Combinatorics

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Course Info

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  • 804 , Wednesday, 4–6pm

• course homepage:
  • http://tcs.nju.edu.cn/wiki/
Combinatorics

- **Enumeration (counting):** How many solutions satisfying the constraints?
- **Existence:** Does there exist a solution?
- **Extremal:** How large/small a solution can be to preserve/avoid certain structure?
- **Ramsey:** When a solution is sufficiently large, some structure must emerge.
- **Optimization:** Find the optimal solution.
- **Construction (design):** Construct a solution.

*combinatorial* ≈ *discrete finite*  
*solution:* *combinatorial object*  
*constraint:* *combinatorial structure*
Textbook

van Lint and Wilson,
A course in Combinatorics,
2nd Edition.

Jukna,
Extremal Combinatorics: with applications in computer science,
2nd Edition.
Reference Books

Stanley,  
*Enumerative Combinatorics, Volume 1*

Graham, Knuth, and Patashnik,  
*Concrete Mathematics: A Foundation for Computer Science*
Reference Books

Aigner and Ziegler.
*Proofs from THE BOOK.*

Alon and Spencer.
*The Probabilistic Method.*

Cook, Cunningham, Pulleyblank, and Schrijver.
*Combinatorial Optimization.*
Enumeration  
(counting)

How many ways are there:

• to rank $n$ people?
• to assign $m$ zodiac signs to $n$ people?
• to choose $m$ people out of $n$ people?
• to partition $n$ people into $m$ groups?
• to distribute $m$ yuan to $n$ people?
• to partition $m$ yuan to $n$ parts?
• ... ...
The Twelvefold Way

Gian-Carlo Rota
(1932-1999)
The twelvefold way

\[ f : N \rightarrow M \quad \mid N \mid = n, \quad \mid M \mid = m \]

<table>
<thead>
<tr>
<th>elements of ( N )</th>
<th>elements of ( M )</th>
<th>any ( f )</th>
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<th>on-to</th>
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Knuth’s version (in *TAOCP* vol.4A)

$n$ balls are put into $m$ bins

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Tuples

\[ [m] = \{0, 1, \ldots, m - 1\} \]

\[ [m]^n = \underbrace{[m] \times \cdots \times [m]}_n \]

\[ |[m]^n| = m^n \]

Product rule:

finite sets \( S \) and \( T \)

\[ |S \times T| = |S| \cdot |T| \]
Functions

Count the # of functions

\( f : [n] \rightarrow [m] \)

\((f(1), f(2), \ldots, f(n)) \in [m]^n\)

One-one correspondence

\([n] \rightarrow [m] \Leftrightarrow [m]^n\)
Functions

count the # of functions

\[ f : [n] \rightarrow [m] \]

one-one correspondence

\[ [n] \rightarrow [m] \Leftrightarrow [m]^n \]

Bijection rule:

finite sets \( S \) and \( T \)

\[ \exists \phi : S \xrightarrow{1-1 \text{ on-to}} T \implies |S| = |T| \]
Functions

Count the number of functions

\[ f : [n] \rightarrow [m] \]

One-one correspondence

\[ [n] \rightarrow [m] \Leftrightarrow [m]^n \]

\[ |[n] \rightarrow [m]| = |[m]^n| = m^n \]

“Combinatorial proof.”
Injections

Count the # of 1-1 functions

\[ f : [n] \xrightarrow{1-1} [m] \]

One-to-one correspondence

\[ \pi = (f(1), f(2), \ldots, f(n)) \]

\textbf{n-permutation:} \quad \pi \in [m]^n \quad \text{of distinct elements}

\[ (m)_n = m(m - 1) \cdots (m - n + 1) = \frac{m!}{(m - n)!} \]

“\textit{m lower factorial n}”
Subsets

subsets of \{ 1, 2, 3 \}:

\[ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \]

\[ [n] = \{1, 2, \ldots, n\} \]

**Power set:** \( 2^{[n]} = \{ S \mid S \subseteq [n] \} \)

\[ \left| 2^{[n]} \right| = \]
Subsets

\[ [n] = \{1, 2, \ldots, n\} \]

