \nabla \mathcal{F}, S \subset T\}$$

\forall S \in \mathcal{F}, \quad n-k \ T \in \binom{[n]}{k+1} \ 	ext{have } T \supset S$$

$$|\mathcal{R}| = (n-k)|\mathcal{F}|$$

\forall T \in \nabla \mathcal{F}, \quad T \ 	ext{has } \binom{k+1}{k} = k+1 \ 	ext{many } k\text{-subsets}$$

$$|\mathcal{R}| \leq (k+1)|\nabla \mathcal{F}|$$
Lemma (Sperner)

Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Then

$$ |\nabla \mathcal{F}| \geq \frac{n-k}{k+1} |\mathcal{F}| \quad \text{ (for } k < n) $$

$$ |\Delta \mathcal{F}| \geq \frac{k}{n-k+1} |\mathcal{F}| \quad \text{ (for } k > 0) $$

Corollary:

If $k \leq \frac{1}{2}(n-1)$, then $|\nabla \mathcal{F}| \geq |\mathcal{F}|$.

If $k \geq \frac{1}{2}(n+1)$, then $|\Delta \mathcal{F}| \geq |\mathcal{F}|$. 
Sperner’s Theorem

\( \mathcal{F} \subseteq 2^{[n]} \) is an antichain. Then \( |\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} \).

![Diagram]

let \( \mathcal{F}_k = \mathcal{F} \cap \binom{[n]}{k} \)

If \( k \leq \frac{1}{2}(n - 1) \), then \( |\nabla \mathcal{F}| \geq |\mathcal{F}| \).

If \( k \geq \frac{1}{2}(n + 1) \), then \( |\Delta \mathcal{F}| \geq |\mathcal{F}| \).

replace \( \mathcal{F}_k \) by \( \begin{cases} \nabla \mathcal{F}_k & \text{if } k < \frac{1}{2}(n - 1) \\ \Delta \mathcal{F}_k & \text{if } k \geq \frac{1}{2}(n + 1) \end{cases} \)

repeat until \( \mathcal{F} \subseteq \binom{[n]}{\lfloor n/2 \rfloor} \) with no decreasing of \( |\mathcal{F}| \)

still antichain!
Sperner’s Theorem
\( \mathcal{F} \subseteq 2^{[n]} \) is an antichain. Then \( |\mathcal{F}| \leq \binom{n}{\lceil n/2 \rceil} \).

**Lubell’s proof** (double counting)

maximal chain:
\[
\emptyset \subset S_1 \subset \cdots \subset S_{n-1} \subset [n]
\]

# of maximal chains in \( 2^{[n]} \): \( n! \)

\( \forall S \subseteq [n], \)

# of maximal chains containing \( S \): \( |S|!(n - |S|)! \)

\( \mathcal{F} \) is antichain \( \Rightarrow \forall \) chain \( C \), \( |\mathcal{F} \cap C| \leq 1 \)

# maximal chains crossing \( \mathcal{F} \) \( \leq \) # all maximal chains
Sperner’s Theorem
\( \mathcal{F} \subseteq 2^{[n]} \) is an antichain. Then \( |\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} \).

Lubell’s proof (double counting)

maximal chain:
\[
\emptyset \subset S_1 \subset \cdots \subset S_{n-1} \subset [n]
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# of maximal chains in \( 2^{[n]} \): \( n! \)

\( \forall S \subseteq [n], \)

# of maximal chains containing \( S \): \( |S|!(n - |S|)! \)

\( \mathcal{F} \) is antichain \( \implies \forall \text{ chain } C, \quad |\mathcal{F} \cap C| \leq 1 \)

\[
\sum_{S \in \mathcal{F}} |S|!(n - |S|)! \leq n!
\]
Sperner’s Theorem

\( \mathcal{F} \subseteq 2^{[n]} \) is an antichain. Then \( |\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} \).

