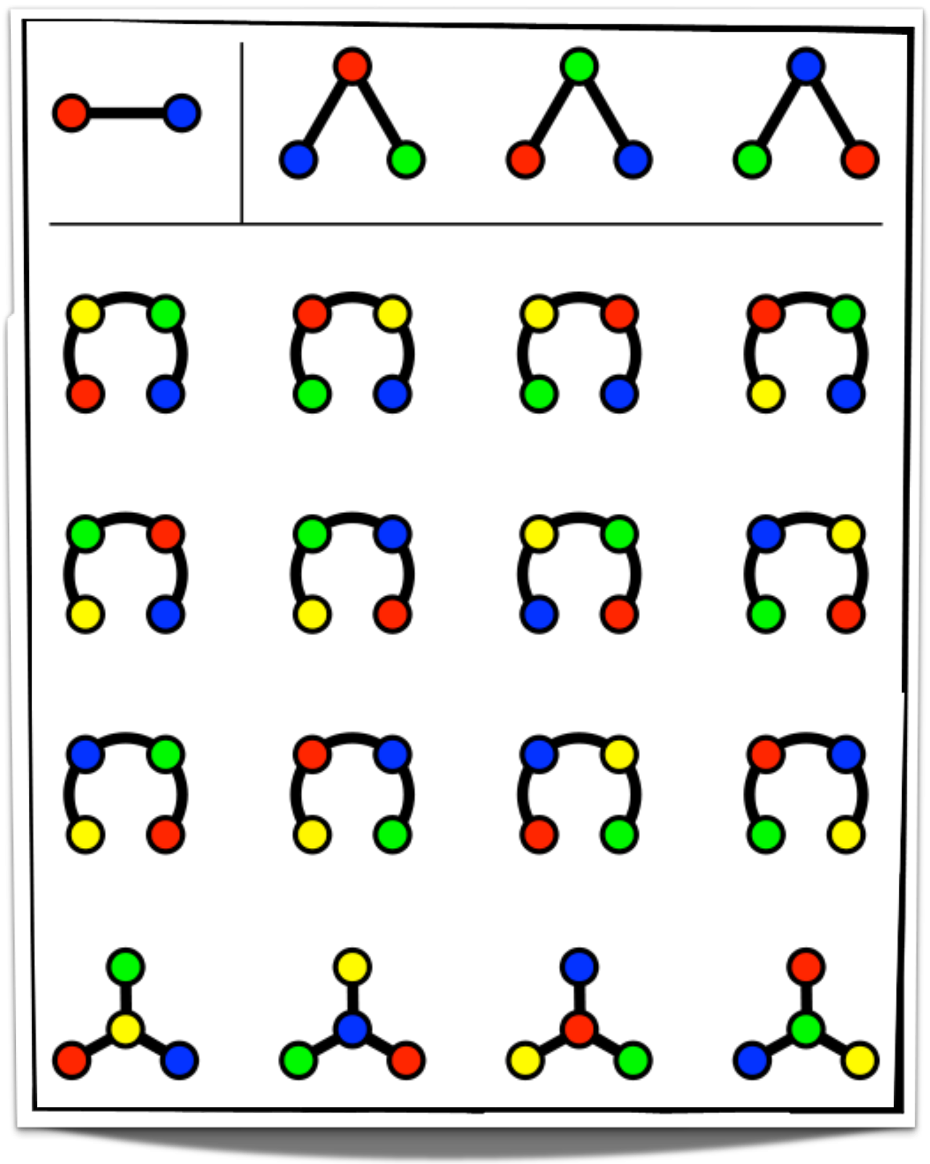
Combinatorics

Cayley Formula

尹一通 Nanjing University, 2023 Spring

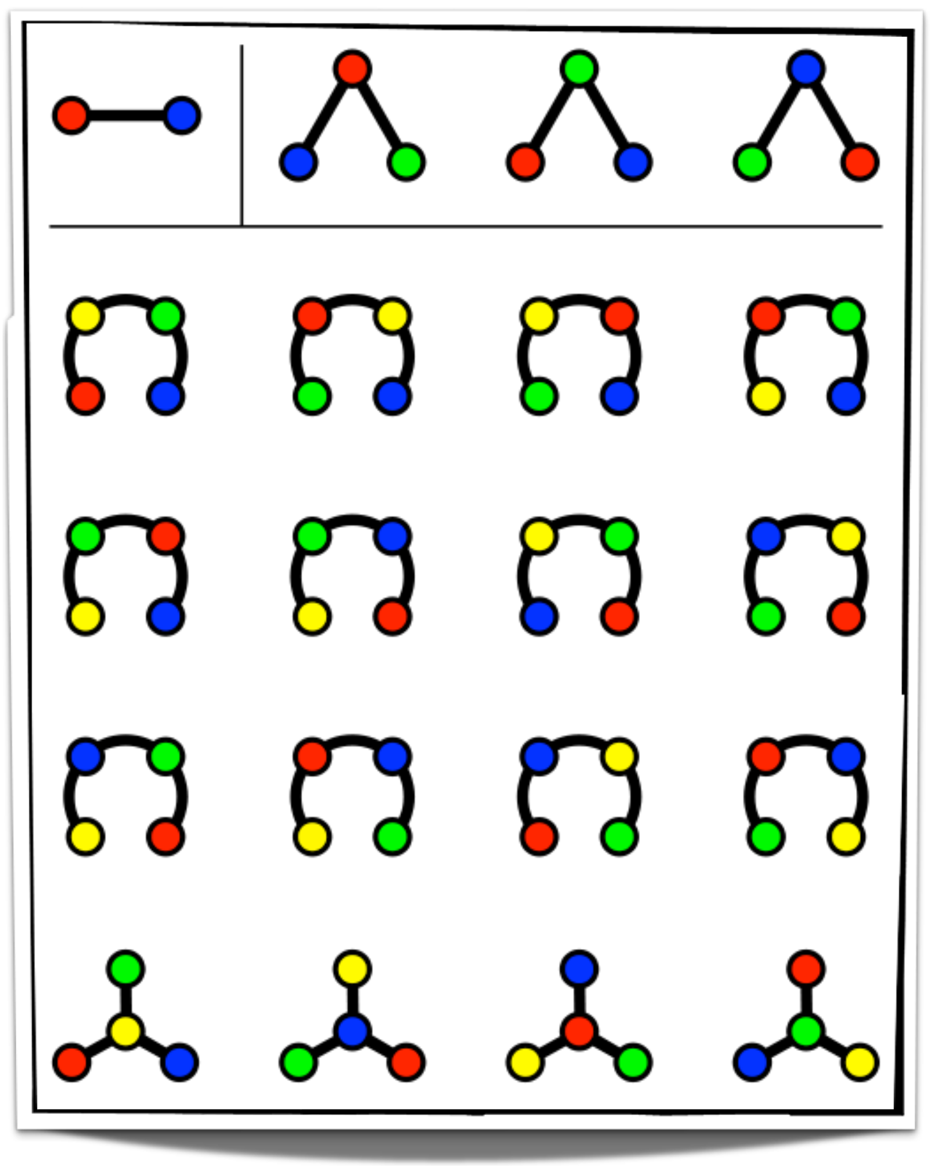
Counting (labeled) trees

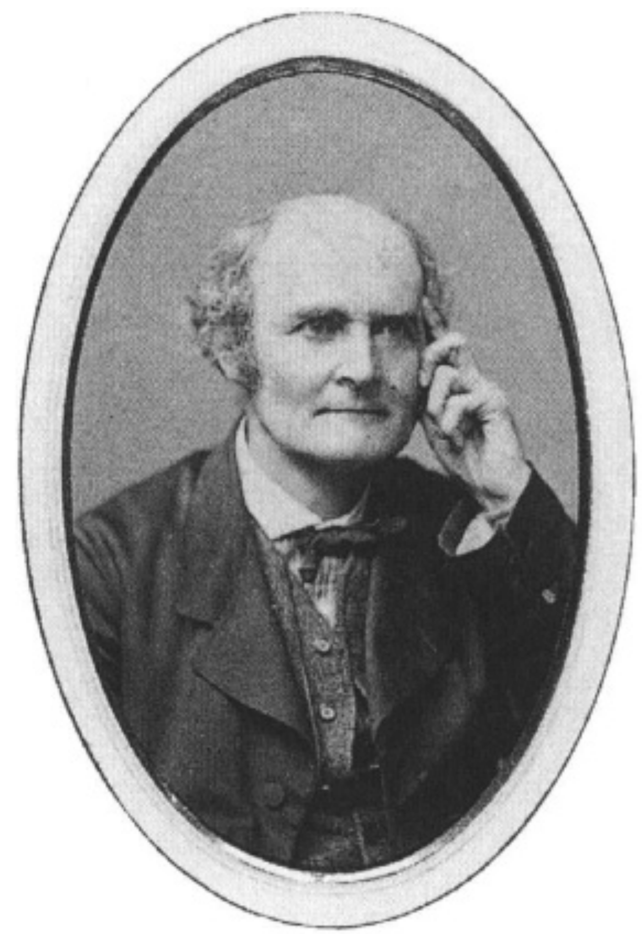


"How many different trees can be formed from n distinct vertices?"

Cayley's formula:

There are n^{n-2} trees on n distinct vertices.





Arthur Cayley

Cayley Formula:

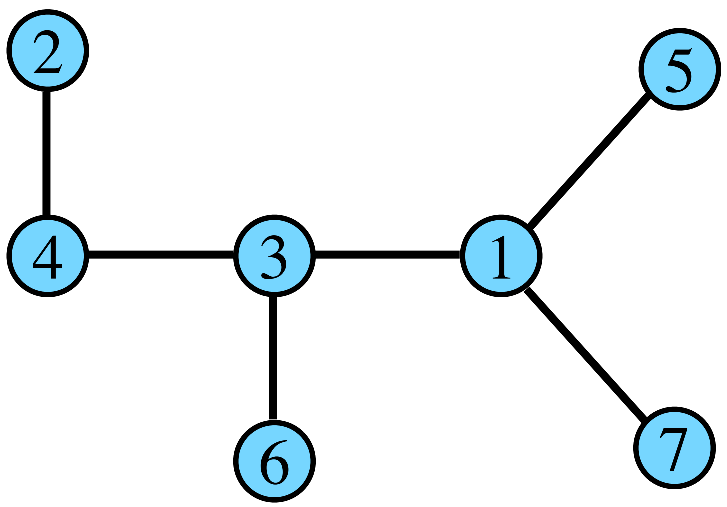
Prüfer Code

Prüfer Code

leaf : vertex of degree 1

removing a leaf from T still gives a tree

T_1 :



$T_1 = T$;
for $i = 1$ to n-1
u_i : smallest leaf in T_i ;
(u_i, v_i) : edge in T_i ;
T_{i+1} = delete u_i from T_i ;

u_i : 2

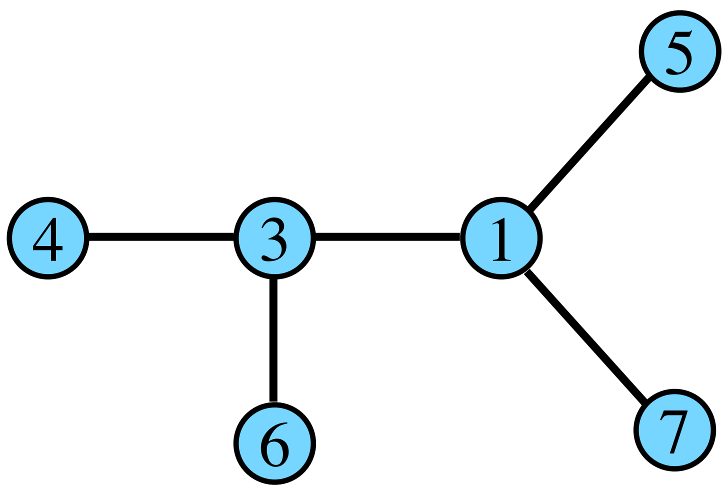
v_i : 4

Prüfer Code

leaf : vertex of degree 1

removing a leaf from *T* still gives a tree

T_2 :



$T_1 = T$;

for $i = 1$ to n-1

u_i : smallest leaf in T_i ;

(u_i, v_i) : edge in T_i ;

T_{i+1} = delete u_i from T_i ;

u_i : 2 , 4

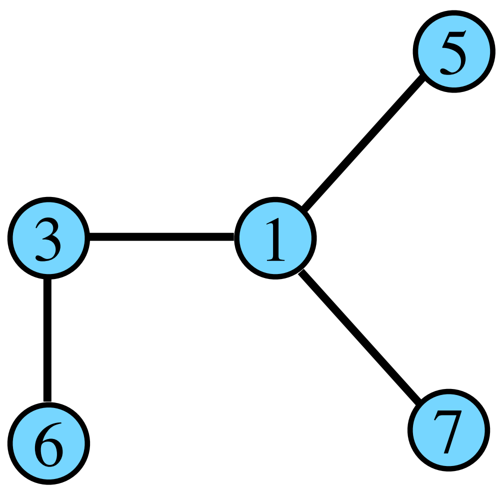
v_i : 4 , 3

