Combinatorics

Ramsey Theory

尹一通 Nanjing University, 2023 Spring

Ramsey's Theorem

Frank P. Ramsey
(1903-1930)

"In any party of six people, either at least three of them are mutual strangers or at least three of them are mutual acquaintances"

Color edges of K_6 with 2 colors.

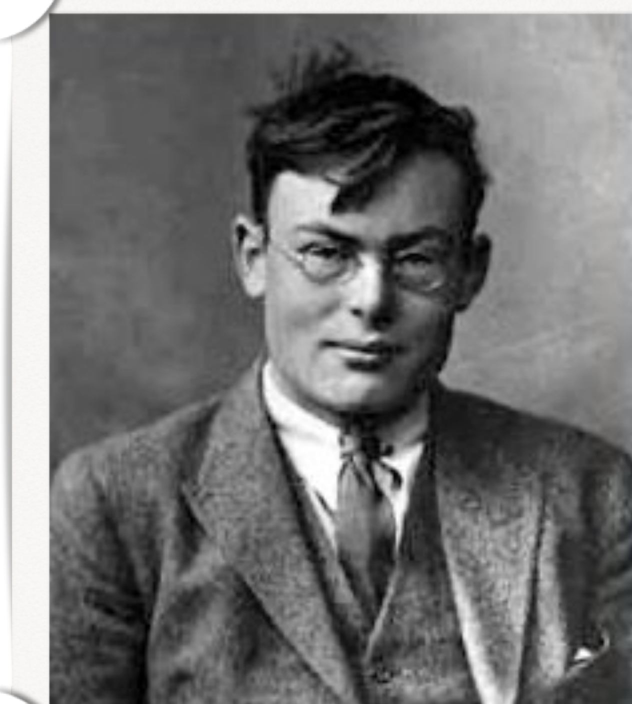
There must be a monochromatic K_3.

ON A PROBLEM OF FORMAL LOGIC

By F. P. Ramsey.

[Received 28 November, 1928.—Read 13 December, 1928.]

This paper is primarily concerned with a special case of one of the leading problems of mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula*. But in the course of this investigation it is necessary to use certain theorems on combinations which have an independent interest and are most conveniently set out by themselves beforehand.



Frank P. Ramsey
(1903-1930)

$R(k,l) \triangleq$ the smallest integer satisfying:

if $n \geq R(k,l)$, then no matter how to color edges of K_n with [red square] and [blue square], there must exist a red K_k or a blue K_l.

2-coloring of K_n

$$f : \binom{[n]}{2} \to \{\texttt{red}, \texttt{blue}\}$$

Ramsey Theorem

$R(k, l)$ is finite.

$R(3,3) = 6$

Frank P. Ramsey
(1903-1930)

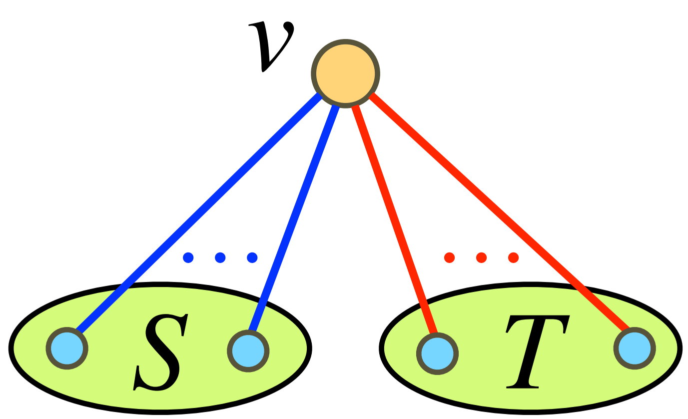
if $n \geq R(k,l)$, then no matter how to color edges of K_n with ■ and ■, there must exist a red K_k or a blue K_l.

$$R(k,2) = k \; ; \quad R(2,l) = l \; ;$$

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

if $n \geq R(k,l)$, then no matter how to color edges of K_n with [red] and [blue], there must exist a red K_k or a blue K_l.

