组合数学 (Fall 2023)/Problem Set 2 Solution Sketch

Problem 1

Difficulty: Easy

> Give n, m, k, find the number of “integer” solutions to the following equation: $a_1 + a_2 + ... + a_n = m, \forall 1 \le i \le n, 0 \le a_i < k$. You should give a formula and explain your answer.

当没有$a_i < k$的限制时，根据**插板法**，解的数量为：$\binom{m+n-1}{n-1}$

考虑容斥，设
$U = \{(a_1, a_2, ..., a_n) \in \mathbb{N}^n : \sum_{i=1}^n a_i = m\}$
$A_i = \{(a_1, a_2, ..., a_n) \in U : a_i \ge k\}$
$A_I = \bigcap_{i\in I} A_i$

同样根据**插板法**，$|A_I| = \binom{m-ik+n-1}{n-1}$。

故答案为：
$\left|\bigcup_{i=1}^n \overline{A_i}\right| = \sum_{I\subseteq[n]} (-1)^{|I|} |A_I| = \sum_{i=0}^n (-1)^i \binom{n}{i} \binom{m-ik+n-1}{n-1}$

Problem 2

Difficulty: Hard

> Let D_n denote the number of derangements of $\{1, 2, \ldots, n\}$.
>
> - Give a “combinatorial” proof that $D(n) = nD(n-1) + (-1)^n$.
> - For a permutation $p_1, p_2, ..., p_n$, for $1 \le i < n$, we call position i an ascent if $p_i < p_{i+1}$. Let $\mathcal{E}_n$ denote the set of permutations whose first ascent is in an even position (where we always count n as an ascent). For instance, $\mathcal{E}(3) = \{213, 312\}$ and $\mathcal{E}(4) = \{2134, 2143, 3124, 3142, 3241, 4123, 4132, 4231, 4321\}$. Set $E(n) = |\mathcal{E}(n)|$. Show that $E(n) = nE(n-1) + (-1)^n$. Hence (since $E(1) = D(1) = 0$) we have$E(n) = D(n)$.
> - Give a bijection between the permutations being counted by $E(n)$ and the derangements of $[n]$.

(1) 参考“A Note on a Recursion for the Number of Derangements”。 链接

(2)对任意$S\subseteq[n]$，令$\mathcal{E}(S)$表示S的所有满足“第一个上升位置为偶数”的排列集合。

令$\times$表示笛卡尔积，可以证明：

- 对于n是奇数，$\mathcal{E}([n])=\left(\bigcup_{i\in[n]}(\mathcal{E}([n]\setminus\{i\})\times\{i\})\right)\setminus\{(n,n-1,...,2,1)\}$
- 对于n是偶数，$\mathcal{E}([n])=\left(\bigcup_{i\in[n]}(\mathcal{E}([n]\setminus\{i\})\times\{i\})\right)\cup\{(n,n-1,...,2,1)\}$

从而$E(n)=nE(n{-}1)+(-1)^n$。

(3) 对于一个错排w，将w中所有cycle按照cycle中的“最小值”从大到小排序，并对于每个cycle，将最小值放在cycle的第二个位置。将所有cycle排在一起后得到排列w'。

例如$w=974382651=(85)(43)(627)(91)$，则$w'=854362791$。

令$\mathcal{D}(n)$表示所有$[n]$的错排集合。可以证明：

- 对于一个$w\in\ \mathcal{D}(n)$，按照上述方法生成的$w'\in\mathcal{E}(n)$。
- 对于一个$w'\in\mathcal{E}(n)$，可以通过解码得到**唯一**的$w\in\mathcal{D}(n)$。