Prüfer Code

leaf : vertex of degree 1

removing a leaf from *T* still gives a tree

T_3 :



$T_1 = T$;

for $i = 1$ to n-1

u_i : smallest leaf in T_i ;

(u_i, v_i) : edge in T_i ;

T_{i+1} = delete u_i from T_i ;

u_i : 2 , 4 , 5

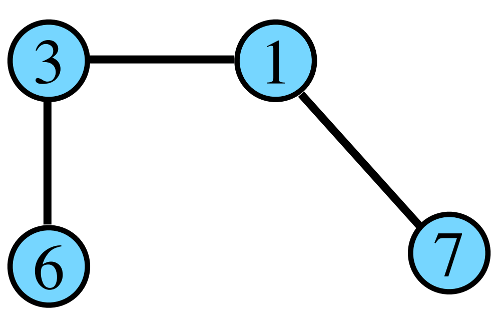
v_i : 4 , 3 , 1

Prüfer Code

leaf : vertex of degree 1

removing a leaf from T still gives a tree

T_4 :



$T_1 = T$;

for $i = 1$ to n-1

u_i : smallest leaf in T_i ;

(u_i, v_i) : edge in T_i ;

T_{i+1} = delete u_i from T_i ;

u_i : 2 , 4 , 5 , 6

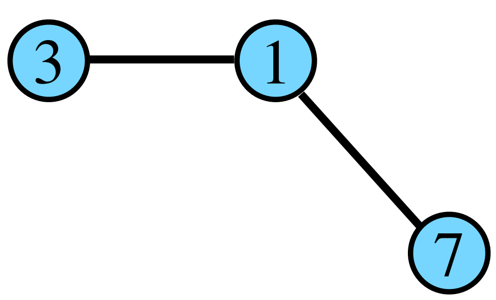
v_i : 4 , 3 , 1 , 3

Prüfer Code

leaf : vertex of degree 1

removing a leaf from *T* still gives a tree

T_5 :



$T_1 = T$;

for $i = 1$ to n-1

u_i : smallest leaf in T_i ;

(u_i, v_i) : edge in T_i ;

T_{i+1} = delete u_i from T_i ;

u_i : 2 , 4 , 5 , 6 , 3

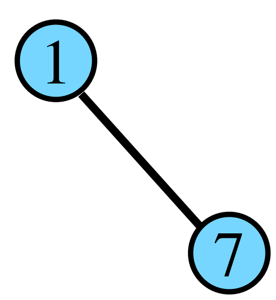
v_i : 4 , 3 , 1 , 3 , 1

Prüfer Code

leaf : vertex of degree 1

removing a leaf from T still gives a tree

T_6 :



