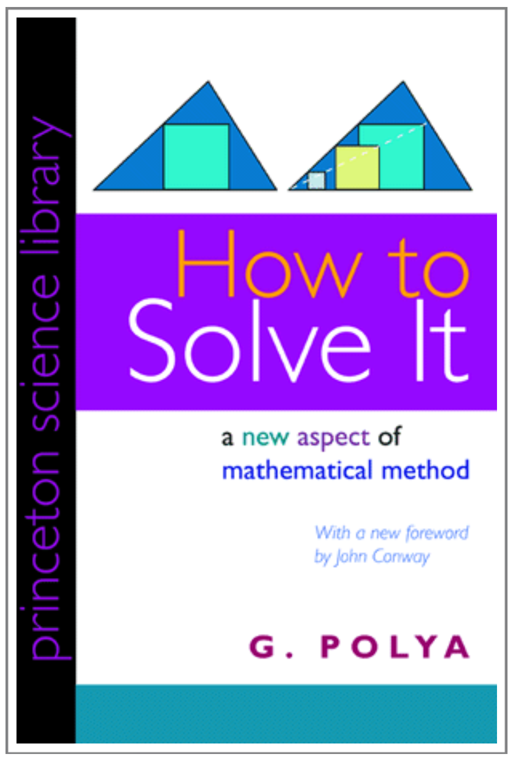
Combinatorics

Pólya’s Theory of Counting

尹一通 Nanjing University, 2023 Spring

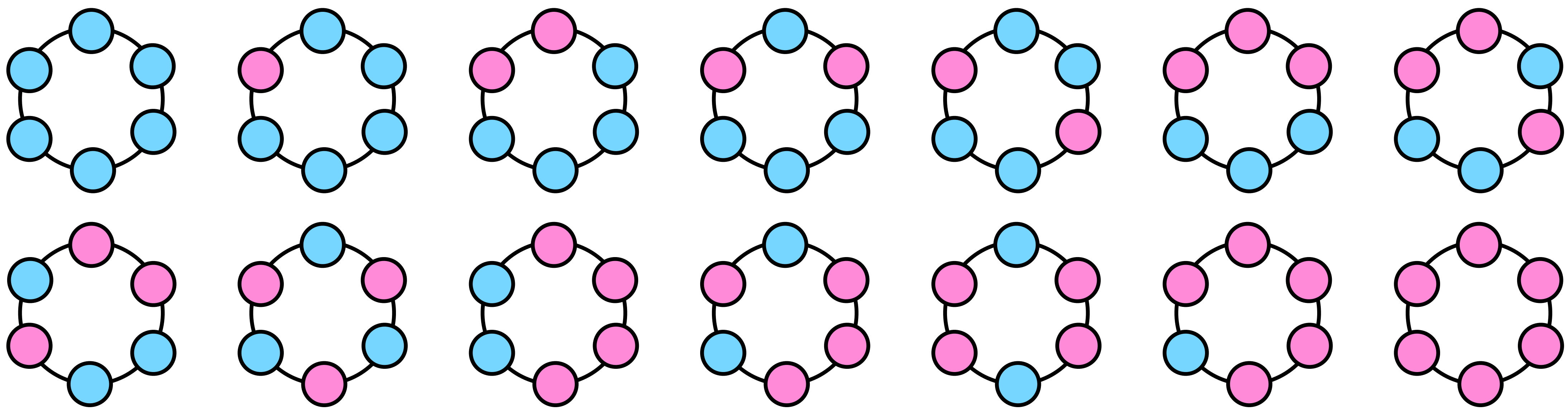
Pólya's Theory of Counting



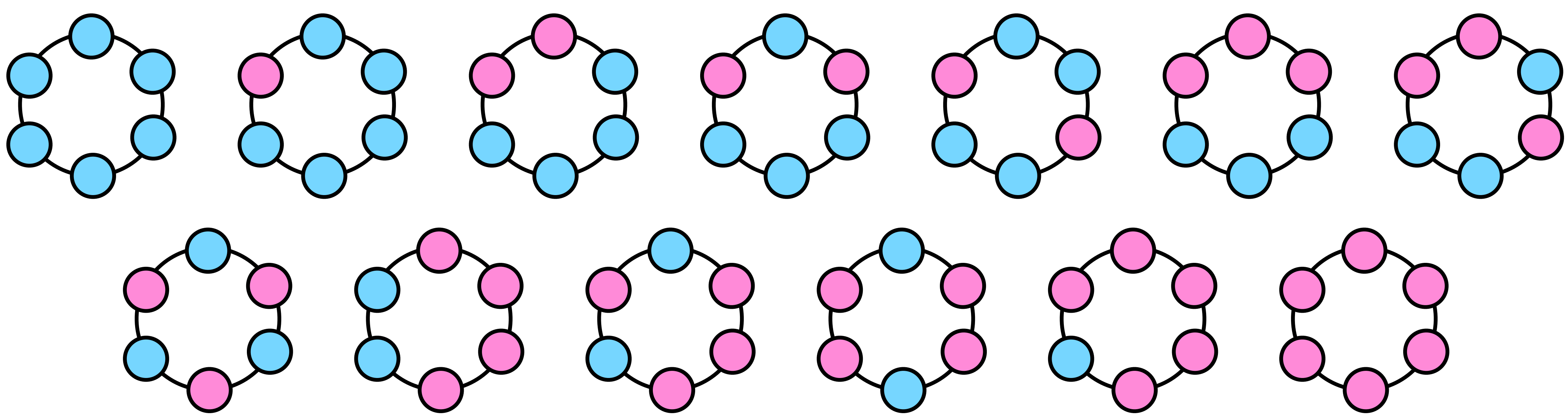
George Pólya
(1887-1985)

Counting with Symmetry

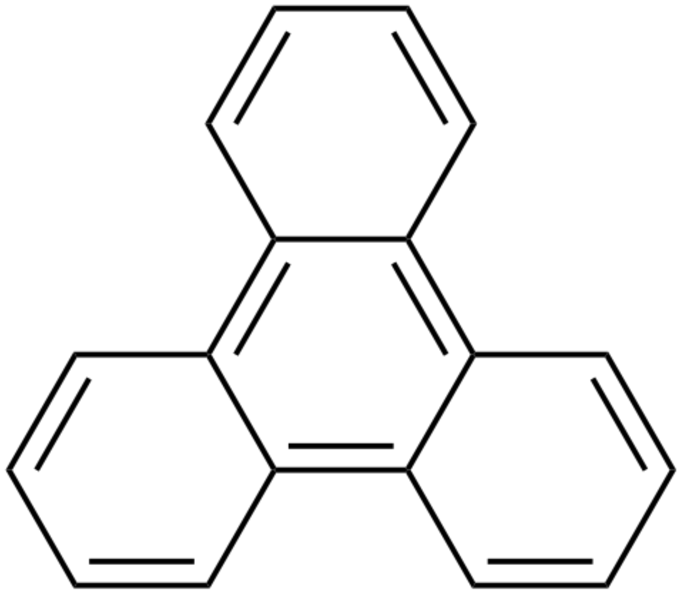
Rotation :

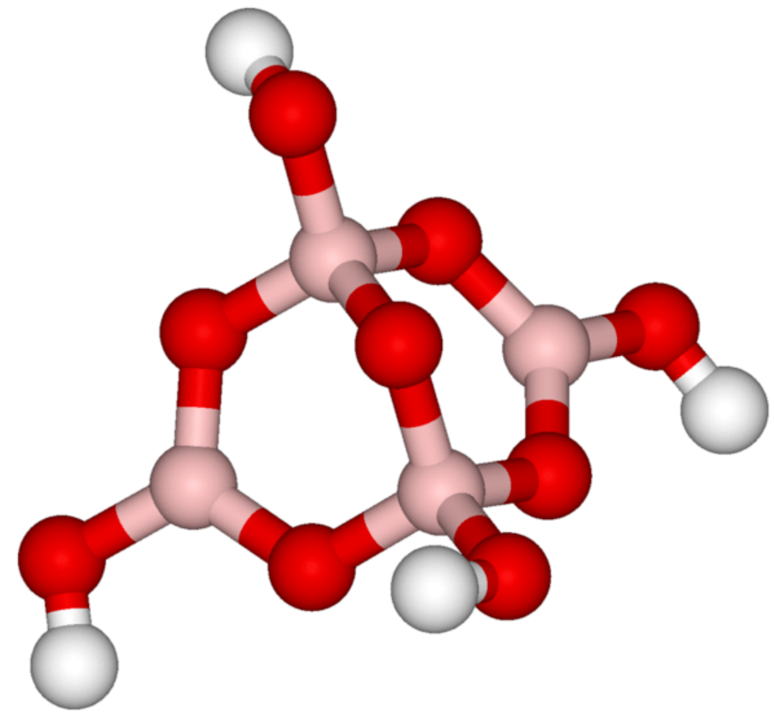


Rotation & Reflection:

