Combinatorics

Generating Function

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Compositions by 1 and 2

of compositions of n with summands from {1,2}

of $(x_1, x_2, \ldots, x_k)$

for some $k \leq n$

$x_1 + \cdots + x_k = n$

$x_i \in \{1, 2\}$

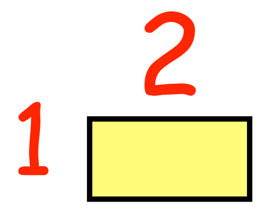
$$a_n = a_{n-1} + a_{n-2} \qquad a_0 = 1 \quad a_1 = 1$$

Case.1 $x_k = 1$ $\quad x_1 + \cdots + x_{k-1} = n - 1$

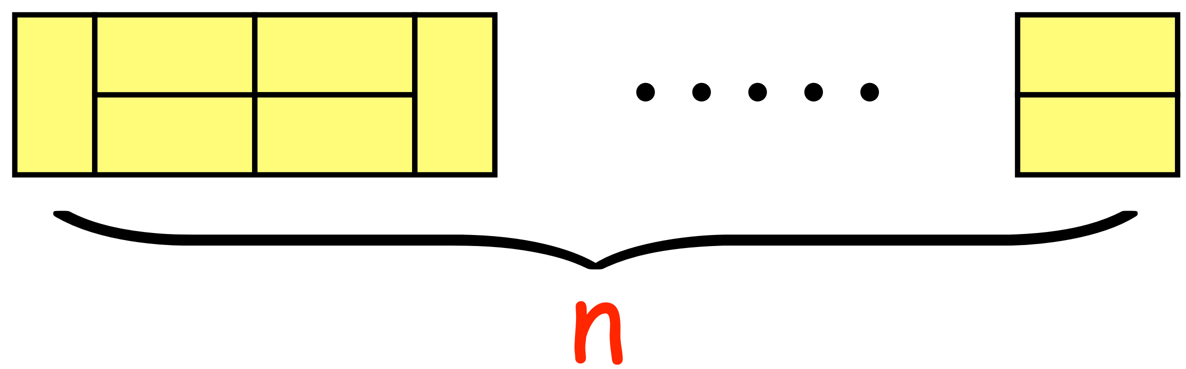
Case.2 $x_k = 2$ $\quad x_1 + \cdots + x_{k-1} = n - 2$

Dominos

domino:

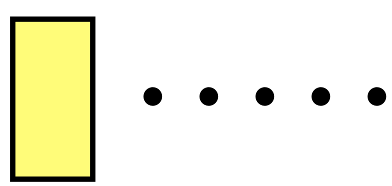


# of tilings 2



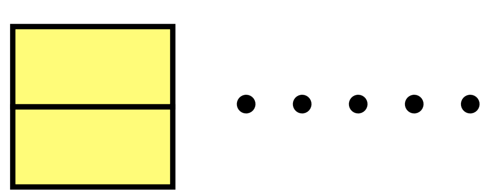
$$a_n = a_{n-1} + a_{n-2} \qquad a_0 = 1 \quad a_1 = 1$$

Case.1



$2\times(n-1)$

Case.2



$2\times(n-2)$

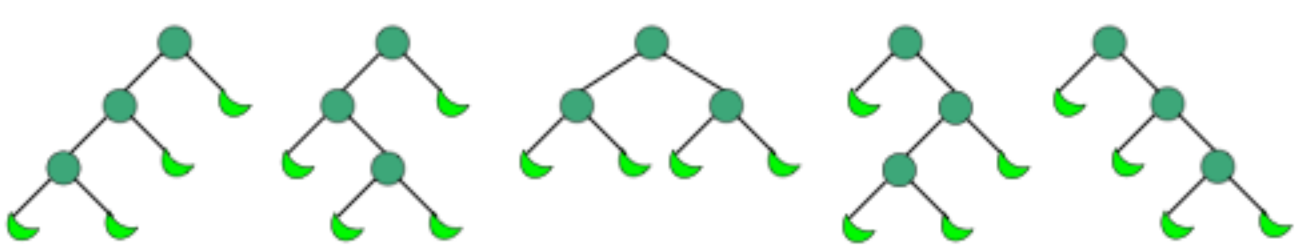
Fibonacci number

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

full parenthesization of $n+1$ factors

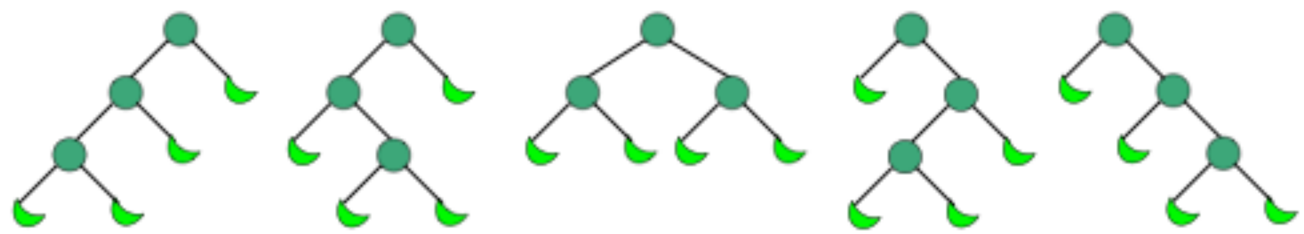
$$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$$

full binary trees with $n+1$ leaves



Catalan Number

C_n : # of full binary trees with $n+1$ leaves



Recursion:

C_k $\quad C_{n-1-k}$

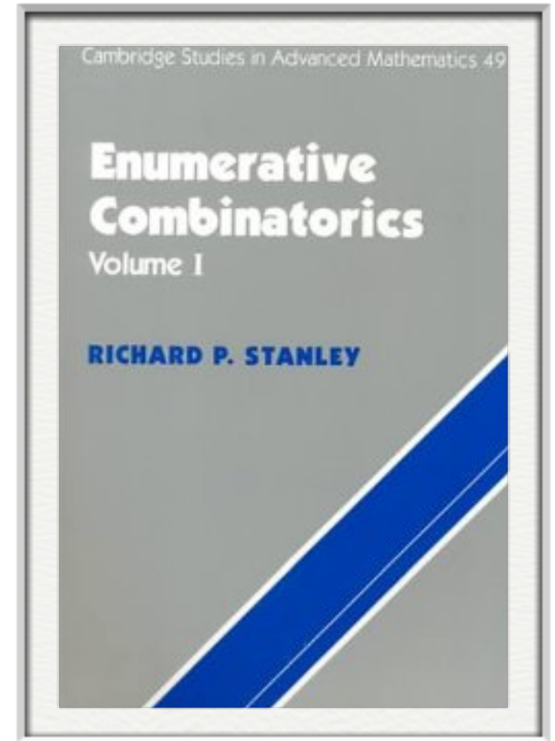
$$C_0 = 1 \quad \text{for } n \geq 1,$$

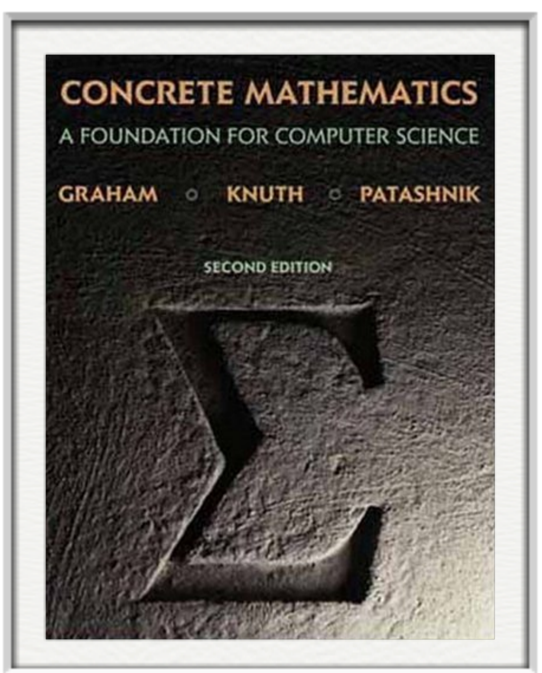
$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Generating Function