Lubell’s proof (double counting)

\[
\sum_{S \in \mathcal{F}} |S|!(n - |S|)! \leq n!
\]

\[
\frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{S \in \mathcal{F}} \frac{1}{\binom{|S|}{n}} = \sum_{S \in \mathcal{F}} \frac{|S|!(n - |S|)!}{n!} \leq 1
\]

\[|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}\]
LYM Inequality

(Lubell-Yamamoto 1954, Meschalkin 1963)

LYM Inequality

\[ \mathcal{F} \subseteq 2^{[n]} \text{ is an antichain.} \]

\[ \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1 \]
\[ \mathcal{F} \subseteq 2^{[n]} \text{ is an antichain.} \quad \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1 \]

**Alon’s proof** (the probabilistic method)

let \( \pi \) be a random permutation \([n]\)

\[ \mathcal{C}_\pi = \{ \{\pi_1\}, \{\pi_1, \pi_2\}, \ldots, \{\pi_1, \ldots, \pi_n\} \} \]

\[ \forall S \in \mathcal{F}, \quad X_S = \begin{cases} 1 & S \in \mathcal{C}_\pi \\ 0 & \text{otherwise} \end{cases} \]

let \( X = \sum_{S \in \mathcal{F}} X_S = |\mathcal{F} \cap \mathcal{C}_\pi| \)

\[ \mathbb{E}[X_S] = \Pr[S \in \mathcal{C}_\pi] = \frac{1}{\binom{n}{|S|}} \]

\( \mathcal{C}_\pi \) contains precisely 1 \(|S|\)-set uniform over all \(|S|\)-sets
Alon’s proof (the probabilistic method)

let \( \pi \) be a random permutation \([n]\)

\[
C_\pi = \{\{\pi_1\}, \{\pi_1, \pi_2\}, \ldots, \{\pi_1, \ldots, \pi_n\}\}
\]

\[
X = \sum_{S \in \mathcal{F}} X_S = |\mathcal{F} \cap C_\pi| \leq 1 \quad \mathcal{F} \text{ is antichain}
\]

\[
\mathbb{E}[X_S] = \frac{1}{\binom{n}{|S|}}
\]

\[
1 \geq \mathbb{E}[X] = \sum_{S \in \mathcal{F}} \mathbb{E}[X_S] = \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}}
\]
Sperner's Theorem

- Sperner's proof (shadows)
- Lubell's proof (counting)

- LYM inequality

- Alon's proof (probabilistic)
Shattering

\[ \mathcal{F} \subseteq 2^{[n]} \quad \text{and} \quad R \subseteq [n] \]

**trace** \( \mathcal{F}|_R \):

\[ \mathcal{F}|_R = \{ S \cap R \mid S \in \mathcal{F} \} \]

\( \mathcal{F} \) shatters \( R \)

\[ \mathcal{F}|_R = 2^R \]
**Sauer’s Lemma**

\[ |\mathcal{F}| > \sum_{0 \leq i < k} \binom{n}{i} \quad \Rightarrow \quad \exists R \subseteq \binom{[n]}{k}, \mathcal{F} \text{ shatters } R \]

Sauer; Shelah-Perles; Vapnik-Cervonenkis;

**VC-dimension of** \( \mathcal{F} \)

size of the largest \( R \) shattered by \( \mathcal{F} \)

\[ \mathcal{F} \subseteq 2^{[n]} \quad \mathcal{F}|_R = \{ S \cap R \mid S \in \mathcal{F} \} \]

\[ \text{VC-dim}(\mathcal{F}) = \max \{ |R| \mid R \subseteq [n], \mathcal{F}|_R = 2^R \} \]
Hereditary (ideal, simplicial complex)

\( \mathcal{F} \) is hereditary if \( \forall B \subseteq A \in \mathcal{F}, \quad B \in \mathcal{F} \)
Heredity  (ideal, simplicial complex)

Sauer’s Lemma

\[ |\mathcal{F}| > \sum_{0 \leq i < k} \binom{n}{i} \quad \Rightarrow \quad \exists R \in \binom{[n]}{k}, \mathcal{F} \text{ shatters } R \]

\[ |\mathcal{F}| > \sum_{0 \leq i < k} \binom{n}{i} \quad \Rightarrow \quad \exists R \in \mathcal{F}, |R| \geq k \]

for hereditary \( \mathcal{F} \): \( \forall B \subseteq A \in \mathcal{F}, \quad B \in \mathcal{F} \)