$T_1 = T$;
for $i = 1$ to n-1
u_i : smallest leaf in T_i ;
(u_i, v_i) : edge in T_i ;
T_{i+1} = delete u_i from T_i ;

u_i : 2 , 4 , 5 , 6 , 3 , 1

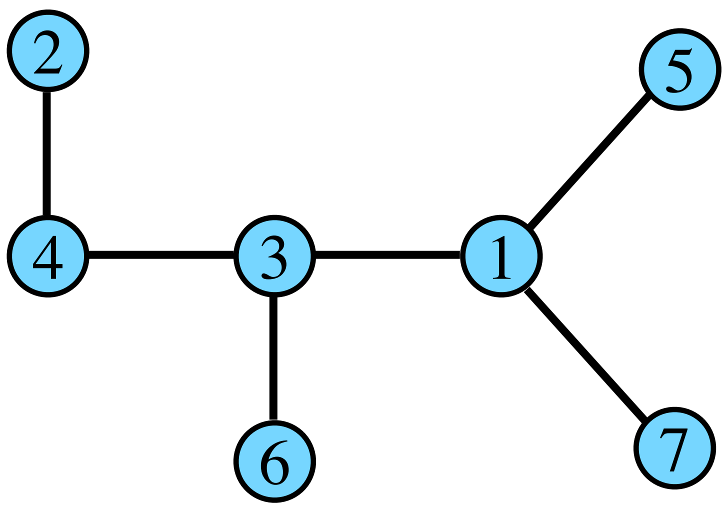
v_i : 4 , 3 , 1 , 3 , 1 , 7

Prüfer Code

leaf : vertex of degree 1

removing a leaf from T still gives a tree

T :



$T_1 = T$;

for $i = 1$ to n-1

u_i : smallest leaf in T_i ;

(u_i, v_i) : edge in T_i ;

T_{i+1} = delete u_i from T_i ;

u_i : 2 , 4 , 5 , 6 , 3 , 1

v_i : 4 , 3 , 1 , 3 , 1 , 7

Prüfer code:

$(v_1, v_2, \ldots, v_{n-2})$

edges of T : (u_i, v_i), $1 \leq i \leq n-1$

$v_{n-1} = n$

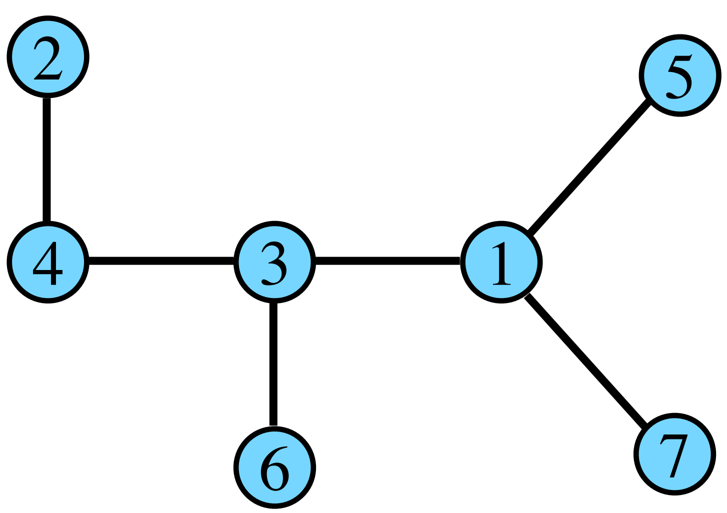
u_i : smallest leaf in T_i

a tree has ≥2 leaves

⇒ n is never deleted

$u_i \neq n$

T :



Only need to recover every u_i from $(v_1, v_1, \ldots, v_{n-2})$.

u_i is the smallest number *not* in

$$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$$

u_i : 2 , 4 , 5 , 6 , 3 , 1

v_i : 4 , 3 , 1 , 3 , 1 , 7

$(v_1, v_2, \ldots, v_{n-2})$

u_i is the smallest number *not* in

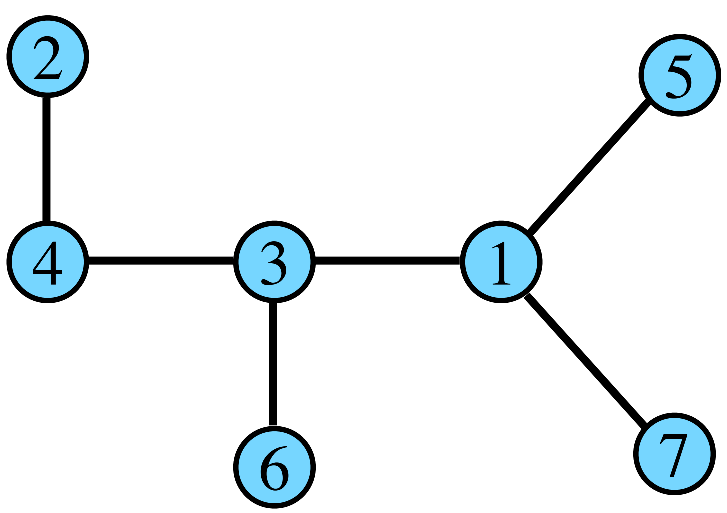
$$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$$

∀ vertex *v* in T,

occurrences of v in $u_1, u_2, \ldots, u_{n-1}, v_{n-1}$: 1

occurrences of v in edges (u_i, v_i), $1 \le i \le n-1$: $\deg_T(v)$

T :



occurrences of v in **Prüfer code**: $(v_1, v_2, \ldots, v_{n-2})$

$\deg_T(v)-1$

u_i: 2, 4, 5, 6, 3, 1

v_i: 4, 3, 1, 3, 1, 7

$(v_1, v_2, \ldots, v_{n-2})$

u_i is the smallest number *not* in

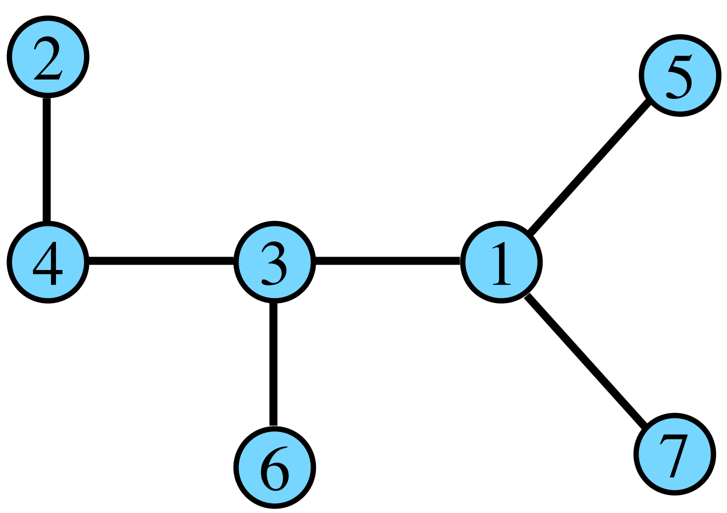
$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$

∀ vertex *v* in T,

occurrences of v in $u_1, u_2, \ldots, u_{n-1}, v_{n-1}$: 1

occurrences of v in edges (u_i, v_i), $1 \le i \le n-1$: $\deg_T(v)$

T_3 :



occurrences of v in $(v_i, \ldots, v_{n-2})$

$\deg_{T_i}(v) - 1$

leaf v of T_i:

in $\{u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1}\}$

not in $\{v_i, v_{i+1}, \ldots, v_{n-2}\}$

u_i : ~~2, 4~~, 5, 6, 3, 1

v_i : ~~4, 3~~, 1, 3, 1, 7

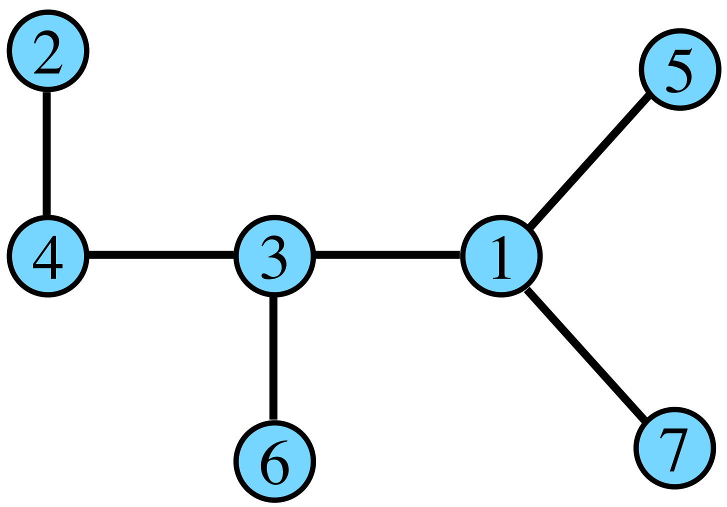
$(v_1, v_2, \ldots, v_{n-2})$

u_i : smallest leaf in T_i

u_i is the smallest number *not* in

$$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$$

T :



u_i : 2, 4, 5, 6, 3, 1

v_i : 4, 3, 1, 3, 1, 7

$(v_1, v_2, \ldots, v_{n-2})$

T = empty graph;

