Combinatorics

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Ramsey Theory

"In any party of six people, either at least three of them are mutual strangers or at least three of them are mutual acquaintances"

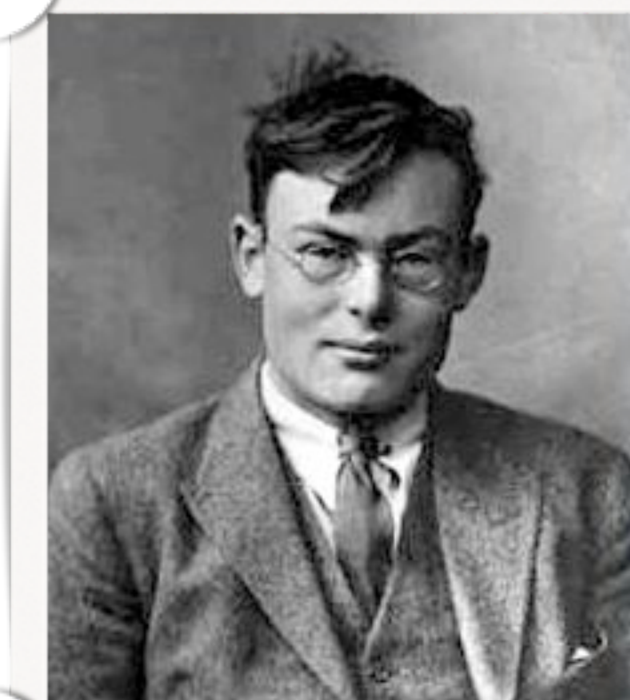
Color edges of K_6 with 2 colors.
There must be a monochromatic K_3.

ON A PROBLEM OF FORMAL LOGIC

By F. P. Ramsey.

[Received 28 November, 1928.—Read 13 December, 1928.]

This paper is primarily concerned with a special case of one of the leading problems of mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula*. But in the course of this investigation it is necessary to use certain theorems on combinations which have an independent interest and are most conveniently set out by themselves beforehand.



Frank P. Ramsey
(1903-1930)

$R(k,l) \triangleq$ the smallest integer satisfying:

if $n \geq R(k,l)$, for any 2-coloring of K_n, there exists a red K_k or a blue K_l.

2-coloring of K_n

$$f : \binom{[n]}{2} \to \{\texttt{red}, \texttt{blue}\}$$

Ramsey Theorem

$R(k,l)$ is finite.

$$R(3,3) = 6$$

Frank P. Ramsey
(1903-1930)

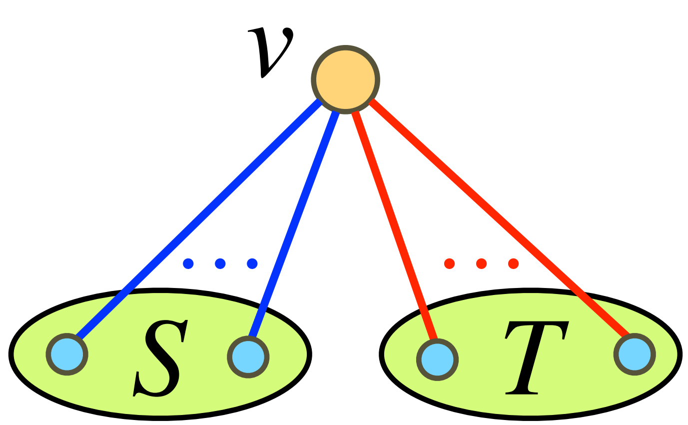
if $n \geq R(k,l)$, for any 2-coloring of K_n, there exists a red K_k or a blue K_l.

$$R(k,2) = k \,; \quad R(2,l) = l \,;$$

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

if $n \geq R(k,l)$, for any 2-coloring of K_n,
there exists a red K_k or a blue K_l.

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$



take $n = R(k,l-1) + R(k-1,l)$

arbitrary vertex v

$$|S| + |T| + 1 = n = R(k,l-1) + R(k-1,l)$$

or

$|S| \geq R(k,l-1)$ → or: K_k in S; K_{l-1} in S $\xrightarrow{+v}$ K_l

$|T| \geq R(k-1,l)$ → or: K_{k-1} in T $\xrightarrow{+v}$ K_k; K_l in T

if $n \geq R(k,l)$, for any 2-coloring of K_n, there exists a red K_k or a blue K_l.

$$R(k,2) = k \,; \quad R(2,l) = l \,;$$

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

Ramsey Theorem $R(k,l)$ is finite.

By induction:

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

$$R(k,k) \geq n$$

"∃ a 2-coloring of K_n, no monochromatic K_k."

a random 2-coloring of K_n :

$\forall \{u,v\} \in K_n$, *uniformly* and *independently* $\begin{cases} uv \\ uv \end{cases}$

