Combinatorics

南京大学
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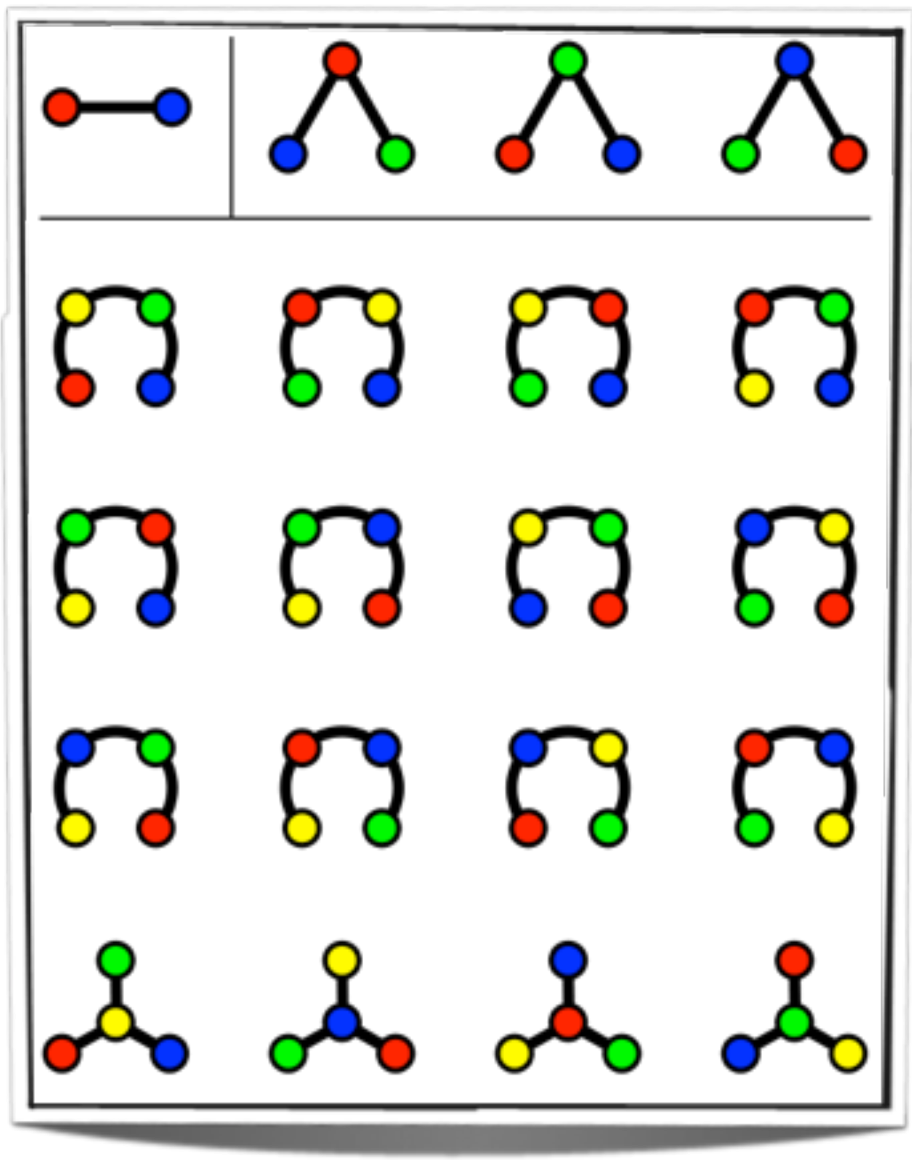
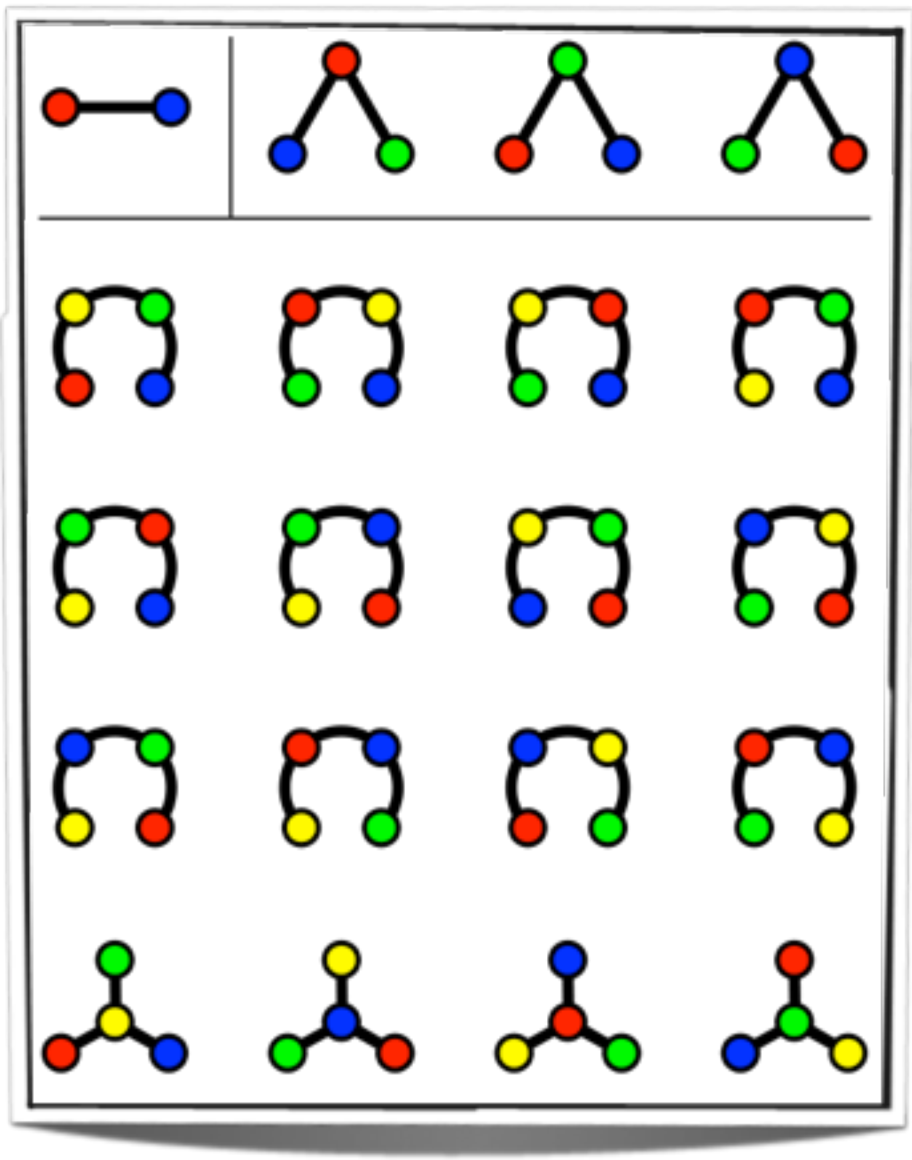
Counting (labeled) trees