**Power set:** \( 2^{[n]} = \{ S \mid S \subseteq [n] \} \)

\[ |2^{[n]}| = \]

**Combinatorial proof:**

A subset \( S \subseteq [n] \) corresponds to a string of \( n \) bit, where bit \( i \) indicates whether \( i \in S \).
Subsets

\[ [n] = \{1, 2, \ldots, n\} \]

**Power set:** \( 2^{[n]} = \{S \mid S \subseteq [n]\} \)

\[ |2^{[n]}| = |\{0, 1\}^n| = 2^n \]

**Combinatorial proof:**

\( S \subseteq [n] \iff \chi_S \in \{0, 1\}^n \quad \chi_S(i) = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases} \)

one-to-one correspondence
Subsets

$[n] = \{1, 2, \ldots, n\}$

**Power set:** $2^{[n]} = \{S \mid S \subseteq [n]\}$

$|2^{[n]}| =$

**A not-so-combinatorial proof:**

Let $f(n) = |2^{[n]}|$

$f(n) = 2f(n - 1)$
\[ f(n) = 2^{[n]} \]

\[ f(n) = 2f(n - 1) \]

\[ 2^{[n]} = \{ S \subseteq [n] \mid n \not\in S \} \cup \{ S \subseteq [n] \mid n \in S \} \]

\[ |2^{[n]}| = |2^{[n-1]}| + |2^{[n-1]}| = 2f(n - 1) \]

**Sum rule:**

finite **disjoint** sets \( S \) and \( T \)

\[ |S \cup T| = |S| + |T| \]
Subsets

\[ [n] = \{1, 2, \ldots, n\} \]

Power set: \( 2^{[n]} = \{S \mid S \subseteq [n]\} \)

\[ \left| 2^{[n]} \right| = 2^n \]

Let \( f(n) = \left| 2^{[n]} \right| \)

\[ f(n) = 2f(n - 1) \]

\[ f(0) = |2^\emptyset| = 1 \]
Three rules

Sum rule:

finite disjoint sets \( S \) and \( T \)

\[ |S \cup T| = |S| + |T| \]

Product rule:

finite sets \( S \) and \( T \)

\[ |S \times T| = |S| \cdot |T| \]

Bijection rule:

finite sets \( S \) and \( T \)

\[ \exists \phi : S \xrightarrow{1-1 \text{ on-to}} T \quad \implies \quad |S| = |T| \]
Subsets of fixed size

2-subsets of \( \{ 1, 2, 3 \} \): \( \{1, 2\}, \{1, 3\}, \{2, 3\} \)

\( k \)-uniform

\[
\binom{S}{k} = \left\{ T \subseteq S \mid |T| = k \right\}
\]

\[
\binom{n}{k} = \left| \binom{[n]}{k} \right|
\]

“\( n \) choose \( k \)”
Subsets of fixed size

\[ \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1} = \frac{n!}{k!(n-k)!} \]

# of ordered k-subsets: \( n(n-1) \cdots (n-k+1) \)

# of permutations of a k-set: \( k(k-1) \cdots 1 \)
Binomial coefficients

Binomial coefficient:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

1. \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)
2. choose a \( k \)-subset \( \Leftrightarrow \)
   choose its compliment
3. 0-subsets + 1-subsets + ... + \( n \)-subsets = all subsets
Binomial theorem

Binomial Theorem

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

Proof:

\[(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\]

\(n\) copies

\# of \(x^k\): choose \(k\) factors out of \(n\)
Binomial Theorem

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

Let \(x = 1\).

\[\sum_{k=0}^{n} \binom{n}{k} = 2^n\]  

Let \(x = 1\).

\[S = \{x_1, x_2, \ldots, x_n\}\]

\[\text{# of subsets of } S \text{ of odd sizes} = \text{# of subsets of } S \text{ of even sizes}\]

Let \(x = -1\).
### The twelvefold way

$n$ balls are put into $m$ bins

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Compositions of an integer

How many ways to assign \( n \) beli to \( k \) pirates?

