Combinatorics
Counting (labeled) trees

“How many different trees can be formed from $n$ distinct vertices?”
Cayley’s formula:
There are $n^{n-2}$ trees on $n$ distinct vertices.

Arthur Cayley
Prüfer Code

leaf: vertex of degree 1

removing a leaf from $T$, still a tree

$T_\mathfrak{a}$:

$T_1 = T$;

for $i = 1$ to $n-1$

$u_i$: smallest leaf in $T_i$;

$(u_i,v_i)$: edge in $T_i$;

$T_{i+1} = \text{delete } u_i \text{ from } T_i$;

Prüfer code:

$(v_1, v_2, \ldots, v_{n-2})$

$u_i$: 2, 4, 5, 6, 3, 1

$v_i$: 4, 3, 1, 3, 1, 7
edges of $T$: $(u_i, v_i), 1 \leq i \leq n-1$

$u_i$: smallest leaf in $T_i$

$v_{n-1} = n$

a tree has $\geq 2$ leaves

$T$

$u_i$: 2, 4, 5, 6, 3, 1

$v_i$: 4, 3, 1, 3, 1, 7

$(v_1, v_2, \ldots, v_{n-2})$

$n$ is never deleted

$u_i \neq n$

Only need to recover every $u_i$ from $(v_1, v_1, \ldots, v_{n-2})$.

$u_i$ is the smallest number not in

$$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$$
$u_i$ is the smallest number not in
\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}

\forall$ vertex $v$ in $T$,

- # occurrences of $v$ in $u_1, u_2, \ldots, u_{n-1}, v_{n-1}$: 1
- # occurrences of $v$ in edges $(u_i, v_i)$, $1 \leq i \leq n-1$: $\deg_T(v)$

$T$:

\begin{center}
\begin{tikzpicture}
    \node[circle, fill=blue!20] (v1) at (2,3) {2};
    \node[circle, fill=blue!20] (v2) at (1,1) {4};
    \node[circle, fill=blue!20] (v3) at (2,-1) {3};
    \node[circle, fill=blue!20] (v4) at (3,1) {5};
    \node[circle, fill=blue!20] (v5) at (3,-1) {1};
    \node[circle, fill=blue!20] (v6) at (0,0) {6};
    \node[circle, fill=blue!20] (v7) at (4,-1) {7};
    \draw (v1) -- (v2);
    \draw (v2) -- (v3);
    \draw (v3) -- (v5);
    \draw (v5) -- (v4);
    \draw (v4) -- (v6);
    \draw (v5) -- (v7);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{itemize}
    \item $u_i$: 2, 4, 5, 6, 3, 1
    \item $v_i$: 4, 3, 1, 3, 1, 7
\end{itemize}
\end{center}

Prüfer code: $(v_1, v_2, \ldots, v_{n-2})$

$\deg_T(v)-1$
∀ vertex ν in $T_i$,

- # occurrences of ν in $u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1}$: 1
- # occurrences of ν in edges $(u_j, v_j)$, $i \leq j \leq n-1$: $\text{deg}_{T_i}(ν)$

$T_3$:

**u_i**: 2, 4, 5, 6, 3, 1

**ν_i**: 4, 3, 1, 3, 1, 7

$(ν_1, ν_2, \ldots, ν_{n-2})$

leaf ν of $T_i$:

- in $\{u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1}\}$
- not in $\{ν_i, ν_{i+1}, \ldots, ν_{n-2}\}$

$u_i$: smallest leaf in $T_i$
\( u_i \) is the smallest number not in \( \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \)

\[ T : \]

\[ u_i : 2, 4, 5, 6, 3, 1 \]

\[ v_i : 4, 3, 1, 3, 1, 7 \]

\( T = \text{empty graph;} \)

\( v_{n-1} = n \);  

for \( i = 1 \) to \( n-1 \)

\( u_i : \) smallest number not in \( \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \)

add edge \( (u_i, v_i) \) to \( T \);
Prüfer code is reversible \[ \rightarrow 1-1 \]

Every \((v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2}\)
is decodable to a tree \[ \rightarrow \text{onto} \]

\[ T : \]

\begin{align*}
  &\quad 2 \\
  &\quad \quad \quad 5 \\
  &4 \quad 3 \quad 1 \\
  &\quad \quad \quad 6 \\
  &\quad \quad \quad 7 \\
\end{align*}

\[ u_i: 2, 4, 5, 6, 3, 1 \]
\[ v_i: 4, 3, 1, 3, 1, 7 \]