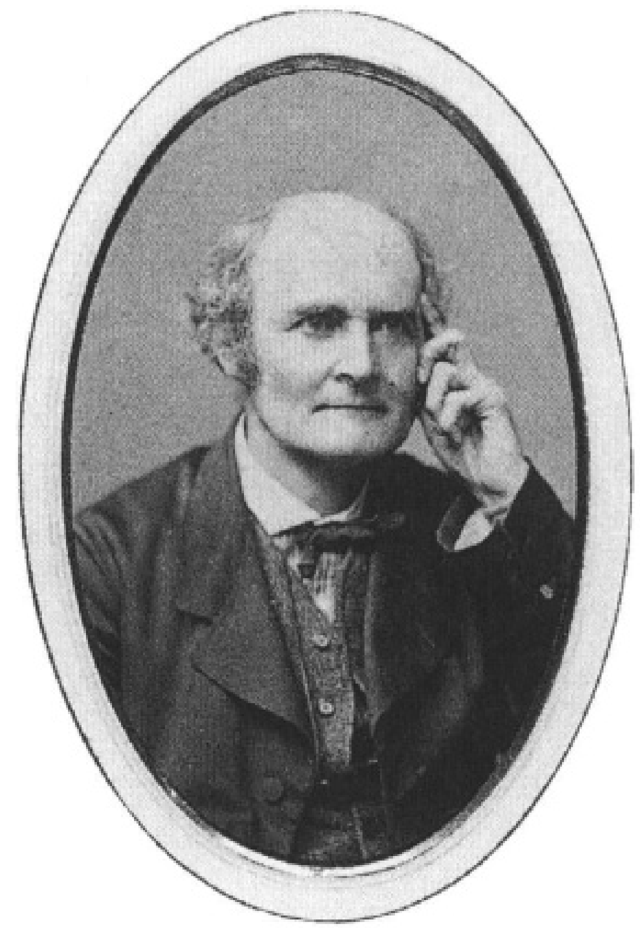


"How many different trees can be formed from n distinct vertices?"

Cayley's formula:

There are n^{n-2} trees on n distinct vertices.





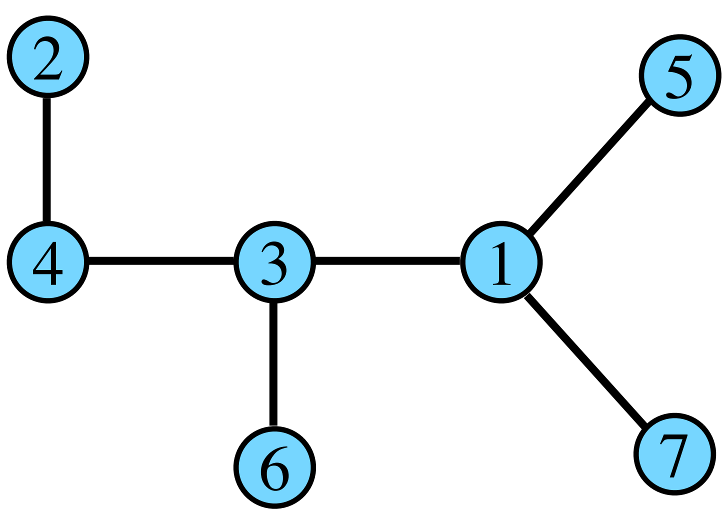
Arthur Cayley

Prüfer Code

leaf : vertex of degree 1

removing a leaf from T, still a tree

T_1 :



$T_1 = T$;

for $i = 1$ to n-1

u_i : smallest leaf in T_i ;

(u_i, v_i) : edge in T_i ;

T_{i+1} = delete u_i from T_i ;

u_i : 2 , 4 , 5 , 6 , 3 , 1

v_i : 4 , 3 , 1 , 3 , 1 , 7

Prüfer code:

$(v_1, v_2, \ldots, v_{n-2})$

edges of T : (u_i, v_i), $1 \le i \le n-1$

$v_{n-1} = n$

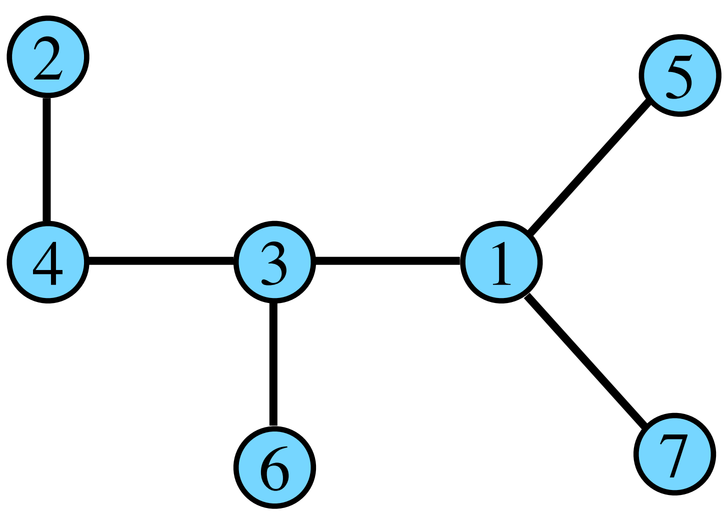
u_i : smallest leaf in T_i

a tree has ≥ 2 leaves

n is never deleted

$u_i \neq n$

T :



Only need to recover every u_i from $(v_1, v_1, \ldots, v_{n-2})$.

u_i : 2 , 4 , 5 , 6 , 3 , 1

v_i : 4 , 3 , 1 , 3 , 1 , 7

$(v_1, v_2, \ldots, v_{n-2})$

u_i is the smallest number not in $\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$

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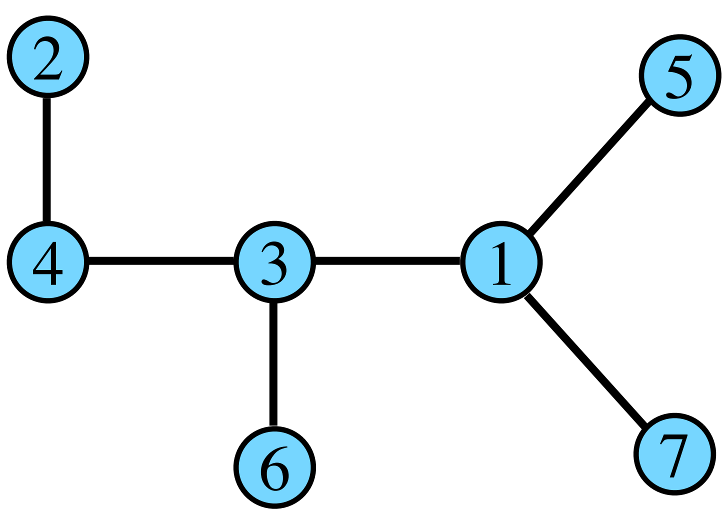
∀ vertex v in T,