$\forall S \in \binom{[n]}{k}$ event A_S : S is a monochromatic K_k

$$\Pr[A_S] = 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

A_S, A_T dependent ⟺ $|S \cap T| \geq 2$

max degree of dependency graph $d \leq \binom{k}{2}\binom{n}{k-2}$

To prove: $\Pr\left[\bigwedge_{S \in \binom{[n]}{k}} \overline{A_S}\right] > 0$

Lovász Local Lemma

- $\forall i,\ \Pr[A_i] \leq p$
- $ep(d+1) \leq 1$

$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$

$$\left.\begin{array}{l} \Pr[A_S] = 2^{1-\binom{k}{2}} \\ d \leq \binom{k}{2}\binom{n}{k-2} \end{array}\right\} \Rightarrow$$

for some $n = ck2^{k/2}$

with constant c

$$e2^{1-\binom{k}{2}}(d+1) \leq 1$$

To prove: $\Pr\left[\bigwedge_{S\in\binom{[n]}{k}} \overline{A_S}\right] > 0$

$$R(k,k) \geq n = \Omega(k2^{k/2})$$

Ramsey Number

Lovász Local Lemma

$$\Omega\left(k2^{k/2}\right) \leq R(k,k) \leq \binom{2k-2}{k-1} = \mathrm{O}\left(\frac{4^k}{\sqrt{k}}\right)$$

k \ l	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–42
4				**18**	25	36–41	49–61	59–84	73–115	92–149
5					**43–48**	58–87	80–143	101–216	133–316	149–442
6						**102–165**	115–298	134–495	183–780	204–1171
7							**205–540**	217–1031	252–1713	292–2826
8								**282–1870**	329–3583	343–6090
9									**565–6588**	581–12677
10										**798–23556**

Multicolor

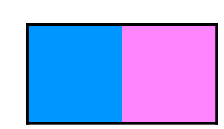
if $n \geq R(k, l)$, for any 2-coloring of K_n,
there exists a red K_k or a blue K_l.

$R(r; k_1, k_2, \ldots, k_r)$

if $n \geq R(r; k_1, k_2, \ldots, k_r)$,
for any r-coloring of K_n, there exists a monochromatic k_i-clique with color i for some $i \in \{1, 2, \ldots, r\}$.

$$R(r; k_1, \ldots, k_{r-2}, k_{r-1}, k_r) \leq R(r\text{-}1; k_1, \ldots, k_{r-2}, R(2; k_{r-1}, k_r))$$

the mixing color trick:

color 

Multicolor

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Ramsey Theorem

$R(r; k_1, k_2, \ldots, k_r)$ is finite.

if $n \geq R(r; k_1, k_2, \ldots, k_r)$,
for any r-coloring of K_n, there exists a monochromatic k_i-clique with color i for some $i \in \{1, 2, \ldots, r\}$.

$$K_n = \binom{[n]}{2}$$

r-coloring $\quad f : \binom{[n]}{2} \to \{1, 2, \ldots, r\}$

Hypergraph

if $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

for any r-coloring of $\binom{[n]}{t}$, there exists a monochromatic $\binom{S}{t}$ with color i and $|S|=k_i$ for some $i \in \{1, 2, \ldots, r\}$.

complete t-uniform hypergraph $\binom{[n]}{t}$

r-coloring $f : \binom{[n]}{t} \to \{1, 2, \ldots, r\}$

Partition of Set Family

If $n \geq R_t(r; k_1, k_2, \dots, k_r)$,

for any r-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$,
there exists an $S \subseteq [n]$ such that $|S| = k_i$
and $\binom{S}{t} \subseteq C_i$ for some $i \in \{1, 2, \dots, r\}$.

Erdős-Rado partition arrow

$$n \rightarrow (k_1, k_2, \dots, k_r)^t$$

If $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

for any r-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$, there exists an $S \subseteq [n]$ such that $|S| = k_i$ and $\binom{S}{t} \subseteq C_i$ for some $i \in \{1, 2, \ldots, r\}$.

mixing color:

$$R_t(r; k_1, \ldots, k_{r-2}, k_{r-1}, k_r) \leq R_t(r-1; k_1, \ldots, k_{r-2}, R_t(2; k_{r-1}, k_r))$$

$$R_t(k,l) \leq R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$n = R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

goal: $\forall f : \binom{[n]}{t} \to \{\texttt{red}, \texttt{blue}\}$

$\exists \binom{X}{t}, |X| = k$ or $\binom{Y}{t}, |Y| = l$

$$R_t(k,l) \le R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$n = R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$\forall f : \binom{[n]}{t} \to \{\texttt{red}, \texttt{blue}\}$$

remove n from $[n]$, consider $\binom{[n-1]}{t-1}$

$([n] = \{1, 2, ..., n\})$

define $f' : \binom{[n-1]}{t-1} \to \{\texttt{red}, \texttt{blue}\}$

$$\forall A \in \binom{[n-1]}{t-1}, \quad f'(A) = f(A \cup \{n\})$$

$$n\text{-}1 = R_{t-1}(R_t(k-1,l), R_t(k,l-1))$$

or $\begin{cases} \exists S \subseteq [n-1], |S| = R_t(k-1,l), \binom{S}{t-1} \text{ by } f' \\ \exists T \subseteq [n-1], |T| = R_t(k,l-1), \binom{T}{t-1} \text{ by } f' \end{cases}$

$$R_t(k,l) \le R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$n = R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$\forall f : \binom{[n]}{t} \to \{\texttt{red}, \texttt{blue}\}$$

define $f' : \binom{[n-1]}{t-1} \to \{\texttt{red}, \texttt{blue}\}$

$$\forall A \in \binom{[n-1]}{t-1}, \quad f'(A) = f(A \cup \{n\})$$

by symmetry $\exists S \subseteq [n-1], |S| = R_t(k-1,l), \binom{S}{t-1}$ by f'

or
$\exists X \subseteq S, |X| = k-1, \binom{X}{t}$ by f

$\exists Y \subseteq S, |Y| = l, \binom{Y}{t}$ by f ✓

✓ $\binom{X \cup \{n\}}{t}$ by f

If $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

$\forall$ r-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$

$\exists i \in \{1,2,\ldots,r\}$ and $S \subseteq [n]$ with $|S| = k_i$

such that $\binom{S}{t} \subseteq C_i$

mixing color:

$$R_t(r; k_1, \ldots, k_{r-2}, k_{r-1}, k_r) \leq R_t(r-1; k_1, \ldots, k_{r-2}, R_t(2; k_{r-1}, k_r))$$

$$R_t(k,l) \leq R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

Theorem (Ramsey 1930)

$R_t(r; k_1, k_2, \ldots, k_r)$ is finite.