How many ways to assign \( n \) beli to \( k \) pirates, so that each pirate receives at least 1 beli?
Compositions of an integer

\[ n \in \mathbb{Z}^+ \]

**k**-composition of \( n \):

an ordered sum of \( k \) positive integers

a \( k \)-tuple \((x_1, x_2, \cdots, x_k)\)

\[ x_1 + x_2 + \cdots + x_k = n \quad \text{and} \quad x_i \in \mathbb{Z}^+ \]
Compositions of an integer

\( n \in \mathbb{Z}^+ \)

**k-composition of \( n \):**

A \( k \)-tuple \((x_1, x_2, \cdots, x_k)\)

\[ x_1 + x_2 + \cdots + x_k = n \quad \text{and} \quad x_i \in \mathbb{Z}^+ \]

**\# of \( k \)-compositions of \( n \)?**

\[ \binom{n-1}{k-1} \]

\( n \) identical balls

\( x_1 \quad x_2 \quad \cdots \quad x_k \)
Compositions of an integer

a $k$-tuple $(x_1, x_2, \ldots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

# of $k$-compositions of $n$?

$\binom{n - 1}{k - 1}$

$\phi(((x_1, x_2, \ldots, x_k))) = \{x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + x_2 + \cdots + x_{k-1}\}$

$\phi$ is a 1-1 correspondence between

$\{k$-compositions of $n\}$ and $\binom{\{1,2,\ldots,n-1\}}{k-1}$
Compositions of an integer

weak $k$-composition of $n$:

an ordered sum of $k$ nonnegative integers

a $k$-tuple $(x_1, x_2, \cdots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ \text{ and } x_i \in \mathbb{N}$
Compositions of an integer

weak $k$-composition of $n$:

a $k$-tuple $(x_1, x_2, \cdots, x_k)$

\[ x_1 + x_2 + \cdots + x_k = n \quad \text{and} \quad x_i \in \mathbb{N} \]

# of weak $k$-compositions of $n$?

\[ \binom{n + k - 1}{k - 1} \]

\[ (x_1 + 1) + (x_2 + 1) + \cdots + (x_k + 1) = n + k \]

a $k$-composition of $n+k$

1-1 correspondence
Multisets

\( k \)-subset of \( S \)

"\( k \)-combination of \( S \) without repetition"

3-combinations of \( \{1, 2, 3, 4\} \)

without repetition:

\( \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\} \)

with repetition:

\( \{1,1,1\}, \{1,1,2\}, \{1,1,3\}, \{1,1,4\}, \{1,2,2\}, \{1,3,3\}, \{1,4,4\}, \{2,2,2\}, \{2,2,3\}, \{2,2,4\}, \{2,3,3\}, \{2,4,4\}, \{3,3,3\}, \{3,3,4\}, \{3,4,4\}, \{4,4,4\} \)
Multisets

multiset $M$ on set $S$:

$$m : S \rightarrow \mathbb{N}$$

multiplicity of $x \in S$

$$m(x) : \# \text{ of repetitions of } x \text{ in } M$$

cardinality

$$|M| = \sum_{x \in S} m(x)$$

“$k$-combination of $S$ with repetition” $\leftrightarrow$ $k$-multiset on $S$

$$\binom{n}{k} : \# \text{ of } k\text{-multisets on an } n\text{-set}$$
Multisets

\[
\binom{n}{k} = \binom{n + k - 1}{n-1} = \binom{n + k - 1}{k}
\]

\(k\)-multiset on \(S = \{x_1, x_2, \ldots, x_n\}\)

\[m(x_1) + m(x_2) + \cdots + m(x_n) = k\]

\(m(x_i) \geq 0\)

a weak \(n\)-composition of \(k\)
Multinomial coefficients

permutations of a multiset of size $n$ with multiplicities $m_1, m_2 \ldots, m_k$