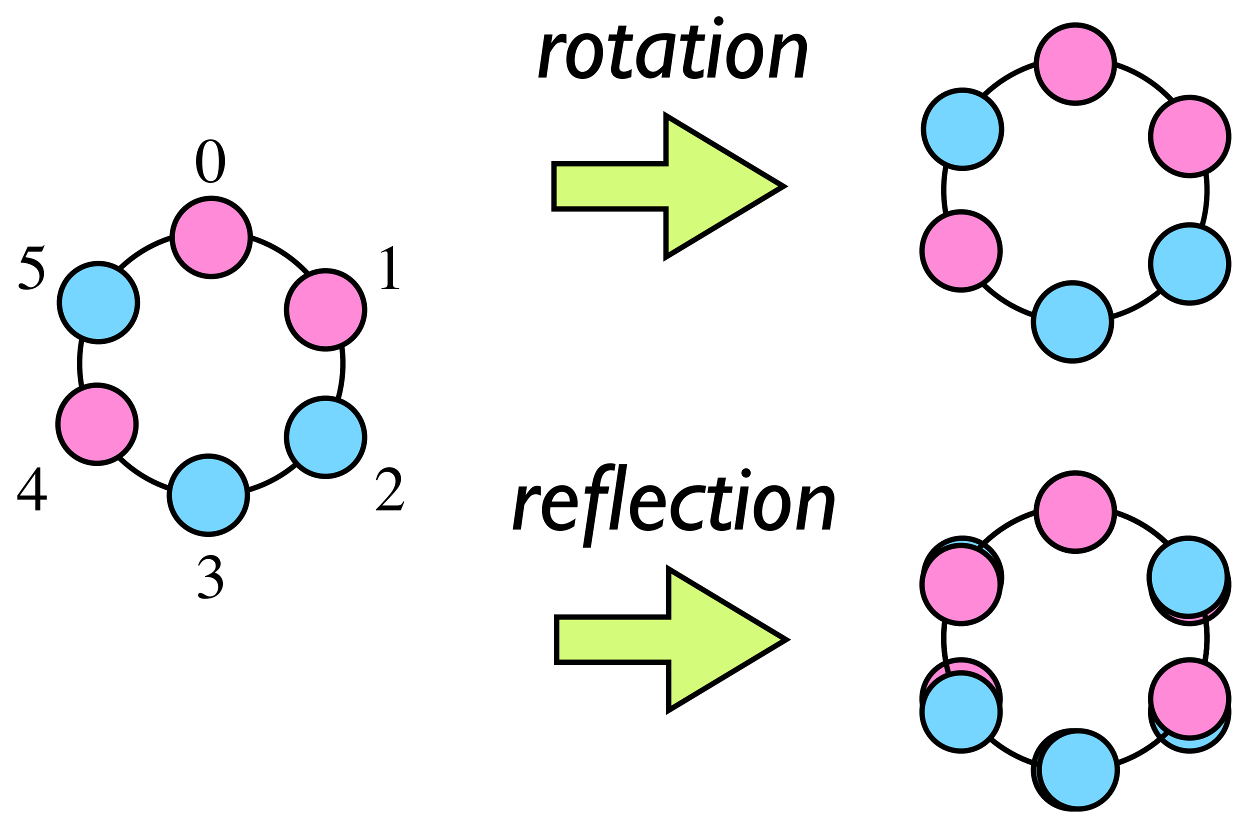


Symmetries





Symmetry



configuration $x : [n] \to [m]$ $X = [m]^{[n]}$

positions colors

permutation $\pi : [n] \xrightarrow[\text{on-to}]{\text{1-1}} [n]$

Permutation Groups

group $(G, \cdot)$ with binary operator $\cdot : G \times G \to G$

- closure: $\pi, \sigma \in G \Rightarrow \pi \cdot \sigma \in G$
- associativity: $\pi \cdot (\sigma \cdot \tau) = (\pi \cdot \sigma) \cdot \tau$
- identity: $\exists e \in G, \forall \pi \in G, e \cdot \pi = \pi$
- inverse: $\forall \pi \in G, \exists \sigma \in G, \pi \cdot \sigma = \sigma \cdot \pi = e$

$$\sigma = \pi^{-1}$$

commutative (abelian) group: $\pi \cdot \sigma = \sigma \cdot \pi$

symmetric group S_n : all permutations

cyclic group C_n : rotations

Dihedral group D_n : rotations & reflections

Permutation Groups

symmetric group S_n : all permutations

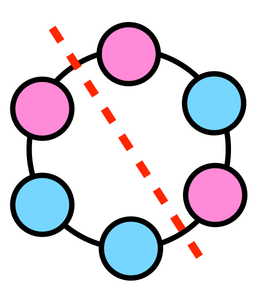
$$\pi : [n] \xrightarrow[\text{on-to}]{\text{1-1}} [n]$$

cyclic group C_n : rotations

$$\pi = (012 \cdots n-1) \qquad \pi(i) = (i+1) \bmod n$$

$\langle(012 \cdots n-1)\rangle$ generated by $(012 \cdots n-1)$

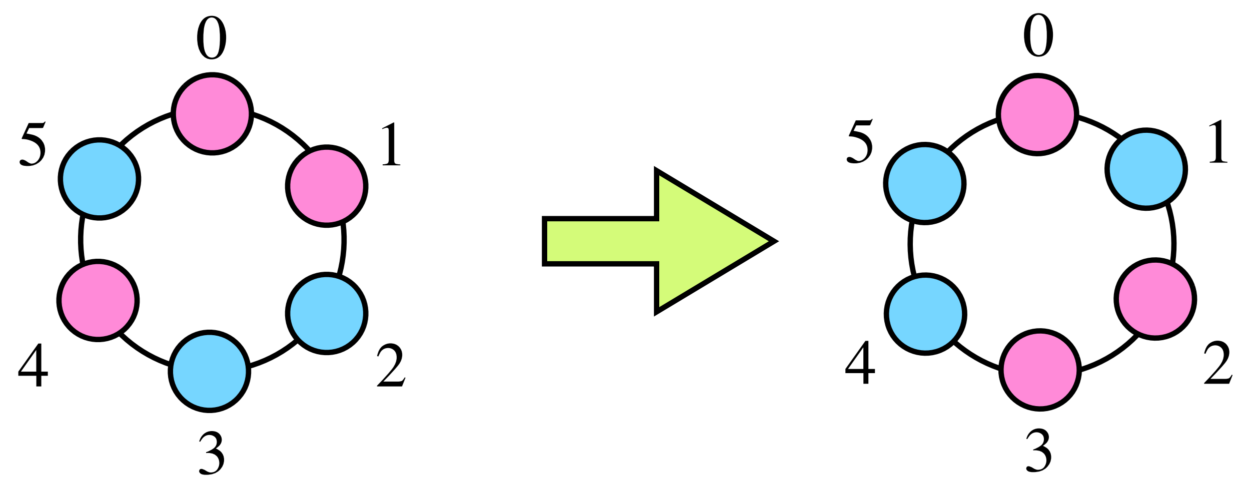
Dihedral group D_n : rotations & reflections



$$\rho(i) = (n-1) - i$$

generated by $(012 \cdots n-1)$ and ρ

Group Action



configuration $x : [n] \to [m]$ $X = [m]^{[n]}$

permutation $\pi : [n] \xrightarrow[\text{on-to}]{\text{1-1}} [n]$ group G

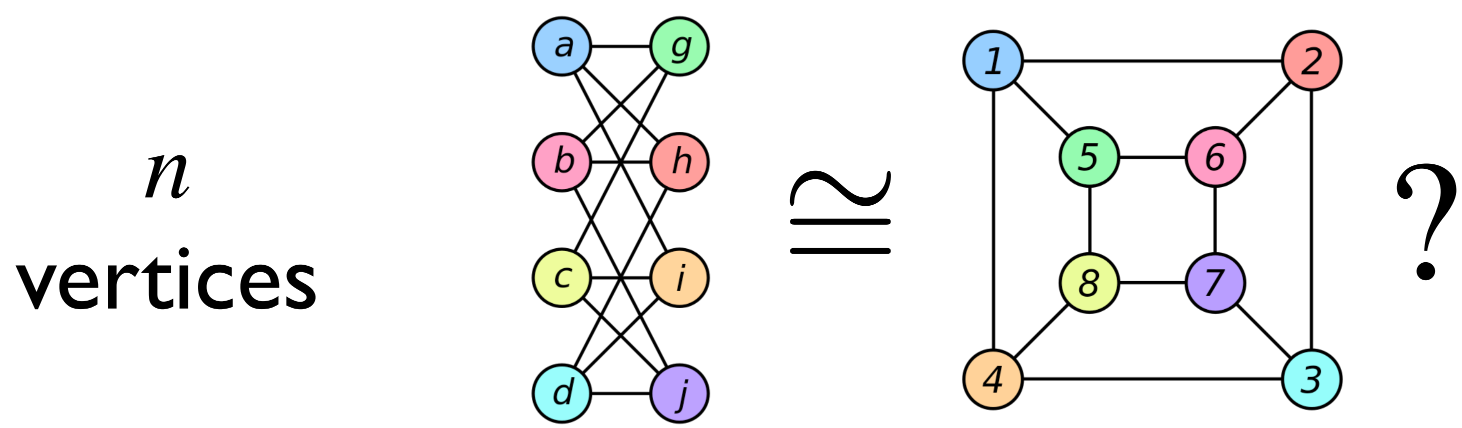
$$(\pi \circ x)(i) = x(\pi(i))$$

group action $\circ : G \times X \to X$

- associativity: $(\pi \cdot \sigma) \circ x = \pi \circ (\sigma \circ x)$
- identity: $e \circ x = x$

Graph Isomorphism (GI) Problem

input: two undirected graphs G and H
output: $G \cong H$?



- GI is in **NP**, but is NOT known to be in **P** or **NPC**
- trivial algorithm: $O(n!)$ time
- Babai-Luks '83: $2^{O\left(\sqrt{n \log n}\right)}$ time

Babai 2017: a *quasi-polynomial* time algorithm!

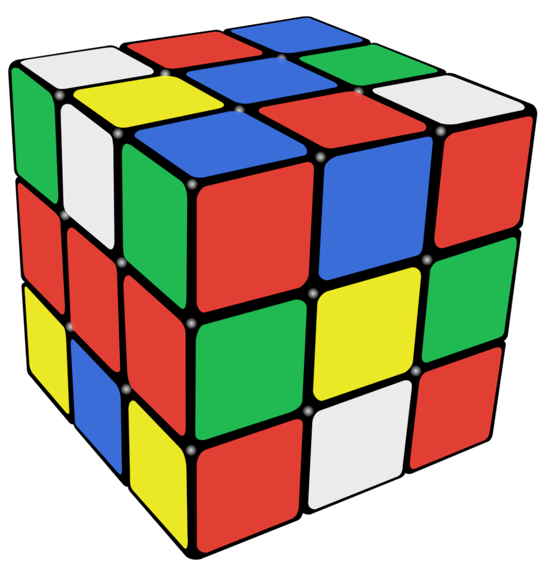
$$n^{\text{polylog}(n)} = 2^{\text{polylog}(n)}$$

String Isomorphism (SI)

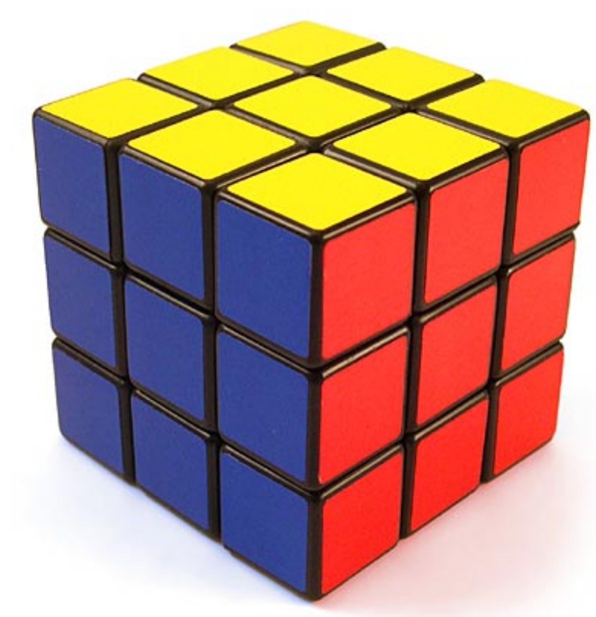
input: two *strings* $x, y : [n] \to [m]$

a permutation group $G \subseteq S_n$

output: $x \cong_G y$? $(\exists \sigma \in G \text{ s.t. } \sigma \circ x = y)$



$\cong$



?

