组合数学 (Fall 2023)/Problem Set 1 Solution Sketch

Problem 1

Difficulty: Easy

> Fix positive integers n and k. Let S be a set with $|S| = n$. Find the numbers of k-tuples $(T_1, T_2, \ldots, T_k)$ of subsets T_i of S subject to each of the following conditions separately. Briefly explain your answer.
>
> - $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$.
> - The T_is are pairwise disjoint.
> - $T_1 \cup T_2 \cup \cdots T_k = S$.

不妨假设$S = \{1, 2, \ldots, n\}$.对于每一小题，令所有满足题意的k元组集合为$\mathcal{T}$。

(1) 构造如下映射$f : \mathcal{T} \to [k+1]^n$,使得对于$f(T_1, T_2, \ldots, T_k) = (a_1, a_2, \ldots, a_n)$,且其中

$$\forall 1 \le i \le n, a_i = \max\{1 \le j \le k \mid j \in T_i\},$$

不难验证f是一个从$\mathcal{T}$到$[k+1]^n$的双射，故有$|\mathcal{T}| = |[k+1]^n| = (k+1)^n$。

(2) 可以构造和(1)中相同的双射，故同样有$|\mathcal{T}| = |[k+1]^n| = (k+1)^n$。

(3) 构造如下映射$f : \mathcal{T} \to [2^{[k]} \setminus \{\varnothing\}]^n$,使得对于$f(T_1, T_2, \ldots, T_k) = (P_1, P_2, \ldots, P_n)$,且其中

$$\forall 1 \le i \le n, P_i = \{1 \le j \le k \mid j \in T_i\},$$

不难验证f是一个从$\mathcal{T}$到$[2^{[k]} \setminus \{\varnothing\}]^n$的双射，故有$|\mathcal{T}| = |[2^{[k]} \setminus \{\varnothing\}]^n| = (2^k - 1)^n$。

Problem 2

Difficulty: Medium

> Let p be a prime integer and a be a positive integer. Show “combinatorially” that $a^p - a$ is divisible by p. (A combinatorial proof would consist of exhibiting a set S with $a^p - a$ elements and a partition of S into pairwise disjoint subsets, each with p elements)

令S表示满足$n_i \in [a]$且不是所有n_i都相同的p元组$(n_1, n_2, \ldots, n_p)$，则有$|S| = a^p - a$。

定义S上的关系R,其中$a \sim_R b$当且仅当a是b的一个的循环移位，不难验证$\sim_R$是一个等价关系。由于p是一个质数，所以不难证明一个元素不全相同的p元组的p种循环移位后得到的p元组互不相同，故S由$\sim_R$划分出的每个等价类都包含恰好p个元素。这样就给出了一个$p \mid (a^p - a)$的组合证明。

Problem 3

Difficulty: Easy

> Fix $1 \le k \le n$. How many integer sequences $1 \le a_1 < a_2 < \cdots < a_k \le n$ satisfy $a_i \equiv i \pmod 2$ for all i? Prove your answer.

令原序列集合为A, 令序列集合B为

$$B = \{(b_1, b_2, \ldots, b_k) \mid 1 \le b_1 \le \cdots \le b_k \le \lfloor \frac{n-k+2}{2} \rfloor\}$$

则根据$b_i = \lfloor \frac{a_i - i + 2}{2} \rfloor$可以构造出一个从$A$到$B$的双射，因此

$$|A| = |B| = \binom{\lfloor \frac{n-k+2}{2} \rfloor + k - 1}{k} = \binom{\lfloor \frac{n+k}{2} \rfloor}{k}$$

Problem 4

Difficulty: Medium

> For any integer $n \ge 1$, let F_n denote the n-th Fibonacci number.
>
> - Show that $F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}$.
> - Let $f(n)$ be the number of ways to choose a subset $S \subseteq [n]$ and a permutation $\pi \in P_n$ such that $\pi(i) \notin S$ whenever $i \in S$. Show that $f(n) = F_{n+1} n!$.

(1) 使用数学归纳法证明。基础情况是$n = 1, 2$。此时有

$$F_2 = 2 = \sum_{k=0}^{1} \binom{1-k}{k}, \quad F_3 = 3 = \sum_{k=0}^{2} \binom{2-k}{k},$$

对于$n \ge 3$, 我们有

$$F_{n+1} = F_n + F_{n-1} = \sum_{k=0}^{n-1} \binom{n-1-k}{k} + \sum_{k=0}^{n-2} \binom{n-2-k}{k},$$

其中第一个等号来自斐波那契数的定义，第二个等号来自归纳假设。因此我们有

$$F_{n+1} = \sum_{k=0}^{n-2} \left(\binom{n-1-k}{k} + \binom{n-2-k}{k} \right) = \sum_{k=0}^{n-2} \binom{n-k}{k} = \sum_{k=0}^{n-1} \binom{n-k}{k},$$

其中第二个等号用到了经典的组合恒等式$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$，第一个和第三个等号$n < k$时$\binom{n}{k} = 0$的性质,因此原命题得证。

(2)考虑按照S的大小分类，可以得到

$$f(n)=\sum_{k=0}^{n}(n-k)_k(n-k)!\binom{n}{k}=n!\sum_{k=0}^{n}\binom{n-k}{k}=n!F_n,$$

其中$(n)_m=n\cdot(n-1)\cdots(n-m+1)$为$n$的$m$次下降幂。这里第一个等式中$\binom{n}{k}$表示选择$S$的方案数, $(n-k)_k$表示给每个$i\in S$选择$\pi(i)$的方案数，$(n-k)!$表示在给每个$i\in S$选择完$\pi(i)$后给每个$j\in[n]\setminus S$选择$\pi(j)$的方案数。第三个等式来自(1)的结论。

Problem 5

Difficulty: Medium

> Let $n-r=2k$. Show that the number $f(n,r,s)$ of compositions of n with r odd parts and s even parts is given by $\binom{r+s}{r}\binom{r+k-1}{r+s-1}$:
>
> - Give a generating function poof.
> - Give a bijective proof.

(1) 固定r和s,则

$$F(x)=\sum_n f(n,r,s)x^n=\binom{r+s}{r}\left(\frac{x}{1-x^2}\right)^r\left(\frac{1}{1-x^2}\right)^s=\binom{r+s}{s}\left(\frac{1}{1-x^2}\right)^{r+s}x^r$$

这是因为$1+x^2+x^4+\cdots=\frac{1}{1-x^2}$.

故

$$[x^n]F(x)=\binom{r+s}{r}[x^{2n+r}]\left(1-x^2\right)^{-r-s}=\binom{r+s}{r}\binom{r+k-1}{r+s-1}$$

(2) 令题目所考虑的集合为T.考虑$r+k$的所有$(r+s)$-composition $C:(a_1,a_2,\ldots,a_{r+s})$,选择其中一个大小为$r$的集合$S\subseteq 2^{[r+s]}$,将所有满足$i\in S$的$a_i$乘以二减一，将所有满足$i\in[r+s]\setminus S$的$a_i$乘以二，可以建立出一个从$C\times S$到$T$的双射，因此有

$$|T|=|C\times S|=|S|\times|C|=\binom{r+s}{r}\binom{r+k-1}{r+s-1}$$