# of reordering of “multinomial”

permutations of $\{a, i,i, l,l, m,m, n, o, t, u\}$

assign $n$ distinct balls to $k$ distinct bins with the $i$-th bin receiving $m_i$ balls

multinomial coefficient \[ \binom{n}{m_1, \ldots, m_k} \]

\[ m_1 + m_2 + \cdots + m_k = n \]
Multinomial coefficients

permutations of a multiset of size $n$ with multiplicities $m_1, m_2 \ldots, m_k$

assign $n$ distinct balls to $k$ distinct bins with the $i$-th bin receiving $m_i$ balls

$$\binom{n}{m_1, \ldots, m_k} = \frac{n!}{m_1!m_2!\cdots m_k!}$$

$$\binom{n}{m, n-m} = \binom{n}{m}$$
Multinomial theorem

**Multinomial Theorem**

\[
(x_1 + x_2 + \cdots + x_k)^n = \sum_{m_1 + \cdots + m_k = n} \binom{n}{m_1, \ldots, m_k} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}
\]

**Proof:**

\[
(x_1 + x_2 + \cdots + x_k)^n = (x_1 + x_2 + \cdots + x_k) \cdots (x_1 + x_2 + \cdots + x_k)
\]

\[
\text{# of } x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}:
\]

assign \(n\) factors to \(k\) groups of sizes \(m_1, m_2, \ldots, m_k\)
The twelvefold way

$n$ balls are put into $m$ bins

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Partitions of a set

\[ P = \{A_1, A_2, \ldots, A_k\} \text{ is a partition of } S: \]

\[ A_i \neq \emptyset \]

\[ A_i \cap A_j = \emptyset \]

\[ A_1 \cup A_2 \cup \cdots \cup A_k = S \]
Partitions of a set

\[ P = \{A_1, A_2, \ldots, A_k\} \text{ is a partition of } S:\]
\[
A_i \neq \emptyset \\
A_i \cap A_j = \emptyset \\
A_1 \cup A_2 \cup \cdots \cup A_k = S
\]

\[ \binom{n}{k} \]  

# of k-partitions of an n-set

“Stirling number of the second kind”

\[ B_n = \sum_{k=1}^{n} \binom{n}{k} \]  

# of partitions of an n-set

“Bell number”
Stirling number of the 2nd kind

\[
\begin{pmatrix} n \\ k \end{pmatrix} \quad \# \text{ of } k\text{-partitions of an } n\text{-set}
\]

\[
\begin{pmatrix} n \\ k \end{pmatrix} = k \begin{pmatrix} n-1 \\ k \end{pmatrix} + \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}
\]

**Case.1** \{n\} is not a partition block

n is in one of the k blocks in a k-partition of \([n-1]\)

**Case.2** \{n\} is a partition block

the remaining k-1 blocks forms a \((k-1)\)-partition of \([n-1]\)
The twelvefold way

\[ f : N \rightarrow M \quad \text{n balls are put into m bins} \]

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Surjections

\[ f : [n] \xrightarrow{\text{on-to}} [m] \]

\[ \forall i \in [m], \quad f^{-1}(i) \neq \emptyset \]

\[ (f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(m)) \]

ordered \( m \)-partition of \([n] \]

\[ m! \binom{n}{m} \]
The twelvefold way

$n$ balls are put into $m$ bins

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Partitions of a number

$n$ beli

$k$ boxes

a partition of $n$ into $k$ parts:

an unordered sum of $k$ positive integers
Partitions of a number

A partition of $n$ into $k$ parts:

$n=7$

\[ p_k(n) \]  
# of partitions of $n$ into $k$ parts

\{7\}

\{1,6\}, \{2,5\}, \{3,4\}

\{1,1,5\}, \{1,2,4\}, \{1,3,3\}, \{2,2,3\}

\{1,1,1,4\}, \{1,1,2,3\}, \{1,2,2,2\}

\{1,1,1,1,3\}, \{1,1,1,2,2\}

\{1,1,1,1,1,2\}

\{1,1,1,1,1,1,1\}

“positive”

“unordered”
$p_k(n)$  \# of partitions of $n$ into $k$ parts

integral solutions to \[
x_1 + x_2 + \cdots + x_k = n \quad \begin{align*}
x_1 &\geq x_2 \geq \cdots \geq x_k \geq 1
\end{align*}
\]