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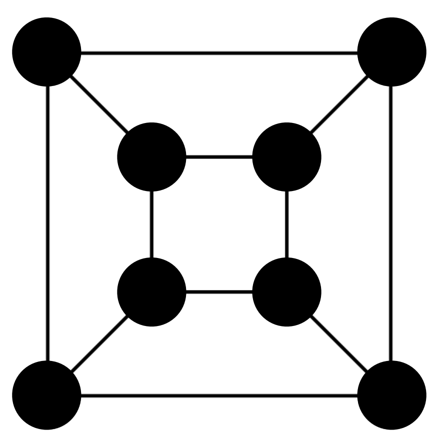


a graph $X(V,E)$ is a string

$$x : \binom{V}{2} \to \{0,1\}$$

1: edge

0: no edge

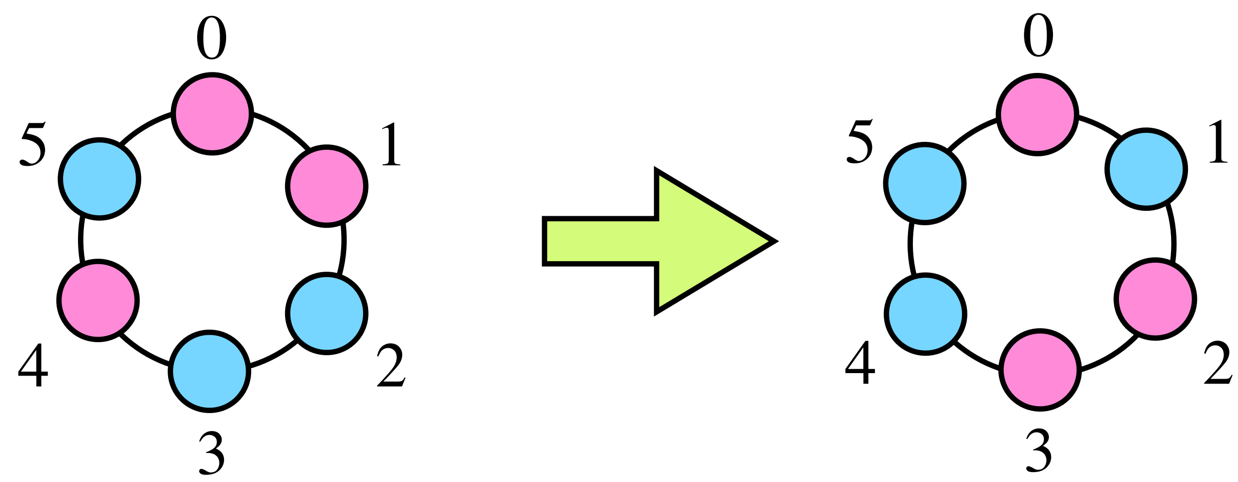
positions $\longleftrightarrow$ vertex-pairs

all permutations on V induces a permutation group on $\binom{V}{2}$

Johnson group: $\mathbb{S}_V^{(2)} \subset S_{\binom{V}{2}}$ on vertex pairs

two graphs $X \cong Y$ iff their string versions $x \cong_{\mathbb{S}_V^{(2)}} y$

Orbits



group action $\circ : G \times X \to X$

orbit of x : $Gx = \{\pi \circ x \mid \pi \in G\}$

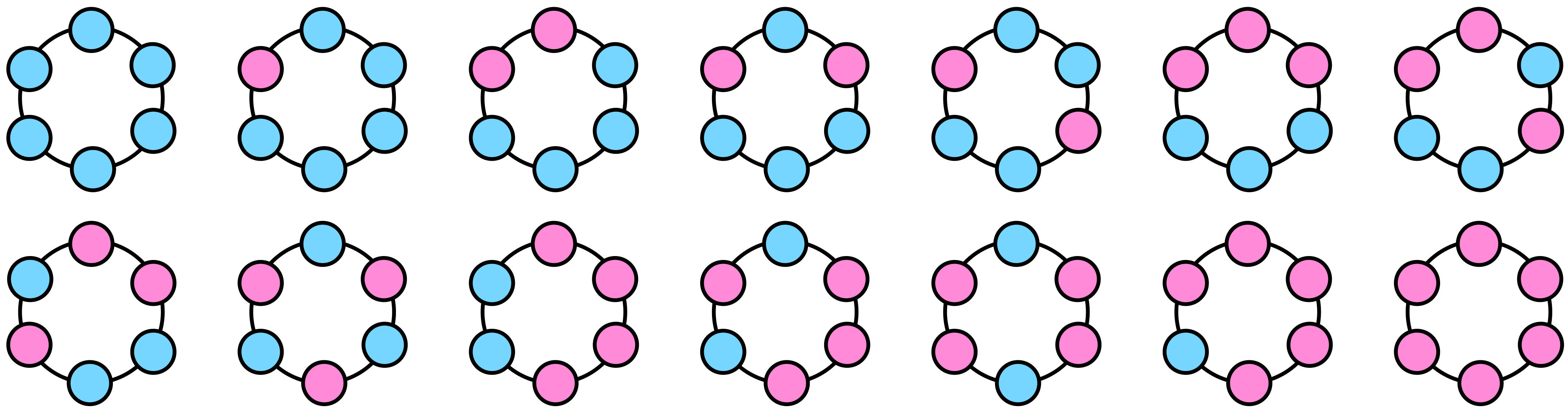
equivalent class of configuration x

$$X/G = \{Gx \mid x \in X\}$$

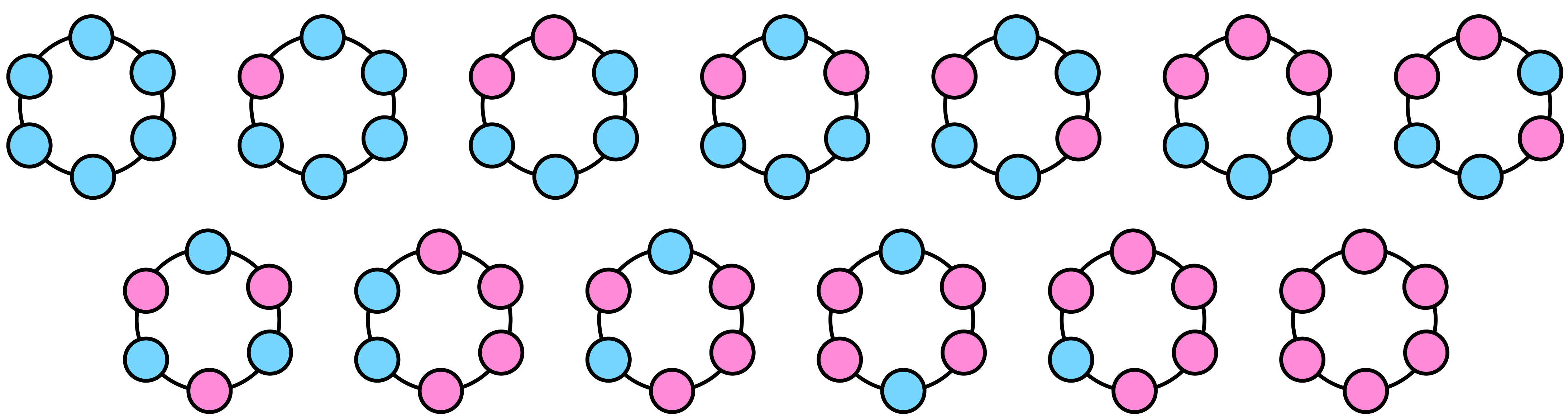
our goal: count $|X/G|$

$$X = [2]^{[n]}$$

$G = C_n$ $\quad$ $X/G:$



$G = D_n$ $\quad$ $X/G:$



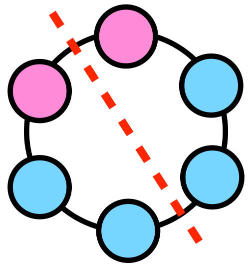
Burnside's Lemma

group action $\circ : G \times X \to X$

orbit of x : $Gx = \{\pi \circ x \mid \pi \in G\}$

$$X/G = \{Gx \mid x \in X\}$$

invariant set of π :



$$X_\pi = \{x \in X \mid \pi \circ x = x\}$$

Burnside's Lemma:

$$|X/G| = \frac{1}{|G|} \sum_{\pi \in G} |X_\pi|$$

group action $\circ : G \times X \to X$

orbit of x : $Gx = \{\pi \circ x \mid \pi \in G\}$

$$X/G = \{Gx \mid x \in X\}$$

invariant set of π : $X_\pi = \{x \in X \mid \pi \circ x = x\}$

stabilizer of x : $G_x = \{\pi \in G \mid \pi \circ x = x\}$

Lemma: $\forall x \in X, \quad |G_x||Gx| = |G|$

$$A(\pi, x) = \begin{cases} 1 & \pi \circ x = x \\ 0 & \text{otherwise} \end{cases}$$

$$|X_\pi| = \sum_{x \in X} A(\pi, x) \qquad |G_x| = \sum_{\pi \in G} A(\pi, x)$$

$$A(\pi, x) = \begin{cases} 1 & \pi \circ x = x \\ 0 & \text{otherwise} \end{cases}$$

$$|X_\pi| = \sum_{x \in X} A(\pi, x) \qquad |G_x| = \sum_{\pi \in G} A(\pi, x)$$

double counting:

$$\sum_{\pi \in G} |X_\pi| = \sum_{\pi \in G} \sum_{x \in X} A(\pi, x) = \sum_{x \in X} \sum_{\pi \in G} A(\pi, x) = \sum_{x \in X} |G_x|$$