从而构建了$\mathcal{D}(n)$和$\mathcal{E}(n)$之间的双射。

Problem 3

Difficulty: Medium

> Let $S=\{P_1,...,P_n\}$ be a set of properties, and let f_k (respectively, $f_{\geq k}$) denote the number of objects in a finite set U that satisfy “exactly” k (respectively, “at least” k) of the properties.
> Let A_i denote the set of objects that satisfy P_i in U, for any $I\subseteq\{1,...,n\}$, we denote $A_I=\bigcap_{i\in I}A_i$ with the convention that $A_\emptyset=U$.
> Show that
>
> - $f_k=\sum_{i=k}^{n}(-1)^{i-k}\binom{i}{k}g_i$
> - $f_{\geq k}=\sum_{i=k}^{n}(-1)^{i-k}\binom{i-1}{k-1}g_i$.
>
> where $g_i=\sum_{I\subseteq\{1,...,n\},|I|=i}|A_I|$.

(1) 令B_I表示**恰好**满足集合I内的性质的元素集合。根据容斥原理,

$|B_I|=\sum_{J\supseteq I}(-1)^{|J|-|I|}|A_J|$

故

$f_k=\sum_{|I|=k}|B_I|=\sum_{|I|=k}\sum_{J\supseteq I}(-1)^{|J|-|I|}|A_J|$
$=\sum_{|J|\geq k}(-1)^{|J|-k}\binom{|J|}{k}|A_J|=\sum_{j=k}^{n}(-1)^{j-k}\binom{j}{k}g_j$

(2)

$f_{\geq k}=\sum_{|I|\geq k}|B_I|=\sum_{|I|\geq k}\sum_{J\supseteq I}(-1)^{|J|-|I|}|A_J|$

$= \sum_{|J|\geq k} \sum_{|I|\geq k, I\subseteq J} (-1)^{|J|-|I|} |A_J|$
$= \sum_{j=k}^{n} \sum_{i=k}^{j} (-1)^{j-i} \binom{j}{i} g_j$

注意到

$\sum_{i=k}^{j} (-1)^{j-i} \binom{j}{i} = \sum_{i=k}^{j} (-1)^{j-i} \binom{j-1}{i-1} + (-1)^{j-i} \binom{j-1}{i}$
$= (-1)^{j-k} \binom{j-1}{k-1}$

从而

$f_{\geq k} = \sum_{j=k}^{n} \sum_{i=k}^{j} (-1)^{j-i} \binom{j}{i} g_j = \sum_{j=k}^{n} (-1)^{j-k} \binom{j-1}{k-1} g_j$

Problem 4

Difficulty: Medium

> For any k-size subset A of vertices set $\{1, 2, \ldots, n\}$, there are $T_{n,k}$ forests on the n vertices with exactly k connected components that each element of A is in a different component.
>
> - Prove $T_{n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} T_{n-1,k+i-1}$.
> - Prove $T_{n,k} = k \cdot n^{n-k-1}$.

(1) 令$S_n(K)$表示所有“n个点的$|K|$个连通块构成的**带标号**森林，且满足集合K内的点属于不同连通块”构成的集合。

令

$U = \bigcup_{i=0}^{n-k} \bigcup_{k+1\leq p_1<p_2<...<p_i\leq n} S_{n-1}(1, 2, ..., k-1, p_1, ..., p_i)$

- 对于$\mathcal{T} \in S_n(\{1, 2, ..., k\})$。设$\mathcal{T}$中顶点$k$的度数为$i$，它的邻居为$p_1, p_2, ..., p_i$。则将$\mathcal{T}$中删去点$k$后，得到一个$n-1$个点的$k+i-1$个连通块构成的**带标号**森林，且顶点$(1, 2, ..., k-1, p_1, ..., p_i)$属于不同的连通块。
- 即$\mathcal{T} - k \in S_{n-1}(\{1, 2, ..., k-1, p_1, ..., p_i\})$
- 上述构造是双向的，从而建立了$S_n(\{1, 2, ..., k\})$与U的双射。

