Course Info

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Textbooks


Combinatorics: A math for finite structures (not really)

- It is difficult to give rigorous definition of Combinatorics.
- According to Wikipedia, subfields of combinatorics include:
  - enumerative combinatorics, extremal combinatorics, combinatorial design
  - algebraic combinatorics, analytic combinatorics, probabilistic combinatorics, geometric combinatorics, topological combinatorics, arithmetic combinatorics
  - partition theory, graph theory, finite geometry, order theory, matroid theory, combinatorics on words, infinitary combinatorics

- In The Princeton Companion to Mathematics, there are two branches of mathematics on the subject of combinatorics:
  - **Enumerative and Algebraic Combinatorics** (Counting)
  - **Extremal and Probabilistic Combinatorics** (Hungarian)
Combinatorics: by the types of problems it studies

- **Enumeration (counting)** of finite structures (e.g. solutions/assignments/arrangements/configurations of finite systems) satisfying certain given constraints.

- **Existence** of finite structures satisfying certain given constraints.

- **Extremal problems**: How large/small a finite structure can be to satisfy certain given constraints?

- **Ramsey problems**: When a finite structure becomes large enough, some regularity must show up somewhere.

- **Construction (design)** of such finite structures.

- **Optimization**: to find the best structure/solution in some sense.
References


• Graham, Knuth, and Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. 
References


• Lovász and Plummer. *Matching Theory*. 
Enumeration
(Counting)

• How many ways are there:
  • to rank $n$ people?
  • to assign $m$ zodiac signs to $n$ people?
  • to choose $m$ people out of $n$ people?
  • to partition $n$ people into $m$ groups?
  • to distribute $m$ yuan to $n$ people?
  • to partition $m$ yuan to $n$ parts?
  • ... ....
The Twelvefold Way

\[ f : N \rightarrow M \quad |N| = n, \quad |M| = m \]

<table>
<thead>
<tr>
<th>elements of N</th>
<th>elements of M</th>
<th>any f</th>
<th>1-1</th>
<th>on-to</th>
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Knuth’s version (in *TAOCP* vol.4A)

$n$ balls are put into $m$ bins

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Tuples

\[ [m] = \{0, 1, \ldots, m-1\} \]

\[ [m]^n = [m] \times \cdots \times [m] \]

\[ |[m]^n| = m^n \]

Product rule:

finite sets \( S \) and \( T \)

\[ |S \times T| = |S| \cdot |T| \]
Functions

Count the number of functions

$$f : [n] \rightarrow [m]$$

$(f(1), f(2), \ldots, f(n)) \in [m]^n$ 

One-to-one correspondence

$$[n] \rightarrow [m] \iff [m]^n$$
Functions

count the # of functions

\[ f : [n] \rightarrow [m] \]

one-one correspondence

\[ [n] \rightarrow [m] \Leftrightarrow [m]^n \]

Bijection rule:

finite sets \( S \) and \( T \)

\[ \exists \phi : S \xrightarrow{1-1} T \implies |S| = |T| \]
Functions

Count the # of functions

\[ f : [n] \rightarrow [m] \]

One-one correspondence

\[ [n] \rightarrow [m] \Leftrightarrow [m]^n \]

\[ |[n] \rightarrow [m]| = |[m]^n| = m^n \]

"Combinatorial proof."
Injections

Count the \# of 1-1 functions

\[ f : [n] \xrightarrow{\text{1-1}} [m] \]

One-to-one correspondence

\[ \pi = (f(1), f(2), \ldots, f(n)) \]

\textbf{n-permutation: } \pi \in [m]^n \text{ of distinct elements}

\[ (m)_n = m(m-1) \cdots (m-n+1) = \frac{m!}{(m-n)!} \]

“m lower factorial n”
Subsets

subsets of \{1, 2, 3\}:

\emptyset, 
\{1\}, \{2\}, \{3\}, 
\{1, 2\}, \{1, 3\}, \{2, 3\}, 
\{1, 2, 3\}

\[n\] = \{1, 2, \ldots, n\}

**Power set:**\[2^{[n]} = \{ S \mid S \subseteq [n]\}\]

\[\left|2^{[n]}\right| =\]
Subsets

\[ [n] = \{1, 2, \ldots, n\} \]

**Power set:**  \[ 2^[n] = \{S \mid S \subseteq [n]\} \]

\[ |2^[n]| = \]