Lemma: $\forall x \in X, \quad |G_x||Gx| = |G|$

$$= |G| \sum_{x \in X} \frac{1}{|Gx|}$$

$$\sum_{\pi \in G} |X_\pi| = |G| \sum_{x \in X} \frac{1}{|Gx|}$$

orbits: $X_1, X_2, \ldots, X_{|X/G|}$

a partition of X

$$= |G| \sum_{i=1}^{|X/G|} \sum_{x \in X_i} \frac{1}{|Gx|} = |G| \sum_{i=1}^{|X/G|} \sum_{x \in X_i} \frac{1}{|X_i|}$$

$$= |G| \sum_{i=1}^{|X/G|} 1 = |G||X/G|$$

$$\Rightarrow \quad |X/G| = \frac{1}{|G|} \sum_{\pi \in G} |X_\pi|$$

Burnside's Lemma

Burnside's Lemma

group action $\circ : G \times X \to X$

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$$X/G = \{Gx \mid x \in X\}$$

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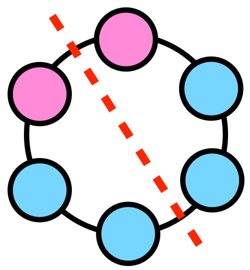


$$X_\pi = \{x \in X \mid \pi \circ x = x\}$$

Burnside's Lemma:

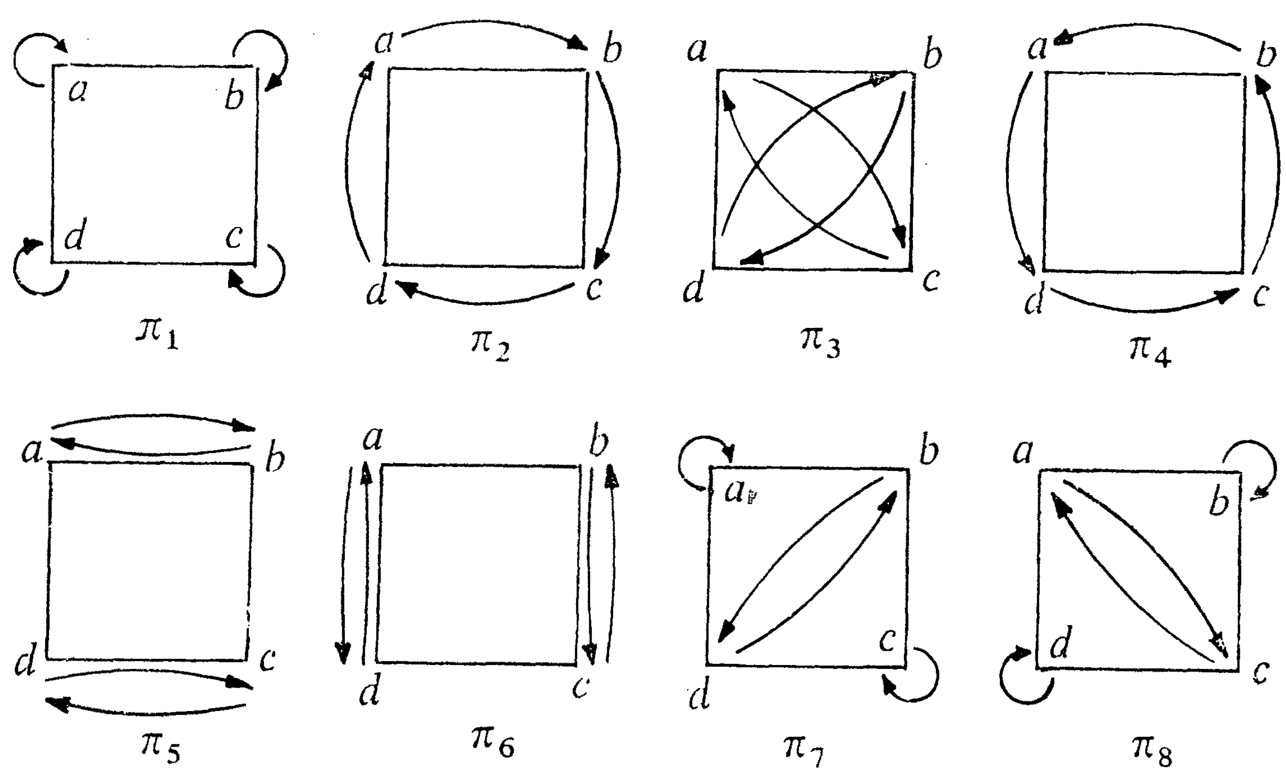
$$|X/G| = \frac{1}{|G|} \sum_{\pi \in G} |X_\pi|$$

Cycle Decomposition

permutation $\pi : [n] \xrightarrow[\text{on-to}]{\text{1-1}} [n]$

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & n-1 \\ \pi(0) & \pi(1) & \pi(2) & \cdots & \pi(n-1) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 4 & 0 & 1 & 3 \end{pmatrix} = (0\ 2)(1\ 4\ 3)$$



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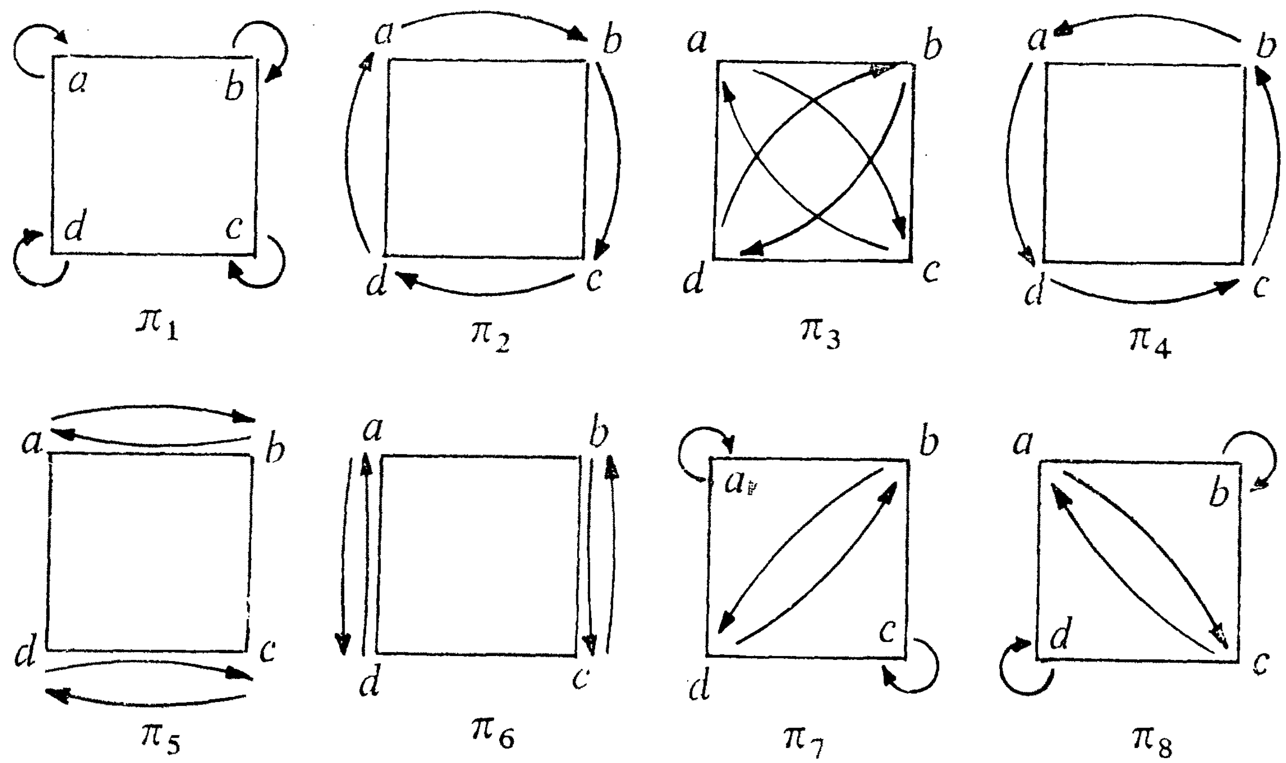
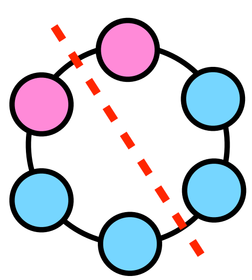


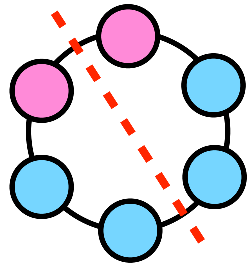


invariant

$\pi \circ x = x$

every cycle of π has the same color

$\forall i \in [n],\ x(\pi(i)) = x(i)$



invariant

$\pi \circ x = x$

⇒ every cycle of π has the same color

$$\forall i \in [n],\ x(\pi(i)) = x(i)$$

$X = [m]^{[n]}$ *m*-colorings of *n* positions

$$\pi = \underbrace{(\cdots)(\cdots)\cdots(\cdots)}_{k \text{ cycles}}$$

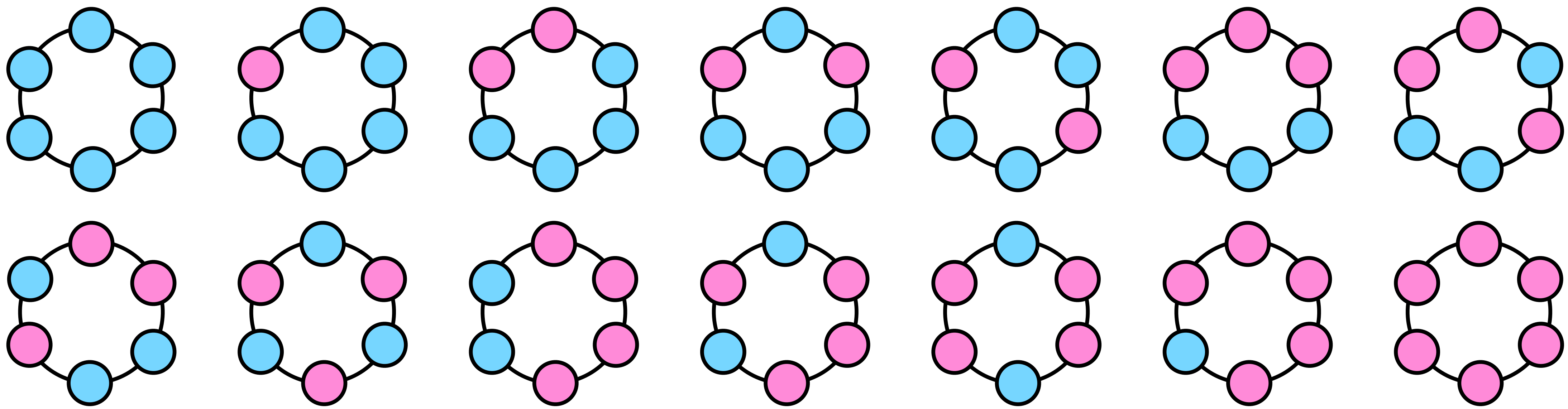
$$|X_\pi| = |\{x \in X \mid \pi \circ x = x\}| = m^k$$