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$



take $n = R(k,l\text{-}1) + R(k\text{-}1,l)$

arbitrary vertex v

$$|S| + |T| + 1 = n = R(k,l\text{-}1) + R(k\text{-}1,l)$$

or

$|S| \geq R(k,l\text{-}1)$ → or
- K_k in S
- $K_{l\text{-}1}$ in S $\xrightarrow{+v}$ K_l

$|T| \geq R(k\text{-}1,l)$ → or
- $K_{k\text{-}1}$ in T $\xrightarrow{+v}$ K_k
- K_l in T

if $n \geq R(k,l)$, then no matter how to color edges of K_n with ■ and ■, there must exist a red K_k or a blue K_l.

$$R(k,2) = k\ ; \quad R(2,l) = l\ ;$$

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

Ramsey Theorem $R(k,l)$ is finite.

By induction:

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

$$R(k,k) \geq n$$

"$\exists$ a 2-coloring of K_n with no monochromatic K_k."

a random 2-coloring of K_n:

$\forall \{u,v\} \in K_n$, *uniformly* and *independently* $\begin{cases} uv \\ uv \end{cases}$

$\forall S \in \binom{[n]}{k}$ event A_S: S is a monochromatic K_k

$$\Pr[A_S] = 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

A_S, A_T dependent $\Longleftrightarrow$ $|S \cap T| \geq 2$

max degree of dependency graph $d \leq \binom{k}{2}\binom{n}{k-2}$

To prove: $\Pr\left[\bigwedge_{S \in \binom{[n]}{k}} \overline{A_S}\right] > 0$

Lovász Local Lemma

- $\forall i,\ \Pr[A_i] \le p$
- $ep(d+1) \le 1$

$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$

$$\Pr[A_S] = 2^{1-\binom{k}{2}}$$

$$d \le \binom{k}{2}\binom{n}{k-2}$$

$\Rightarrow$ for some $n = ck2^{k/2}$ with constant c

$$e2^{1-\binom{k}{2}}(d+1) \le 1$$

To prove: $\Pr\left[\bigwedge_{S\in\binom{[n]}{k}} \overline{A_S}\right] > 0$

$$R(k,k) \ge n = \Omega(k2^{k/2})$$

Ramsey Number

$$\Omega\left(k2^{k/2}\right) \leq R(k,k) \leq \binom{2k-2}{k-1} = \mathrm{O}\left(\frac{4^k}{\sqrt{k}}\right)$$

R(*k*,*l*)	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	[40,42]
4				**18**	25	[36,40]	[49,58]	[59,79]	[73,106]	[92,136]
5					**[43,48]**	[58,85]	[80,133]	[101,194]	[133,282]	[149,381]
6						**[102,161]**	[115,273]	[134,427]	[183,656]	[204,949]
7							**[205,497]**	[219,840]	[252,1379]	[292,2134]
8								**[282,1532]**	[329,2683]	[343,4432]
9									**[565,6588]**	[581,12677]
10										**[798,23556]**

Multicolor

if $n \geq R(k,l)$, for any 2-coloring of edges of K_n,
there exists a red K_k or a blue K_l.

$R(r; k_1, k_2, \ldots, k_r)$

if $n \geq R(r; k_1, k_2, \ldots, k_r)$,
for any r-coloring of edges of K_n, for some $i \in [r]$
there exists a k_i-clique **monochromatic** with color i.

$$R(r; k_1, \ldots, k_{r\text{-}2}, k_{r\text{-}1}, k_r) \leq R(r\text{-}1; k_1, \ldots, k_{r\text{-}2}, R(2; k_{r\text{-}1}, k_r))$$

the mixing color trick:

color

Multicolor

if $n \geq R(k,l)$, for any 2-coloring of edges of K_n,
there exists a red K_k or a blue K_l.

$$R(r; k_1, k_2, \ldots, k_r)$$

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Ramsey Theorem

$R(r; k_1, k_2, \ldots, k_r)$ is finite.

Hypergraph

if $n \geq R(r; k_1, k_2, \ldots, k_r)$,

for any r-coloring of edges of K_n, for some $i \in [r]$

there exists a k_i-clique **monochromatic** with color i.

complete t-uniform **hypergraph** $\binom{[n]}{t}$

r-coloring $\quad f : \binom{[n]}{t} \to \{1, 2, \ldots, r\}$

Hypergraph

if $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

for any r-coloring of $\binom{[n]}{t}$, there exists

a monochromatic $\binom{S}{t}$ with color i and

$|S| = k_i$ for some $i \in \{1,2,\ldots,r\}$

complete t-uniform **hypergraph** $\binom{[n]}{t}$

r-coloring $\quad f : \binom{[n]}{t} \to \{1, 2, \ldots, r\}$

Partition of Set Family

If $n \geq R_t\left(r; k_1, k_2, \ldots, k_r\right)$,

for any r-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$,
there exists an $S \subseteq [n]$ such that $|S| = k_i$
and $\binom{S}{t} \subseteq C_i$ for some $i \in \{1, 2, \ldots, r\}$.