**Combinatorial proof:**

A subset \( S \subseteq [n] \) corresponds to a string of \( n \) bit, where bit \( i \) indicates whether \( i \in S \).
Subsets

\[ [n] = \{1, 2, \ldots, n\} \]

**Power set:** \( 2^{[n]} = \{ S \mid S \subseteq [n] \} \)

\[ \left| 2^{[n]} \right| = \left| \{0, 1\}^n \right| = 2^n \]

**Combinatorial proof:**

\( S \subseteq [n] \iff \chi_S \in \{0, 1\}^n \)

\( \chi_S(i) = \begin{cases} 
1 & i \in S \\
0 & i \notin S 
\end{cases} \)

one-to-one correspondence
Subsets

\[ [n] = \{1, 2, \ldots, n\} \]

**Power set:** \( 2^{[n]} = \{ S \mid S \subseteq [n]\} \)

A not-so-combinatorial proof:

Let \( f(n) = |2^{[n]}| \)

\[ f(n) = 2 f(n - 1) \]
\[ f(n) = \left\lvert 2^{[n]} \right\rvert \]
\[ f(n) = 2f(n - 1) \]

\[ 2^{[n]} = \{ S \subseteq [n] \mid n \not\in S \} \cup \{ S \subseteq [n] \mid n \in S \} \]

\[ \left\lvert 2^{[n]} \right\rvert = \left\lvert 2^{[n-1]} \right\rvert + \left\lvert 2^{[n-1]} \right\rvert = 2f(n - 1) \]

**Sum rule:**

finite **disjoint** sets \( S \) and \( T \)

\[ |S \cup T| = |S| + |T| \]
Subsets

\([n] = \{1, 2, \ldots, n\}\)

**Power set:** \(2^{[n]} = \{S \mid S \subseteq [n]\}\)

\[\left|2^{[n]}\right| = 2^n\]

**A not-so-combinatorial proof:**

Let \(f(n) = \left|2^{[n]}\right|\)

\[f(n) = 2f(n - 1)\quad f(0) = |2^\emptyset| = 1\]
Three Rules

Sum rule:

finite disjoint sets $S$ and $T$

$$|S \cup T| = |S| + |T|$$

Product rule:

finite sets $S$ and $T$

$$|S \times T| = |S| \cdot |T|$$

Bijection rule:

finite sets $S$ and $T$

$$\exists \phi : S \xrightarrow{1-1} \text{on-to} T \implies |S| = |T|$$
2-subsets of \{1, 2, 3\}: \{1, 2\}, \{1, 3\}, \{2, 3\}

\(k\)-uniform \(\binom{S}{k} = \{T \subseteq S \mid |T| = k\}\)

\(\binom{n}{k} = \left| \binom{[n]}{k} \right| \)

“\(n\) choose \(k\)”
Subsets of fixed size

\[
\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1} = \frac{n!}{k!(n-k)!}
\]

# of ordered k-subsets: \( n(n-1) \cdots (n-k+1) \)

# of permutations of a k-set: \( k(k-1) \cdots 1 \)
Binomial Coefficients

Binomial coefficient:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

1. \[
\binom{n}{k} = \binom{n}{n-k}
\]
2. \[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

choose a k-subset \(\Leftrightarrow\) choose its compliment

0-subsets + 1-subsets + ... + n-subsets = all subsets
Binomial Theorem

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

Proof:

\[(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)\]

# of \(x^k\): choose \(k\) factors out of \(n\)
Binomial Theorem

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

Let \(x = 1\).

\[\sum_{k=0}^{n} \binom{n}{k} = 2^n\]

\(S = \{x_1, x_2, \ldots, x_n\}\)

\[
\begin{align*}
\text{# of subsets of } S \text{ of odd sizes} &= \text{# of subsets of } S \text{ of even sizes} \\
\text{Let } x &= -1.
\end{align*}
\]
# The Twelvefold Way

$n$ balls are put into $m$ bins

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Compositions of an Integer

How many ways to assign $n$ beli to $k$ pirates?

