组合数学 (Fall 2023)/Problem Set 3 Solution Sketch

Problem 1

Difficulty: Hard

> Solve the following two existence problems:
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> - You are given n integers $a_1, a_2, \ldots, a_n$, such that for each $1 \le i \le n$ it holds that $i - n \le a_i \le i - 1$. Show that there exists a “nonempty” subsequence (not necessarily consecutive) of these integers, whose sum is equal to 0. (Hint: Consider $b_i = a_i - i$)
> - You are given two “multisets” A and B, both with n integers from 1 to n. Show that there exists two “nonempty” subsets $A' \subseteq A$ and $B' \subseteq B$ with equal sum, i.e. $\sum_{x \in A'} x = \sum_{y \in B'} y$ (Hint: Replace the term “multiset” by “sequence”, the term “subset” by “consecutive subsequence”, and the statement is still true.)

(1)令$b_i = a_i - i$, 则有$-n \le b_i \le -1$。考虑有向图$G = (V, E)$,其中$V = [n], E = \{(i, -b_i) \mid 1 \le i \le n\}$, 则每个顶点的出度均为1, 故图中一定存在环$C = (v_0, v_1, \ldots, v_\ell)$,其中对于$0 \le i \le \ell$都有$(v_i, v_{(i+1) \bmod \ell}) \in E$,故

$$\sum_{0 \le i \le \ell} a_{v_i} = \sum_{0 \le i \le \ell} (v_i - b_{v_i}) = \sum_{0 \le i \le \ell} v_i - \sum_{0 \le i \le \ell} b_{v_i} = 0,$$

即原命题得证。

(2) 不妨设$A = \{A_1, A_2, \ldots, A_n\}, B = \{B_1, B_2, \ldots, B_n\}$。对于$0 \le i \le n$, 令$a_i = \sum_{j=1}^{i} A_j, b_i = \sum_{j=1}^{n} B_j$, 则如果能够找到$0 \le i < i' \le n, 0 \le j < j' \le n$使得$A_{i'} - A_i = B_{j'} - B_j$, 则令$A' = \{A_{i'+1}, A_{i'+2}, \ldots, A_i\}, B' = \{B_{j'+1}, B_{j'+2}, \ldots, B_j\}$即可得到一组合法解。

我们接下来证明满足条件的四元组(i,i',j,j')一定存在。不妨设$a_n \le b_n$。对于每个$0 \le i \le n$, 我们都可以找到最大的j使得$b_j \le a_i$,记这样的j为$f(i)$,由于A和B中每个整数都在1到n之间，我们不难证明对于每个$0 \le i \le n$都有$0 \le a_i - b_{f(i)} \le n-1$,故根据鸽巢原理，一定存在$0 \le i < i' \le n$使得$a_i - b_{f(i)} = a_{i'} - b_{f(i')}$,此时$(i,i',f(i),f(i'))$即为一组满足条件的四元组。

Problem 2

Difficulty: Easy

> Suppose $n \ge 4$, and let H be an n-uniform hypergraph with at most $\frac{4^{n-1}}{3^n}$ (hyper)edges. Prove that there is a coloring of the vertices of H by four colors so that in every (hyper)edge all four colors are represented.

假设$H = (V, \mathcal{E})$。考虑对V中每个顶点进行均匀随机染色, 对所有$e \in \mathcal{E}$,令A_e表示事件“e中出现了少于四种颜色”，令A表示事件“$\mathcal{E}$中存在超边中出现了少于四种颜色”,那么有

$$\Pr[A] \le \sum_{e\in\mathcal{E}} \Pr[A_e] < |\mathcal{E}| \cdot 4 \times \left(\frac{3}{4}\right)^n \le 1,$$

其中第一个不等式来自union bound,第二个不等式来自考虑枚举不出现的颜色当作上界，最后一个不等号来自于题设。故一定存在一种染色使得$\mathcal{E}$中每条超边上都出现了四种颜色。

Problem 3

Difficulty: Medium

> The “girth” of a graph is the length of its smallest cycle. Show that for any integer $k \ge 3$, for n “sufficiently large” there is a simple graph with n vertices, at least $\frac{1}{4}n^{1+1/k}$ edges, and girth at least k. (Hint: Try to generate a random graph with n vertices and then fix things up!)