Erdős-Rado partition arrow

$$n \longrightarrow (k_1, k_2, \ldots, k_r)^t$$

If $n \geq R_t\left(r; k_1, k_2, \ldots, k_r\right)$,

for any r-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$, there exists an $S \subseteq [n]$ such that $|S| = k_i$ and $\binom{S}{t} \subseteq C_i$ for some $i \in \{1, 2, \ldots, r\}$.

mixing color:

$$R_t(r; k_1, \ldots, k_{r-2}, k_{r-1}, k_r) \leq R_t(r-1; k_1, \ldots, k_{r-2}, R_t(2; k_{r-1}, k_r))$$

$$R_t(k,l) \leq R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$n = R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

goal: $\forall f : \binom{[n]}{t} \to \{\texttt{red}, \texttt{blue}\}$

$\exists \binom{X}{t}, |X| = k$ or $\binom{Y}{t}, |Y| = l$

$$R_t(k,l) \leq R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$n = R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$\forall f : \binom{[n]}{t} \to \{\texttt{red}, \texttt{blue}\}$$

remove n from $[n]$, consider $\binom{[n-1]}{t-1}$

($[n] = \{1, 2, ..., n\}$)

define $f' : \binom{[n-1]}{t-1} \to \{\texttt{red}, \texttt{blue}\}$

$$\forall A \in \binom{[n-1]}{t-1}, \quad f'(A) = f(A \cup \{n\})$$

$$n\text{-}1 = R_{t-1}(R_t(k-1,l), R_t(k,l-1))$$

$$\text{or} \begin{cases} \exists S \subseteq [n-1], |S| = R_t(k-1,l), \binom{S}{t-1} \text{ by } f' \\ \exists T \subseteq [n-1], |T| = R_t(k,l-1), \binom{T}{t-1} \text{ by } f' \end{cases}$$

$$R_t(k,l) \leq R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$n = R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$\forall f : \binom{[n]}{t} \to \{\texttt{red}, \texttt{blue}\}$$

define $f' : \binom{[n-1]}{t-1} \to \{\texttt{red}, \texttt{blue}\}$

$$\forall A \in \binom{[n-1]}{t-1}, \quad f'(A) = f(A \cup \{n\})$$

by symmetry $\exists S \subseteq [n-1], |S| = R_t(k-1, l), \binom{S}{t-1}$ by f'

or

$\exists X \subseteq S, |X| = k-1, \binom{X}{t}$ by f

$\exists Y \subseteq S, |Y| = l, \binom{Y}{t}$ by f ✓

✓ $\binom{X \cup \{n\}}{t}$ by f

If $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

$\forall r$-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$

$\exists i \in [r]$ and $S \subseteq [n]$ with $|S| = k_i$

such that $\binom{S}{t} \subseteq C_i$

mixing color:

$$R_t(r; k_1, \ldots, k_{r-2}, k_{r-1}, k_r) \leq R_t(r-1; k_1, \ldots, k_{r-2}, R_t(2; k_{r-1}, k_r))$$

$$R_t(k,l) \leq R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

Theorem (Ramsey 1930)

$R_t(r; k_1, k_2, \ldots, k_r)$ is finite.

Ramsey Theorem

If $n \geq R_t\left(r; k_1, k_2, \ldots, k_r\right)$,

$\forall r$-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$

$\exists i \in [r]$ and $S \subseteq [n]$ with $|S| = k_i$

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Theorem (Ramsey 1930)

$R_t(r; k_1, k_2, \ldots, k_r)$ is finite.

Ramsey Theorem

If $n \geq R_t\left(r; k_1, k_2, \ldots, k_r\right)$,

$\forall\, r$-coloring $f : \binom{[n]}{t} \to [r]$

$\exists\, i \in [r]$ and $S \subseteq [n]$ with $|S| = k_i$

such that $f\left(\binom{S}{t}\right) = \{i\}$

Theorem (Ramsey 1930)

$R_t(r; k_1, k_2, \ldots, k_r)$ is finite.