Generating Function

4. The most useful but most difficult to understand method for evaluating $f(i)$ is to give its *generating function*. We will not develop in this chapter a rigorous abstract theory of generating functions, but will instead content ourselves with an informal discussion and some examples. Informally, a generating function is



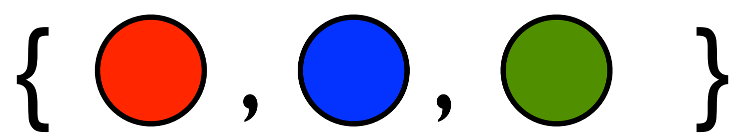


Generating Functions

THE MOST POWERFUL WAY to deal with sequences of numbers, as far as anybody knows, is to manipulate infinite series that "generate" those sequences. We've learned a lot of sequences and we've seen a few generating functions; now we're ready to explore generating functions in depth, and to see how remarkably useful they are.

generate

~~enumerate~~ all subsets of



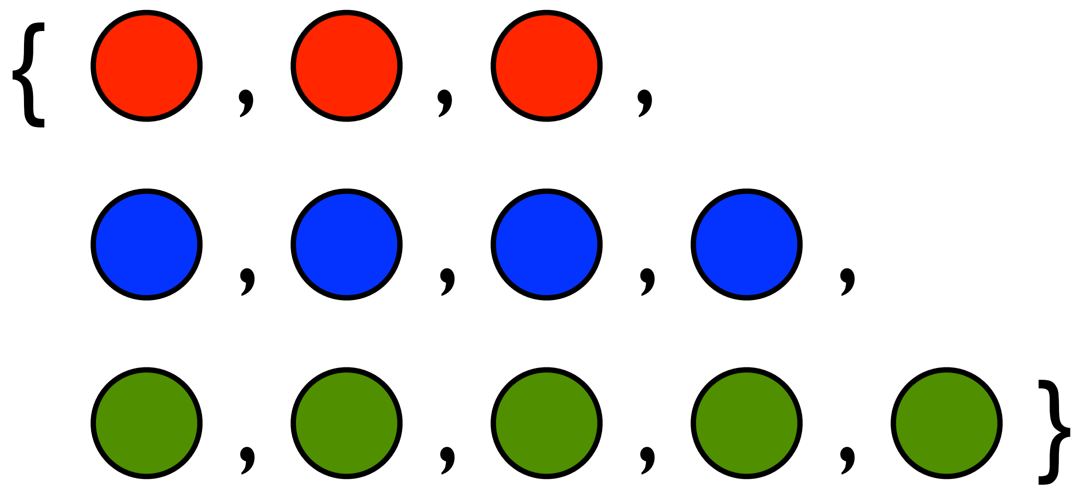
$$\color{red}{(x^0 + x^1)}\color{blue}{(x^0 + x^1)}\color{green}{(x^0 + x^1)}$$

$$= \color{red}{x^0}\color{blue}{x^0}\color{green}{x^0} + \color{red}{x^0}\color{blue}{x^0}\color{green}{x^1} + \color{red}{x^0}\color{blue}{x^1}\color{green}{x^0} + \color{red}{x^0}\color{blue}{x^1}\color{green}{x^1}$$

$$+ \color{red}{x^1}\color{blue}{x^0}\color{green}{x^0} + \color{red}{x^1}\color{blue}{x^0}\color{green}{x^1} + \color{red}{x^1}\color{blue}{x^1}\color{green}{x^0} + \color{red}{x^1}\color{blue}{x^1}\color{green}{x^1}$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

coefficient of x^k : # of k-subsets



$$(1 + x + x^2 + x^3)$$

$$(1 + x + x^2 + x^3 + x^4)$$

$$(1 + x + x^2 + x^3 + x^4 + x^5)$$

$$\begin{aligned} = \quad & 1 + 3x + 6x^2 + 10x^3 + 14x^4 + 17x^5 + 18x^6 \\ & +17x^7 + 14x^8 + 10x^9 + 6x^{10} + 3x^{11} + x^{12} \end{aligned}$$

Double Decks

choose m cards from

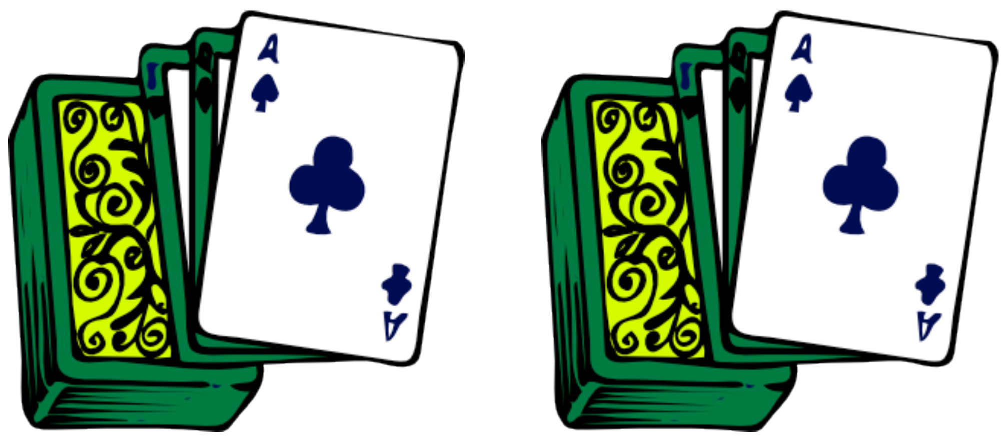
2 decks of n cards

$$(x_1^0 + x_1^1 + x_1^2)\,(x_2^0 + x_2^1 + x_2^2)\cdots(x_n^0 + x_n^1 + x_n^2)$$

\# of m-order terms

coefficient of x^m in $(1 + x + x^2)^n$

Double Decks



choose m cards from

2 decks of n cards

$$(1+(x+x^2))^n = \sum_k \binom{n}{k} (x+x^2)^k$$

$$= \sum_k \binom{n}{k} x^k \sum_{\ell \leq k} \binom{k}{\ell} x^\ell = \sum_k \sum_{\ell \leq k} \binom{n}{k} \binom{k}{\ell} x^{k+\ell}$$

$$= \sum_m \left(\sum_\ell \binom{n}{m-\ell} \binom{m-\ell}{\ell} \right) x^m$$

Multisets

multisets on $S = \{x_1, x_2, \ldots, x_n\}$

$$(1 + x_1 + x_1^2 + \cdots)(1 + x_2 + x_2^2 + \cdots)\cdots(1 + x_n + x_n^2 + \cdots)$$

$$= \sum_{m:S\to\mathbb{N}} \prod_{x_i\in S} x_i^{m(x_i)}$$

$$(1 + x + x^2 + \cdots)^n = \sum_{m\in\mathbb{N}^n} x^{m_1+\cdots+m_n} = \sum_{k\geq 0} \left(\!\!\binom{n}{k}\!\!\right) x^k$$

|| geometric

$$(1-x)^{-n} = \sum_{k\geq 0} \frac{(-n)(-n-1)\cdots(-n-k+1)(-1)^k}{k!} x^k$$

Taylor

Multisets

multisets on $S = \{x_1, x_2, \ldots, x_n\}$

$$(1 + x_1 + x_1^2 + \cdots)(1 + x_2 + x_2^2 + \cdots)\cdots(1 + x_n + x_n^2 + \cdots)$$

$$= \sum_{m:S\to\mathbb{N}} \prod_{x_i \in S} x_i^{m(x_i)}$$

$$(1 + x + x^2 + \cdots)^n = \sum_{m\in\mathbb{N}^n} x^{m_1+\cdots+m_n} = \sum_{k\geq 0} \left(\!\!\binom{n}{k}\!\!\right) x^k$$

$$\parallel$$

$$(1 - x)^{-n} = \sum_{k\geq 0} \binom{n+k-1}{k} x^k$$

$$\boxed{\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}}$$

Ordinary Generating Function (OGF)