If $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

$\forall$ r-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$

$\exists i \in \{1,2,\ldots,r\}$ and $S \subseteq [n]$ with $|S| = k_i$

such that $\binom{S}{t} \subseteq C_i$

Theorem (Ramsey 1930)

$R_t(r; k_1, k_2, \ldots, k_r)$ is finite.

If $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

∀ r-coloring $f : \binom{[n]}{t} \to [r]$

$\exists i \in \{1, 2, \ldots, r\}$ and $S \subseteq [n]$ with $|S| = k_i$

such that entire $\binom{S}{t}$ is colored by i

Theorem (Ramsey 1930)

$R_t(r; k_1, k_2, \ldots, k_r)$ is finite.

If $n \geq R_t(r; k) \triangleq R_t(r; \underbrace{k, \ldots, k}_{r})$

$\forall$ r-coloring $f : \binom{[n]}{t} \to [r]$

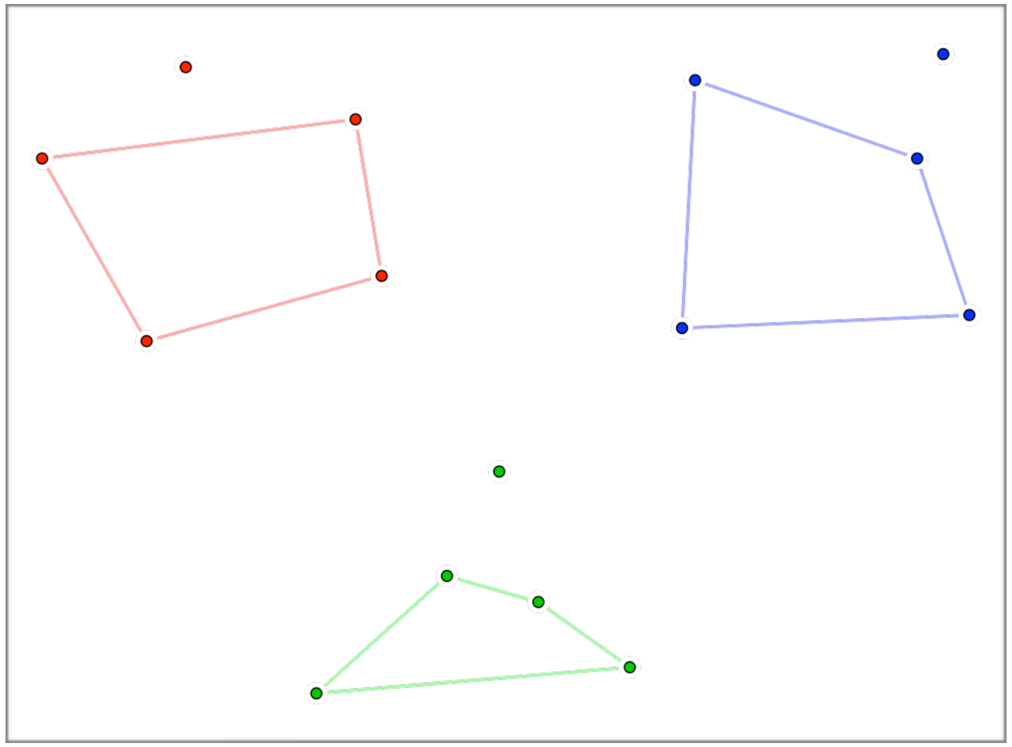
$\exists$ a monochromatic $\binom{S}{t}$ with $S \subseteq [n]$ and $|S|=k$

Theorem (Ramsey 1930)

$R_t(r; k)$ is finite.

Happy Ending Problem

Any 5 points in the plane, no three on a line, has a subset of 4 points that form a convex quadrilateral.

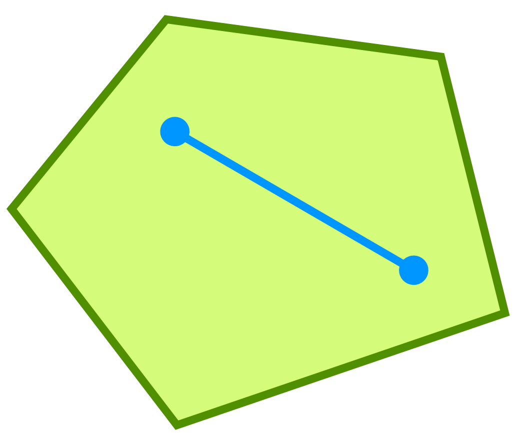


Theorem (Erdős-Szekeres 1935)

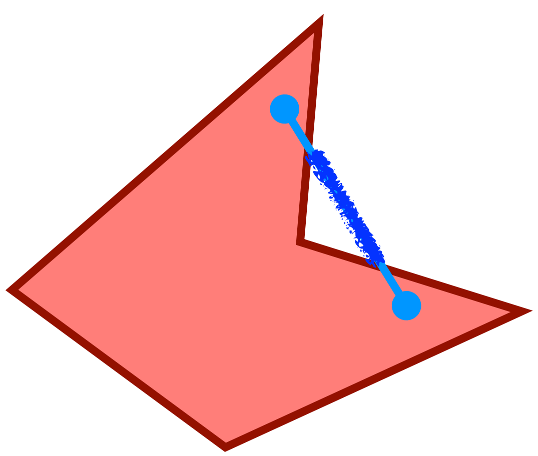
$\forall\, m \geq 3, \exists\, N(m)$ such that any set of $n \geq N(m)$ points in the plane, no three on a line, contains m points that are the vertices of a convex m-gon.