Burnside's Lemma:

$$|X/G| = \frac{1}{|G|}\sum_{\pi \in G} |X_\pi| = \frac{1}{|G|}\sum_{\pi \in G} m^{\#\text{cycle}(\pi)}$$

$$X = [2]^{[n]}$$

$G = C_n$ $X/G:$



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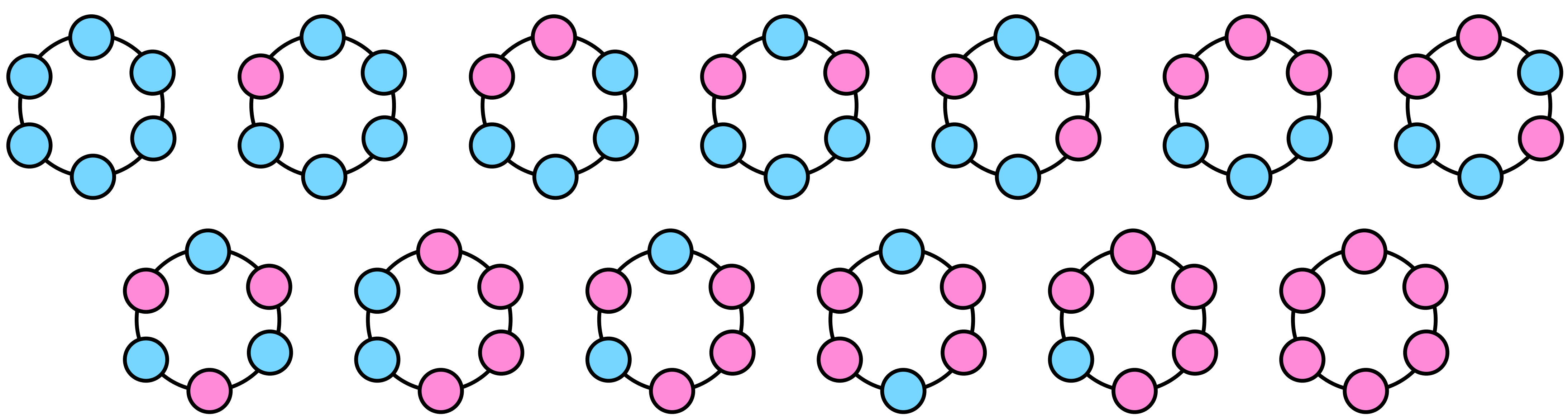
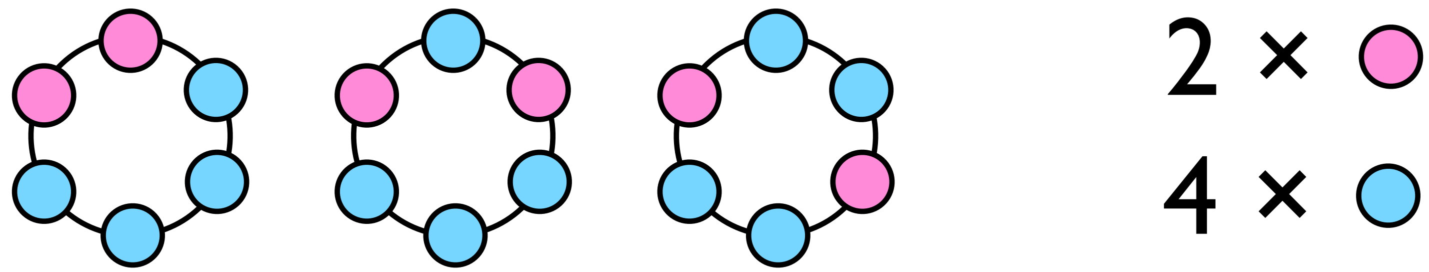


configuration $x : [n] \to [m]$



$$a_{(2,4)} = 3$$

$\vec{v} = (n_1, n_2, \ldots, n_m)$ **s.t.** $n_1 + n_2 + \cdots + n_m = n$

$a_{\vec{v}}$: # of config. (up to symmetry) with n_i many color i

pattern inventory :

(multi-variate) generating function

$$F_G(y_1, y_2, \ldots, y_m) = \sum_{\substack{\vec{v}=(n_1,\ldots,n_m) \\ n_1+\cdots+n_m=n}} a_{\vec{v}} \, y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

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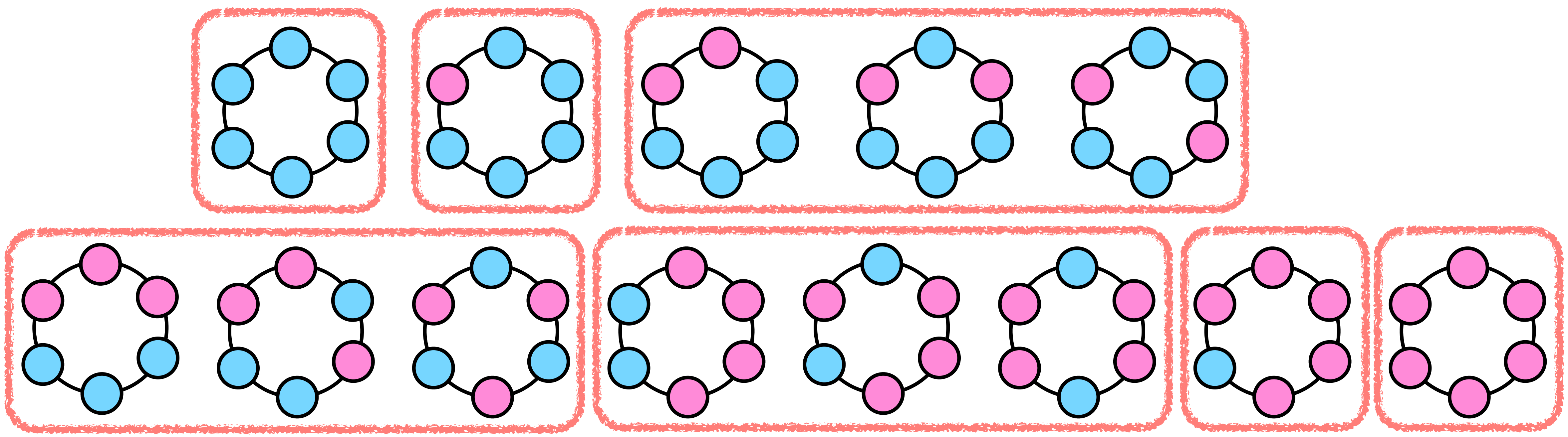
$G = D_n$ X/G :



$$F_{D_6}(y_1, y_2) = y_1^6 + y_1^5 y_2 + 3y_1^4 y_2^2 + 3y_1^3 y_2^3 + 3y_1^2 y_2^4 + y_1 y_2^5 + y_2^6$$

Cycle index

permutation $\pi : [n] \xrightarrow[\text{on-to}]{\text{1-1}} [n]$ from group G

$$\pi = \underbrace{\overbrace{(\cdots)}^{\ell_1}\overbrace{(\cdots)}^{\ell_2}\cdots\overbrace{(\cdots)}^{\ell_k}}_{k \text{ cycles}}$$

monomial: $M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{k} x_{\ell_i}$

cycle index:

$$P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|}\sum_{\pi \in G} M_\pi(x_1, x_2, \ldots, x_n)$$

Burnside's Lemma:

$$|X/G| = P_G(\underbrace{m, m, \ldots, m}_{n}) = \frac{1}{|G|}\sum_{\pi \in G} M_\pi(\underbrace{m, m, \ldots, m}_{n})$$

$$\pi = \overbrace{\underbrace{\overset{\ell_1}{\overbrace{(\cdots)}}\overset{\ell_2}{\overbrace{(\cdots)}}\cdots\overset{\ell_k}{\overbrace{(\cdots)}}}_{k\text{ cycles}}} \qquad M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{k} x_{\ell_i}$$

cycle index: $P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|}\sum_{\pi\in G} M_\pi(x_1, x_2, \ldots, x_n)$

pattern inventory : (multi-variate) generating function

$$F_G(y_1, y_2, \ldots, y_m) = \sum_{\substack{\vec{v}=(n_1,\ldots,n_m) \\ n_1+\cdots+n_m=n}} a_{\vec{v}}\; y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

$a_{\vec{v}}$: # of config. (up to symmetry) with n_i many color i

Pólya's enumeration formula (1937,1987):

$$F_G(y_1, y_2, \ldots, y_m) = P_G\left(\sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \ldots, \sum_{i=1}^{m} y_i^n\right)$$