$\{a_n\}$ $\qquad a_0, a_1, a_2, \ldots$

generating function:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$[x^n]G(x) = a_n$$

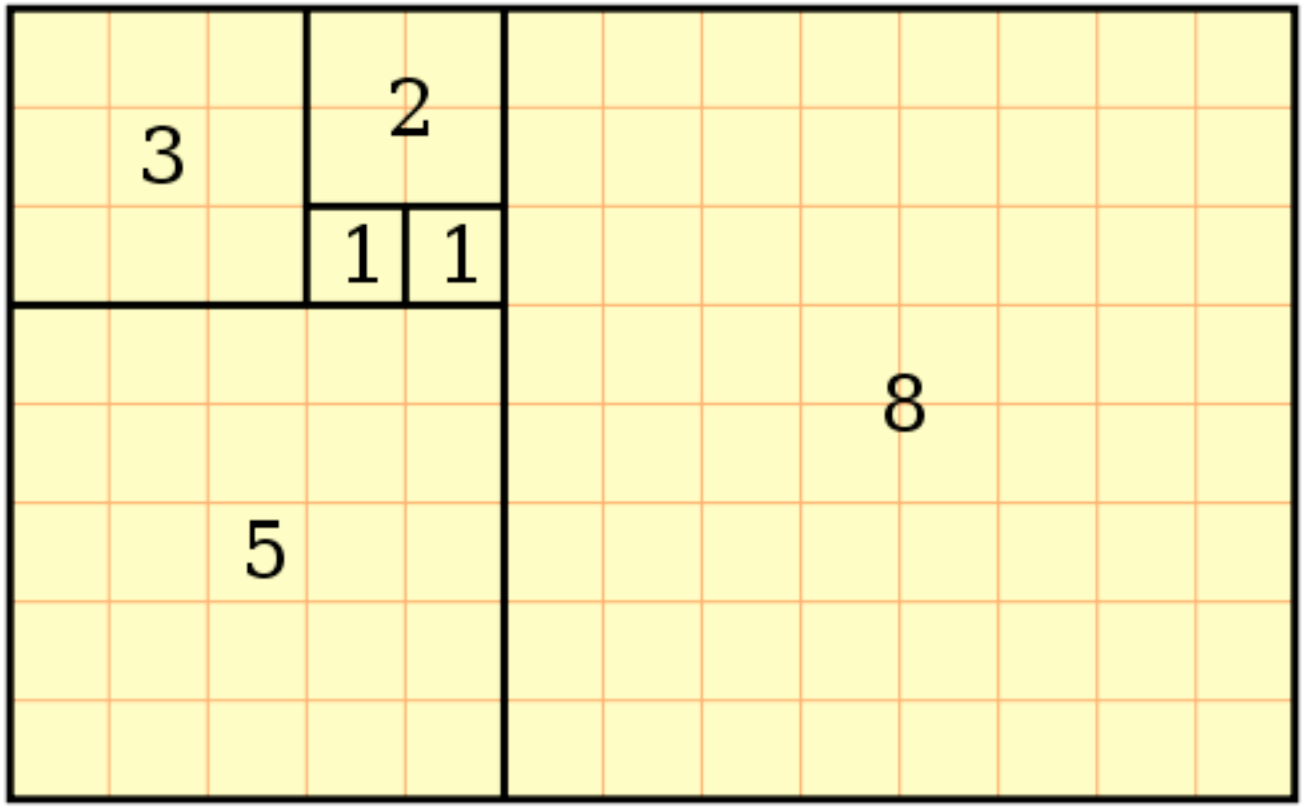
Fibonacci number

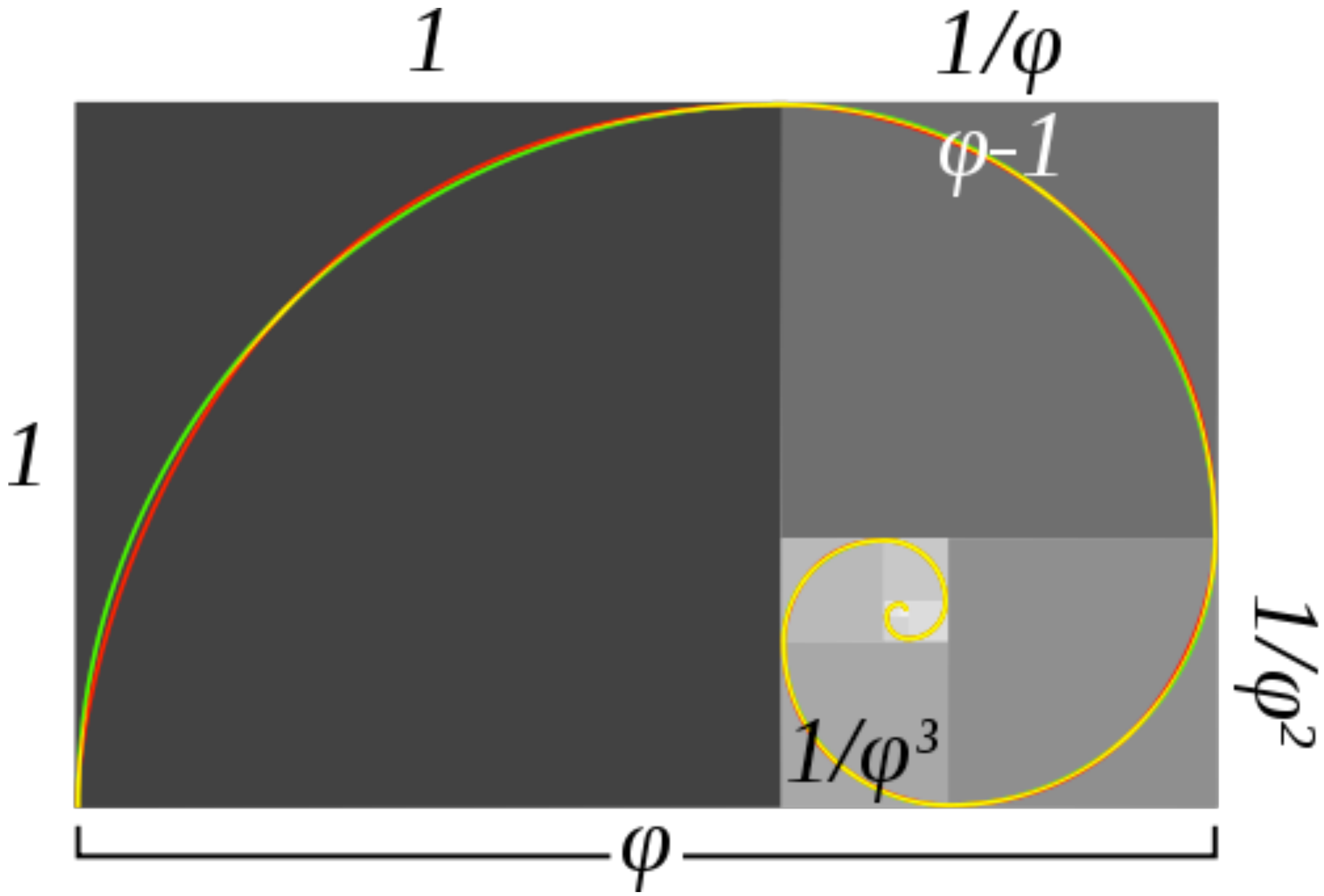
$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

Fibonacci number

$$\phi = \frac{1 + \sqrt{5}}{2} \qquad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right)$$