$v_{n-1} = n$;

for $i = 1$ to n-1

u_i : smallest number not in $\{u_1,\ldots,u_{i-1}\} \cup \{v_i,\ldots,v_{n-1}\}$

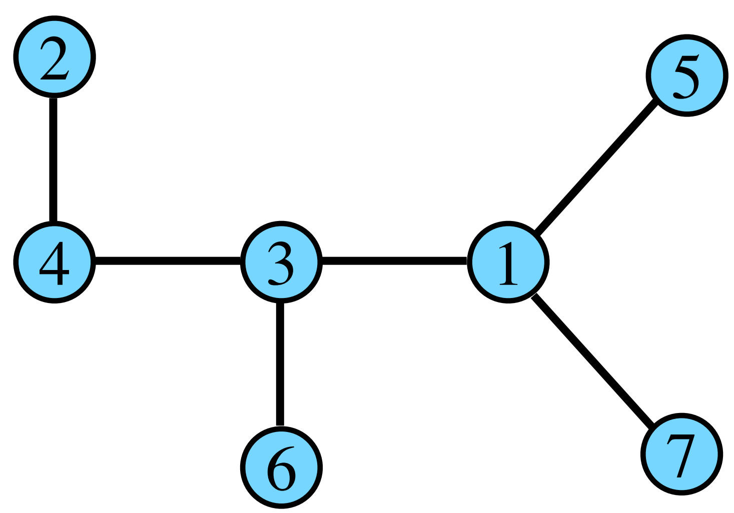
add edge (u_i, v_i) to T ;

Prüfer code is reversible → 1-1

every $(v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2}$

is decodable to a tree → onto

T :



u_i : 2 , 4 , 5 , 6 , 3 , 1

v_i : 4 , 3 , 1 , 3 , 1 , 7

$(v_1, v_2, \ldots, v_{n-2})$

T = empty graph;

$v_{n-1} = n$;

for $i = 1$ to $n-1$

u_i : smallest number not in $\{u_1,\ldots,u_{i-1}\} \cup \{v_i,\ldots,v_{n-1}\}$

add edge (u_i, v_i) to T ;

Prüfer code is reversible ⇨ 1-1

every $(v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2}$

is decodable to a tree ⇨ onto

Cayley's formula:

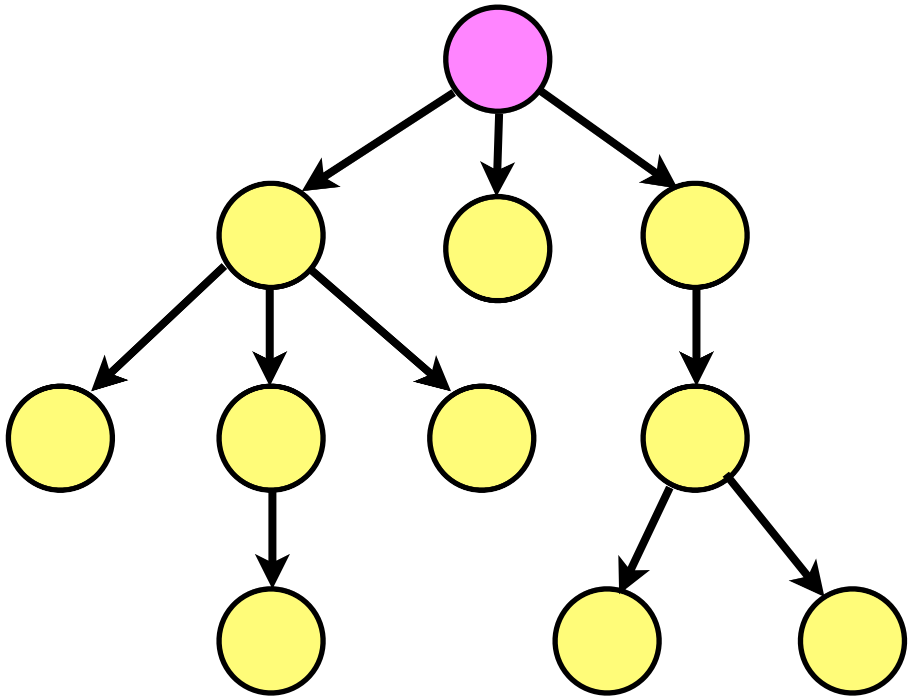
There are n^{n-2} trees on n distinct vertices.

Cayley Formula:

Double Counting

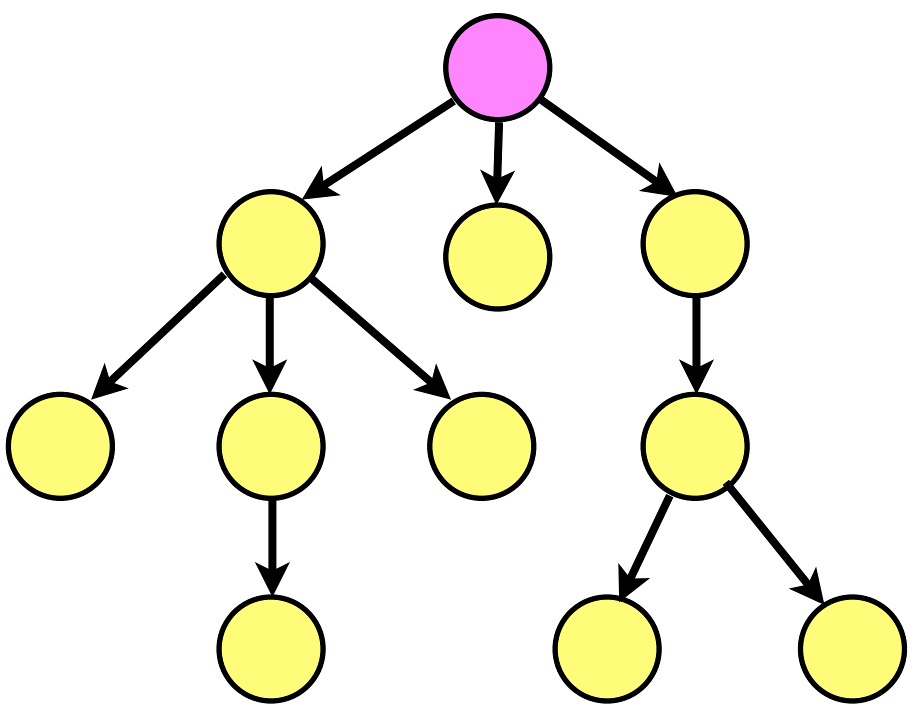
Double Counting

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*



T_n : # of trees on *n* distinct vertices.