general positioned

Polygon:



convex



concave

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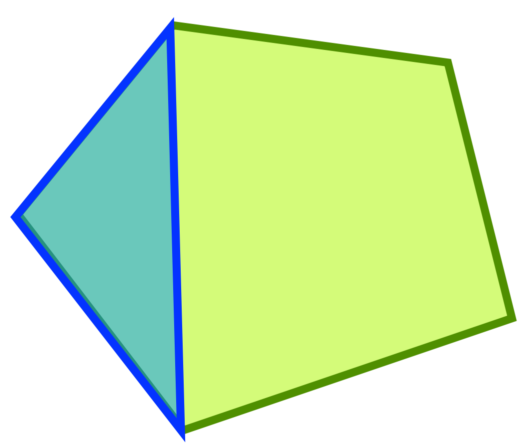
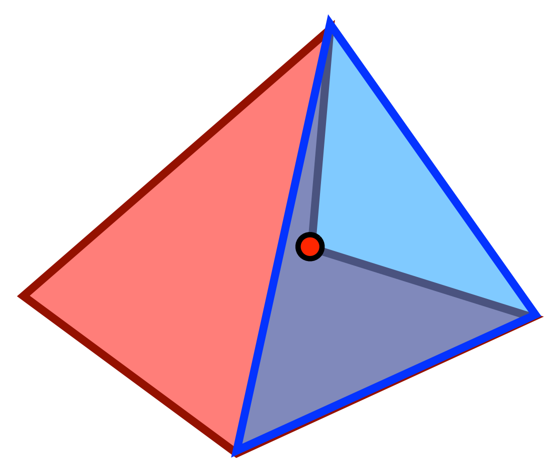
Polygon:



convex



concave

Theorem (Erdős-Szekeres 1935)

$\forall\, m \geq 3, \exists\, N(m)$ such that any set of $n \geq N(m)$ points in the plane, no three on a line, contains m points that are the vertices of a convex m-gon.

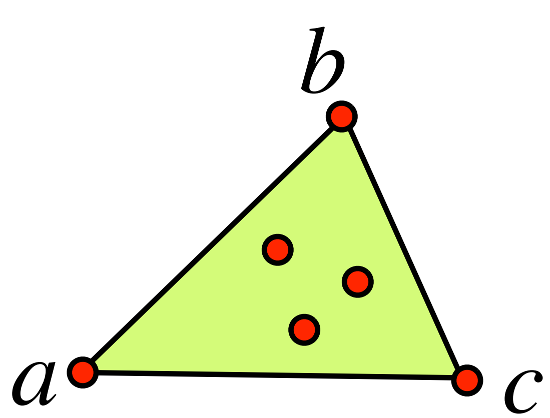
$N(m) = R_3(2; m, m)$

$|X| \geq N(m)$

$\forall f : \binom{X}{3} \to \{0, 1\}$ $\quad \exists S \subseteq X, |S| = m$

monochromatic $\binom{S}{3}$

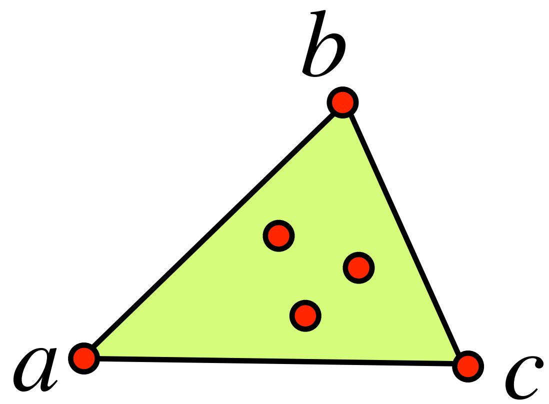
X: set of points in the plane, no 3 on a line



$\forall a, b, c \in X$, $\quad \triangle_{abc}$: points in triangle abc

$$f(\{a, b, c\}) = |\triangle_{abc}| \bmod 2$$

X: set of points in the plane, no 3 on a line



$\forall a, b, c \in X$, $\triangle_{abc}$: points in triangle abc

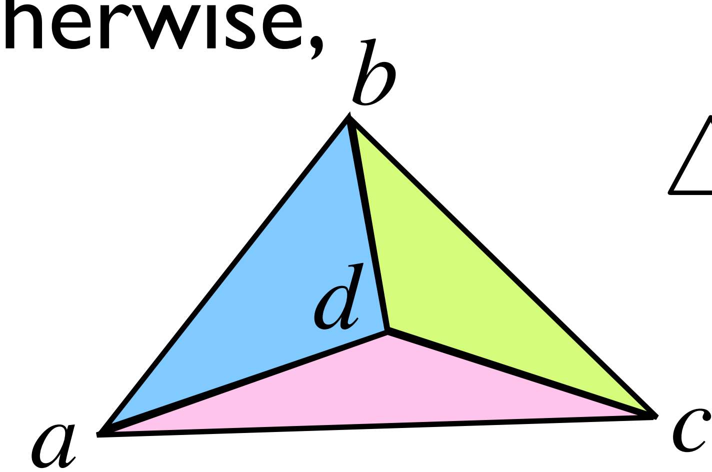
$$f(\{a, b, c\}) = |\triangle_{abc}| \bmod 2$$

$|X| \geq R_3(2; m, m)$ $\quad \forall f : \binom{X}{3} \to \{0, 1\}$

$\exists S \subseteq X, |S| = m$ monochromatic $\binom{S}{3}$

S is a convex m-gon

Otherwise,



disjoint union:

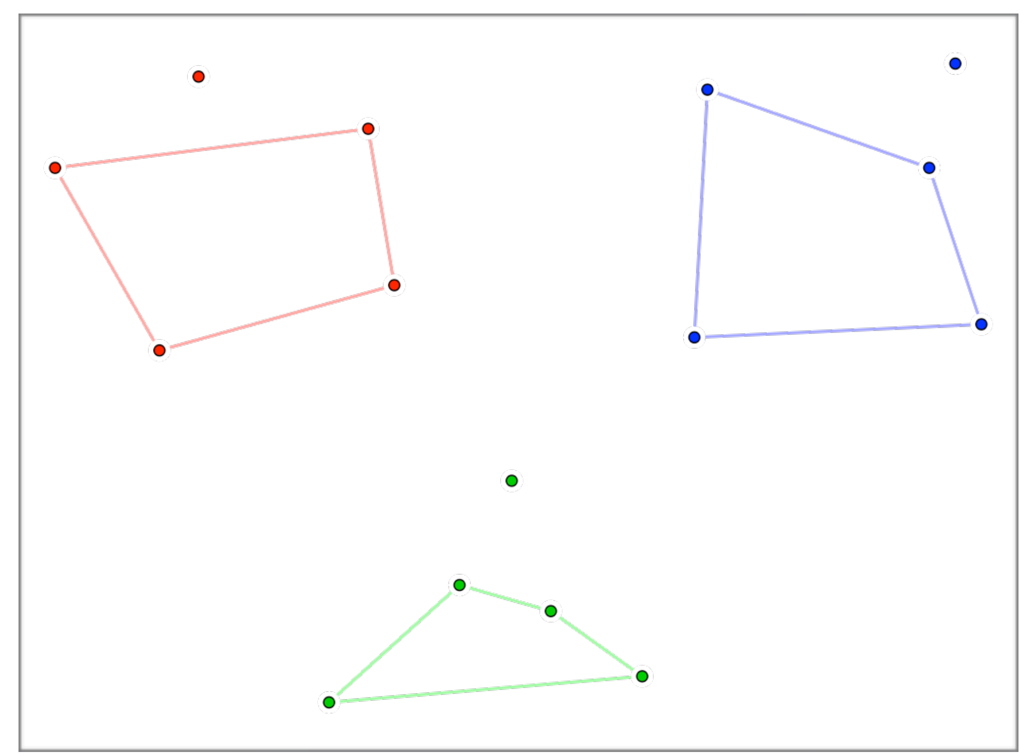
$$\triangle_{abc} = \triangle_{abd} \cup \triangle_{acd} \cup \triangle_{bcd} \cup \{d\}$$

$$f(abc) = f(abd) + f(acd) + f(bcd) + 1$$

Contradiction!

Happy Ending Problem

Any 5 points in the plane, no three on a line, has a subset of 4 points that form a convex quadrilateral.



Data Structures

Problem: "Is $x \in S$?"

data set $S \in \binom{[N]}{n}$ key $x \in [N]$ data universe $[N]$

Solution:

Data structure:

sorted table

Search Alg:

~~binary search~~

Complexity:

$\geq \log_2 n$

memory accesses

in the worst-case

"Is $x \in S$?" $x \in [N]$ $S \in \binom{[N]}{n}$

Theorem (Yao 1981)
If $N \geq 2n$, on sorted table, any search Alg requires $\Omega(\log n)$ accesses in the worst-case.