G. Pólya (1937). Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Mathematica* 68 (1): 145–254.

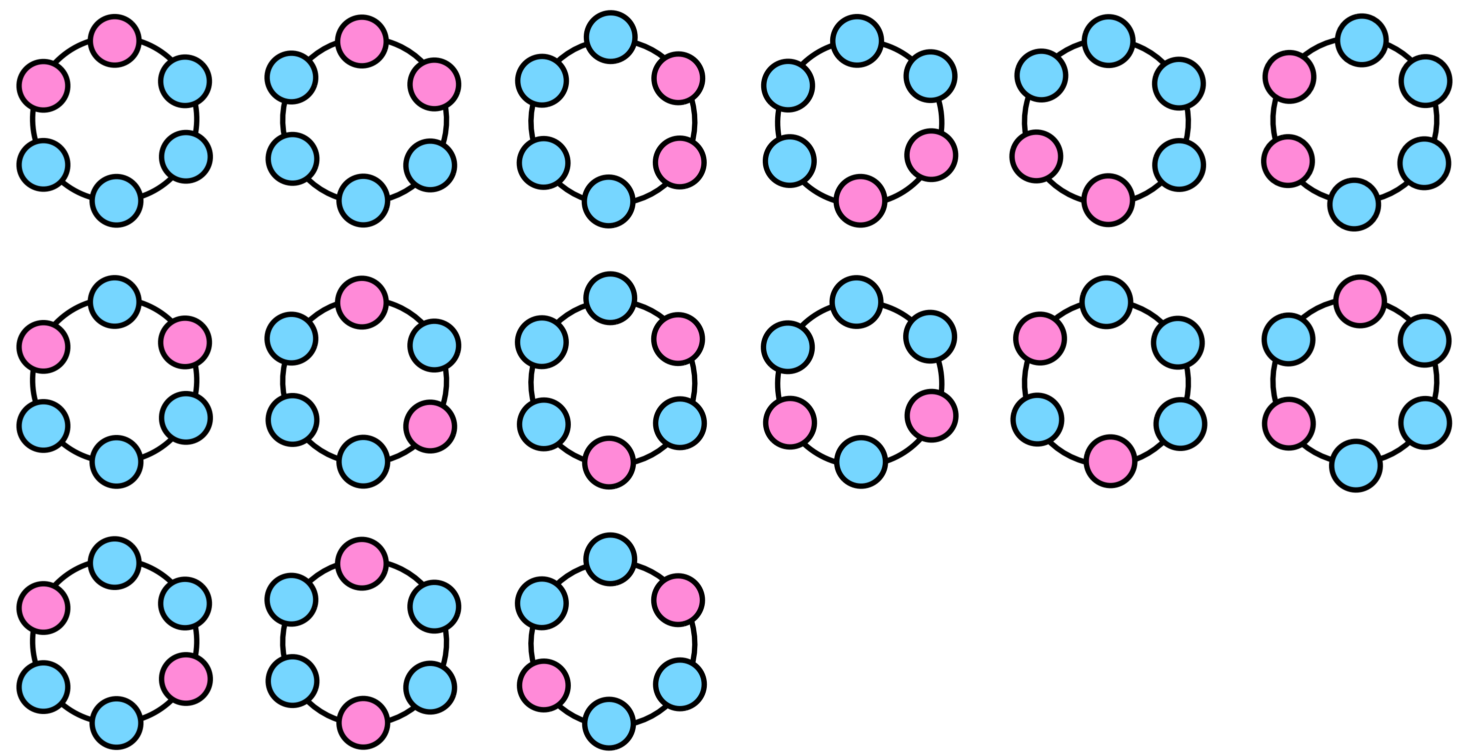
G. Pólya; R. C. Read (1987). *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*. Springer-Verlag.

$$\vec{v} = (n_1, n_2, \ldots, n_m) \qquad n_1 + n_2 + \cdots + n_m = n$$

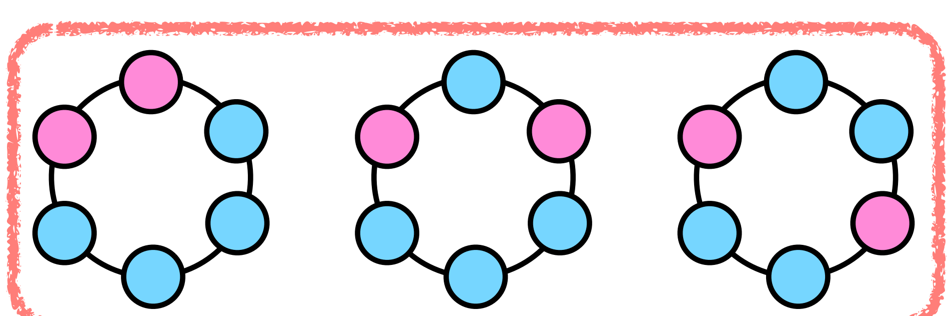
$$X = [m]^{[n]} \qquad X^{\vec{v}} = \{x \in [m]^{[n]} \mid \forall i \in [m], x^{-1}(i) = n_i\}$$

invariant set of π: $X_\pi^{\vec{v}} = \{x \in X^{\vec{v}} \mid \pi \circ x = x\}$

$X^{(2,4)}$:



$$a_{(2,4)} = 3$$

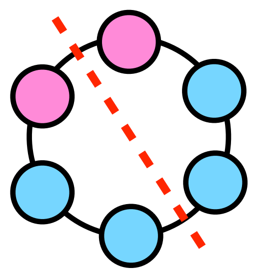


Burnside's Lemma:

$$a_{\vec{v}} = \frac{1}{|G|} \sum_{\pi \in G} |X_\pi^{\vec{v}}|$$

$$\pi = \overbrace{(\cdots)}^{\ell_1}\overbrace{(\cdots)}^{\ell_2}\cdots\overbrace{(\cdots)}^{\ell_k}$$
k cycles

$$X_\pi^{\vec{v}} = \{x \in X^{\vec{v}} \mid \pi \circ x = x\}$$



invariant

$$\pi \circ x = x$$

$$\forall i \in [n],\ x(\pi(i)) = x(i)$$

every cycle of π has the same color

$$M_\pi\left(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \ldots, \sum_{i=1}^m y_i^n\right) =$$

$$(y_1^{\ell_1} + y_2^{\ell_1} + \cdots + y_m^{\ell_1})(y_1^{\ell_2} + y_2^{\ell_2} + \cdots + y_m^{\ell_2})\cdots(y_1^{\ell_k} + y_2^{\ell_k} + \cdots + y_m^{\ell_k})$$

$$= \sum_{\substack{\vec{v}=(n_1,\ldots,n_m)\\ n_1+\cdots+n_m=n}} |X_\pi^{\vec{v}}| y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

recall: $M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^k x_{\ell_i}$

$$X_\pi^{\vec{v}} = \{x \in X^{\vec{v}} \mid \pi \circ x = x\}$$

$$M_\pi\left(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \ldots, \sum_{i=1}^m y_i^n\right)$$

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Burnside's Lemma:

$$a_{\vec{v}} = \frac{1}{|G|} \sum_{\pi \in G} |X_\pi^{\vec{v}}|$$

pattern inventory : $F_G(y_1, y_2, \ldots, y_m)$

$$= \sum_{\substack{\vec{v}=(n_1,\ldots,n_m) \\ n_1+\cdots+n_m=n}} a_{\vec{v}}\, y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m} = \sum_{\substack{\vec{v}=(n_1,\ldots,n_m) \\ n_1+\cdots+n_m=n}} \left(\frac{1}{|G|}\sum_{\pi \in G} |X_\pi^{\vec{v}}|\right) y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

$$= \frac{1}{|G|} \sum_{\pi \in G} \sum_{\substack{\vec{v}=(n_1,\ldots,n_m) \\ n_1+\cdots+n_m=n}} |X_\pi^{\vec{v}}| y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

cycle index :

$$= \frac{1}{|G|} \sum_{\pi \in G} M_\pi\left(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \ldots, \sum_{i=1}^m y_i^n\right) = P_G\left(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \ldots, \sum_{i=1}^m y_i^n\right)$$

$$\pi = \underbrace{\overbrace{(\cdots)}^{\ell_1}\overbrace{(\cdots)}^{\ell_2}\cdots\overbrace{(\cdots)}^{\ell_k}}_{k\text{ cycles}} \qquad M_\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{k} x_{\ell_i}$$

cycle index: $P_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|}\sum_{\pi\in G} M_\pi(x_1, x_2, \ldots, x_n)$