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*



From a tree:

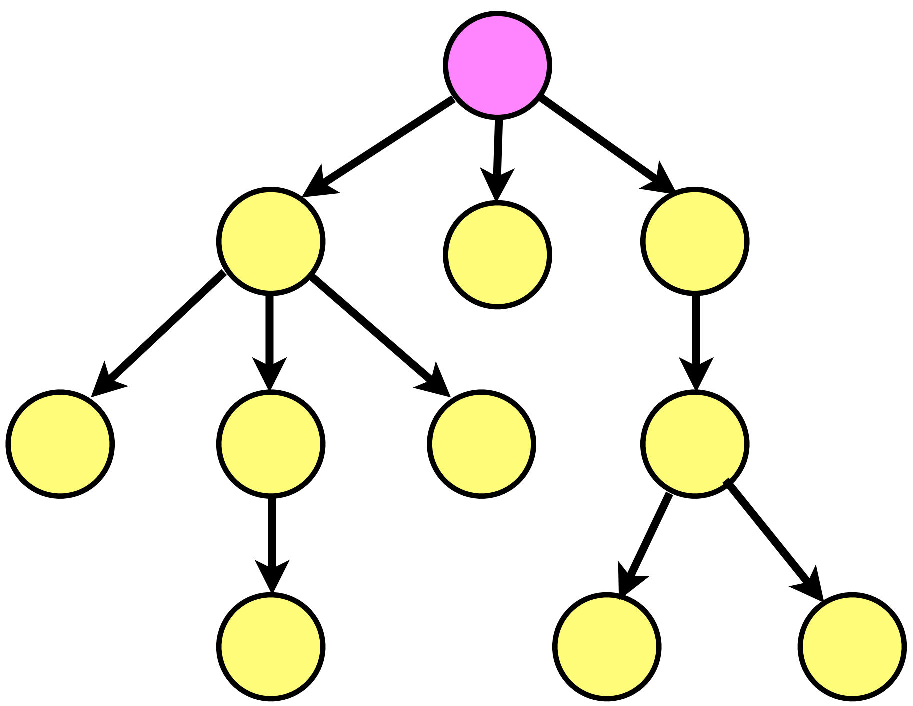
- pick a root;
- pick an order of edges.

$$T_n n(n-1)!$$

$$= n! T_n$$

T_n : # of trees on *n* distinct vertices.

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*



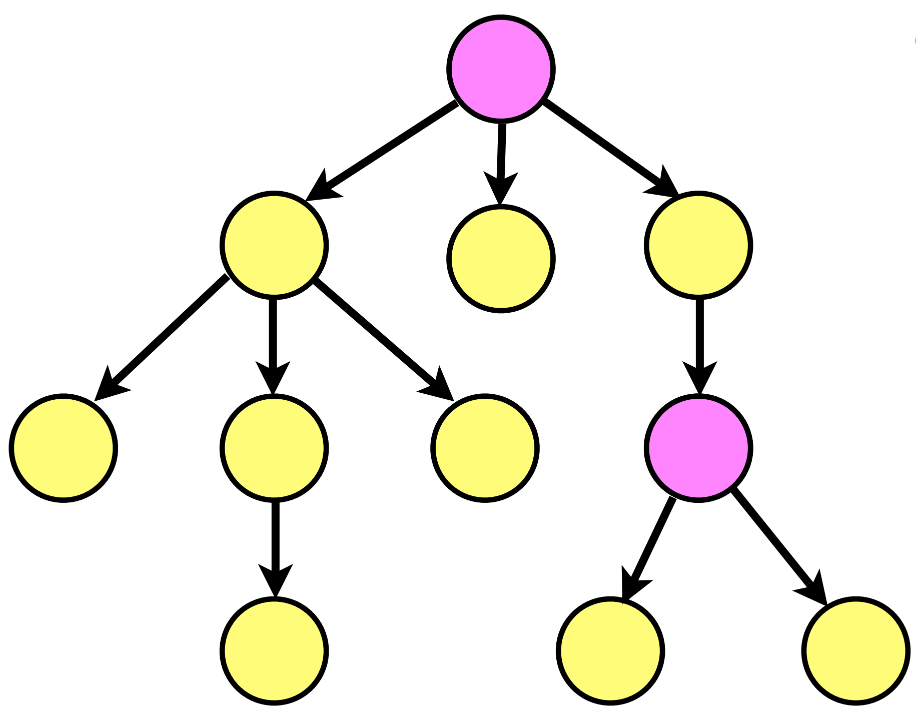
From an empty graph:

- add edges one by one

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

From an empty graph:

- add edges one by one



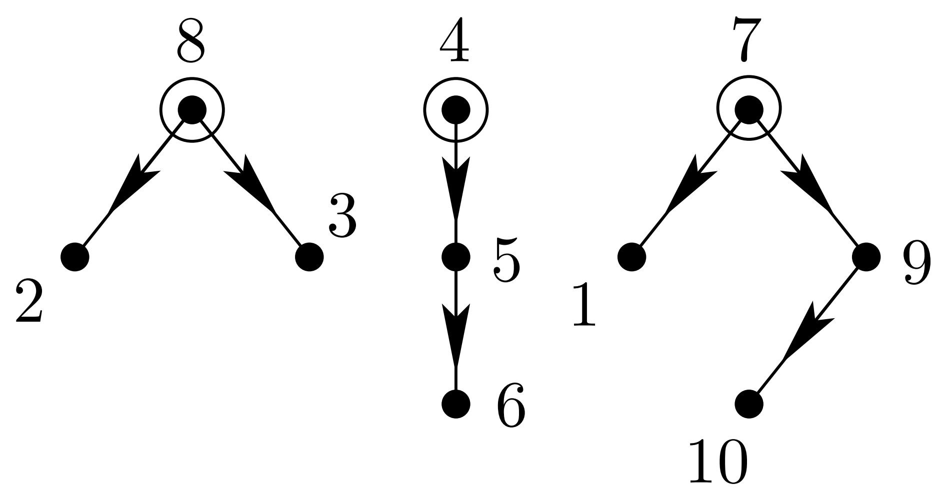
Start from n ~~isolated vertices~~ rooted trees

Each step joins 2 trees.

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

From an empty graph:

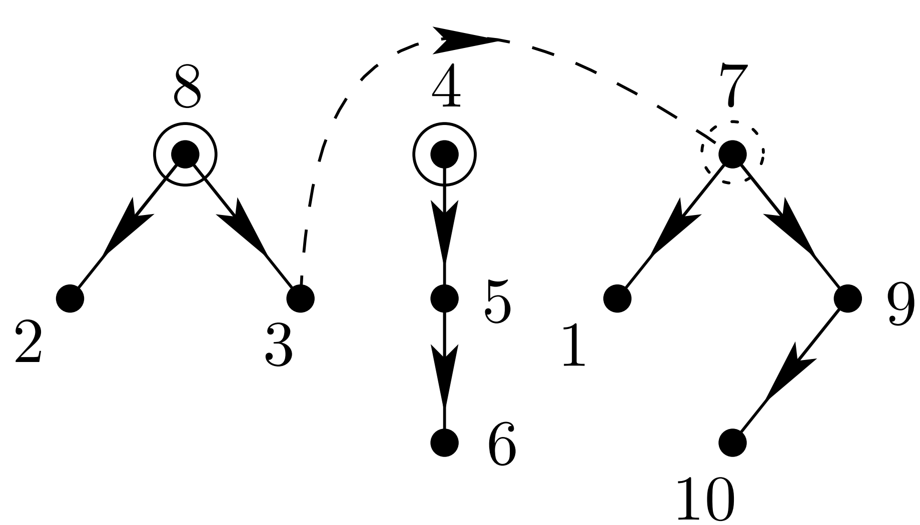
- add edges one by one



Start from n rooted trees.

After adding k edges

n-k rooted trees



add an edge

any vertex → root of another tree

n $\qquad$ $n-k-1$

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

From an empty graph:

- add edges one by one

$$\prod_{k=0}^{n-2} n(n-k-1)$$

$$= n^{n-1}\prod_{k=1}^{n-1} k$$

$$= n^{n-2}n!$$

Start from *n* rooted trees.