3
2
1
1
8
5


1
1/φ
φ-1
1
1/φ²
1/φ³
φ

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

generating function:

$$G(x) = \sum_{n\geq 0} F_n x^n$$

recursion:

$$G(x) = F_0 + F_1 x + \sum_{n\geq 2} F_n x^n = x + \sum_{\boldsymbol{n\geq 2}} (F_{n-1} x^n + F_{n-2} x^n)$$

$$x + \sum_{\boldsymbol{n\geq 2}} F_{n-1} x^n + \sum_{n\geq 2} F_{n-2} x^n$$

$$\sum_{n\geq 2} F_{n-1} x^n = \sum_{n\geq 1} F_{n-1} x^n = \sum_{n\geq 0} F_n x^{n+1} = xG(x)$$

$$\sum_{n\geq 2} F_{n-2} x^n = \sum_{n\geq 0} F_n x^{n+2} = x^2 G(x)$$

identity: $G(x) = x + (x + x^2) G(x)$

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

generating function:

$$G(x) = \sum_{n \geq 0} F_n x^n$$

identity: $G(x) = x + (x + x^2)G(x)$

solution: $G(x) = \dfrac{x}{1 - x - x^2}$ =? Taylor ?

denote $\phi = \dfrac{1+\sqrt{5}}{2}$ $\hat{\phi} = \dfrac{1-\sqrt{5}}{2}$

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\hat{\phi} x}$$

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

generating function:

$$G(x) = \sum_{n \geq 0} F_n x^n$$

identity: $G(x) = x + (x + x^2) G(x)$

solution:

$$G(x) = \frac{x}{1 - x - x^2}$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x}$$

$$= \frac{1}{\sqrt{5}} \sum_{n \geq 0} (\phi x)^n - \frac{1}{\sqrt{5}} \sum_{n \geq 0} (\hat{\phi} x)^n$$

$$= \sum_{n \geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) x^n$$

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

$$\phi = \frac{1 + \sqrt{5}}{2} \qquad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right)$$

Ordinary Generating Function (OGF)

$\{a_n\}$ $\qquad$ $a_0, a_1, a_2, \ldots$

generating function:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\frac{1}{1-\alpha x} = \sum_{n=0}^{\infty} \alpha^n x^n$$

Formal Power Series

formal power series: $$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$\mathbb{C}[[x]]$: ring of formal power series with complex coefficient

inverse

$$F(x)G(x) = 1 \quad \Rightarrow \quad F(x) = G(x)^{-1} = \frac{1}{G(x)}$$

$$(1 - \alpha x)\left(\sum_{n=0}^{\infty} \alpha^n x^n\right) = 1 \quad \Rightarrow \quad \frac{1}{1 - \alpha x} = \sum_{n=0}^{\infty} \alpha^n x^n$$

Generatingfunctionology

“Generatingfunctionology”

1. Recurrence:

$$a_0 = 0 \qquad a_1 = 1 \qquad a_n = a_{n-1} + a_{n-2}$$

2. Manipulation:

$$\begin{aligned} G(x) &= \sum_{n\geq 0} a_n x^n = x + \sum_{n\geq 1} a_{n-1} x^n + \sum_{n\geq 2} a_{n-2} x^n \\ &= x + (x + x^2) G(x) \end{aligned}$$

3. Solving:

$$G(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x}$$

$$= \sum_{n\geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) x^n$$

Generating Function Algebra

$$G(x) = \sum_{n\geq 0} g_n x^n \qquad F(x) = \sum_{n\geq 0} f_n x^n$$

shift:

$$x^k G(x) = \sum_{n\geq k} g_{n-k} x^n$$

addition:

$$F(x) + G(x) = \sum_{n\geq 0} (f_n + g_n) x^n$$

convolution:

$$F(x)G(x) = \sum_{n\geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n$$

differentiation:

$$G'(x) = \sum_{n\geq 0} (n+1) g_{n+1} x^n$$

“Generatingfunctionology”

1. Recurrence:

$$a_0 = 0 \qquad a_1 = 1 \qquad a_n = a_{n-1} + a_{n-2}$$

2. Manipulation:

$$\begin{aligned} G(x) &= \sum_{n \geq 0} a_n x^n = x + \sum_{n \geq 1} a_{n-1} x^n + \sum_{n \geq 2} a_{n-2} x^n \\ &= x + (x + x^2) G(x) \end{aligned}$$

3. Solving:

expending!

$$G(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x}$$

$$= \sum_{n \geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) x^n$$

Generating Function Expansion

Taylor's expansion:

$$G(x) = \sum_{n \geq 0} \frac{G^{(n)}(0)}{n!} x^n$$

Geometric sequence:

$$\frac{a}{1 - bx} = a \sum_{n \geq 0} (bx)^n$$

$$G(x) = \frac{a_1}{1 - b_1 x} + \frac{a_2}{1 - b_2 x} + \cdots + \frac{a_k}{1 - b_k x}$$

$$[x^n]G(x) = a_1 b_1^n + a_2 b_2^n + \cdots + a_k b_k^n$$

Generating Function Expansion

Binomial theorem: Newton's formula

$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n$$

$$((1+x)^\alpha)^{(n)} = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$$

generalized binomial coefficient:

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$

Changing Money

$$(\text{壹}, \text{伍})_n = \sum_{k=0}^{n} \text{壹}_k \text{伍}_{n-k}$$

壹_n : # of ways to change n yuan using $\boxed{\text{壹圆}}$

$$\sum_{n \geq 0} \text{壹}_n x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

伍_n : # of ways to change n yuan using $\boxed{\text{伍圆}}$

$$\sum_{n \geq 0} \text{伍}_n x^n = 1 + x^5 + x^{10} + \cdots = \frac{1}{1-x^5}$$

Changing Money

$$(壹, 伍)_n = \sum_{k=0}^{n} 壹_k 伍_{n-k}$$

$$\sum_{n\geq 0} 壹_n x^n = \frac{1}{1-x} \qquad \sum_{n\geq 0} 伍_n x^n = \frac{1}{1-x^5}$$

$$\sum_{n\geq 0} (壹, 伍)_n x^n = \sum_{n\geq 0} \sum_{k=0}^{n} 壹_k 伍_{n-k} x^n$$

convolution!

$$= \frac{1}{(1-x)(1-x^5)}$$

Changing Money

$$\sum_{n\geq 0}(\text{壹}, \text{伍}, \text{拾}, \text{贰拾}, \text{伍拾}, \text{壹佰})_n x^n$$

$$= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})}$$

$$\sum_{n\geq 0}(\text{壹, 伍, 拾, 贰拾, 伍拾, 壹佰})_n x^n$$

$$= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})}$$

$$= (1+x+x^2+x^3+\cdots+x^{99})\cdot(1+x^5+x^{10}+x^{15}+\cdots+x^{95})$$

$$\cdot(1+x^{10}+x^{20}+x^{30}+\cdots+x^{90})\cdot(1+x^{20}+x^{40}+x^{60}+x^{80})$$