Problem: "Is $n \in S$?" $\forall S \in \binom{[N]}{n}$ $N \geq 2n$

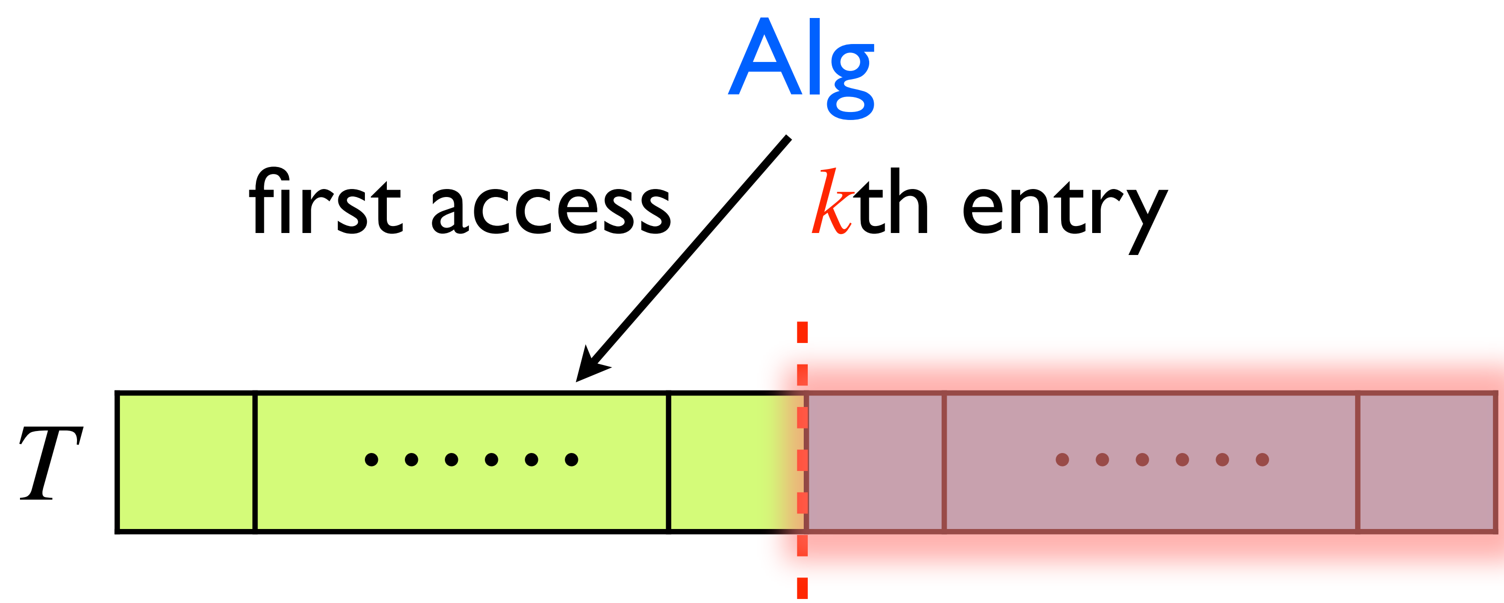
Induction on n, $n=2$, trivial

Suppose it is true for any smaller n.

adversarial argument + self-reduction

Problem: "Is $n \in S$?" $\forall S \in \binom{[N]}{n}$ $N \geq 2n$

sorted table



$$T[k] = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ N - (n-k) & \text{if } k > \frac{n}{2} \end{cases}$$

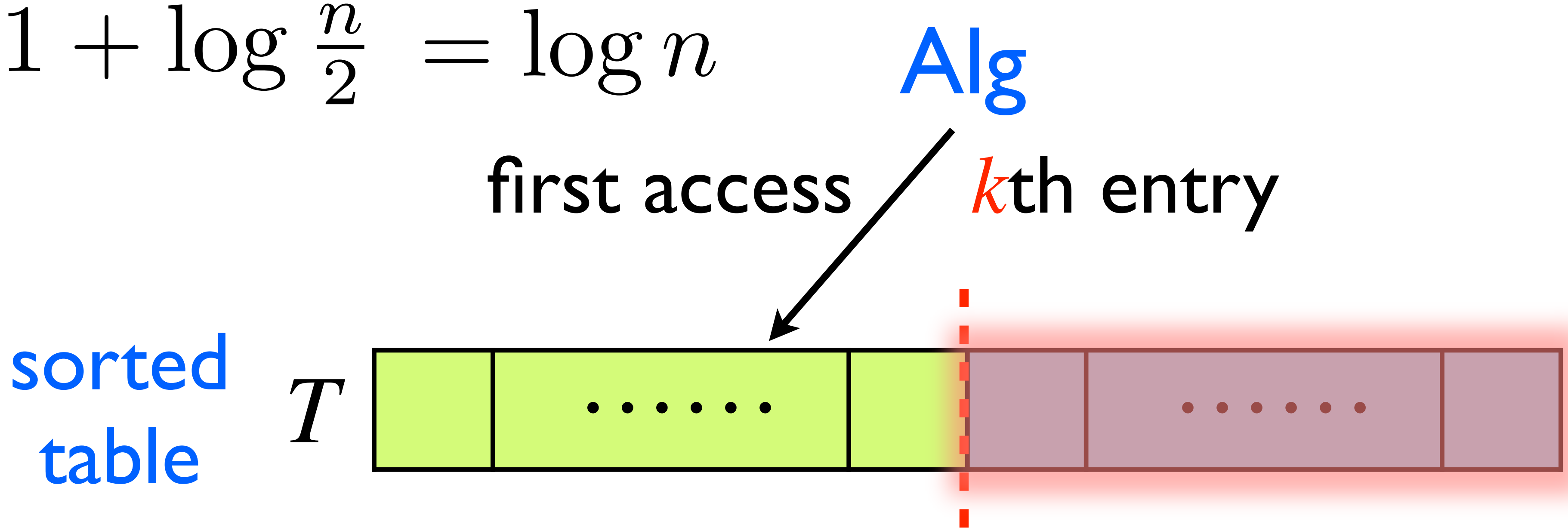
$$\binom{\{\frac{n}{2}, \ldots, N - \frac{n}{2}\}}{\frac{n}{2}} \subseteq \binom{\{\frac{n}{2}, \ldots, N\}}{\frac{n}{2}} \subseteq$$ possible $\{\, T[\frac{n}{2}+1], \ldots, T[n] \,\}$

$$n' = \frac{n}{2} \qquad N' = |\{\tfrac{n}{2}, \ldots, N - \tfrac{n}{2}\}| \geq n \geq 2n'$$