从而有

$T_{n,k} = |S_n(\{1, 2, ..., k\}| = \sum_{i=0}^{n-k} \sum_{k+1\leq p_1<p_2<...<p_i\leq n} |S_{n-1}(1, 2, ..., k-1, p_1, ..., p_i)| = \sum_{i=0}^{n-k} \binom{n-k}{i} T_{n-1,k+i-1}$

(2)

思路1：根据上一问得到的公式，对n进行归纳证明。

- 当$n=1$时，$k=1, T_{n,k}=1$。
- 设对于$m<n, 1\leq k\leq m$，$T_{m,k} = k\cdot m^{m-k-1}$均成立，则

$$
\begin{aligned}
T_{n,k} &= \sum_{i=0}^{n-k}\binom{n}{k}T_{n-1,k+i-1} \\
&= \sum_{i=0}^{n-k}\binom{n-k}{i}(k+i-1)(n-1)^{n-1-(k+i-1)-1} \\
&= \sum_{i=0}^{n-k}\binom{n-k}{i}(n-i-1)(n-1)^{i-1} \\
&= \frac{n-k}{n-1}\sum_{i=0}^{n-k}\binom{n-k}{i}(n-i-1)(n-1)^{i-1} + \frac{k-1}{n-1}\sum_{i=0}^{n-k}\binom{n-k}{i}(n-i-1)(n-1)^{i-1} \\
&= \frac{n-k}{n-1}\cdot n^{n-k-1} + \frac{k-1}{n-1}\cdot n^{n-k} = k\cdot n^{n-k-1}
\end{aligned}
$$

思路2：按照Prufer code的思路做组合证明。

考虑$\mathcal{T}\in S_n(\{n-k+1,...,n-1,n\})$，其Prufer code$(v_1,v_2,...,v_{n-k})$按如下方式构造：

- 每次删除$\mathcal{T}$中编号最小的度数为1的节点和它所连的边，将它所连的点加入Prufer code。
- 可以发现，v_{n-k}一定在$\{n-k+1,...,n-1,n\}$中。

参考Prufer code证明Cayley定理的方式，证明上述构造的code$(v_1,v_2,...,v_{n-k-1},v_k)\in [n]^{n-k-1}\times\{n-k+1,...,n\}$与$S_n(\{n-k+1,...,n-1,n\})$形成双射。

Problem 5

Difficulty: Medium

> Identify necklaces of two types of colored beads with their duals obtained by switching the colors of the beads. (We can now distinguish between the two colors, but we can't tell which is which.) Count the number of dintinct configurations with $n=12$ and dihedral equivalence.
>
> For example, with $n=6$ and dihedral equivalence, there are 8 distinct configurations: $000000, 000001, 000011, 000101, 001001, 000111, 001011, 010101.$

令所有染色方案的集合为$|X|$。考虑由旋转，反射和按位翻转生成的群G。$|G| = 4n$。

根据Burnside's Lemma，染色方案数为$|X/G| = \frac{1}{|G|}\sum_{\pi\in G}|X_\pi|$。

我们需要计算每个X_π的大小，即不动点个数。

- 旋转$i\cdot\frac{360}{n}$度：$|X_\pi| = 2^{gcd(n,i)}$；
- 旋转$i\cdot\frac{360}{n}$度+反射：$|X_\pi| = 2^{n/2+[2|i]}$；
- 旋转$i\cdot\frac{360}{n}$度+按位翻转：$|X_\pi| = [2|n/gcd(n,i)]\cdot 2^{n/2+[2|i]}$；
- 旋转$i\cdot\frac{360}{n}$度+反射+按位翻转：$|X_\pi| = [2|i]\cdot 2^{n/2+[2|i]}$。

答案为：
$\sum_{i=0}^{n-1}\left(2^{gcd(n,i)}+2^{n/2+[2|i]}+[2|n/gcd(n,i)]\cdot 2^{n/2+[2|i]}+[2|i]\cdot 2^{n/2+[2|i]}\right)/48=122$