occurrences of v in $u_1, u_2, \ldots, u_{n-1}, v_{n-1}$: 1

occurrences of v in edges (u_i, v_i), $1 \le i \le n-1$: $\deg_T(v)$

T :



occurrences of v in Prüfer code: $(v_1, v_2, \ldots, v_{n-2})$

$\deg_T(v)$-1

u_i: 2, 4, 5, 6, 3, 1

v_i: $\underline{4, 3, 1, 3, 1}$, 7

$(v_1, v_2, \ldots, v_{n-2})$

u_i is the smallest number not in

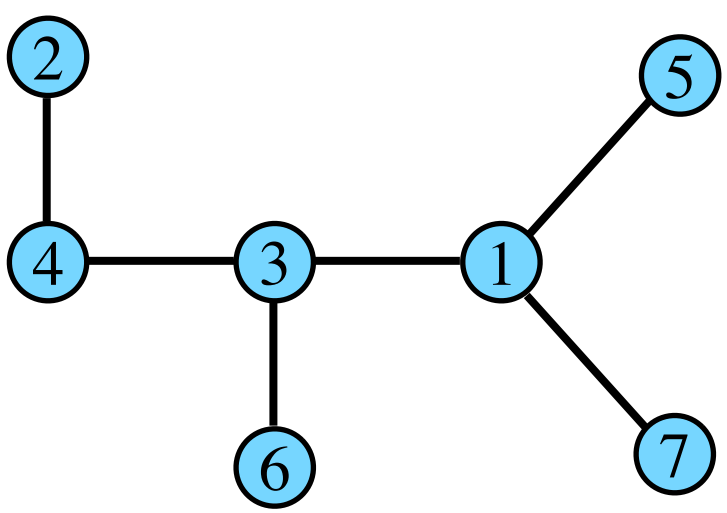
$$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$$

$\forall$ vertex v in T_i,

occurrences of v in $u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1}$: 1

occurrences of v in edges (u_j, v_j), $i \le j \le n-1$: $\deg_{T_i}(v)$

T_3 :



occurrences of v in $(v_i, \ldots, v_{n-2})$

$\deg_{T_i}(v) - 1$

leaf v of T_i :

in $\{u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1}\}$

not in $\{v_i, v_{i+1}, \ldots, v_{n-2}\}$

u_i : smallest leaf in T_i

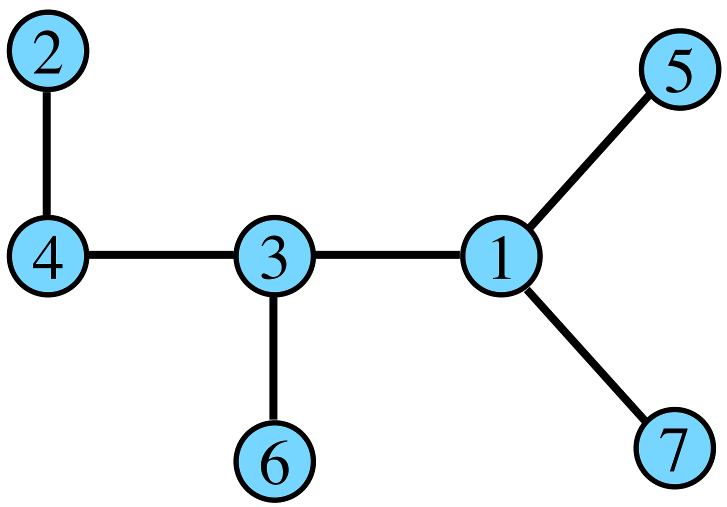
u_i : $\cancel{2}, \cancel{4}, 5, 6, 3, 1$

v_i : $\underline{\cancel{4}, \cancel{3}, 1, 3, 1}, 7$

$(v_1, v_2, \ldots, v_{n-2})$

u_i is the smallest number not in $\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$

T :



u_i : 2 , 4 , 5 , 6 , 3 , 1

v_i : 4 , 3 , 1 , 3 , 1 , 7

$(v_1, v_2, \ldots, v_{n-2})$

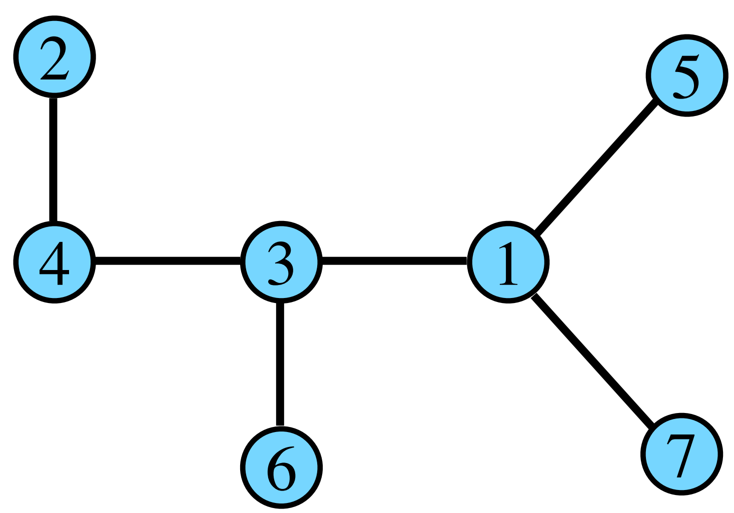
T = empty graph;
$v_{n-1} = n$;
for $i = 1$ to $n-1$
  u_i : smallest number not in
    $\{u_1,\ldots,u_{i-1}\} \cup \{v_i,\ldots,v_{n-1}\}$
  add edge (u_i, v_i) to T;

Prüfer code is reversible $\Rightarrow$ 1-1

every $(v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2}$

is decodable to a tree $\Rightarrow$ onto

T :