After adding *k* edges

n-*k* rooted trees

add an edge

any vertex → root of another tree

n — *n*-*k*-1

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

From a tree:

- pick a root;
- pick an order of edges.

$$T_n n(n-1)!$$
$$= n!T_n$$

=

From an empty graph:

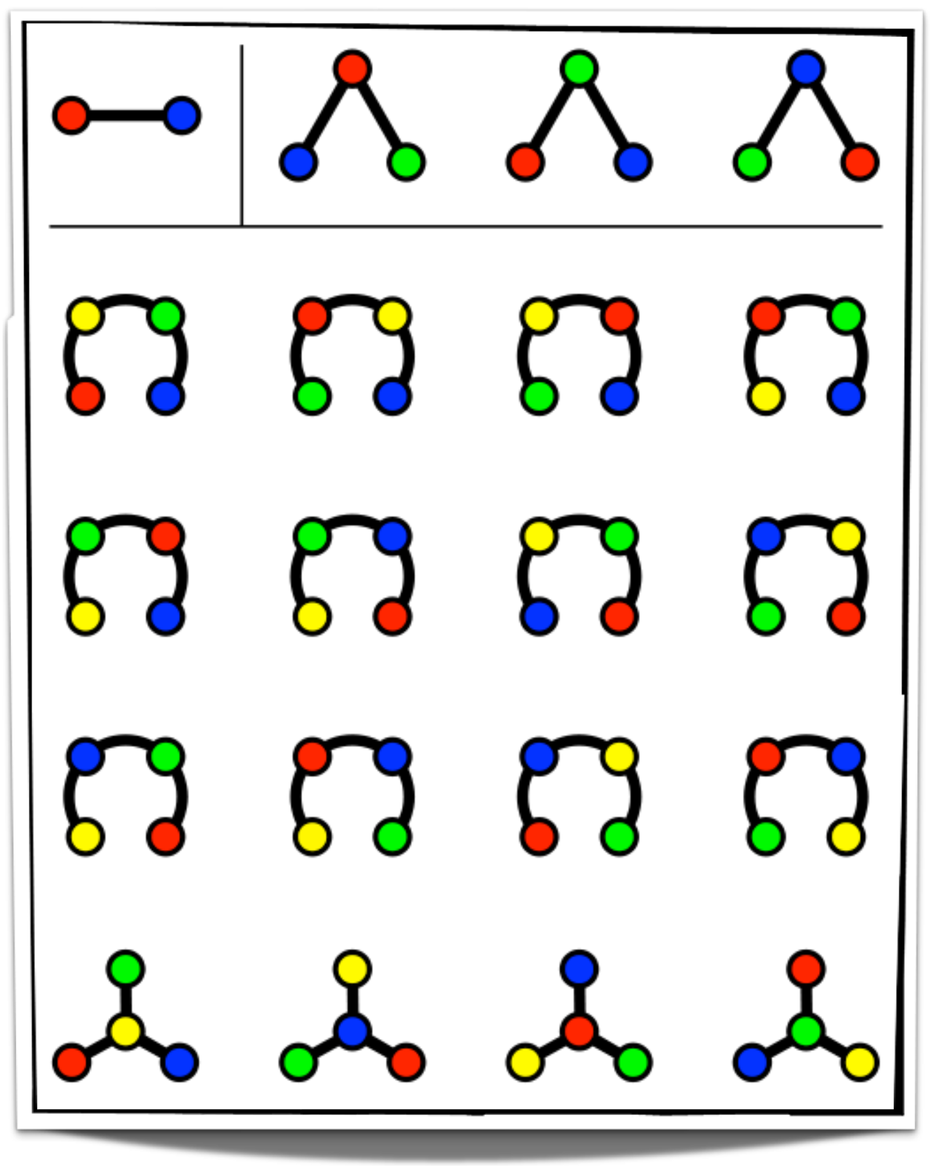
- add edges one by one

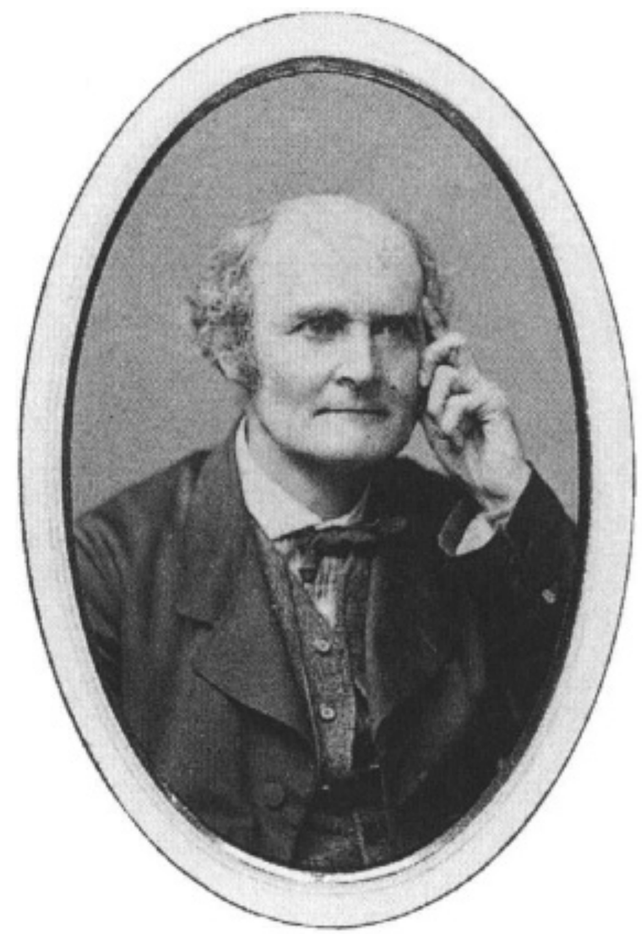
$$\prod_{k=0}^{n-2} n(n-k-1)$$
$$= n^{n-2} n!$$

$$T_n = n^{n-2}$$

Cayley's formula:

There are n^{n-2} trees on n distinct vertices.