relative key in $[N']$: $n - \frac{n}{2} = n'$

Problem: "Is $n \in S$?" $\forall S \in \binom{[N]}{n}$ $N \geq 2n$

$\geq 1 + \log \frac{n}{2} = \log n$



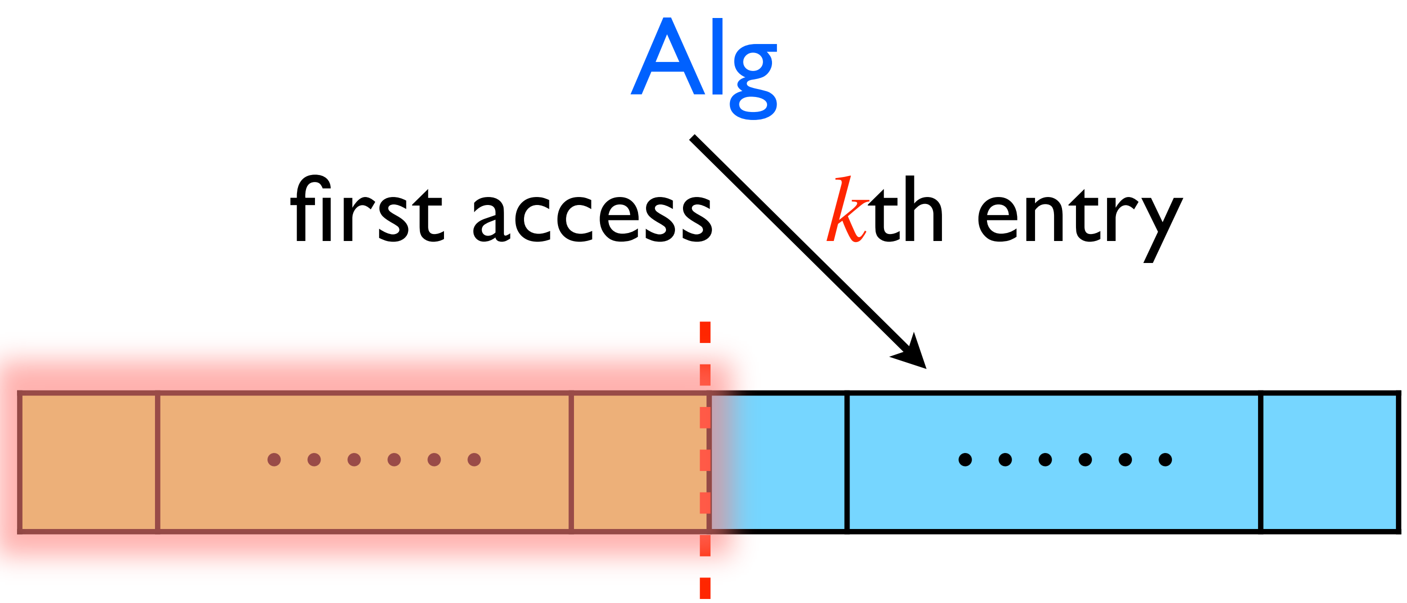
$$T[k] = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ N - (n-k) & \text{if } k > \frac{n}{2} \end{cases}$$

$n' = \frac{n}{2}$ "Is $n' \in S'$?" $\forall S' \in \binom{[N']}{n'}$ $N' \geq 2n'$

I.H. require $\log \frac{n}{2}$ memory accesses

Problem: "Is $n \in S$?" $\forall S \in \binom{[N]}{n}$ $N \geq 2n$

sorted table T



$$T[k] = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ N - (n-k) & \text{if } k > \frac{n}{2} \end{cases}$$

$$\binom{\{\frac{n}{2}, \ldots, N - \frac{n}{2}\}}{\frac{n}{2}} \subseteq \binom{\{1, \ldots, N - \frac{n}{2}\}}{\frac{n}{2}} \subseteq \text{possible } \{\, T[1], \ldots, T[\tfrac{n}{2}] \,\}$$

$$n' = \frac{n}{2} \qquad N' = \left|\{\tfrac{n}{2}, \ldots, N - \tfrac{n}{2}\}\right| \geq 2n'$$

relative key in $[N']$: $n - \frac{n}{2} = n'$

"Is $x \in S$?" $x \in [N]$ $S \in \binom{[N]}{n}$

Theorem (Yao 1981)
If $N \geq 2n$, on sorted table, any search Alg requires $\Omega(\log n)$ accesses in the worst-case.

implicit data structure:

each $S \in \binom{[N]}{n}$ is stored as a permutation of S

$$f : \binom{[N]}{n} \to [n!]$$

$f(S) = \pi$
dataset S: $x_1 < \cdots < x_n$
table: $(x_{\pi(1)}, \ldots, x_{\pi(n)})$

$\binom{[N]}{n}$ is mapped to the same π → same as sorted

$\exists X \subseteq [N], |X| \geq 2n$, $\binom{X}{n}$ monochromatic → same as sorted

"Is $x \in S$?" $x \in [N]$ $S \in \binom{[N]}{n}$

Theorem (Yao 1981)
For sufficiently large N, on any implicit data structure, any search Alg requires $\Omega(\log n)$ accesses in the worst-case.

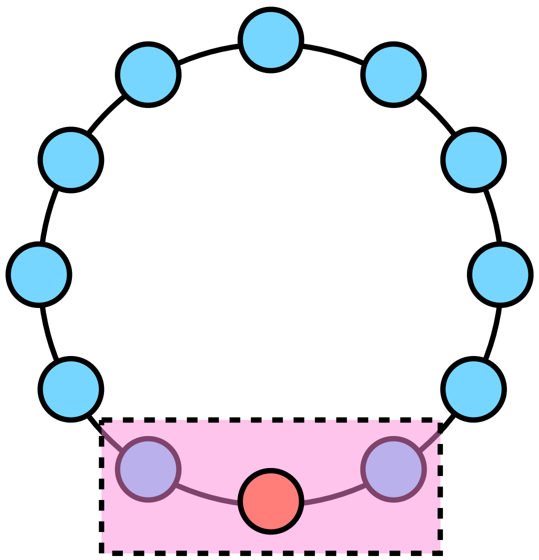
implicit data structure:

$$f : \binom{[N]}{n} \to [n!]$$

$N \geq R_n(n!; \underbrace{2n, \ldots, 2n}_{n!})$ or equivalently $N \to (\underbrace{2n, \ldots, 2n}_{n!})^n$

$\exists X \subseteq [N], |X| \geq 2n,$ $\binom{X}{n}$ monochromatic ⇒ $\geq \log n$ accesses