我们考虑Erdős-Rényi图$G \in G(n,p)$,其中每条边出现概率$p = n^{1/k-1}$。令X表示G中的边数，则有

$$\mathbb{E}[X] = p\binom{n}{2} = \frac{1}{2}\left(1 - \frac{1}{n}\right) n^{1/k+1}.$$

令Y表示表示G中长度至多为$k-1$的简单环的个数。注意任何长度为$3 \le i \le k-1$的简单环的出现概率都是p^i，且长度为i的简单环的个数为$\binom{n}{i}\frac{(i-1)!}{2}$，因为选择环上点集有$\binom{n}{i}$种方案，然后环上旋转和反转都是等价的。因此有

$$\mathbb{E}[Y] = \sum_{i=3}^{k-1}\binom{n}{i}\frac{(i-1)!}{2}p^i \le \sum_{i=3}^{k-1} n^i p^i = \sum_{i=3}^{k-1} n^{i/k} < kn^{(k-1)/k}.$$

我们修改图G，从图中每个长度至多为$k-1$的简单环中选择一条边并删去所有被选中至少一次的边，最终得到图G'。显然图G'的围长(girth)至少为k。令Z表示G'中的边数，那么当n足够大时有

$$\mathbb{E}[Z] \ge \mathbb{E}[X - Y] > \frac{1}{2}\left(1 - \frac{1}{n}\right) n^{1/k+1} - kn^{(k-1)/k} \ge \frac{1}{4}n^{1/k+1},$$

即存在一个围长至少为k且边数至少为$\frac{1}{4}n^{1/k+1}$的图。

Problem 4

Difficulty: Medium

> Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of at least $2er$ colors, where $r \ge 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most r neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colors assigned to u and v are different.

首先将所有$S(v)$只保留$\lceil 2er \rceil$种颜色。接下来考虑对每个$v \in V$从$S(v)$中均匀随机染上一种颜色，对每个$e \in E$和颜色$c \in S(u) \cap S(v)$定义坏事件$A_{e,c}$:e的两个端点染的颜色同为c,那么显然有

$$p = \Pr[A_{e,c}] = \frac{1}{|S(u)|}\frac{1}{|S(v)|} = \frac{1}{\lceil 2er \rceil^2},$$

对于任意$A_{e,c}$，可以发现$A_{e,c}$和除了满足以下条件的$A_{e',c'}$无关(mutually independent):

- $e' = e$
- e'和e在G中有公共顶点且$c' = c$,

考虑所有坏事件的依赖图，可以知道每个坏事件最多和$D = \lceil 2er \rceil + 2(r - 1)\lceil 2er \rceil = (2r - 1)\lceil 2er \rceil$个坏事件相邻，因此$ep(D + 1) \leq 1$,根据Lovász local lemma可知所有坏事件都不发生的概率大于0，即存在一种合法染色。

Problem 5

Difficulty: Medium

> Prove the following theorem on extremal graph theory:
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> - (Graham–Kleitman 1973). If the edges of a complete graph on n vertices are labeled arbitrarily with the integers $1, 2, \ldots, \frac{n(n-1)}{2}$, each edge receiving its own integer, then there is a trail (i.e., a walk without repeated edges) of length at least $n-1$ with an increasing sequence of edge-labels. (Hint: To each vertex v, assign its weight w_v equal to the length of the longest increasing trail ending at v. Can you show that $\sum_{i=1}^{n} w_i \geq n(n-1)$?)

考虑从空图$G_0 = (V, \emptyset)$开始按照编号为$1, 2, \ldots, \frac{n(n-1)}{2}$的顺序将边添加进去,令添加了前$i$条边后的图为$G_i$。对$0 \leq i \leq \frac{n(n-1)}{2}$和每个顶点$v \in V$令$w_{i,v}$表示在$G_i$中以顶点$v$终止的最长上升编号路径的长度。我们令$W_i = \sum_{v \in V} w_{i,v}$,我们想要证明

$$(\star) \qquad W_{\frac{n(n-1)}{2}} = \sum_{v \in V} w_{\frac{n(n-1)}{2},v} \geq n(n-1),$$

那么根据平均原理，在$G_{\frac{n(n-1)}{2}}$即题给的完全图中存在一条长度为$n - 1$的编号递增的路径，即原命题得证。

我们接下来证明($\star$)。注意到对于$1 \leq i \leq \frac{n(n-1)}{2}$从$G_{i-1}$到$G_i$添加的边的编号大于$G_i$中所有边的编号。不妨设添加的边是$(u, v)$，那么有$w_{i,u} \geq w_{i-1,v} + 1$且$w_{i,v} \geq w_{i-1,u} + 1$,因此有

$$W_i = \sum_{v\in V} w_{i,v} \geq 2 + \sum_{v\in V} w_{i-1,v} = 2 + W_{i-1}$$

由于对于空图有$W_0 = 0$,因此$(\star)$及原命题得证。