$p_k(n) = ?$
\[
\begin{cases}
    x_1 + x_2 + \cdots + x_k = n \\
    x_1 \geq x_2 \geq \cdots \geq x_k \geq 1
\end{cases}
\]

\[
p_k(n) = p_{k-1}(n - 1) + p_k(n - k)
\]

**Case.1** \(x_k = 1\)

\((x_1, \ldots, x_{k-1})\) is a \((k - 1)\)-partition of \(n - 1\)

**Case.2** \(x_k > 1\)

\((x_1 - 1, \ldots, x_k - 1)\) is a \(k\)-partition of \(n - k\)
partition \[ \begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases} \]

composition \[ \begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_i \geq 1 \]

\( \pi \) is a permutation of \( \{x_1, \cdots, x_k\} \)

“on-to”

\[ k!p_k(n) \geq \binom{n - 1}{k - 1} \]
partition \( \{x_1, \cdots, x_k\} \) \quad y_i = x_i + k - i

\[
x_1 \geq x_2 \geq \cdots \geq x_{k-2} \geq x_{k-1} \geq x_k \geq 1
\]

\[
+k - 1 \quad +k - 2 \quad +2 \quad +1
\]

\[
y_1 > y_2 > \cdots > y_{k-2} > y_{k-1} > y_k > 1
\]

composition of \( n + \frac{k(k-1)}{2} \)

\[
(y_1, y_2, \cdots, y_k)
\]

permutation \( \pi \)

“1-1”

\[
k!p_k(n) \leq \left( n + \frac{k(k-1)}{k - 1} - 1 \right)
\]
If $k$ is fixed,

$$p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!} \quad \text{as} \quad n \to \infty$$
\[ p(n) = \sum_{k=1}^{n} p_k(n) \]
\[ \approx \frac{1}{4n\sqrt{3}} \exp \left\{ \pi \sqrt{\frac{2n}{3}} \right\} \]

Srinivasa Ramanujan
(1887-1920)

G. H. Hardy
(1877-1947)

The Man Who Knew Infinity
(2015 film)
Ferrers diagram

(Young diagram)

\[ x_1 + x_2 + \cdots + x_k = n \]
\[ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \]
conjugate

one-to-one correspondence
\# of partitions of \( n \) into \( k \) parts \quad = \quad \# of partitions of \( n \) with largest part \( k \)
The number of partitions of $n$ into $k$ parts is equal to the number of partitions of $n-k$ into at most $k$ parts.

$$p_k(n) = \sum_{j=1}^{k} p_j(n - k)$$
# The twelvefold way

$n$ balls are put into $m$ bins

<table>
<thead>
<tr>
<th>balls per bin:</th>
<th>unrestricted</th>
<th>$\leq 1$</th>
<th>$\geq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ distinct balls, $m$ distinct bins</td>
<td>$m^n$</td>
<td>$(m)_n$</td>
<td>$m! \binom{n}{m}$</td>
</tr>
<tr>
<td>$n$ identical balls, $m$ distinct bins</td>
<td>$\binom{m}{n}$</td>
<td>$\binom{m}{n}$</td>
<td>$(n-1)\binom{m-1}{m-1}$</td>
</tr>
<tr>
<td>$n$ distinct balls, $m$ identical bins</td>
<td>$\sum_{k=1}^{m} \binom{n}{k}$</td>
<td>$\begin{cases} 1 &amp; \text{if } n \leq m \ 0 &amp; \text{if } n &gt; m \end{cases}$</td>
<td>$\binom{n}{m}$</td>
</tr>
<tr>
<td>$n$ identical balls, $m$ identical bins</td>
<td>$\sum_{k=1}^{m} p_k(n)$</td>
<td>$\begin{cases} 1 &amp; \text{if } n \leq m \ 0 &amp; \text{if } n &gt; m \end{cases}$</td>
<td>$p_m(n)$</td>
</tr>
</tbody>
</table>