u_i: 2 , 4 , 5 , 6 , 3 , 1

v_i: $\underbrace{4 , 3 , 1 , 3 , 1}_{}$, 7

$(v_1, v_2, \ldots, v_{n-2})$

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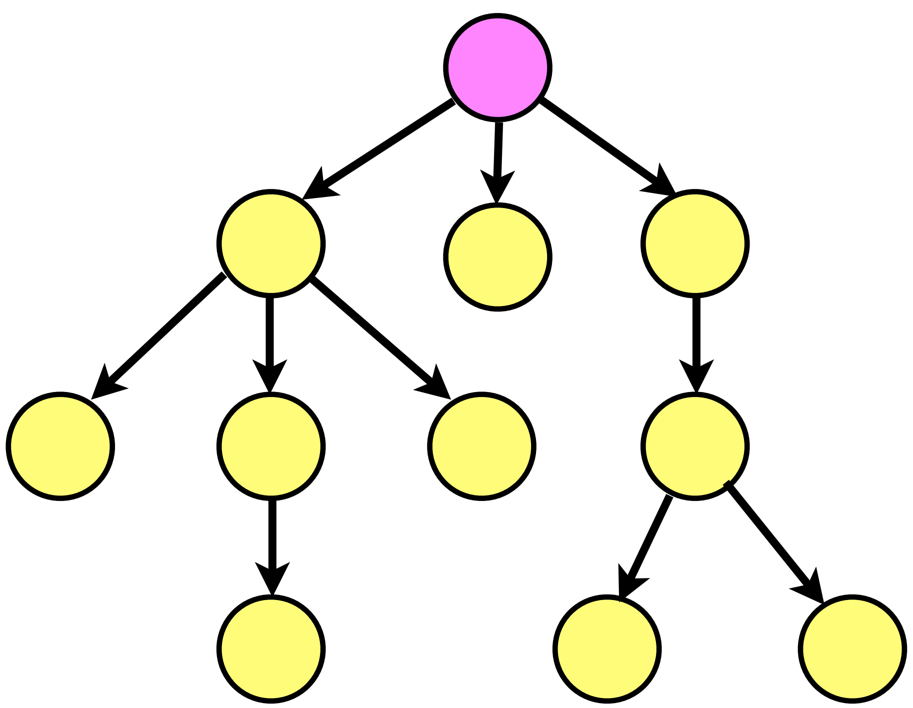
is decodable to a tree $\Rightarrow$ onto

Cayley's formula:

There are n^{n-2} trees on n distinct vertices.

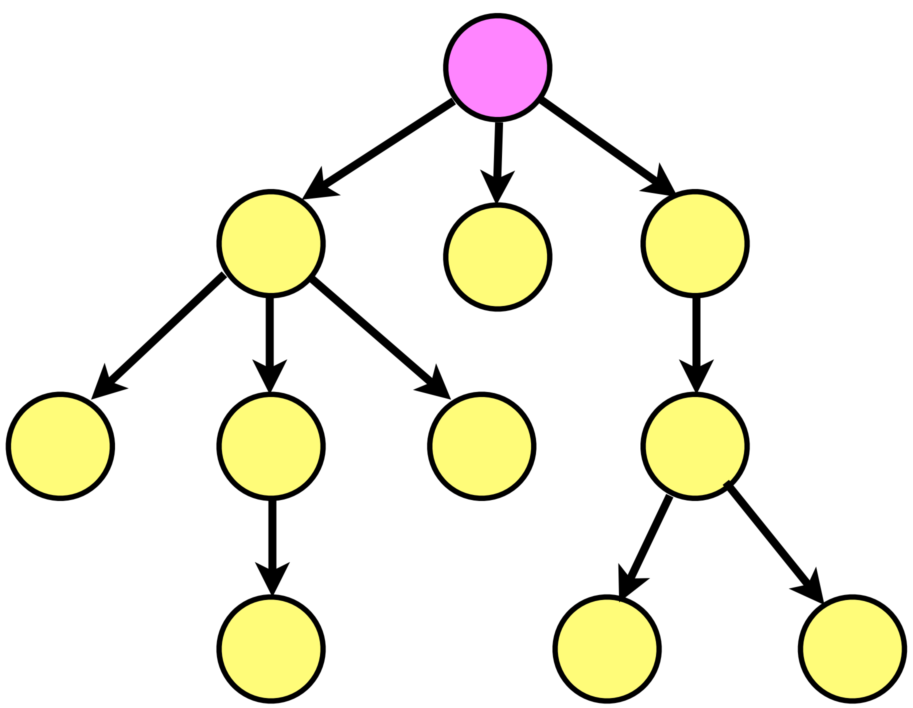
Double Counting

\# of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*



T_n : # of trees on *n* distinct vertices.

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*



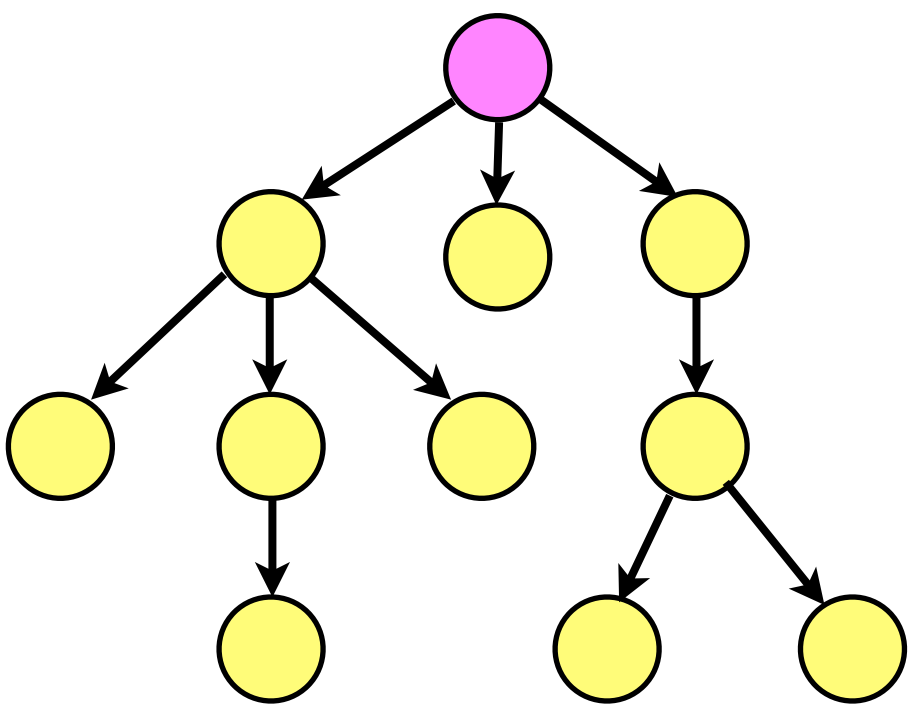
From a tree:

- pick a root;
- pick an order of edges.

$$T_n n(n-1)!$$
$$= n! T_n$$

T_n : # of trees on *n* distinct vertices.

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

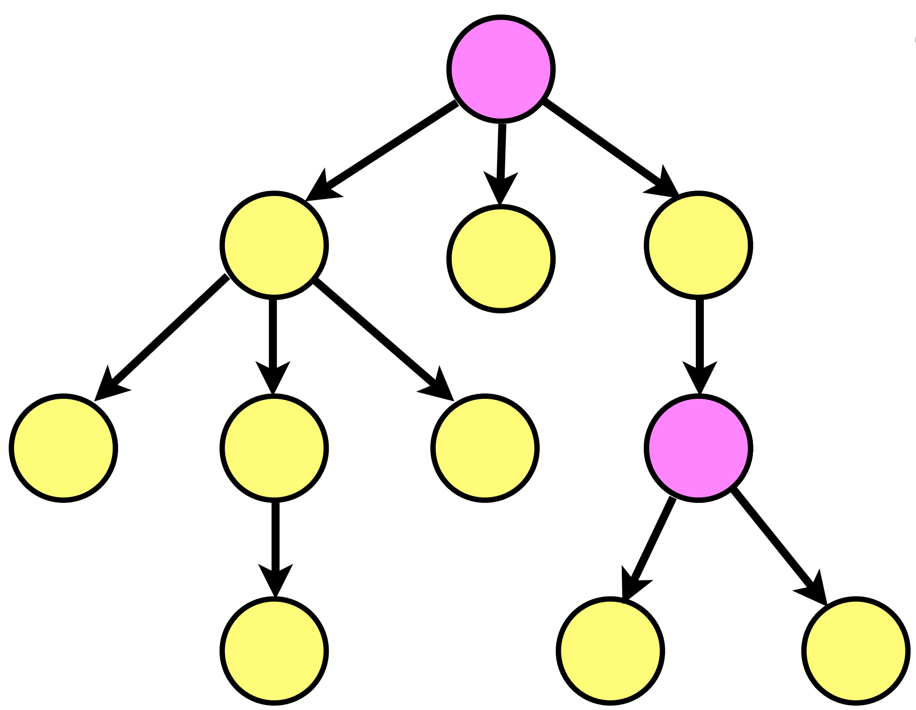


From an empty graph:

- add edges one by one

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

From an empty graph: • add edges one by one

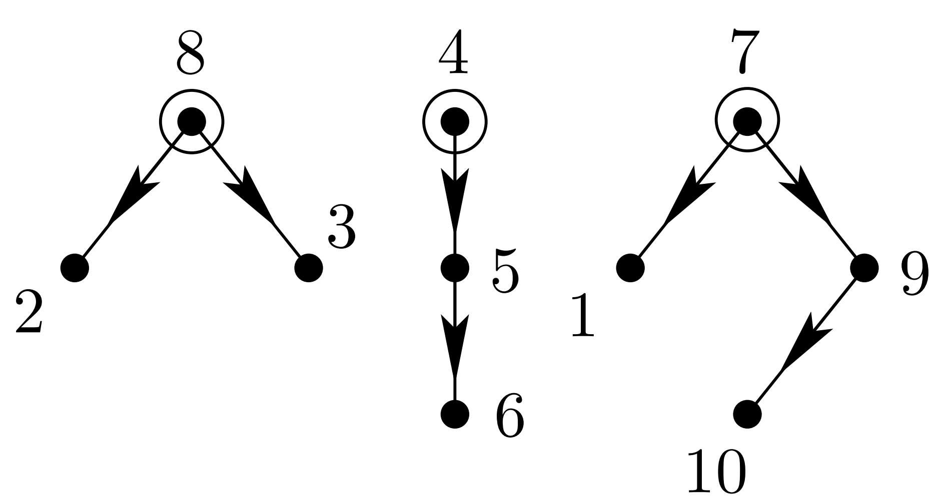


Start from *n* ~~isolated vertices~~ rooted trees

Each step joins 2 trees.

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

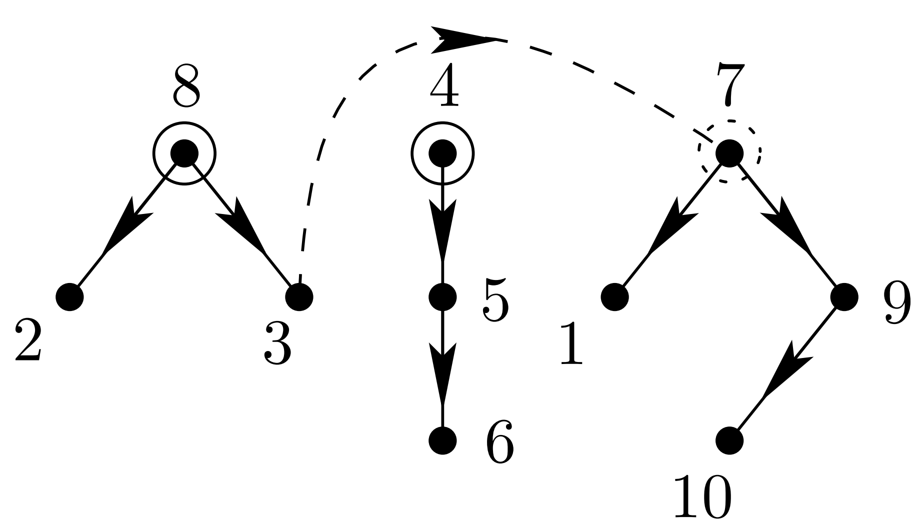
From an empty graph:
- add edges one by one



Start from n rooted trees.

After adding k edges

n-k rooted trees



add an edge

any vertex → root of another tree

n → n-k-1

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

From an empty graph:

- add edges one by one

$$\prod_{k=0}^{n-2} n(n-k-1)$$

$$= n^{n-1}\prod_{k=1}^{n-1} k$$

$$= n^{n-2}n!$$

Start from *n* rooted trees.

After adding *k* edges

n-k rooted trees

add an edge

any vertex $\longrightarrow$ root of another tree

n $\qquad$ $n-k-1$

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

From a tree:

- pick a root;
- pick an order of edges.

$$T_n n(n-1)!$$

$$= n! T_n$$

=

From an empty graph:

- add edges one by one

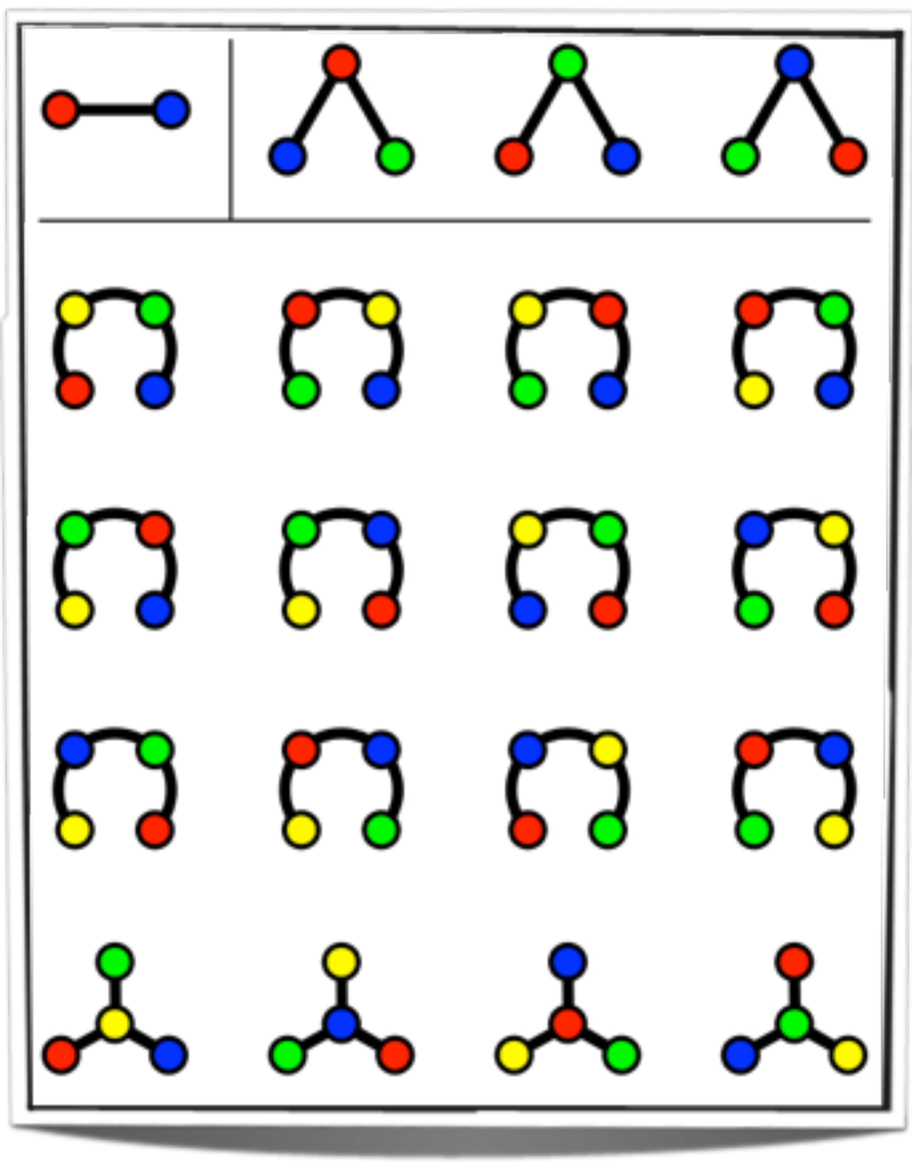
$$\prod_{k=2}^{n} n(k-1)$$

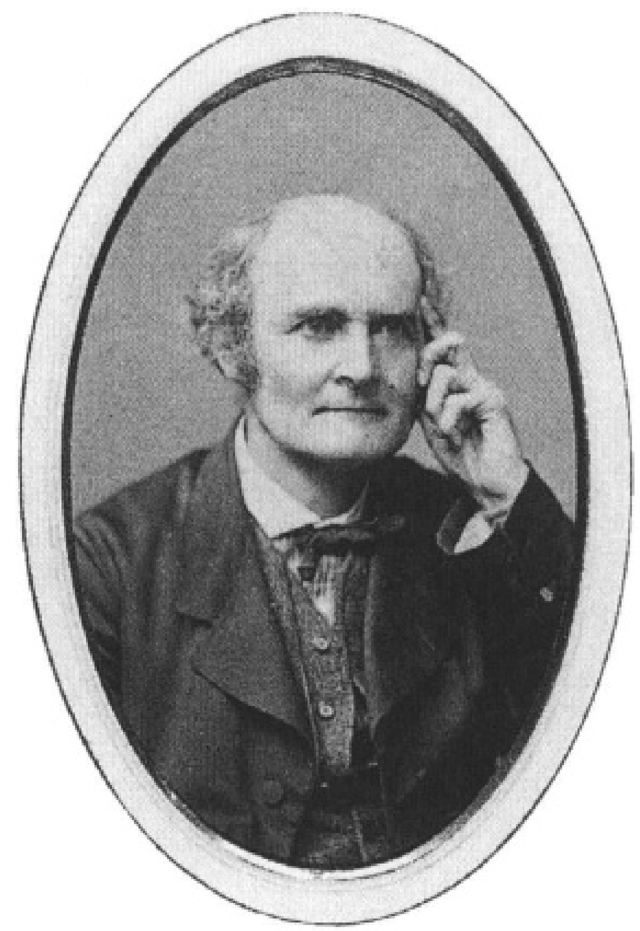
$$= n^{n-2} n!$$

$$T_n = n^{n-2}$$

Cayley's formula:

There are n^{n-2} trees on n distinct vertices.

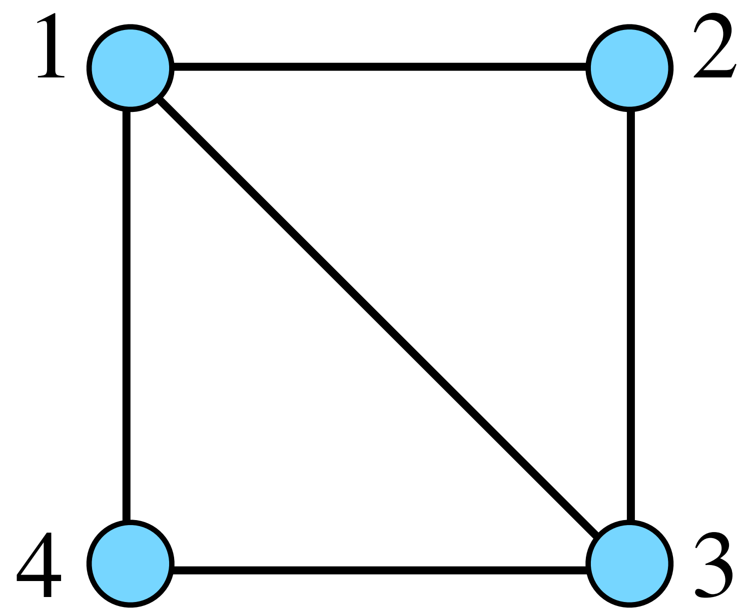




Arthur Cayley

Graph *Laplacian*

Graph $G(V,E)$



adjacency matrix A

$$A(i,j) = \begin{cases} 1 & \{i,j\} \in E \\ 0 & \{i,j\} \notin E \end{cases}$$

diagonal matrix D

$$D(i,j) = \begin{cases} \deg(i) & i = j \\ 0 & i \neq j \end{cases} \qquad D = \begin{bmatrix} d_1 & & & \\ & d_2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & d_n \end{bmatrix}$$

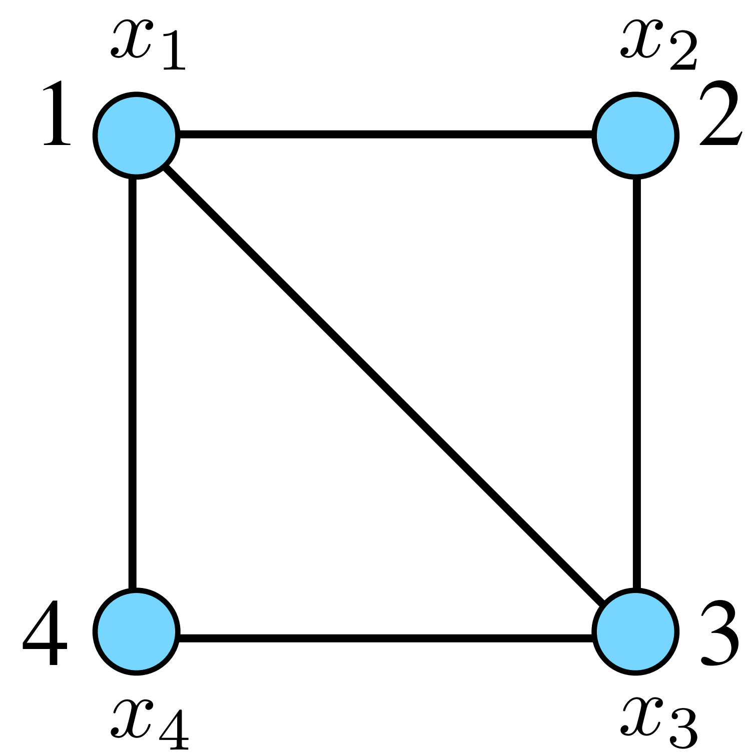
graph *Laplacian* L

$$L = D - A \qquad L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Graph *Laplacian*