Ramsey Theorem (diagonal)

If $n \geq R_t(r; k) \triangleq R_t(r; \underbrace{k, k, \ldots, k}_{r})$,

$\forall\, r$-coloring $f : \binom{[n]}{t} \to [r]$

$\exists$ a **monochromatic** $\binom{S}{t}$ with $S \in \binom{[n]}{k}$

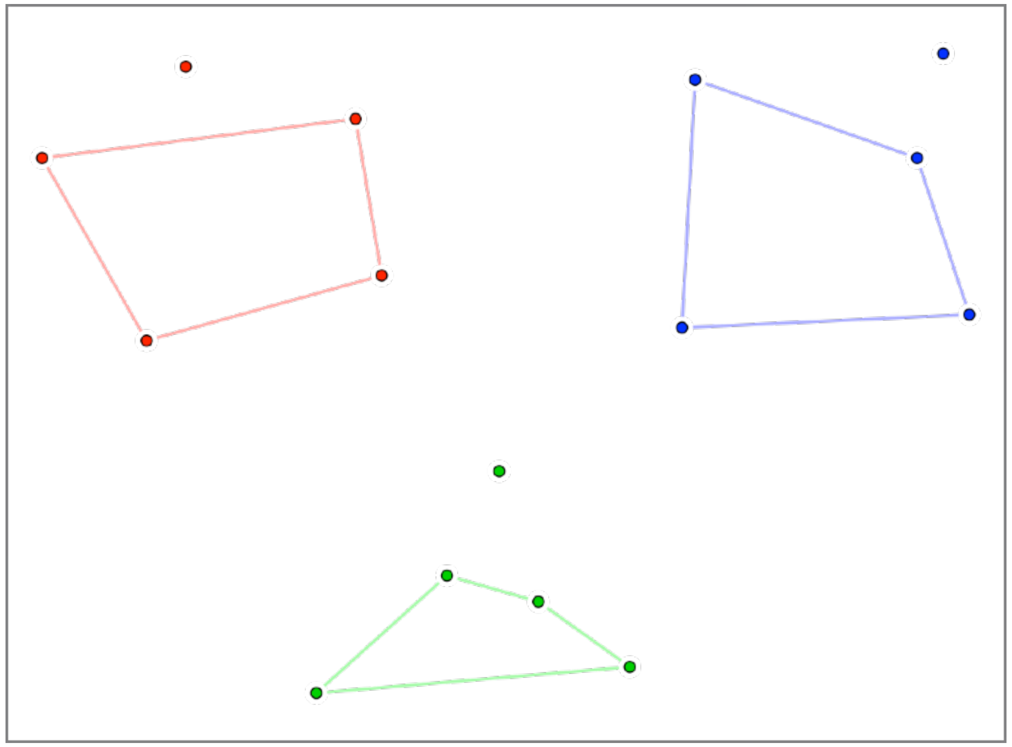
Theorem (Ramsey 1930)

$R_t(r; k)$ is finite.

Applications of Ramsey's Theorem

Happy Ending Problem

Any 5 points in the plane, no three on a line, has a subset of 4 points that form a convex quadrilateral.