$$\cdot(1+x^{50}) \cdot \frac{1}{(1-x^{100})^6}$$

Newton's formula

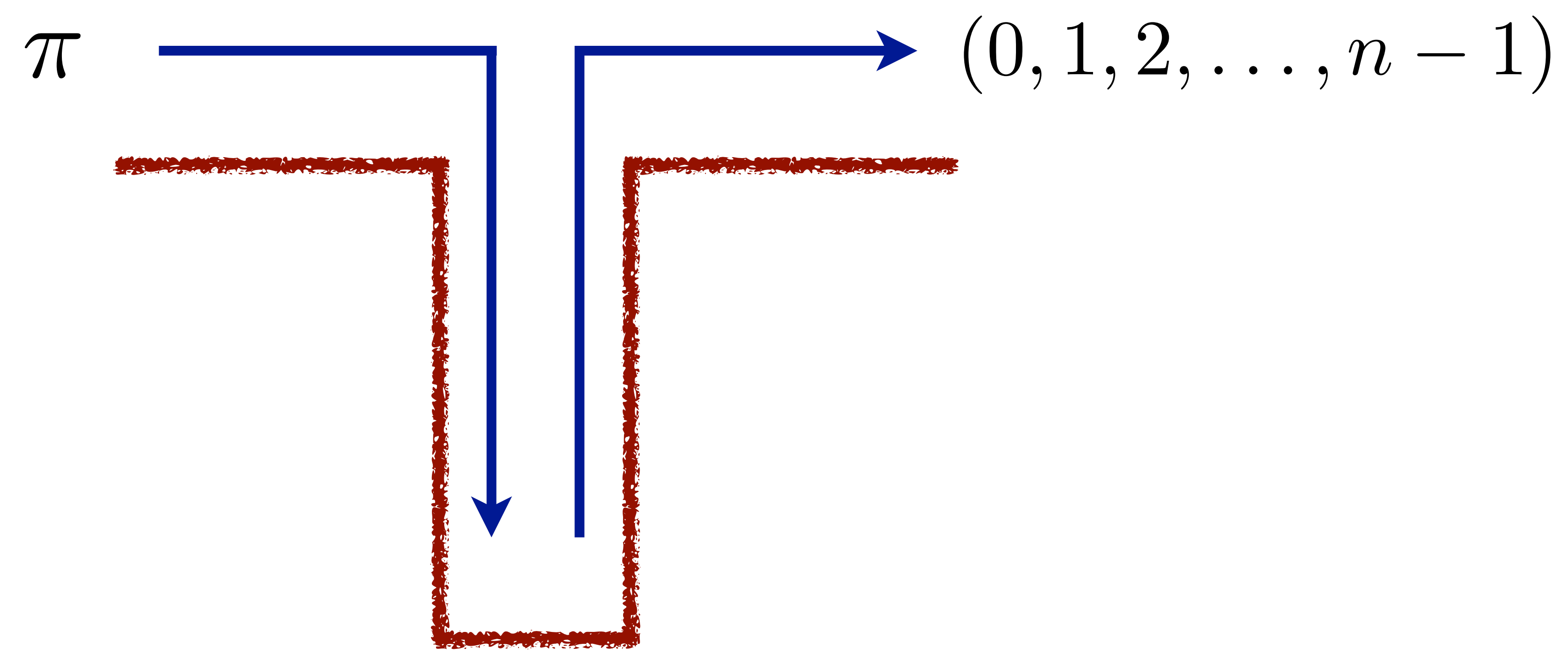
$$\frac{1}{(1-x^{100})^6} = \sum_{n\geq 0}\binom{-6}{n}\left(-x^{100}\right)^n$$

Catalan Number

n pairs of matching parenthesis

((())) ()(()) ()()() (())() (()())

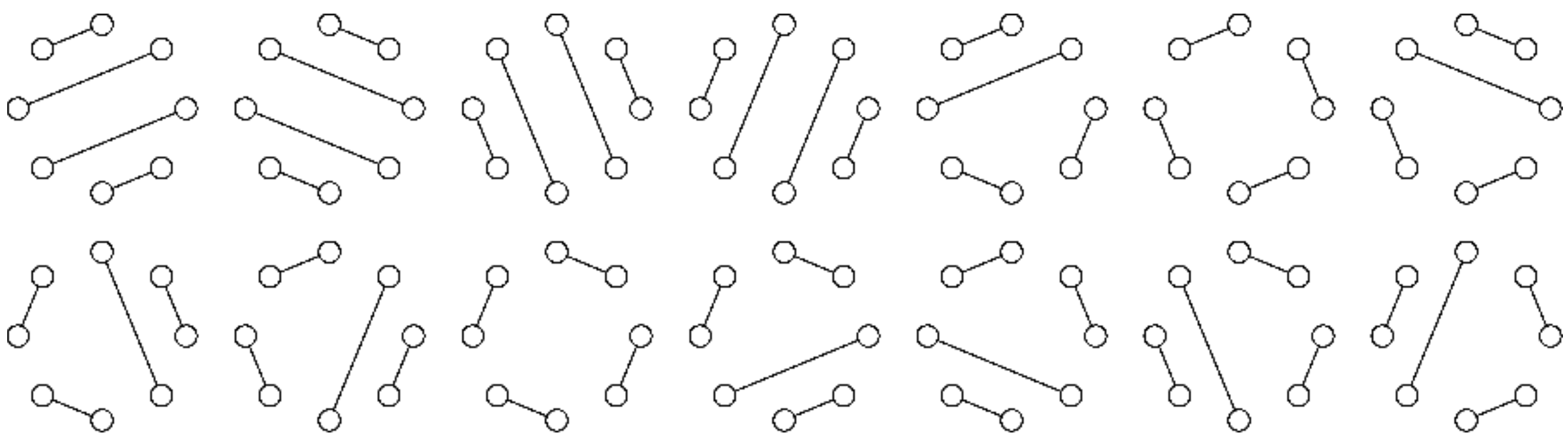
stack-sortable permutations of $[n]$



n pairs of matching parenthesis

((()))   ()(())   ()()()   (())()   (()())

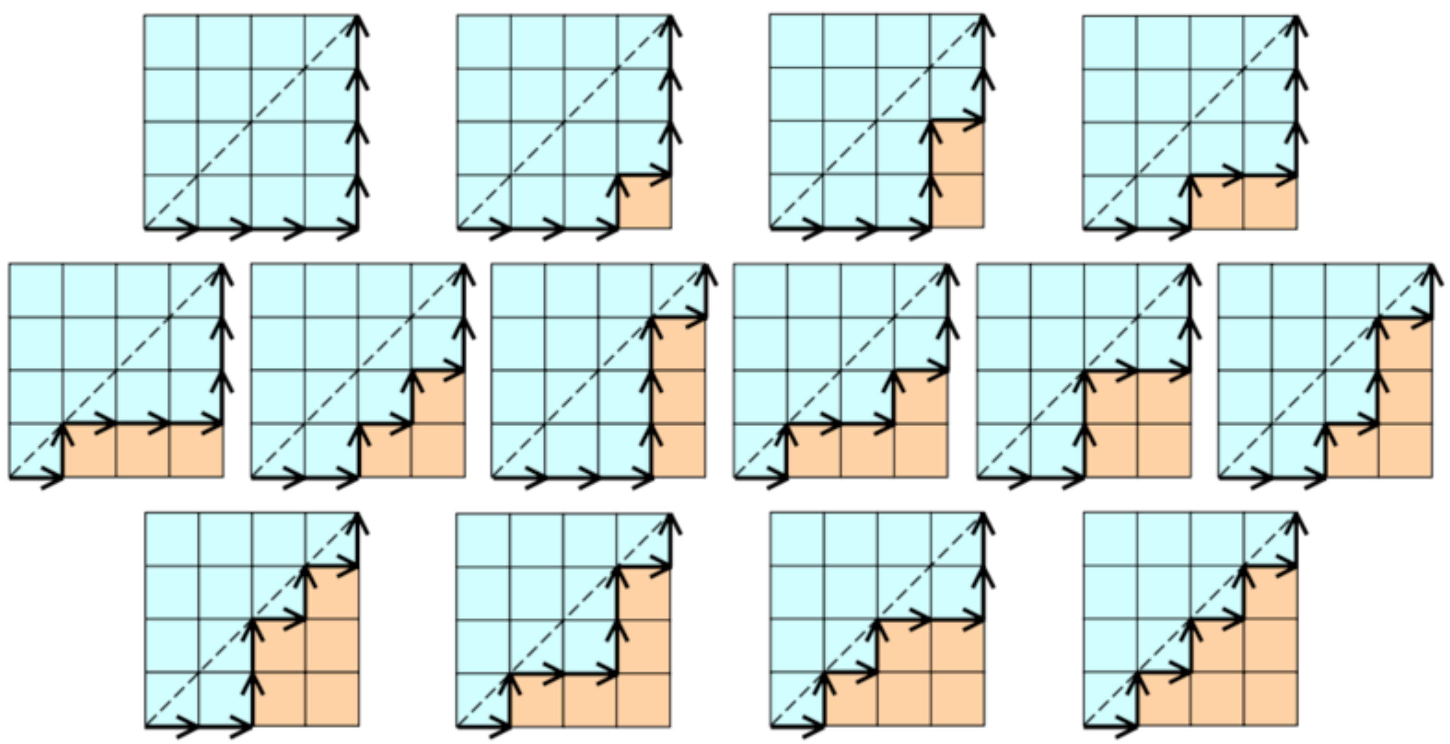
$2n$ people around a circular table shake hands,
no arms cross each other