How many ways to assign $n$ beli to $k$ pirates, so that each pirate receives at least 1 beli?
Compositions of an Integer

\[ n \in \mathbb{Z}^+ \]

\textit{k}-composition of \( n \):

an ordered sum of \( k \) positive integers

a \textit{k}-tuple \((x_1, x_2, \cdots, x_k)\)

\[ x_1 + x_2 + \cdots + x_k = n \quad \text{and} \quad x_i \in \mathbb{Z}^+ \]
Compositions of an Integer

\[ n \in \mathbb{Z}^+ \]

**k-composition of \( n \):**

a \( k \)-tuple \((x_1, x_2, \cdots, x_k)\)  
\[ x_1 + x_2 + \cdots + x_k = n \quad \text{and} \quad x_i \in \mathbb{Z}^+ \]

# of \( k \)-compositions of \( n \)? \[
\binom{n-1}{k-1}
\]

\( n \) identical balls

\[ \underbrace{\text{\( x_1 \)}}_{\text{n identical balls}} \underbrace{\text{\( x_2 \)}}_{\text{\( x_2 \)}} \cdots \underbrace{\text{\( x_k \)}}_{\text{\( x_k \)}} \]
Compositions of an Integer

a $k$-tuple $(x_1, x_2, \ldots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

# of $k$-compositions of $n$? $\binom{n-1}{k-1}$

$\phi((x_1, x_2, \ldots, x_k)) = \{x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + x_2 + \cdots + x_{k-1}\}$

$\phi$ is a 1-1 correspondence between $
\{k$-compositions of $n\}$ and $\binom{\{1,2,\ldots,n-1\}}{k-1}$
Compositions of an Integer

weak $k$-composition of $n$:

an ordered sum of $k$ nonnegative integers

a $k$-tuple $(x_1, x_2, \cdots, x_k)$

$$x_1 + x_2 + \cdots + x_k = n \quad \text{and} \quad x_i \in \mathbb{N}$$
Compositions of an Integer

weak $k$-composition of $n$:

a $k$-tuple $(x_1, x_2, \cdots, x_k)$

\[ x_1 + x_2 + \cdots + x_k = n \quad \text{and} \quad x_i \in \mathbb{N} \]

# of weak $k$-compositions of $n$?

\[ \binom{n + k - 1}{k - 1} \]

\[ (x_1 + 1) + (x_2 + 1) + \cdots + (x_k + 1) = n + k \]

a $k$-composition of $n+k$

1-1 correspondence
Multisets

$k$-subset of $S$  “$k$-combination of $S$
without repetition”

3-combinations of \{ 1, 2, 3, 4 \}  
without repetition:

\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}

with repetition:

\{1,1,1\}, \{1,1,2\}, \{1,1,3\}, \{1,1,4\}, \{1,2,2\}, \{1,3,3\},
\{1,4,4\}, \{2,2,2\}, \{2,2,3\}, \{2,2,4\}, \{2,3,3\}, \{2,4,4\},
\{3,3,3\}, \{3,3,4\}, \{3,4,4\}, \{4,4,4\}
Multisets

multiset $M$ on set $S$:

$$m : S \rightarrow \mathbb{N}$$

multiplicity of $x \in S$

$m(x)$: # of repetitions of $x$ in $M$

cardinality $|M| = \sum_{x \in S} m(x)$

“$k$-combination of $S$ with repetition” $\leftrightarrow$ $k$-multiset on $S$

$$\binom{n}{k} : \# \text{ of } k\text{-multisets on an } n\text{-set}$$
Multisets

\[ \binom{n}{k} = \binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k} \]

A \( k \)-multiset on \( S = \{x_1, x_2, \ldots, x_n\} \)

\[ m(x_1) + m(x_2) + \cdots + m(x_n) = k \]

\[ m(x_i) \geq 0 \]

a weak \( n \)-composition of \( k \)
Multinomial Coefficients

permutations of a multiset of size \( n \) with multiplicities \( m_1, m_2, \ldots, m_k \)

\# of reordering of “multinomial”

permutations of \( \{a, i, i, l, l, m, m, n, o, t, u\} \)

assign \( n \) distinct balls to \( k \) distinct bins with the \( i \)-th bin receiving \( m_i \) balls

multinomial coefficient \( \binom{n}{m_1, \ldots, m_k} \)

\[ m_1 + m_2 + \cdots + m_k = n \]
Multinomial Coefficients

permutations of a multiset of size $n$ with multiplicities $m_1, m_2 \ldots, m_k$

assign $n$ distinct balls to $k$ distinct bins with the $i$-th bin receiving $m_i$ balls

\[
\binom{n}{m_1, \ldots, m_k} = \frac{n!}{m_1!m_2!\cdots m_k!}
\]

\[
\binom{n}{m, n-m} = \binom{n}{m}
\]
Multinomial Theorem

\[(x_1 + x_2 + \cdots + x_k)^n = \sum_{m_1 + \cdots + m_k = n} \binom{n}{m_1, \ldots, m_k} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}\]