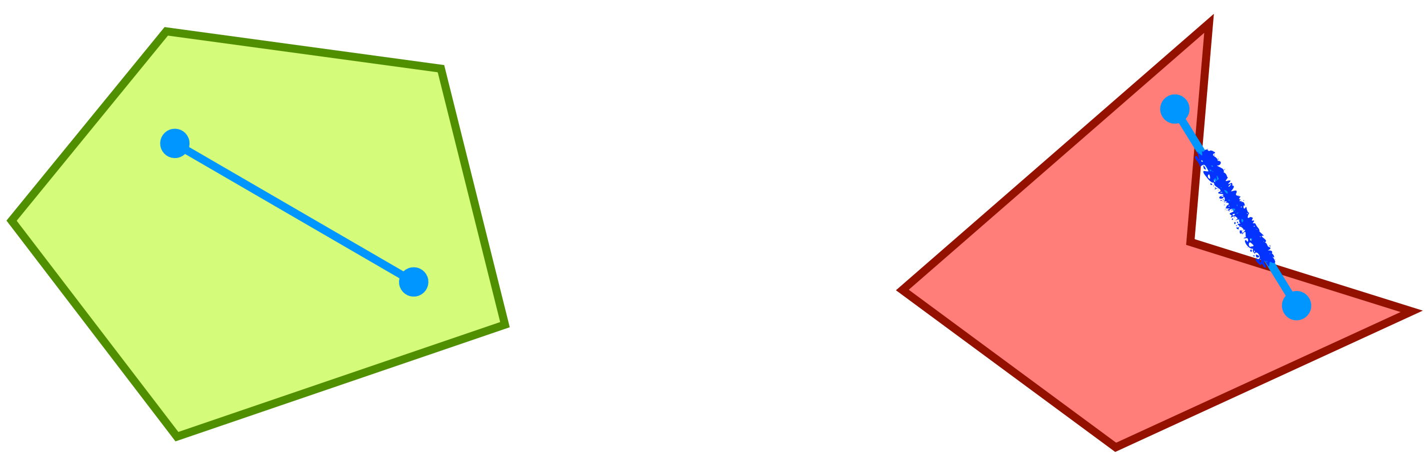


Theorem (Erdős-Szekeres 1935)

$\forall m \geq 3$, $\exists N(m)$ such that any set of $n \geq N(m)$ points in the plane, no three on a line, contains m points that are the vertices of a convex m-gon.

general positioned

Polygon:



convex

concave

Theorem (Erdős-Szekeres 1935)

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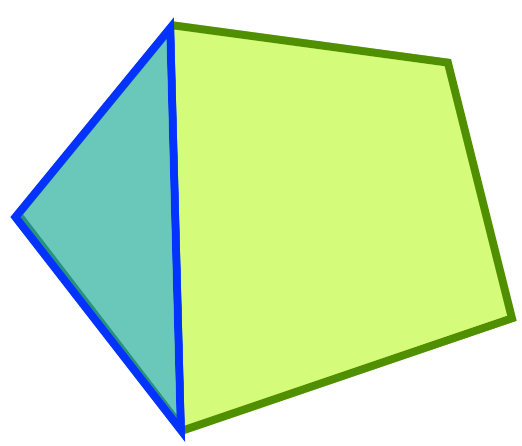
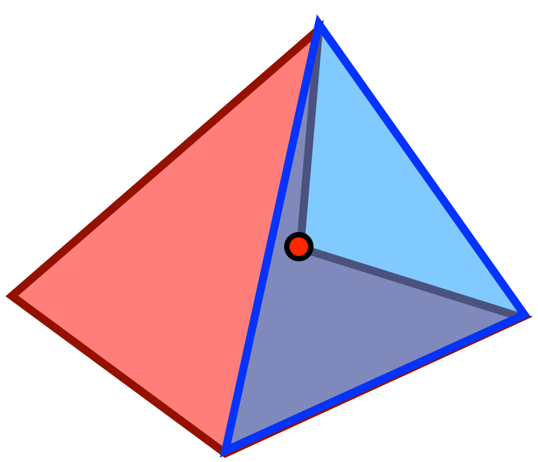
Polygon:



convex



concave

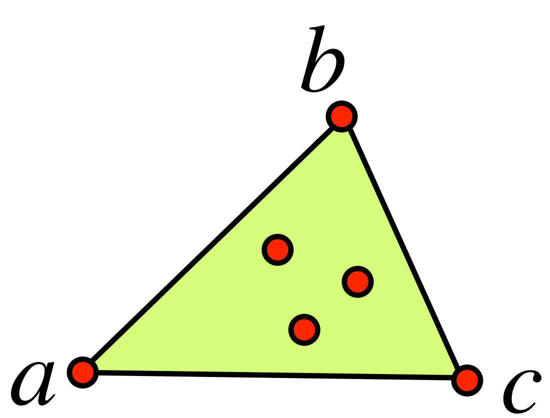
Theorem (Erdős-Szekeres 1935)
$\forall m \geq 3$, $\exists N(m)$ such that any set of $n \geq N(m)$ points in the plane, no three on a line, contains m points that are the vertices of a convex m-gon.