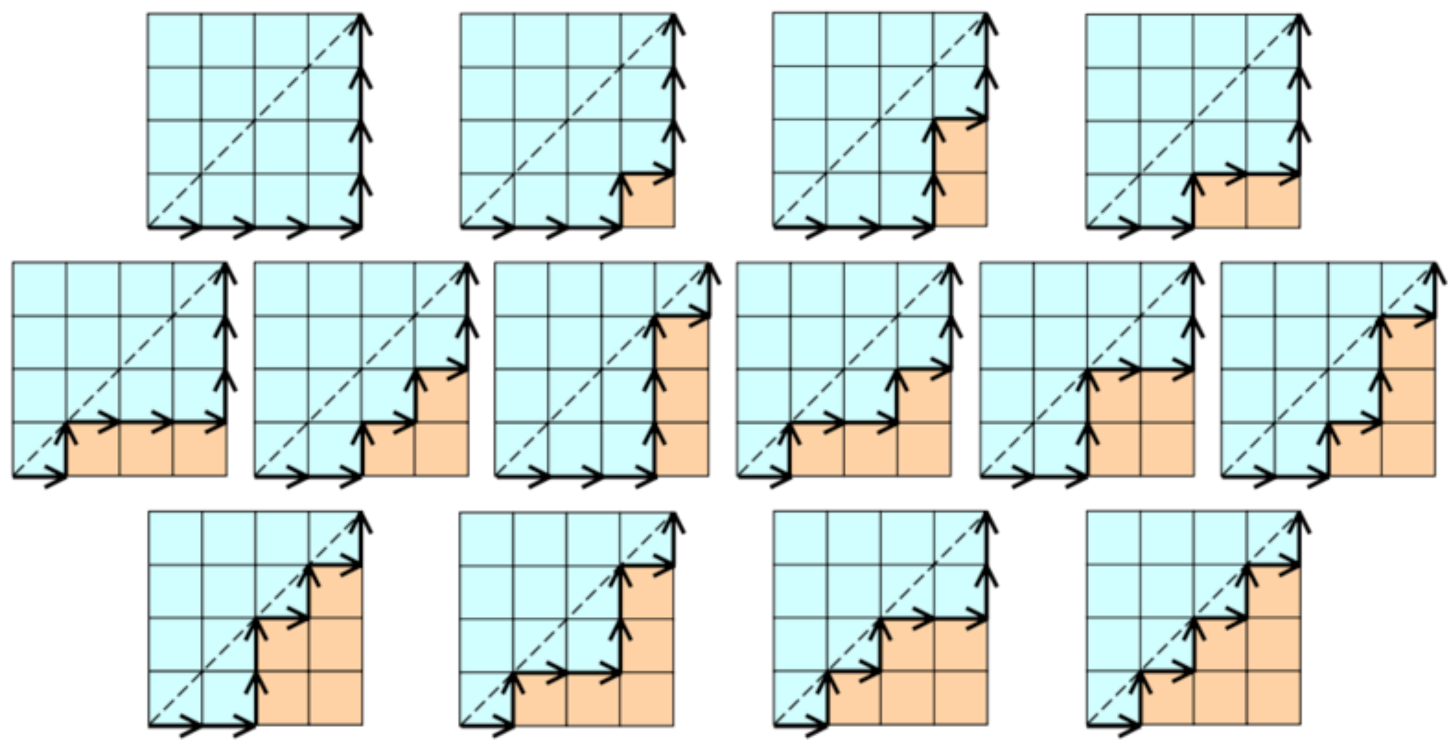
n pairs of matching parenthesis

((())) ()(()) ()()() (())() (()())

monotone paths along $n \times n$ grids
below diagonal



monotone paths along $n \times n$ grids
below diagonal

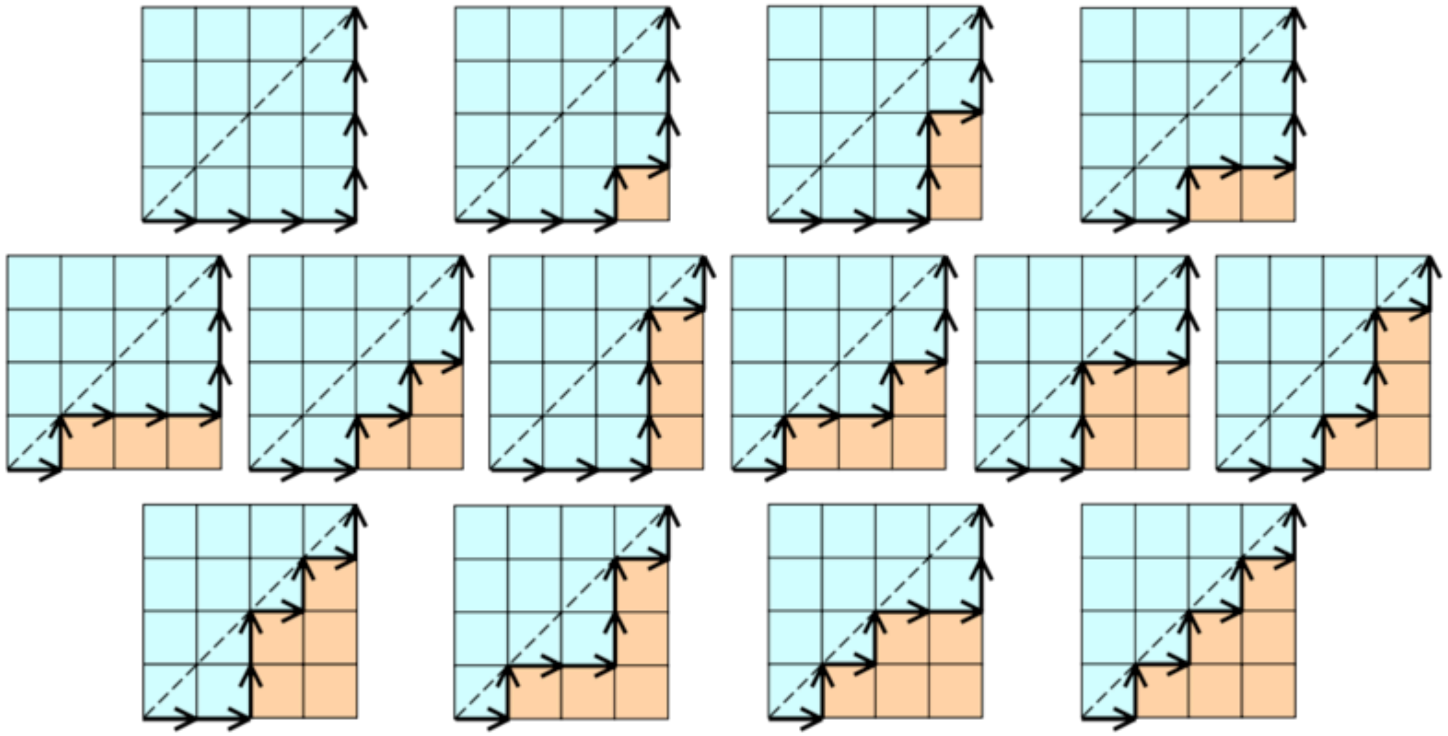


monotonically non-decreasing function:

$$f : [n] \to [n]$$

satisfying $f(i) \leq i$ for all $i \in [n]$

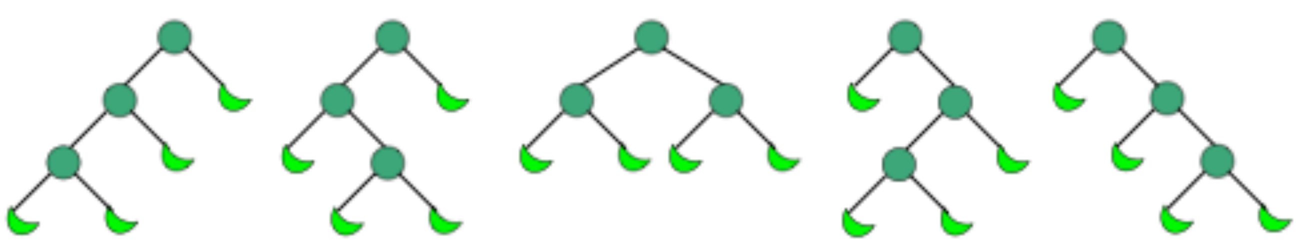
monotone paths along $n \times n$ grids
below diagonal



sequence of n $(+1)$s and n (-1)s,
every prefix-sum is nonnegative

sequence of n $(+1)$s and n (-1)s,
every prefix-sum is nonnegative

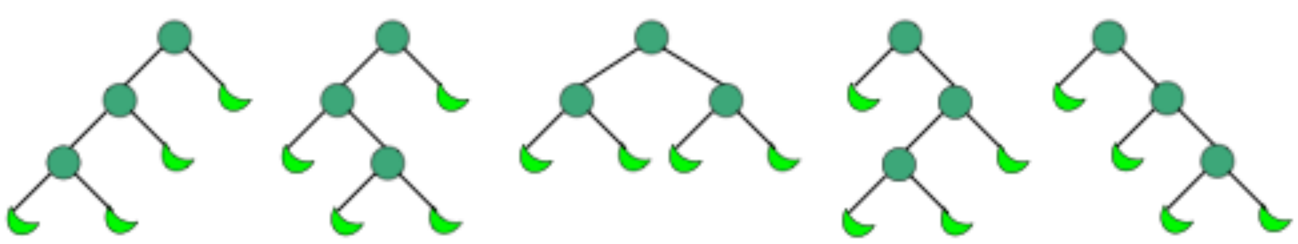
full binary trees with $n + 1$ leaves



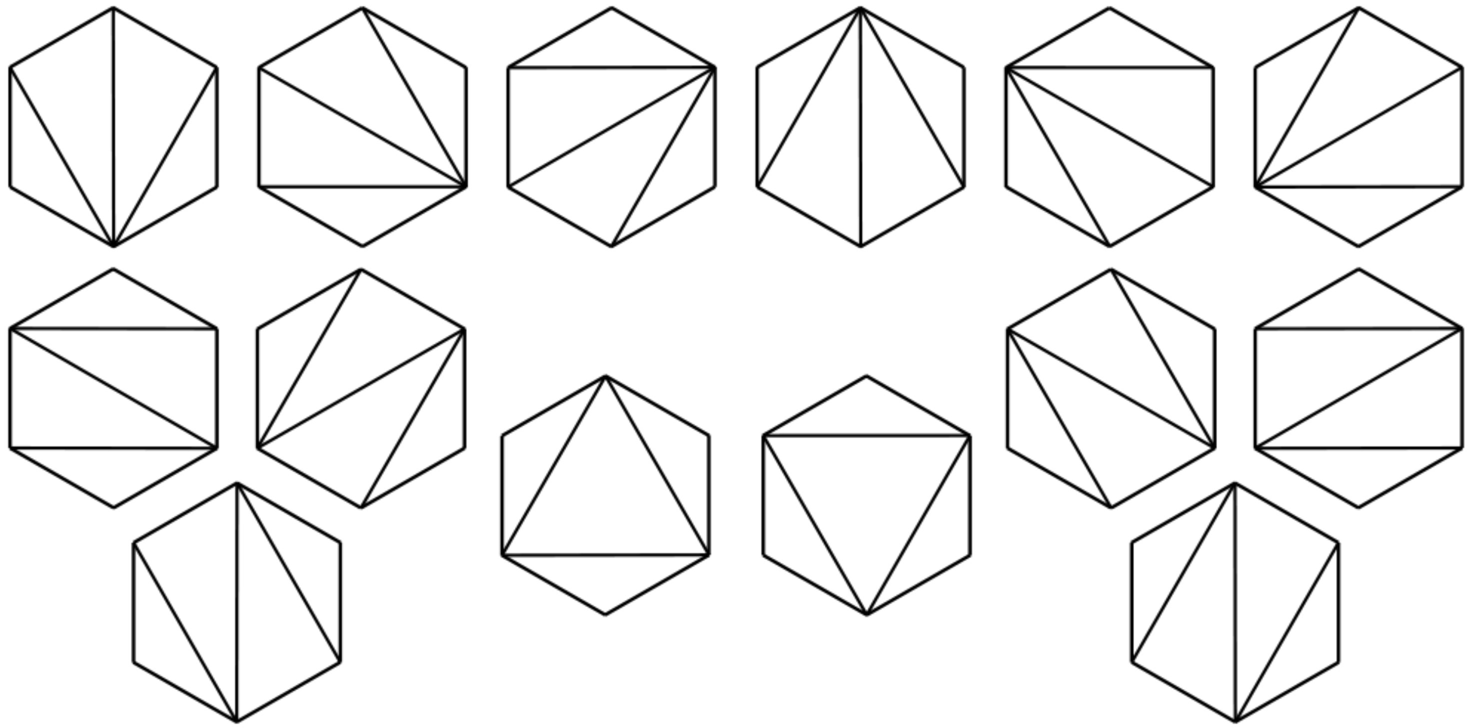
full parenthesization of $n+1$ factors

$$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$$

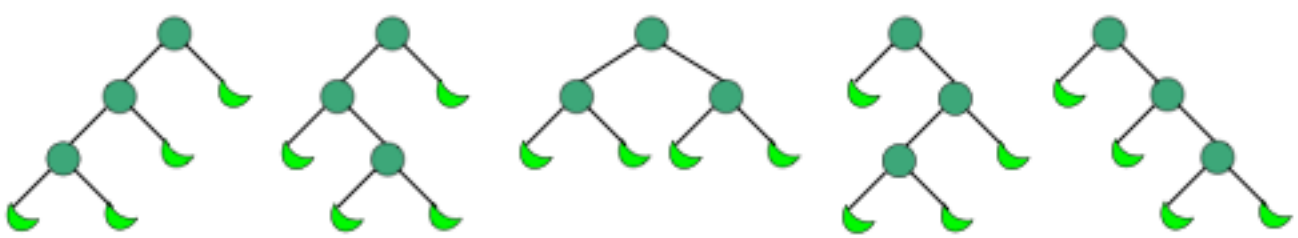
full binary trees with $n+1$ leaves



triangulations of a convex $(n+2)$-gon:

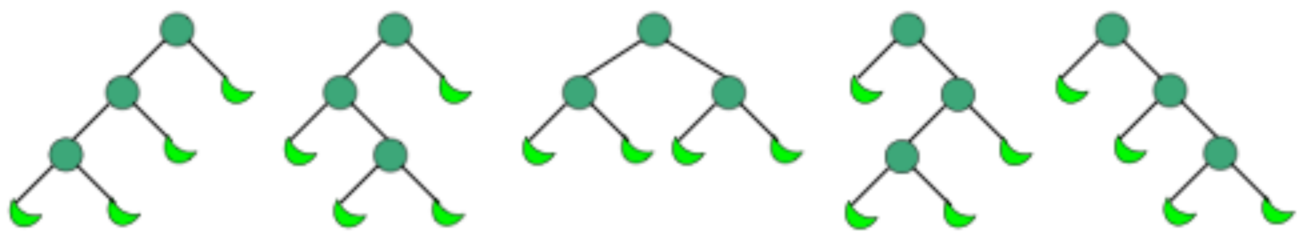


full binary trees with $n+1$ leaves



Catalan Number

C_n : # of full binary trees with $n+1$ leaves



Recursion:

C_k $\qquad C_{n-1-k}$

$$C_0 = 1 \quad \text{for } n \geq 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$$