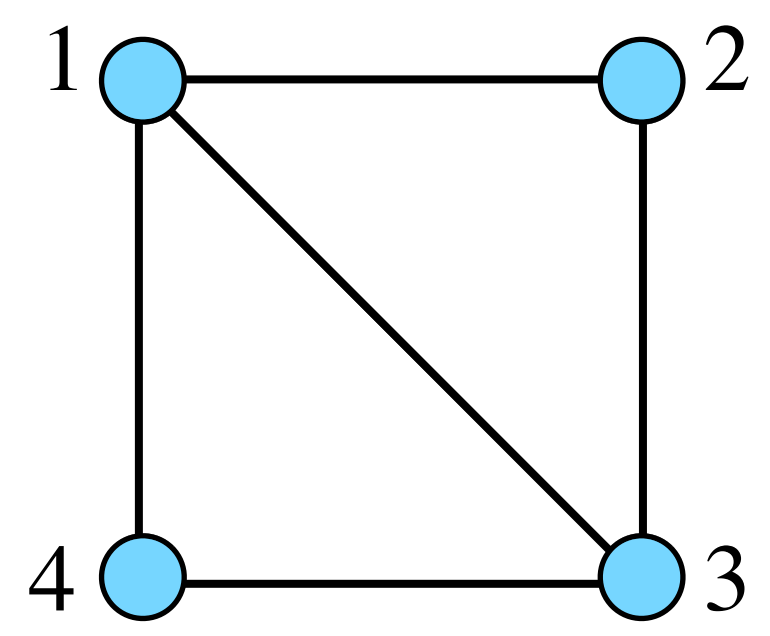


Arthur Cayley

Cayley Formula:
Matrix-Tree Theorem

Graph *Laplacian*

Graph $G(V,E)$



adjacency matrix A

$$A(i,j) = \begin{cases} 1 & \{i,j\} \in E \\ 0 & \{i,j\} \notin E \end{cases}$$

diagonal matrix D

$$D(i,j) = \begin{cases} \deg(i) & i = j \\ 0 & i \neq j \end{cases}$$

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & d_n \end{bmatrix}$$

graph *Laplacian* L

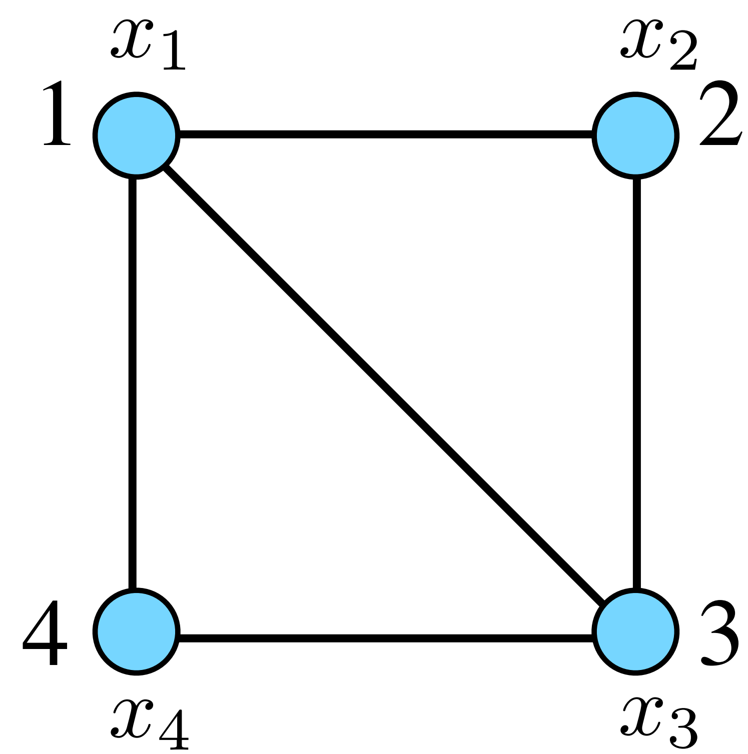
$$L = D - A$$

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Graph *Laplacian*

graph *Laplacian* L

$$L(i,j) = \begin{cases} \deg(i) & i = j \\ -1 & i \neq j, \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$



quadratic form:

$$xLx^T = \sum_i d_i x_i^2 - \sum_{ij \in E} x_i x_j = \frac{1}{2} \sum_{ij \in E} (x_i - x_j)^2$$

incidence matrix B : $n \times m$

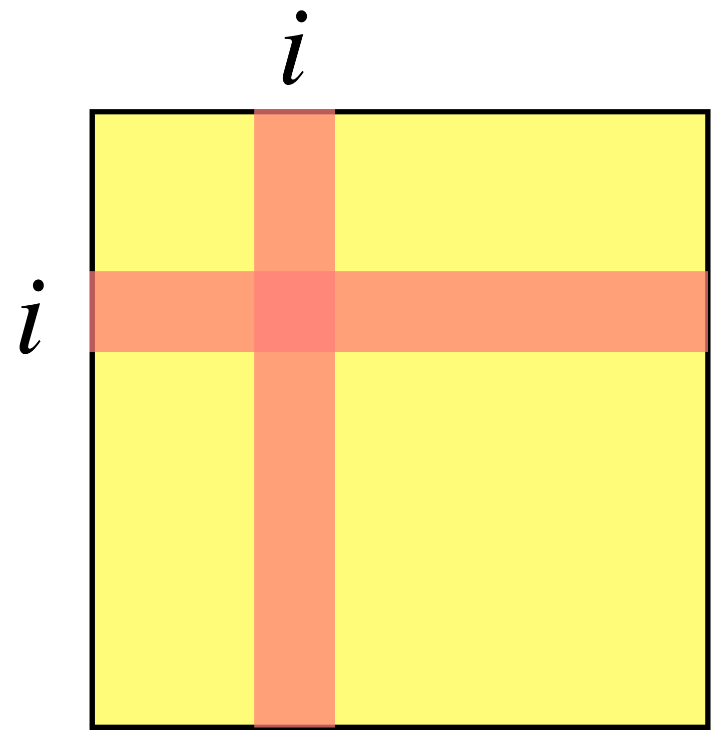
$i \in V, e \in E$

$$B(i,e) = \begin{cases} 1 & e = \{i,j\}, i < j \\ -1 & e = \{i,j\}, i > j \\ 0 & \text{otherwise} \end{cases}$$

$$L = BB^T$$

Kirchhoff's matrix-tree theorem

$L_{i,i}$: submatrix of L obtained by removing the ith row and ith column



$t(G)$: number of spanning trees in G

Kirchhoff's matrix-tree theorem

$L_{i,i}$: submatrix of L obtained by removing the ith row and ith column

$t(G)$: number of spanning trees in G

Kirchhoff's Matrix-Tree Theorem:

$$\forall i, \quad t(G) = \det(L_{i,i})$$

Kirchhoff's Matrix-Tree Theorem:

$$\forall i, \quad t(G) = \det(L_{i,i})$$

$B_i: \ (n-1) \times m$

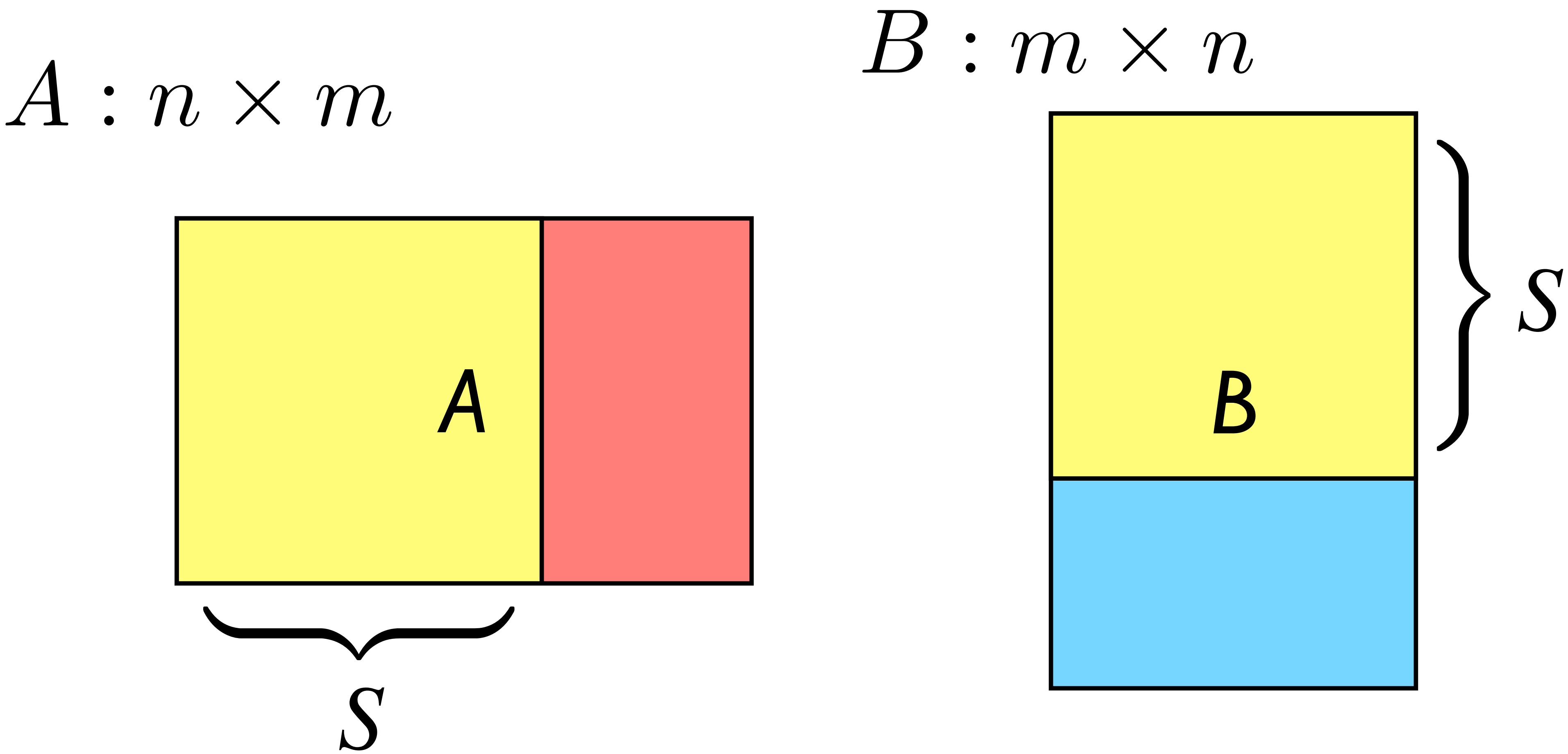
incidence matrix B removing ith row

$$L = BB^T$$

$$L_{i,i} = B_i B_i^T \qquad \det(L_{i,i}) = \det(B_i B_i^T) = ?$$

Cauchy-Binet Theorem:

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],S}) \det(B_{S,[n]})$$



Cauchy-Binet Theorem:

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],S}) \det(B_{S,[n]})$$

$$\det(L_{i,i}) = \det(B_i B_i^T)$$

$$= \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S}) \det(B^T_{S,[n]\setminus\{i\}})$$

$$= \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S})^2$$

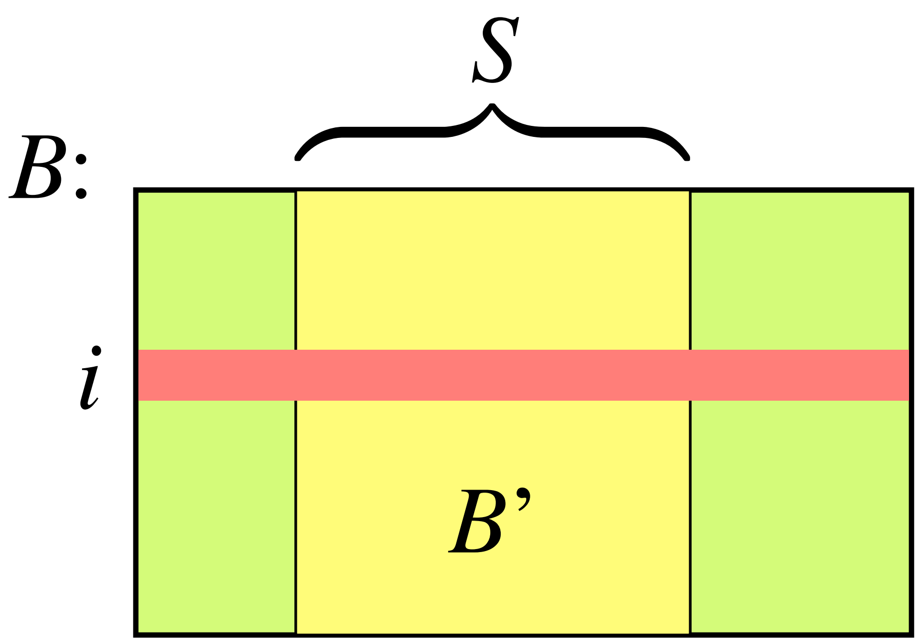
$$\det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S})^2$$

$j \in [n] \setminus \{i\}, e \in S$

$$B_{[n]\setminus\{i\},S}(j,e) = \begin{cases} 1 & e = \{j,k\}, j < k \\ -1 & e = \{j,k\}, j > k \\ 0 & \text{otherwise} \end{cases}$$

$$\det(B_{[n]\setminus\{i\},S}) = \begin{cases} \pm 1 & S \text{ is a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}$$

$$\det(B_{[n]\setminus\{i\},S}) = \begin{cases} \pm 1 & S \text{ is a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}$$



$$B' = B_{[n]\setminus\{i\},S}$$

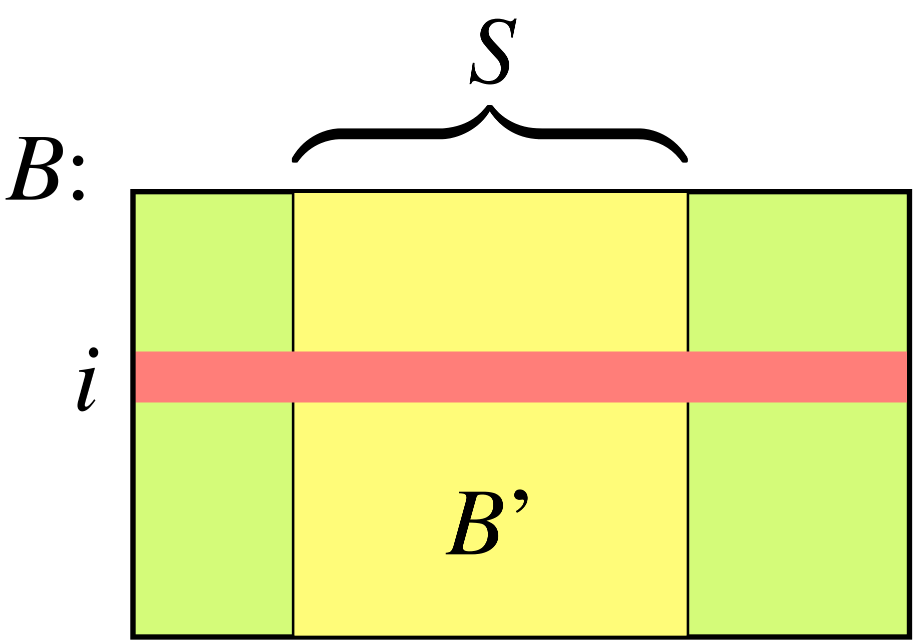
$(n-1) \times (n-1)$ matrix:

every column contains
at most one 1 and at most one -1
and all other entries are 0

$$\Rightarrow \det(B') \in \{-1, 0, 1\}$$

$$\det(B') \neq 0 \text{ iff } S \text{ is a spanning tree}$$

$\det(B') \neq 0$ iff S is a spanning tree



S is not a spanning tree:

$\exists$ a connected component R

s.t. $i \notin R$

$\Rightarrow \det(B') = 0$

S is a spanning tree:

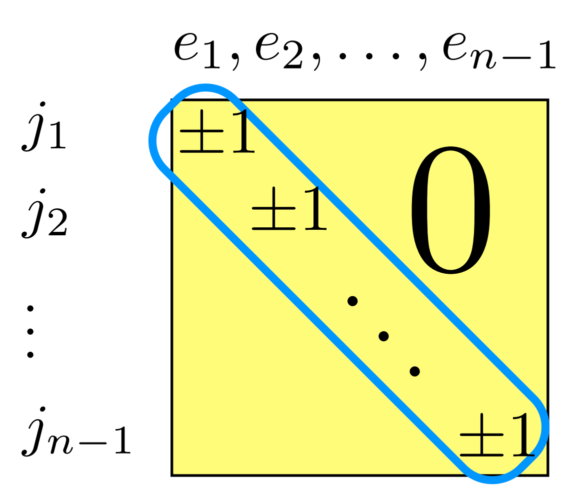
$\exists$ a leaf $j_1 \neq i$ with incident edge e_1, delete e_1

$\exists$ a leaf $j_2 \neq i$ with incident edge e_2, delete e_2

$\vdots$

vertices: $j_1, j_2, \ldots, j_{n-1}$

edges: $e_1, e_2, \ldots, e_{n-1}$



$\Rightarrow \det(B') = \pm 1$

Cauchy-Binet

$$\det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S})^2$$

$j \in [n] \setminus \{i\}, e \in S$

$$B_{[n]\setminus\{i\},S}(j,e) = \begin{cases} 1 & e = \{j,k\}, j < k \\ -1 & e = \{j,k\}, j > k \\ 0 & \text{otherwise} \end{cases}$$

$$\det(B_{[n]\setminus\{i\},S}) = \begin{cases} \pm 1 & S \text{ is a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}$$

Kirchhoff's Matrix-Tree Theorem:

$$\forall i, \quad t(G) = \det(L_{i,i})$$

all n-vertex trees: spanning trees of K_n

$$L_{i,i} = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$

Cayley formula:

$$T_n = t(K_n) = \det(L_{i,i}) = n^{n-2}$$