$N(m) = R_3(2; m, m)$ $|X| \geq N(m)$

$\forall f : \binom{X}{3} \to \{0, 1\}$ $\exists S \subseteq X, |S| = m$

monochromatic $\binom{S}{3}$

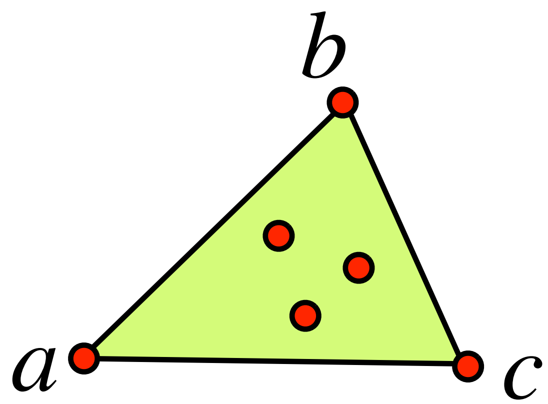
X: set of points in the plane, no 3 on a line



$\forall a, b, c \in X,$ $\triangle_{abc}$: points in triangle abc

$$f(\{a, b, c\}) = |\triangle_{abc}| \bmod 2$$

X: set of points in the plane, no 3 on a line



$\forall a, b, c \in X,\quad \triangle_{abc}$: points in triangle abc

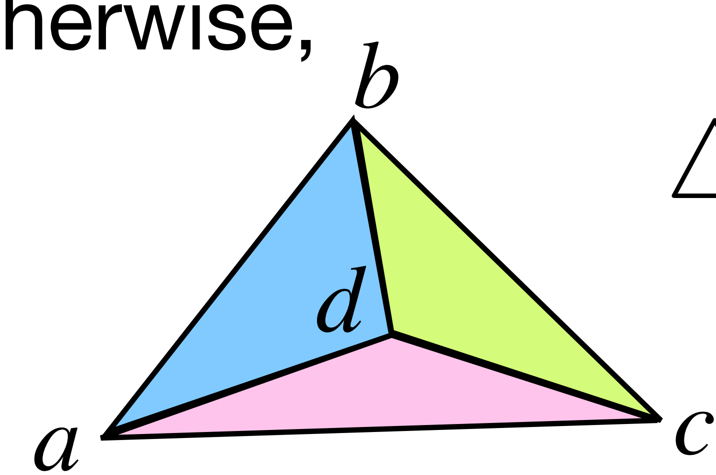
$$f(\{a, b, c\}) = |\triangle_{abc}| \bmod 2$$

$|X| \geq R_3(2; m, m)$ $\quad \forall f : \binom{X}{3} \to \{0, 1\}$

$\exists S \subseteq X, |S| = m$ monochromatic $\binom{S}{3}$

S is a convex m-gon

Otherwise,



disjoint union:

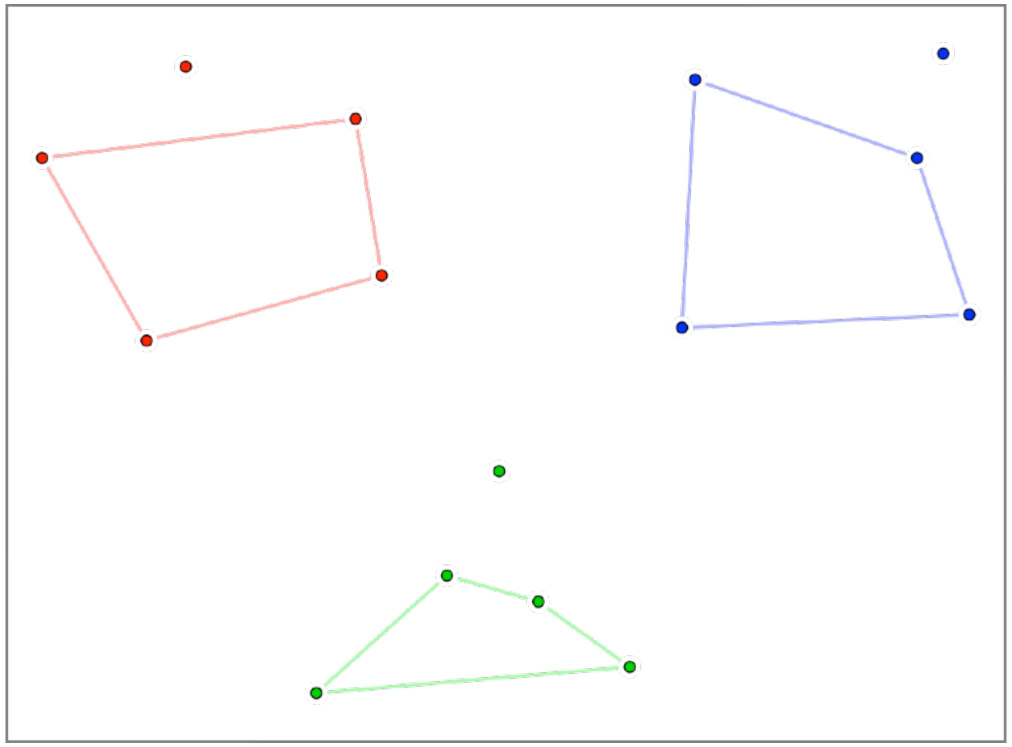
Contradiction!