Generating Function Algebra

$$G(x) = \sum_{n \geq 0} g_n x^n \qquad F(x) = \sum_{n \geq 0} f_n x^n$$

shift:

$$x^k G(x) = \sum_{n \geq k} g_{n-k} x^n$$

addition:

$$F(x) + G(x) = \sum_{n \geq 0} (f_n + g_n) x^n$$

convolution:

$$F(x)G(x) = \sum_{n \geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n$$

differentiation:

$$G'(x) = \sum_{n \geq 0} (n+1) g_{n+1} x^n$$

Recursion:

$$G(x)^2 = \sum_{n \geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^n$$

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$$

Generating Function Algebra

$$G(x) = \sum_{n \geq 0} g_n x^n \qquad F(x) = \sum_{n \geq 0} f_n x^n$$

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Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

$$G(x)^2 = \sum_{n \geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^n$$

$$xG(x)^2 = \sum_{n \geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^{n+1}$$

$$= \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$$

Manipulation:

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n = 1 + xG(x)^2$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = 1 + xG(x)^2$$

Solving:

$$G(x) = \frac{1 \pm (1-4x)^{1/2}}{2x}$$

$$G(x) = \sum_{n \geq 0} C_n x^n \quad \Rightarrow \quad \lim_{x \to 0} G(x) = C_0 = 1$$

$$\Rightarrow \quad G(x) = \frac{1 - (1-4x)^{1/2}}{2x}$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = 1 + xG(x)^2$$

Solving:

$$G(x) = \frac{1-(1-4x)^{1/2}}{2x}$$

Expanding:

Newton's formula

$$(1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} (-4x)^n = 1 + \sum_{n\geq 1} \binom{1/2}{n} (-4x)^n$$

$$= 1 + \sum_{n\geq 0} \binom{1/2}{n+1} (-4x)^{n+1} = 1 - 4x \sum_{n\geq 0} \binom{1/2}{n+1} (-4x)^n$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = 1 + xG(x)^2$$

Solving:

$$G(x) = \frac{1 - (1-4x)^{1/2}}{2x}$$

Expanding:

$$(1-4x)^{1/2} = 1 - 4x \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n$$

$$G(x) = \frac{1 - (1-4x)^{1/2}}{2x} = 2 \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n$$

$$= \sum_{n \geq 0} 2 \binom{1/2}{n+1} (-4)^n x^n$$

$$G(x) = \sum_{n\geq 0} 2\binom{1/2}{n+1}(-4)^n x^n = \sum_{n\geq 0} C_n x^n$$

$$\begin{aligned}
C_n &= 2\binom{1/2}{n+1}(-4)^n \\
&= 2\cdot\left(\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdots\frac{-(2n-1)}{2}\right)\cdot\frac{1}{(n+1)!}\cdot(-4)^n \\
&= \frac{2^n}{(n+1)!}\prod_{k=1}^{n}(2k-1) \quad = \frac{2^n}{(n+1)!}\prod_{k=1}^{n}\frac{(2k-1)2k}{2k} \\
&= \frac{1}{n!(n+1)!}\prod_{k=1}^{n}(2k-1)2k \quad = \frac{(2n)!}{n!(n+1)!} \\
&= \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$

Catalan Number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Average-Case Analysis of *QuickSort*

Quicksort

input: an array A of n numbers

Qsort(A):
choose a pivot $x = A[1]$;
partition A into L with all $L[i] < x$,
R with all $R[i] > x$;
Qsort(L) and Qsort(R);

Complexity: number of comparisons

worst-case: $\Theta(n^2)$

average-case: ?

Qsort(A):

choose a pivot $x = A[1]$;
partition A into L with all $L[i] < x$,
R with all $R[i] > x$;
Qsort(L) and Qsort(R);

T_n :

average # of comparisons used by Qsort

taken over all $n!$

total orders of A

pivot: the k-th smallest number in A

$|L| = k\text{-}1$ $|R| = n\text{-}k$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n - 1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$nT_n = \sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0}\left(\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})\right)x^n$$

$$= \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1}T_k\right)x^n$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

$$= x^2 \sum_{n\geq 0} n(n-1)x^{n-2} = \frac{2x^2}{(1-x)^3}$$

$$= 2x\sum_{n\geq 0}\left(\sum_{k=0}^{n} T_k\right)x^n$$

Generating Function Algebra

$$G(x) = \sum_{n \geq 0} g_n x^n \qquad F(x) = \sum_{n \geq 0} f_n x^n$$

shift: $$x^k G(x) = \sum_{n \geq k} g_{n-k} x^n$$

addition: $$F(x) + G(x) = \sum_{n \geq 0} (f_n + g_n) x^n$$

convolution: $$F(x)G(x) = \sum_{n \geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n$$

differentiation: $$G'(x) = \sum_{n \geq 0} (n+1) g_{n+1} x^n$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

(blue) $$= x^2 \sum_{n\geq 0} n(n-1)x^{n-2} = \frac{2x^2}{(1-x)^3}$$

(orange) $$= 2x\sum_{n\geq 0}\left(\sum_{k=0}^{n} T_k\right)x^n = 2x\sum_{n\geq 0} x^n \sum_{n\geq 0} T_n x^n = \frac{2xG(x)}{1-x}$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

$$= \frac{2x^2}{(1-x)^3}$$

$$= \frac{2xG(x)}{1-x}$$

Generating Function Algebra

$$G(x) = \sum_{n\geq 0} g_n x^n \qquad F(x) = \sum_{n\geq 0} f_n x^n$$

shift: $$x^k G(x) = \sum_{n\geq k} g_{n-k} x^n$$

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differentiation: $$G'(x) = \sum_{n\geq 0} (n+1) g_{n+1} x^n$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n - 1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

$$= \frac{2x^2}{(1-x)^3}$$

$$= \frac{2xG(x)}{1-x}$$

$$= x\sum_{n\geq 0}(n+1)T_{n+1}x^n = xG'(x)$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

$$= xG'(x)$$

$$= \frac{2x^2}{(1-x)^3}$$

$$= \frac{2xG(x)}{1-x}$$

$$xG'(x) = \frac{2x^2}{(1-x)^3} + \frac{2x}{1-x}G(x)$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Solving: first-order linear differential equation

$$y' + P(x)y = Q(x)$$

$$y(x) = \frac{1}{u(x)}\int u(x)Q(x)\,\mathrm{d}x \quad \textbf{with} \quad u(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x}$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Solving: first-order linear differential equation

$$G(x) = e^{\int \frac{2}{1-x}\,\mathrm{d}x}\int \frac{2x}{(1-x)^3}e^{-\int \frac{2}{1-x}\,\mathrm{d}x}\,\mathrm{d}x$$

$$= \frac{2}{(1-x)^2}\ln\frac{1}{1-x}$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Solving:

$$G(x) = \frac{2}{(1-x)^2}\ln\frac{1}{1-x}$$

Expanding:

$$\frac{2}{(1-x)^2} = 2\sum_{n\geq 0}(n+1)x^n \qquad \ln\frac{1}{1-x} = \sum_{n\geq 1}\frac{x^n}{n}$$

Taylor

$$G(x) = 2\sum_{n\geq 0}(n+1)x^n \sum_{n\geq 1}\frac{x^n}{n} = 2\sum_{n\geq 1}\left(\sum_{k=1}^{n}(n-k+1)\frac{1}{k}\right)x^n$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Solving: $$G(x) = \frac{2}{(1-x)^2}\ln\frac{1}{1-x}$$

Expanding: $$G(x) = 2\sum_{n\geq 1}\left(\sum_{k=1}^{n}(n-k+1)\frac{1}{k}\right)x^n$$

$$T_n = 2\sum_{k=1}^{n}(n-k+1)\frac{1}{k} = 2(n+1)\sum_{k=1}^{n}\frac{1}{k} - 2\sum_{k=1}^{n}k\cdot\frac{1}{k}$$

$$= 2(n+1)H(n) - 2n = 2n\ln n + O(n)$$