\((v_1, v_2, \ldots, v_{n-2})\)

\[ T = \text{empty graph;} \]
\[ v_{n-1} = n; \]

for \(i = 1\) to \(n-1\)

\[ u_i: \text{smallest number not in} \]
\[ \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\} \]

add edge \((u_i, v_i)\) to \(T\);
Prüfer code is reversible \( 1-1 \)
every \( (v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2} \)
is decodable to a tree \( \Rightarrow \) onto

**Cayley’s formula:**

There are \( n^{n-2} \) trees on \( n \) distinct vertices.
Double Counting

# of sequences of adding directed edges to an empty graph to form a rooted tree
\( T_n \): \# of trees on \( n \) distinct vertices.

\textbf{From a tree:}

- pick a root;
- pick an order of edges.

\[
T_n n(n - 1)! = n! T_n
\]
$T_n$ : # of trees on $n$ distinct vertices.

# of sequences of adding directed edges to an empty graph to form a rooted tree

From an empty graph:
- add edges one by one
# of sequences of adding directed edges to an empty graph to form a rooted tree

From an empty graph: • add edges one by one

Start from \( n \) isolated vertices rooted trees

Each step joins 2 trees.
## # of sequences of adding directed edges to an empty graph to form a rooted tree

From an empty graph: • add edges one by one

Start from $n$ rooted trees.

After adding $k$ edges

$n$-k rooted trees

add an edge

any vertex → root of another tree

$n$ → $n-k-1$
# of sequences of adding directed edges to an empty graph to form a rooted tree

From an empty graph: • add edges one by one

\[
\prod_{k=0}^{n-2} n(n - k - 1) \\
= n^{n-1} \prod_{k=1}^{n-1} k \\
= n^{n-2} n!
\]

Start from \( n \) rooted trees.

After adding \( k \) edges \( n-k \) rooted trees

add an edge

any vertex \( \rightarrow \) root of another tree

\( n \) \( n-k-1 \)
From a tree:
• pick a root;
• pick an order of edges.

\[ T_n n(n - 1)! = n!T_n \]

From an empty graph:
• add edges one by one

\[ \prod_{k=2}^{n} n(k - 1) = n^{n-2} n! \]

\[ T_n = n^{n-2} \]
Cayley’s formula:

There are $n^{n-2}$ trees on $n$ distinct vertices.
Graph Laplacian

Graph \( G(V,E) \)

adjacency matrix \( A \)

\[
A(i,j) = \begin{cases} 
1 & \{i,j\} \in E \\
0 & \{i,j\} \not\in E 
\end{cases}
\]

diagonal matrix \( D \)

\[
D(i,j) = \begin{cases} 
\deg(i) & i = j \\
0 & i \neq j 
\end{cases}
\]

graph Laplacian \( L \)

\[
L = D - A
\]

\[
L = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 2 & -1 & 0 \\
-1 & -1 & 3 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}
\]
Graph Laplacian

**graph Laplacian** $L$

$$L(i, j) = \begin{cases} \text{deg}(i) & i = j \\ -1 & i \neq j, \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

**quadratic form:**

$$xLx^T = \sum_i d_i x_i^2 - \sum_{ij \in E} x_i x_j = \frac{1}{2} \sum_{ij \in E} (x_i - x_j)^2$$

**incidence matrix $B$** : $n \times m$

$$B(i, e) = \begin{cases} 1 & e = \{i, j\}, i < j \\ -1 & e = \{i, j\}, i > j \\ 0 & \text{otherwise} \end{cases}$$

$$L = BB^T$$
Kirchhoff’s matrix-tree theorem

$L_{i,i}$ : submatrix of $L$ obtained by removing the $i$th row and $i$th column

$t(G)$ : number of spanning trees in $G$
Kirchhoff’s matrix-tree theorem

$L_{i,i}$ : submatrix of $L$ obtained by removing the $i$th row and $i$th column

t($G$) : number of spanning trees in $G$


colored text

Kirchhoff’s Matrix-Tree Theorem:

$\forall i, \ t(G) = \det(L_{i,i})$
Kirchhoff’s Matrix-Tree Theorem:
\[ \forall i, \quad t(G) = \det(L_{i,i}) \]

\[ B_i : (n - 1) \times m \]
incidence matrix \( B \) removing \( i \)th row

\[ L = BB^T \]

\[ L_{i,i} = B_i B_i^T \quad \det(L_{i,i}) = \det(B_i B_i^T) = ? \]
Cauchy-Binet Theorem:

\[
\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],[S]}) \det(B_{S,[n]})
\]
Cauchy-Binet Theorem:
\[
\det(AB) = \sum_{S \in \left(\begin{array}{c} m \\ n \end{array}\right)} \det(A_{[n], S}) \det(B_{S, [n]})
\]

\[
\det(L_{i, i}) = \det(B_i B_i^T)
\]

\[
= \sum_{S \in \left(\begin{array}{c} m \\ n-1 \end{array}\right)} \det(B_{[n] \setminus \{i\}, S}) \det(B_{S, [n] \setminus \{i\}}^T)
\]

\[
= \sum_{S \in \left(\begin{array}{c} m \\ n-1 \end{array}\right)} \det(B_{[n] \setminus \{i\}, S})^2
\]
\[
\det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\{i\},S})^2
\]

\[j \in [n] \setminus \{i\}, e \in S\]

\[
B_{[n]\{i\},S}(j, e) = \begin{cases} 
1 & e = \{j, k\}, j < k \\
-1 & e = \{j, k\}, j > k \\
0 & \text{otherwise}
\end{cases}
\]

\[
\det(B_{[n]\{i\},S}) = \begin{cases} 
\pm1 & S \text{ is a spanning tree of } G \\
0 & \text{otherwise}
\end{cases}
\]
\[
\text{det}(B_{[n]\setminus\{i\},S}) = \begin{cases} 
\pm 1 & \text{if } S \text{ is a spanning tree of } G \\
0 & \text{otherwise}
\end{cases}
\]

\[B' = B_{[n]\setminus\{i\},S}\]

\((n-1) \times (n-1)\) matrix:
- every column contains at most one 1 and at most one -1
- and all other entries are 0

\[\text{det}(B') \in \{-1, 0, 1\}\]

\[
\text{det}(B') \neq 0 \text{ iff } S \text{ is a spanning tree}
\]
\( \det(B') \neq 0 \) iff \( S \) is a spanning tree

\[\begin{align*}
S \text{ is a spanning tree:} & \\
\exists \text{ a leaf } j_1 \neq i \text{ with incident edge } e_1, \text{ delete } e_1 \\
\exists \text{ a leaf } j_2 \neq i \text{ with incident edge } e_2, \text{ delete } e_2 \\
\vdots \\
\text{vertices: } j_1, j_2, \ldots, j_{n-1} \\
\text{edges: } e_1, e_2, \ldots, e_{n-1}
\end{align*}\]

\[\begin{bmatrix}
\pm 1 & & & \\
& \pm 1 & & \\
& & \ddots & \\
& & & \pm 1
\end{bmatrix}
\]

\( \Rightarrow \det(B') = \pm 1 \)

\( S \text{ is not a spanning tree:} \)

\[\begin{align*}
\exists \text{ a connected component } R \\
\text{s.t. } i \notin R \\
\Rightarrow \det(B') = 0
\end{align*}\]
**Cauchy-Binet**

\[ \det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\{i\},S})^2 \]

\[ j \in [n] \setminus \{i\}, e \in S \]

\[ B_{[n]\{i\},S}(j, e) = \begin{cases} 
1 & e = \{j, k\}, j < k \\
-1 & e = \{j, k\}, j > k \\
0 & \text{otherwise}
\end{cases} \]

\[ \det(B_{[n]\{i\},S}) = \begin{cases} 
\pm 1 & S \text{ is a spanning tree of } G \\
0 & \text{otherwise}
\end{cases} \]
Kirchhoff’s Matrix-Tree Theorem:
\[ \forall i, \quad t(G) = \det(L_{i,i}) \]

all \( n \)-vertex trees: spanning trees of \( K_n \)

\[
L_{i,i} = \begin{bmatrix}
n - 1 & -1 & \cdots & -1 \\
-1 & n - 1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & n - 1
\end{bmatrix}
\]

Cayley formula:
\[
T_n = t(K_n) = \det(L_{i,i}) = n^{n-2}
\]