$$\triangle_{abc} = \triangle_{abd} \cup \triangle_{acd} \cup \triangle_{bcd} \cup \{d\}$$

$$f(abc) = f(abd) + f(acd) + f(bcd) + 1$$

Happy Ending Problem

Any 5 points in the plane, no three on a line, has a subset of 4 points that form a convex quadrilateral.



Data Structures

Problem: "Is $x \in S$?"

data set $S \in \binom{[N]}{n}$ key $x \in [N]$ data universe $[N]$

Solution:

Data structure:

sorted table

Search Alg:

~~binary search~~

Complexity:

$\geq \log_2 n$

memory accesses

in the worst-case

“Is $x \in S$?” $x \in [N]$ $S \in \binom{[N]}{n}$

Theorem (Yao 1981)
If $N \geq 2n$, on sorted table, any search Alg requires $\Omega(\log n)$ accesses in the worst-case.

Problem: “Is $n \in S$?” $\forall S \in \binom{[N]}{n}$ $N \geq 2n$

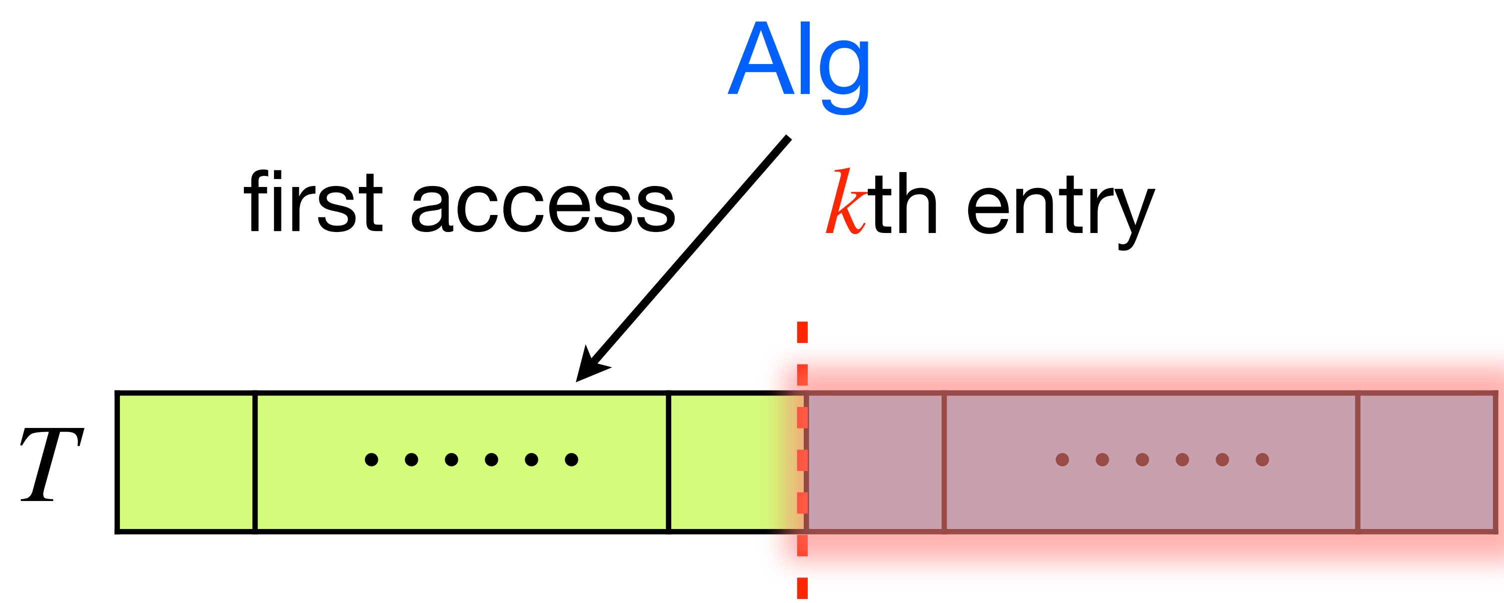
Induction on n, $n = 2$, trivial

Suppose it is true for any smaller n.