Local Computation



distributed computing in a ring

n nodes, ID from $[n]$

maximal independent set (MIS)

for any input ring,
locally compute the MIS

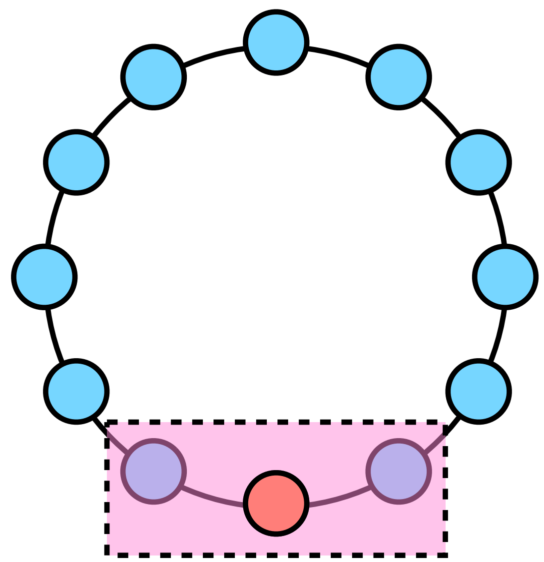
t-local algorithm:

$$\mathcal{L} : \{t\text{-permutations of } [n]\} \to \{0, 1\}$$

$$\Rightarrow \quad f : \binom{[n]}{t} \to \{0, 1\}$$

$$f(\{a_1, \ldots, a_t\}) = \mathcal{L}(a_1, \ldots, a_t)$$
$$a_1 < a_2 < \ldots < a_t$$

Local Computation



distributed computing in a ring

n nodes, ID from $[n]$

maximal independent set (MIS)

t-local algorithm:

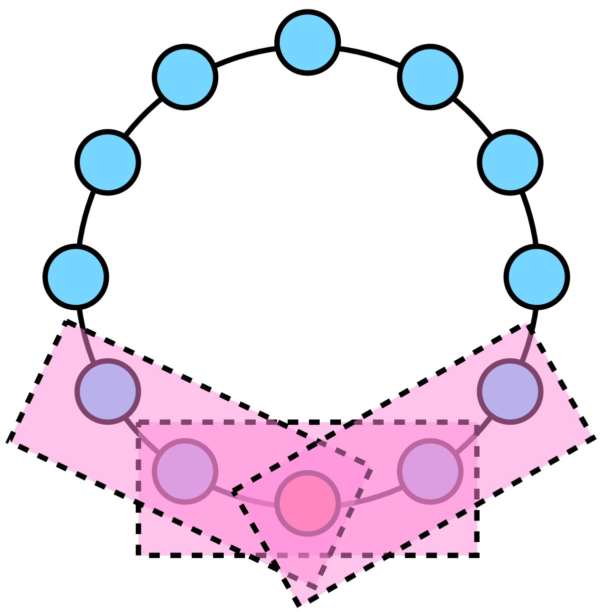
$$f : \binom{[n]}{t} \to \{0, 1\}$$

$$n \geq R_t(2; t+2, t+2)$$

$\exists$ a monochromatic $\binom{S}{t}$ $\qquad |S| = t+2$

$S = \{a_1, \ldots, a_t, a_{t+1}, a_{t+2}\}$ $\qquad a_1 < \ldots < a_t < a_{t+1} < a_{t+2}$

Local Computation



distributed computing in a ring

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maximal independent set (MIS)

t-local algorithm:

$$\mathcal{L} : \{t\text{-permutations of } [n]\} \to \{0, 1\}$$

$$n \geq R_t(2; t+2, t+2) \Rightarrow \exists S = \{a_1, \ldots, a_t, a_{t+1}, a_{t+2}\}$$

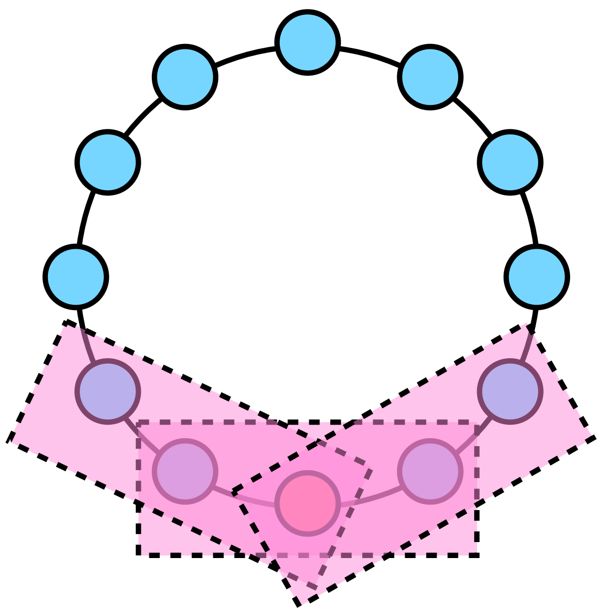
$$\mathcal{L}(a_1, \ldots, a_t) = \mathcal{L}(a_2, \ldots, a_{t+1}) = \mathcal{L}(a_3, \ldots, a_{t+2})$$

construct a bad ring starting with

$$(a_1, a_2, \ldots, a_t, a_{t+1}, a_{t+2})$$

Contradiction!

Local Computation



distributed computing in a ring

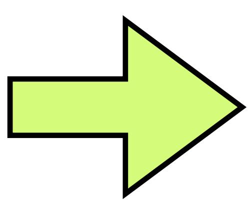
n nodes, ID from $[n]$

maximal independent set (MIS)

t-local algorithm:

$$\mathcal{L} : \{t\text{-permutations of } [n]\} \to \{0, 1\}$$

$$n < R_t(2; t+2, t+2) \quad \le \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{ct}}}}}}}_{t}$$



$$t = \Omega(\log^* n)$$