pattern inventory : (multi-variate) generating function

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$$F_{D_{20}}(r, q, l)$$

$$
\begin{aligned}
F_{D_{20}}(r,q,l) = {} & r^{20} + r^{19}q + r^{19}l + 10r^{18}q^2 + 10r^{18}ql + 10r^{18}l^2 + 33r^{17}q^3 + 90r^{17}q^2l \\
& + 90r^{17}ql^2 + 33r^{17}l^3 + 145r^{16}q^4 + 489r^{16}q^3l + 774r^{16}q^2l^2 + 489r^{16}ql^3 + 145r^{16}l^4 \\
& + 406r^{15}q^5 + 1956r^{15}q^4l + 3912r^{15}q^3l^2 + 3912r^{15}q^2l^3 + 1956r^{15}ql^4 + 406r^{15}l^5 \\
& + 1032r^{14}q^6 + 5832r^{14}q^5l + 14724r^{14}q^4l^2 + 19416r^{14}q^3l^3 + 14724r^{14}q^2l^4 \\
& + 5832r^{14}ql^5 + 1032r^{14}l^6 + 1980r^{13}q^7 + 13608r^{13}q^6l + 40824r^{13}q^5l^2 \\
& + 67956r^{13}q^4l^3 + 67956r^{13}q^3l^4 + 40824r^{13}q^2l^5 + 13608r^{13}ql^6 + 1980r^{13}l^7 \\
& + 3260r^{12}q^8 + 25236r^{12}q^7l + 88620r^{12}q^6l^2 + 176484r^{12}q^5l^3 + 221110r^{12}q^4l^4 \\
& + 176484r^{12}q^3l^5 + 88620r^{12}q^2l^6 + 25236r^{12}ql^7 + 3260r^{12}l^8 + 4262r^{11}q^9 \\
& + 37854r^{11}q^8l + 151416r^{11}q^7l^2 + 352968r^{11}q^6l^3 + 529452r^{11}q^5l^4 + 529452r^{11}q^4l^5 \\
& + 352968r^{11}q^3l^6 + 151416r^{11}q^2l^7 + 37854r^{11}ql^8 + 4262r^{11}l^9 + 4752r^{10}q^{10} \\
& + 46252r^{10}q^9l + 208512r^{10}q^8l^2 + 554520r^{10}q^7l^3 + 971292r^{10}q^6l^4 + 1164342r^{10}q^5l^5 \\
& + 971292r^{10}q^4l^6 + 554520r^{10}q^3l^7 + 208512r^{10}q^2l^8 + 46252r^{10}ql^9 + 4752r^{10}l^{10} \\
& + 4262r^9q^{11} + 46252r^9q^{10}l + 231260r^9q^9l^2 + 693150r^9q^8l^3 + 1386300r^9q^7l^4 \\
& + 1940568r^9q^6l^5 + 1940568r^9q^5l^6 + 1386300r^9q^4l^7 + 693150r^9q^3l^8 + 231260r^9q^2l^9 \\
& + 46252r^9ql^{10} + 4262r^9l^{11} + 3260r^8q^{12} + 37854r^8q^{11}l + 208512r^8q^{10}l^2 \\
& + 693150r^8q^9l^3 + 1560534r^8q^8l^4 + 2494836r^8q^7l^5 + 2912112r^8q^6l^6 + 2494836r^8q^5l^7 \\
& + 1560534r^8q^4l^8 + 693150r^8q^3l^9 + 208512r^8q^2l^{10} + 37854r^8ql^{11} + 3260r^8l^{12} \\
& + 1980r^7q^{13} + 25236r^7q^{12}l + 151416r^7q^{11}l^2 + 554520r^7q^{10}l^3 + 1386300r^7q^9l^4 \\
& + 2494836r^7q^8l^5 + 3326448r^7q^7l^6 + 3326448r^7q^6l^7 + 2494836r^7q^5l^8 + 1386300r^7q^4l^9 \\
& + 554520r^7q^3l^{10} + 151416r^7q^2l^{11} + 25236r^7ql^{12} + 1980r^7l^{13} + 1032r^6q^{14} \\
& + 13608r^6q^{13}l + 88620r^6q^{12}l^2 + 352968r^6q^{11}l^3 + 971292r^6q^{10}l^4 + 1940568r^6q^9l^5 \\
& + 2912112r^6q^8l^6 + 3326448r^6q^7l^7 + 2912112r^6q^6l^8 + 1940568r^6q^5l^9 + 971292r^6q^4l^{10} \\
& + 352968r^6q^3l^{11} + 88620r^6q^2l^{12} + 13608r^6ql^{13} + 1032r^6l^{14} + 406r^5q^{15} + 5832r^5q^{14}l \\
& + 40824r^5q^{13}l^2 + 176484r^5q^{12}l^3 + 529452r^5q^{11}l^4 + 1164342r^5q^{10}l^5 + 1940568r^5q^9l^6 \\
& + 2494836r^5q^8l^7 + 2494836r^5q^7l^8 + 1940568r^5q^6l^9 + 1164342r^5q^5l^{10} + 529452r^5q^4l^{11} \\
& + 176484r^5q^3l^{12} + 40824r^5q^2l^{13} + 5832r^5ql^{14} + 406r^5l^{15} + 145r^4q^{16} + 1956r^4q^{15}l \\
& + 14724r^4q^{14}l^2 + 67956r^4q^{13}l^3 + 221110r^4q^{12}l^4 + 529452r^4q^{11}l^5 + 971292r^4q^{10}l^6 \\
& + 1386300r^4q^9l^7 + 1560534r^4q^8l^8 + 1386300r^4q^7l^9 + 971292r^4q^6l^{10} + 529452r^4q^5l^{11} \\
& + 221110r^4q^4l^{12} + 67956r^4q^3l^{13} + 14724r^4q^2l^{14} + 1956r^4ql^{15} + 145r^4l^{16} + 33r^3q^{17} \\
& + 489r^3q^{16}l + 3912r^3q^{15}l^2 + 19416r^3q^{14}l^3 + 67956r^3q^{13}l^4 + 176484r^3q^{12}l^5 \\
& + 352968r^3q^{11}l^6 + 554520r^3q^{10}l^7 + 693150r^3q^9l^8 + 693150r^3q^8l^9 + 554520r^3q^7l^{10} \\
& + 352968r^3q^6l^{11} + 176484r^3q^5l^{12} + 67956r^3q^4l^{13} + 19416r^3q^3l^{14} + 3912r^3q^2l^{15} \\
& + 489r^3ql^{16} + 33r^3l^{17} + 10r^2q^{18} + 90r^2q^{17}l + 774r^2q^{16}l^2 + 3912r^2q^{15}l^3 \\
& + 14724r^2q^{14}l^4 + 40824r^2q^{13}l^5 + 88620r^2q^{12}l^6 + 151416r^2q^{11}l^7 + 208512r^2q^{10}l^8 \\
& + \mathbf{231260r^2q^9l^9} + 208512r^2q^8l^{10} + 151416r^2q^7l^{11} + 88620r^2q^6l^{12} + 40824r^2q^5l^{13} \\
& + 14724r^2q^4l^{14} + 3912r^2q^3l^{15} + 774r^2q^2l^{16} + 90r^2ql^{17} + 10r^2l^{18} + rq^{19} + 10rq^{18}l \\
& + 90rq^{17}l^2 + 489rq^{16}l^3 + 1956rq^{15}l^4 + 5832rq^{14}l^5 + 13608rq^{13}l^6 + 25236rq^{12}l^7 \\
& + 37854rq^{11}l^8 + 46252rq^{10}l^9 + 46252rq^9l^{10} + 37854rq^8l^{11} + 25236rq^7l^{12} \\
& + 13608rq^6l^{13} + 5832rq^5l^{14} + 1956rq^4l^{15} + 489rq^3l^{16} + 90rq^2l^{17} + 10rql^{18} + rl^{19} \\
& + q^{20} + q^{19}l + 10q^{18}l^2 + 33q^{17}l^3 + 145q^{16}l^4 + 406q^{15}l^5 + 1032q^{14}l^6 + 1980q^{13}l^7 \\
& + 3260q^{12}l^8 + 4262q^{11}l^9 + 4752q^{10}l^{10} + 4262q^9l^{11} + 3260q^8l^{12} + 1980q^7l^{13} \\
& + 1032q^6l^{14} + 406q^5l^{15} + 145q^4l^{16} + 33q^3l^{17} + 10q^2l^{18} + ql^{19} + l^{20}
\end{aligned}
$$

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\begin{aligned}
F_{D_{20}}(r,q,l) = {} & r^{20} + r^{19}q + r^{19}l + 10r^{18}q^2 + 10r^{18}ql + 10r^{18}l^2 + 33r^{17}q^3 + 90r^{17}q^2l \\
& + 90r^{17}ql^2 + 33r^{17}l^3 + 145r^{16}q^4 + 489r^{16}q^3l + 774r^{16}q^2l^2 + 489r^{16}ql^3 + 145r^{16}l^4 \\
& + 406r^{15}q^5 + 1956r^{15}q^4l + 3912r^{15}q^3l^2 + 3912r^{15}q^2l^3 + 1956r^{15}ql^4 + 406r^{15}l^5 \\
& + 1032r^{14}q^6 + 5832r^{14}q^5l + 14724r^{14}q^4l^2 + 19416r^{14}q^3l^3 + 14724r^{14}q^2l^4 \\
& + 5832r^{14}ql^5 + 1032r^{14}l^6 + 1980r^{13}q^7 + 13608r^{13}q^6l + 40824r^{13}q^5l^2 \\
& + 67956r^{13}q^4l^3 + 67956r^{13}q^3l^4 + 40824r^{13}q^2l^5 + 13608r^{13}ql^6 + 1980r^{13}l^7 \\
& + 3260r^{12}q^8 + 25236r^{12}q^7l + 88620r^{12}q^6l^2 + 176484r^{12}q^5l^3 + 221110r^{12}q^4l^4 \\
& + 176484r^{12}q^3l^5 + 88620r^{12}q^2l^6 + 25236r^{12}ql^7 + 3260r^{12}l^8 + 4262r^{11}q^9 \\
& + 37854r^{11}q^8l + 151416r^{11}q^7l^2 + 352968r^{11}q^6l^3 + 529452r^{11}q^5l^4 + 529452r^{11}q^4l^5 \\
& + 352968r^{11}q^3l^6 + 151416r^{11}q^2l^7 + 37854r^{11}ql^8 + 4262r^{11}l^9 + 4752r^{10}q^{10} \\
& + 46252r^{10}q^9l + 208512r^{10}q^8l^2 + 554520r^{10}q^7l^3 + 971292r^{10}q^6l^4 + 1164342r^{10}q^5l^5 \\
& + 971292r^{10}q^4l^6 + 554520r^{10}q^3l^7 + 208512r^{10}q^2l^8 + 46252r^{10}ql^9 + 4752r^{10}l^{10} \\
& + 4262r^9q^{11} + 46252r^9q^{10}l + 231260r^9q^9l^2 + 693150r^9q^8l^3 + 1386300r^9q^7l^4 \\
& + 1940568r^9q^6l^5 + 1940568r^9q^5l^6 + 1386300r^9q^4l^7 + 693150r^9q^3l^8 + 231260r^9q^2l^9 \\
& + 46252r^9ql^{10} + 4262r^9l^{11} + 3260r^8q^{12} + 37854r^8q^{11}l + 208512r^8q^{10}l^2 \\
& + 693150r^8q^9l^3 + 1560534r^8q^8l^4 + 2494836r^8q^7l^5 + 2912112r^8q^6l^6 + 2494836r^8q^5l^7 \\
& + 1560534r^8q^4l^8 + 693150r^8q^3l^9 + 208512r^8q^2l^{10} + 37854r^8ql^{11} + 3260r^8l^{12} \\
& + 1980r^7q^{13} + 25236r^7q^{12}l + 151416r^7q^{11}l^2 + 554520r^7q^{10}l^3 + 1386300r^7q^9l^4 \\
& + 2494836r^7q^8l^5 + 3326448r^7q^7l^6 + 3326448r^7q^6l^7 + 2494836r^7q^5l^8 + 1386300r^7q^4l^9 \\
& + 554520r^7q^3l^{10} + 151416r^7q^2l^{11} + 25236r^7ql^{12} + 1980r^7l^{13} + 1032r^6q^{14} \\
& + 13608r^6q^{13}l + 88620r^6q^{12}l^2 + 352968r^6q^{11}l^3 + 971292r^6q^{10}l^4 + 1940568r^6q^9l^5
\end{aligned}
$$