Proof: \[(x_1 + x_2 + \cdots + x_k)^n = (x_1 + x_2 + \cdots + x_k) \cdots (x_1 + x_2 + \cdots + x_k)\]

\# of \(x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}\): assign \(n\) factors to \(k\) groups of sizes \(m_1, m_2, \ldots, m_k\)
The Twelvefold Way

$n$ balls are put into $m$ bins

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### The Twelvefold Way

$n$ balls are put into $m$ bins

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Partitions of a Set

$n$ pirates

$P = \{A_1, A_2, \ldots, A_k\}$ is a partition of $S$:

$A_i \neq \emptyset$

$A_i \cap A_j = \emptyset$

$A_1 \cup A_2 \cup \cdots \cup A_k = S$

$k$ boats
Partitions of a Set

\[ P = \{A_1, A_2, \ldots, A_k\} \] is a partition of \( S \):

\[
\begin{align*}
A_i &\neq \emptyset \\
A_i \cap A_j &= \emptyset \\
A_1 \cup A_2 \cup \cdots \cup A_k &= S
\end{align*}
\]

\[ \binom{n}{k} \] \# of \( k \)-partitions of an \( n \)-set

“Stirling number of the second kind”

\[
B_n = \sum_{k=1}^{n} \binom{n}{k} \] \# of partitions of an \( n \)-set

“Bell number”
Stirling Number of the 2nd Kind

\[ \binom{n}{k} \]  
\[ \text{# of } k\text{-partitions of an } n\text{-set} \]

\[ \binom{n}{k} = k \binom{n-1}{k} + \binom{n-1}{k-1} \]

Case 1  \{n\} is not a partition block

\[ n \text{ is in one of the } k \text{ blocks in a } k\text{-partition of } [n-1] \]

Case 2  \{n\} is a partition block

the remaining \( k-1 \) blocks forms a \( (k-1)\)-partition of \([n-1]\)
# The Twelvefold Way

Let $f : N \rightarrow M$ be a function where $n$ balls are put into $m$ bins.

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<td>$(n-1)\binom{m-1}{n}$</td>
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Surjections

\[ f : [n] \text{ on-to } [m] \]

\[ \forall i \in [m] \quad f^{-1}(i) \neq \emptyset \]

\[ (f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(m)) \]

ordered \( m \)-partition of \([n]\)

\[ m! \binom{n}{m} \]
### The Twelvefold Way

$n$ balls are put into $m$ bins

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Partitions of a Number

$n$ beli

$k$ boxes

a **partition** of $n$ into $k$ parts:

an **unordered** sum of $k$ **positive** integers
Partitions of a Number

A partition of $n$ into $k$ parts:

$p_k(n)$ \# of partitions of $n$ into $k$ parts

$n=7$ \[ \{7\} \]

\[ \{1,6\}, \{2,5\}, \{3,4\} \]

\[ \{1,1,5\}, \{1,2,4\}, \{1,3,3\}, \{2,2,3\} \]

\[ \{1,1,1,4\}, \{1,1,2,3\}, \{1,2,2,2\} \]

\[ \{1,1,1,1,3\}, \{1,1,1,2,2\} \]

\[ \{1,1,1,1,1,2\} \]

\[ \{1,1,1,1,1,1,1\} \]

“positive”

“unordered”
\( p_k(n) \) \# of partitions of \( n \) into \( k \) parts

integral solutions to

\[
\begin{aligned}
x_1 + x_2 + \cdots + x_k &= n \\
x_1 \geq x_2 \geq \cdots \geq x_k \geq 1
\end{aligned}
\]

\( p_k(n) = ? \)
\begin{equation*}
\begin{cases}
x_1 + x_2 + \cdots + x_k = n \\
x_1 \geq x_2 \geq \cdots \geq x_k \geq 1
\end{cases}
\end{equation*}

\[ p_k(n) = p_{k-1}(n-1) + p_k(n-k) \]