graph *Laplacian* L

$$L(i,j) = \begin{cases} \deg(i) & i = j \\ -1 & i \neq j, \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$



quadratic form:

$$xLx^T = \sum_i d_i x_i^2 - \sum_{ij \in E} x_i x_j = \frac{1}{2} \sum_{ij \in E} (x_i - x_j)^2$$

incidence matrix B : $n \times m$

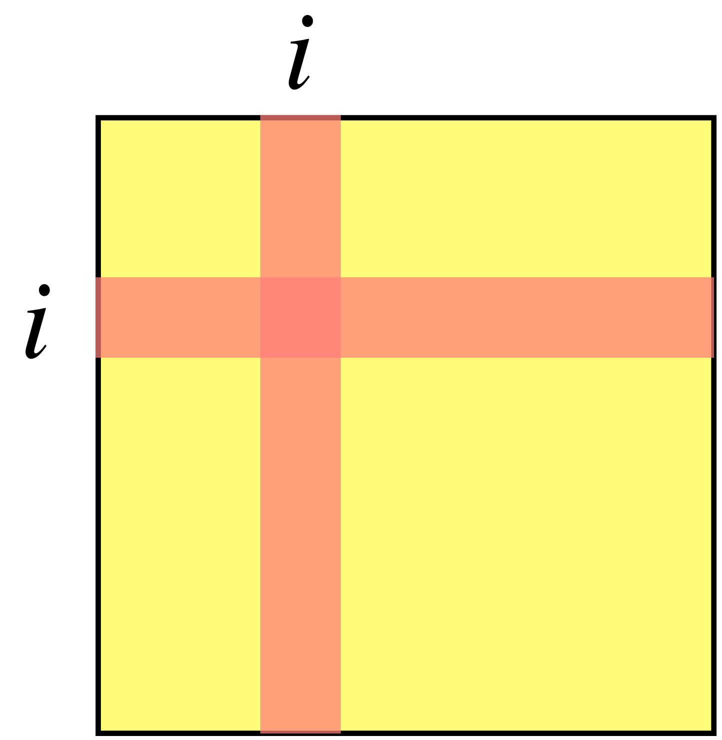
$i \in V, e \in E$

$$B(i,e) = \begin{cases} 1 & e = \{i,j\}, i < j \\ -1 & e = \{i,j\}, i > j \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{L = BB^T}$$

Kirchhoff's matrix-tree theorem

$L_{i,i}$: submatrix of L by removing ith row and ith collumn



$t(G)$: number of spanning trees in G

Kirchhoff's matrix-tree theorem

$L_{i,i}$: submatrix of L by removing ith row and ith collumn

$t(G)$: number of spanning trees in G

Kirchhoff's Matrix-Tree Theorem:

$$\forall i, \quad t(G) = \det(L_{i,i})$$

Kirchhoff's Matrix-Tree Theorem:

$$\boxed{\forall i, \quad t(G) = \det(L_{i,i})}$$

B_i : $(n-1) \times m$

incidence matrix B removing ith row

$$\boxed{L = BB^T}$$

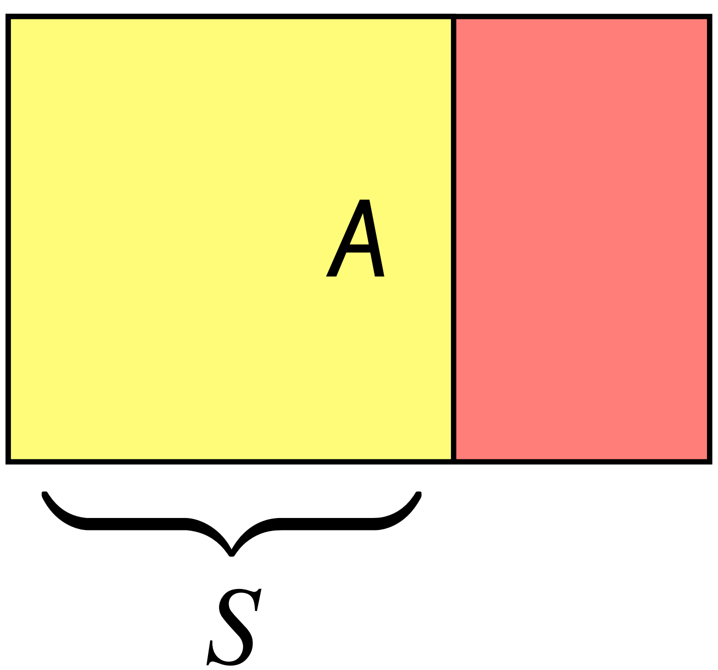
$$L_{i,i} = B_i B_i^T \qquad \det(L_{i,i}) = \det(B_i B_i^T) = ?$$

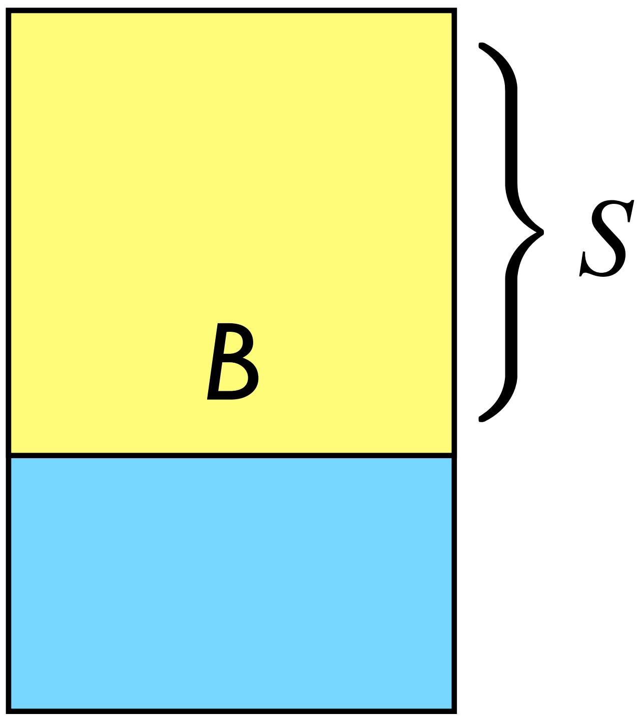
Cauchy-Binet Theorem:

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],S}) \det(B_{S,[n]})$$

$A : n \times m$

$B : m \times n$





Cauchy-Binet Theorem:

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],S}) \det(B_{S,[n]})$$

$$\det(L_{i,i}) = \det(B_i B_i^T)$$

$$= \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S}) \det(B^T_{S,[n]\setminus\{i\}})$$

$$= \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S})^2$$

$$\det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S})^2$$

$j \in [n] \setminus \{i\}, e \in S$

$$B_{[n]\setminus\{i\},S}(j,e) = \begin{cases} 1 & e = \{j,k\}, j < k \\ -1 & e = \{j,k\}, j > k \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\det(B_{[n]\setminus\{i\},S}) = \begin{cases} \pm 1 & S \text{ is a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}}$$

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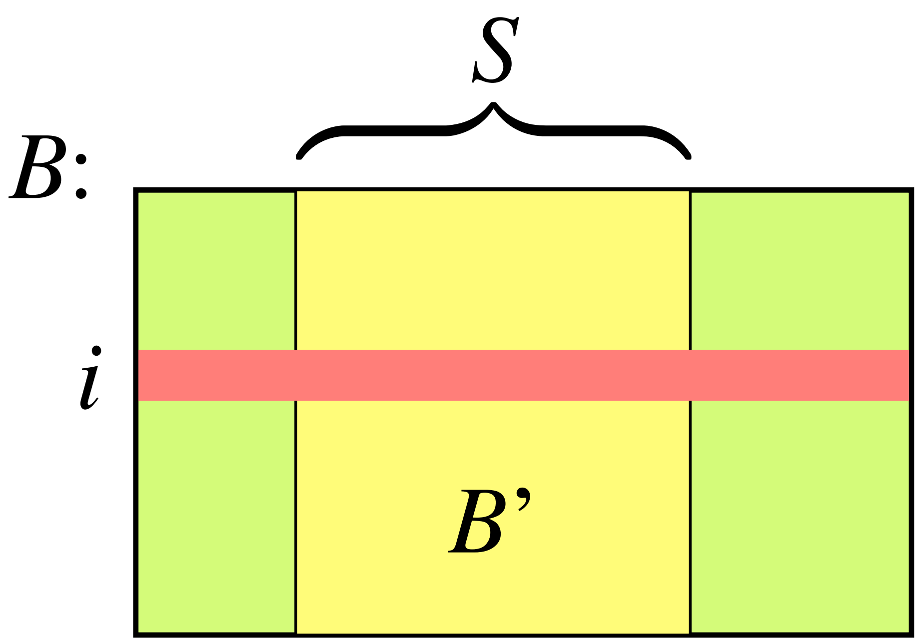


$$B' = B_{[n]\setminus\{i\},S}$$

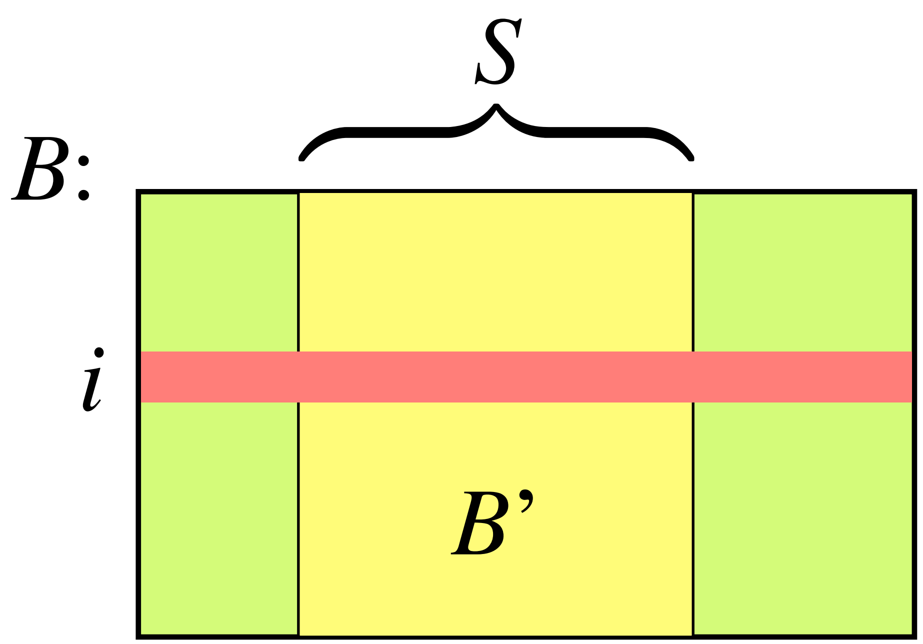
$(n-1)\times(n-1)$ **matrix:**

every column contains at most one 1 and at most one -1 and all other entries are 0

$$\Rightarrow \det(B') \in \{-1, 0, 1\}$$

$$\det(B') \neq 0 \text{ iff } S \text{ is a spanning tree}$$

$$\boxed{\det(B') \neq 0 \text{ iff } S \text{ is a spanning tree}}$$



S is not a spanning tree:

$\exists$ a connected component R

s.t. $i \notin R$

$$\det(B') = 0$$

S is a spanning tree:

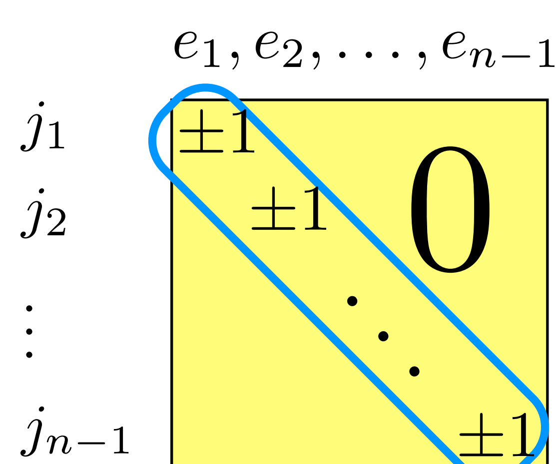
$\exists$ a leaf $j_1 \neq i$ with incident edge e_1, delete e_1

$\exists$ a leaf $j_2 \neq i$ with incident edge e_2, delete e_2

$\vdots$

vertices: $j_1, j_2, \ldots, j_{n-1}$

edges: $e_1, e_2, \ldots, e_{n-1}$



$$\det(B') = \pm 1$$

Cauchy-Binet

$$\det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S})^2$$

$j \in [n] \setminus \{i\}, e \in S$

$$B_{[n]\setminus\{i\},S}(j,e) = \begin{cases} 1 & e = \{j,k\}, j < k \\ -1 & e = \{j,k\}, j > k \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\det(B_{[n]\setminus\{i\},S}) = \begin{cases} \pm 1 & S \text{ is a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}}$$

Kirchhoff's Matrix-Tree Theorem:

$$\forall i, \quad t(G) = \det(L_{i,i})$$

all n-vertex trees: spanning trees of K_n

$$L_{i,i} = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$

Cayley formula:

$$T_n = t(K_n) = \det(L_{i,i}) = n^{n-2}$$