\( R \in \mathcal{F} \quad \Rightarrow \quad \mathcal{F} \text{ shatters } R \)
**Sauer’s Lemma**

\[ |\mathcal{F}| > \sum_{0 \leq i < k} \binom{n}{i} \implies \exists R \in \binom{[n]}{k}, \mathcal{F} \text{ shatters } R \]

\[ |\mathcal{F}| \leq |\mathcal{F}'| \]

arbitrary \( \mathcal{F} \) ➔ hereditary \( \mathcal{F}' \)

\[ \text{VC-dim}(\mathcal{F}) \geq \text{VC-dim}(\mathcal{F}') \]

\( \mathcal{F} \) shatters a \( k \)-set ➔ \( \mathcal{F}' \) shatters a \( k \)-set
Down Shift

\[ F \subseteq 2^{[n]} \quad \text{for} \quad i \in [n] \]

down-shift: \( S_i(\cdot) \)

\[
S_i(T) = \begin{cases} 
T \setminus \{i\} & \text{if } i \in T \in F, \text{ and } T \setminus \{i\} \notin F, \\
T & \text{otherwise.} 
\end{cases}
\]

\[
S_i(F) = \{ S_i(T) \mid T \in F \}
\]
\( \mathcal{F} \subseteq 2^{[n]} \)

\( \mathcal{F}|_R = \{ S \cap R \mid S \in \mathcal{F} \} \quad \text{for } i \in [n] \)

\( S_i(T) = \begin{cases} 
T \setminus \{i\} & \text{if } i \in T \in \mathcal{F}, \text{ and } T \setminus \{i\} \notin \mathcal{F}, \\
T & \text{otherwise.}
\end{cases} \)

\( S_i(\mathcal{F}) = \{ S_i(T) \mid T \in \mathcal{F} \} \)

1. \(|S_i(\mathcal{F})| = |\mathcal{F}| \checkmark\)

2. \(|S_i(\mathcal{F})|_R \leq |\mathcal{F}|_R| \text{ for all } R \subseteq [n] \)

\( S_i(\mathcal{F})|_R \subseteq S_i(\mathcal{F}|_R) \)

**by-case analysis**

\( A \in S_i(\mathcal{F}) \)

\( \left\{ \begin{array}{l}
A = S_i(A \cup \{i\}) \\
A = S_i(A)
\end{array} \right\} \quad A \cap R \in S_i(\mathcal{F}|_R) \)
\[ F \subseteq 2^{[n]} \quad F|_R = \{ S \cap R \mid S \in F \} \quad \text{for} \quad i \in [n] \]

\[ S_i(T) = \begin{cases} 
T \setminus \{i\} & \text{if } i \in T \in F, \text{ and } T \setminus \{i\} \notin F, \\
T & \text{otherwise.} 
\end{cases} \]

\[ S_i(F) = \{ S_i(T) \mid T \in F \} \]

1. \[ |S_i(F)| = |F| \]
2. \[ |S_i(F)|_R \leq |F|_R \] for all \( R \subseteq [n] \)

repeat applying down-shifting \( S_i(F) \) for \( i \in [n] \)

eventually, \( F \) is unchanged by any \( S_i(F) \)

\[ \forall A \in F \quad \text{if } B \subseteq A \quad \rightarrow \quad B \in F \]

\( F \) is hereditary
Sauer’s Lemma

\[ |\mathcal{F}| > \sum_{0 \leq i < k} \binom{n}{i} \quad \rightarrow \quad \exists R \in \binom{[n]}{k}, \mathcal{F} \text{ shatters } R \]

repeat down-shift \(\mathcal{F}\) until unchanged

\(\mathcal{F}\) is hereditary

\[ |\mathcal{F}| > \sum_{0 \leq i < k} \binom{n}{i} \quad \rightarrow \quad \exists S \in \binom{[n]}{\ell} \text{ with } \ell \geq k \]

\[ 2^S \subseteq \mathcal{F} \]

take any \(R \in \binom{S}{k}\) \quad \(\mathcal{F}\) shatters \(R\)