**Case.1** \( x_k = 1 \)

\((x_1, \ldots, x_{k-1})\) is a \((k - 1)\)-partition of \(n - 1\)

**Case.2** \( x_k > 1 \)

\((x_1 - 1, \ldots, x_k - 1)\) is a \(k\)-partition of \(n - k\)
partition \[ \left\{ \begin{aligned} x_1 + x_2 + \cdots + x_k &= n \\ x_1 &\geq x_2 \geq \cdots \geq x_k \geq 1 \end{aligned} \right. \]

composition \[ \left\{ \begin{aligned} x_1 + x_2 + \cdots + x_k &= n \\ x_i &\geq 1 \end{aligned} \right. \]

permutation \( \pi \)

k! p_k(n) \geq \binom{n-1}{k-1} \]
partition \( \{x_1, \cdots, x_k\} \) \hspace{1cm} y_i = x_i + k - i

\[ x_1 \geq x_2 \geq \cdots \geq x_{k-2} \geq x_{k-1} \geq x_k \geq 1 \]
\[ +k - 1 \hspace{1cm} +k - 2 \hspace{1cm} +2 \hspace{1cm} +1 \]

\[ y_1 > y_2 > \cdots > y_{k-2} > y_{k-1} > y_k > 1 \]

\( \pi \) \hspace{1cm} \text{permutation}

\( \text{composition of } \ n + \frac{k(k-1)}{2} \)

\( (y_1, y_2, \cdots, y_k) \)

\( \text{“1-1”} \)

\[ k! p_k(n) \leq \left( n + \frac{k(k-1)}{2} - 1 \right) \]
\[
\frac{(n-1)^{k-1}}{k!} \leq p_k(n) \leq \frac{(n+\frac{k(k-1)}{2})^{k-1}}{k!} - 1
\]

If \( k \) is fixed,

\[
p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!}
\]

as \( n \to \infty \)
The Man Who Knew Infinity
(2015 film)

G. H. Hardy
(1877-1947)

Srinivasa
Ramanujan
(1887-1920)

\[ p(n) = \sum_{k=1}^{n} p_k(n) \]

\[ \approx \frac{1}{4n\sqrt{3}} \exp \left\{ \pi \sqrt{\frac{2n}{3}} \right\} \]
Ferrers diagram
(Young diagram)

\[
x_1 + x_2 + \cdots + x_k = n
\]

\[
x_1 \geq x_2 \geq \cdots \geq x_k \geq 1
\]
conjugate

(6,4,4,2,1)  \rightarrow  (5,4,3,3,1,1)

one-to-one correspondence
# of partitions of $n$ into $k$ parts = # of partitions of $n$ with largest part $k$
# of partitions of \( n \) into \( k \) parts

\[ p_{k}(n) = \sum_{j=1}^{k} p_{j}(n - k) \]
# The Twelvefold Way

$n$ balls are put into $m$ bins

<table>
<thead>
<tr>
<th>balls per bin:</th>
<th>unrestricted</th>
<th>$\leq 1$</th>
<th>$\geq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ distinct balls, $m$ distinct bins</td>
<td>$m^n$</td>
<td>$(m)_n$</td>
<td>$m! \left{ \begin{array}{c} n \ m \end{array} \right}$</td>
</tr>
<tr>
<td>$n$ identical balls, $m$ distinct bins</td>
<td>$\binom{m}{n}$</td>
<td>$\binom{m}{n}$</td>
<td>$\binom{n-1}{m-1}$</td>
</tr>
<tr>
<td>$n$ distinct balls, $m$ identical bins</td>
<td>$\sum_{k=1}^{m} \left{ \begin{array}{c} n \ k \end{array} \right}$</td>
<td>$\begin{cases} 1 &amp; \text{if } n \leq m \ 0 &amp; \text{if } n &gt; m \end{cases}$</td>
<td>$\left{ \begin{array}{c} n \ m \end{array} \right}$</td>
</tr>
<tr>
<td>$n$ identical balls, $m$ identical bins</td>
<td>$\sum_{k=1}^{m} p_k(n)$</td>
<td>$\begin{cases} 1 &amp; \text{if } n \leq m \ 0 &amp; \text{if } n &gt; m \end{cases}$</td>
<td>$p_m(n)$</td>
</tr>
</tbody>
</table>