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\begin{aligned}
&+2912112r^6q^8l^6+3326448r^6q^7l^7+2912112r^6q^6l^8+1940568r^6q^5l^9+971292r^6q^4l^{10}\\
&+352968r^6q^3l^{11}+88620r^6q^2l^{12}+13608r^6ql^{13}+1032r^6l^{14}+406r^5q^{15}+5832r^5q^{14}l\\
&+40824r^5q^{13}l^2+176484r^5q^{12}l^3+529452r^5q^{11}l^4+1164342r^5q^{10}l^5+1940568r^5q^9l^6\\
&+2494836r^5q^8l^7+2494836r^5q^7l^8+1940568r^5q^6l^9+1164342r^5q^5l^{10}+529452r^5q^4l^{11}\\
&+176484r^5q^3l^{12}+40824r^5q^2l^{13}+5832r^5ql^{14}+406r^5l^{15}+145r^4q^{16}+1956r^4q^{15}l\\
&+14724r^4q^{14}l^2+67956r^4q^{13}l^3+221110r^4q^{12}l^4+529452r^4q^{11}l^5+971292r^4q^{10}l^6\\
&+1386300r^4q^9l^7+1560534r^4q^8l^8+1386300r^4q^7l^9+971292r^4q^6l^{10}+529452r^4q^5l^{11}\\
&+221110r^4q^4l^{12}+67956r^4q^3l^{13}+14724r^4q^2l^{14}+1956r^4ql^{15}+145r^4l^{16}+33r^3q^{17}\\
&+489r^3q^{16}l+3912r^3q^{15}l^2+19416r^3q^{14}l^3+67956r^3q^{13}l^4+176484r^3q^{12}l^5\\
&+352968r^3q^{11}l^6+554520r^3q^{10}l^7+693150r^3q^9l^8+693150r^3q^8l^9+554520r^3q^7l^{10}\\
&+352968r^3q^6l^{11}+176484r^3q^5l^{12}+67956r^3q^4l^{13}+19416r^3q^3l^{14}+3912r^3q^2l^{15}\\
&+489r^3ql^{16}+33r^3l^{17}+10r^2q^{18}+90r^2q^{17}l+774r^2q^{16}l^2+3912r^2q^{15}l^3\\
&+14724r^2q^{14}l^4+40824r^2q^{13}l^5+88620r^2q^{12}l^6+151416r^2q^{11}l^7+208512r^2q^{10}l^8\\
&+\mathbf{231260r^2q^9l^9}+208512r^2q^8l^{10}+151416r^2q^7l^{11}+88620r^2q^6l^{12}+40824r^2q^5l^{13}\\
&+14724r^2q^4l^{14}+3912r^2q^3l^{15}+774r^2q^2l^{16}+90r^2ql^{17}+10r^2l^{18}+rq^{19}+10rq^{18}l\\
&+90rq^{17}l^2+489rq^{16}l^3+1956rq^{15}l^4+5832rq^{14}l^5+13608rq^{13}l^6+25236rq^{12}l^7\\
&+37854rq^{11}l^8+46252rq^{10}l^9+46252rq^9l^{10}+37854rq^8l^{11}+25236rq^7l^{12}\\
&+13608rq^6l^{13}+5832rq^5l^{14}+1956rq^4l^{15}+489rq^3l^{16}+90rq^2l^{17}+10rql^{18}+rl^{19}\\
&+q^{20}+q^{19}l+10q^{18}l^2+33q^{17}l^3+145q^{16}l^4+406q^{15}l^5+1032q^{14}l^6+1980q^{13}l^7\\
&+3260q^{12}l^8+4262q^{11}l^9+4752q^{10}l^{10}+4262q^9l^{11}+3260q^8l^{12}+1980q^7l^{13}\\
&+1032q^6l^{14}+406q^5l^{15}+145q^4l^{16}+33q^3l^{17}+10q^2l^{18}+ql^{19}+l^{20}
\end{aligned}
$$

$$
\begin{aligned}
F_{D_{20}}(r,q,l) = {} & r^{20} + r^{19}q + r^{19}l + 10r^{18}q^2 + 10r^{18}ql + 10r^{18}l^2 + 33r^{17}q^3 + 90r^{17}q^2l \\
& + 90r^{17}ql^2 + 33r^{17}l^3 + 145r^{16}q^4 + 489r^{16}q^3l + 774r^{16}q^2l^2 + 489r^{16}ql^3 + 145r^{16}l^4 \\
& + 406r^{15}q^5 + 1956r^{15}q^4l + 3912r^{15}q^3l^2 + 3912r^{15}q^2l^3 + 1956r^{15}ql^4 + 406r^{15}l^5 \\
& + 1032r^{14}q^6 + 5832r^{14}q^5l + 14724r^{14}q^4l^2 + 19416r^{14}q^3l^3 + 14724r^{14}q^2l^4 \\
& + 5832r^{14}ql^5 + 1032r^{14}l^6 + 1980r^{13}q^7 + 13608r^{13}q^6l + 40824r^{13}q^5l^2 \\
& + 67956r^{13}q^4l^3 + 67956r^{13}q^3l^4 + 40824r^{13}q^2l^5 + 13608r^{13}ql^6 + 1980r^{13}l^7 \\
& + 3260r^{12}q^8 + 25236r^{12}q^7l + 88620r^{12}q^6l^2 + 176484r^{12}q^5l^3 + 221110r^{12}q^4l^4 \\
& + 176484r^{12}q^3l^5 + 88620r^{12}q^2l^6 + 25236r^{12}ql^7 + 3260r^{12}l^8 + 4262r^{11}q^9 \\
& + 37854r^{11}q^8l + 151416r^{11}q^7l^2 + 352968r^{11}q^6l^3 + 529452r^{11}q^5l^4 + 529452r^{11}q^4l^5 \\
& + 352968r^{11}q^3l^6 + 151416r^{11}q^2l^7 + 37854r^{11}ql^8 + 4262r^{11}l^9 + 4752r^{10}q^{10} \\
& + 46252r^{10}q^9l + 208512r^{10}q^8l^2 + 554520r^{10}q^7l^3 + 971292r^{10}q^6l^4 + 1164342r^{10}q^5l^5 \\
& + 971292r^{10}q^4l^6 + 554520r^{10}q^3l^7 + 208512r^{10}q^2l^8 + 46252r^{10}ql^9 + 4752r^{10}l^{10} \\
& + 4262r^9q^{11} + 46252r^9q^{10}l + 231260r^9q^9l^2 + 693150r^9q^8l^3 + 1386300r^9q^7l^4 \\
& + 1940568r^9q^6l^5 + 1940568r^9q^5l^6 + 1386300r^9q^4l^7 + 693150r^9q^3l^8 + 231260r^9q^2l^9 \\
& + 46252r^9ql^{10} + 4262r^9l^{11} + 3260r^8q^{12} + 37854r^8q^{11}l + 208512r^8q^{10}l^2 \\
& + 693150r^8q^9l^3 + 1560534r^8q^8l^4 + 2494836r^8q^7l^5 + 2912112r^8q^6l^6 + 2494836r^8q^5l^7 \\
& + 1560534r^8q^4l^8 + 693150r^8q^3l^9 + 208512r^8q^2l^{10} + 37854r^8ql^{11} + 3260r^8l^{12} \\
& + 1980r^7q^{13} + 25236r^7q^{12}l + 151416r^7q^{11}l^2 + 554520r^7q^{10}l^3 + 1386300r^7q^9l^4 \\
& + 2494836r^7q^8l^5 + 3326448r^7q^7l^6 + 3326448r^7q^6l^7 + 2494836r^7q^5l^8 + 1386300r^7q^4l^9 \\
& + 554520r^7q^3l^{10} + 151416r^7q^2l^{11} + 25236r^7ql^{12} + 1980r^7l^{13} + 1032r^6q^{14} \\
& + 13608r^6q^{13}l + 88620r^6q^{12}l^2 + 352968r^6q^{11}l^3 + 971292r^6q^{10}l^4 + 1940568r^6q^9l^5 \\
& + 2912112r^6q^8l^6 + 3326448r^6q^7l^7 + 2912112r^6q^6l^8 + 1940568r^6q^5l^9 + 971292r^6q^4l^{10} \\
& + 352968r^6q^3l^{11} + 88620r^6q^2l^{12} + 13608r^6ql^{13} + 1032r^6l^{14} + 406r^5q^{15} + 5832r^5q^{14}l \\
& + 40824r^5q^{13}l^2 + 176484r^5q^{12}l^3 + 529452r^5q^{11}l^4 + 1164342r^5q^{10}l^5 + 1940568r^5q^9l^6 \\
& + 2494836r^5q^8l^7 + 2494836r^5q^7l^8 + 1940568r^5q^6l^9 + 1164342r^5q^5l^{10} + 529452r^5q^4l^{11} \\
& + 176484r^5q^3l^{12} + 40824r^5q^2l^{13} + 5832r^5ql^{14} + 406r^5l^{15} + 145r^4q^{16} + 1956r^4q^{15}l \\
& + 14724r^4q^{14}l^2 + 67956r^4q^{13}l^3 + 221110r^4q^{12}l^4 + 529452r^4q^{11}l^5 + 971292r^4q^{10}l^6 \\
& + 1386300r^4q^9l^7 + 1560534r^4q^8l^8 + 1386300r^4q^7l^9 + 971292r^4q^6l^{10} + 529452r^4q^5l^{11} \\
& + 221110r^4q^4l^{12} + 67956r^4q^3l^{13} + 14724r^4q^2l^{14} + 1956r^4ql^{15} + 145r^4l^{16} + 33r^3q^{17} \\
& + 489r^3q^{16}l + 3912r^3q^{15}l^2 + 19416r^3q^{14}l^3 + 67956r^3q^{13}l^4 + 176484r^3q^{12}l^5 \\
& + 352968r^3q^{11}l^6 + 554520r^3q^{10}l^7 + 693150r^3q^9l^8 + 693150r^3q^8l^9 + 554520r^3q^7l^{10} \\
& + 352968r^3q^6l^{11} + 176484r^3q^5l^{12} + 67956r^3q^4l^{13} + 19416r^3q^3l^{14} + 3912r^3q^2l^{15} \\
& + 489r^3ql^{16} + 33r^3l^{17} + 10r^2q^{18} + 90r^2q^{17}l + 774r^2q^{16}l^2 + 3912r^2q^{15}l^3 \\
& + 14724r^2q^{14}l^4 + 40824r^2q^{13}l^5 + 88620r^2q^{12}l^6 + 151416r^2q^{11}l^7 + 208512r^2q^{10}l^8 \\
& + \mathbf{231260r^2q^9l^9} + 208512r^2q^8l^{10} + 151416r^2q^7l^{11} + 88620r^2q^6l^{12} + 40824r^2q^5l^{13} \\
& + 14724r^2q^4l^{14} + 3912r^2q^3l^{15} + 774r^2q^2l^{16} + 90r^2ql^{17} + 10r^2l^{18} + rq^{19} + 10rq^{18}l \\
& + 90rq^{17}l^2 + 489rq^{16}l^3 + 1956rq^{15}l^4 + 5832rq^{14}l^5 + 13608rq^{13}l^6 + 25236rq^{12}l^7 \\
& + 37854rq^{11}l^8 + 46252rq^{10}l^9 + 46252rq^9l^{10} + 37854rq^8l^{11} + 25236rq^7l^{12} \\
& + 13608rq^6l^{13} + 5832rq^5l^{14} + 1956rq^4l^{15} + 489rq^3l^{16} + 90rq^2l^{17} + 10rql^{18} + rl^{19} \\
& + q^{20} + q^{19}l + 10q^{18}l^2 + 33q^{17}l^3 + 145q^{16}l^4 + 406q^{15}l^5 + 1032q^{14}l^6 + 1980q^{13}l^7 \\
& + 3260q^{12}l^8 + 4262q^{11}l^9 + 4752q^{10}l^{10} + 4262q^9l^{11} + 3260q^8l^{12} + 1980q^7l^{13} \\
& + 1032q^6l^{14} + 406q^5l^{15} + 145q^4l^{16} + 33q^3l^{17} + 10q^2l^{18} + ql^{19} + l^{20}
\end{aligned}
$$

Orbits of Group Actions