adversarial argument + self-reduction

Problem: "Is $n \in S$?" $\forall S \in \binom{[N]}{n}$ $N \geq 2n$

sorted table



$$T[k] = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ N-(n-k) & \text{if } k > \frac{n}{2} \end{cases}$$

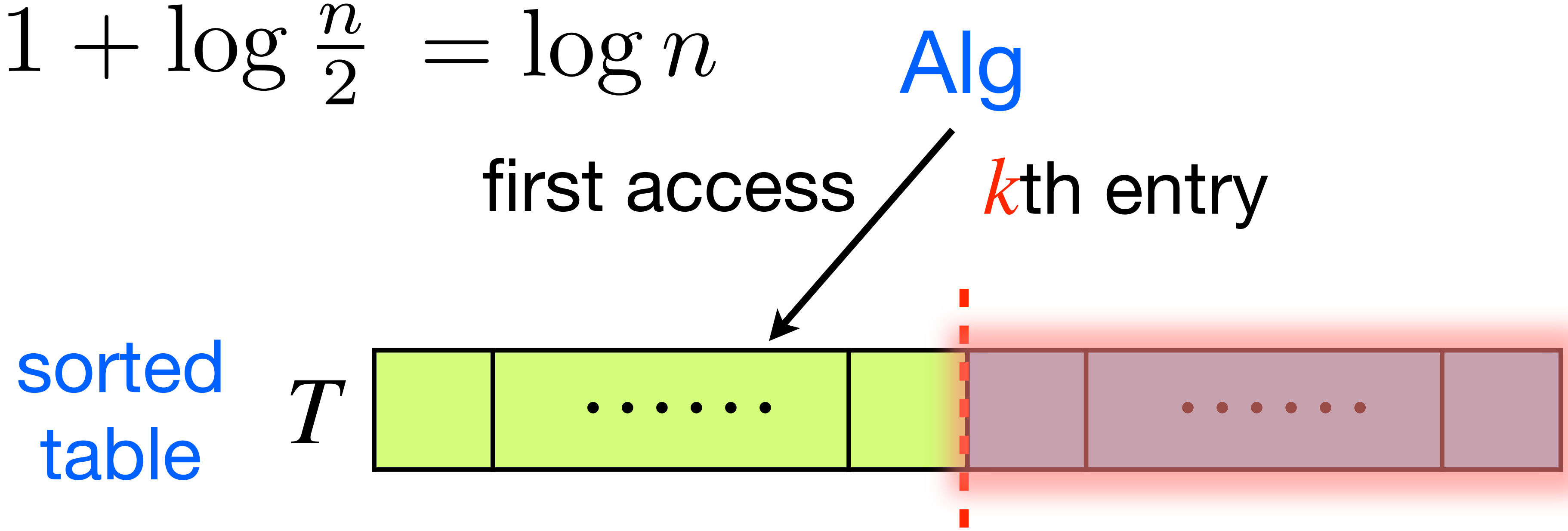
$$\binom{\{\frac{n}{2},\ldots,N-\frac{n}{2}\}}{\frac{n}{2}} \subseteq \binom{\{\frac{n}{2},\ldots,N\}}{\frac{n}{2}} \subseteq$$ possible $\{\,T[\frac{n}{2}+1],\ldots,T[n]\,\}$

$$n' = \frac{n}{2} \qquad N' = |\{\tfrac{n}{2},\ldots,N-\tfrac{n}{2}\}| \geq n \geq 2n'$$

relative key in $[N']$: $n - \frac{n}{2} = n'$

Problem: “Is $n \in S$?” $\forall S \in \binom{[N]}{n}$ $\quad N \geq 2n$

$$\geq 1 + \log \tfrac{n}{2} \;= \log n$$

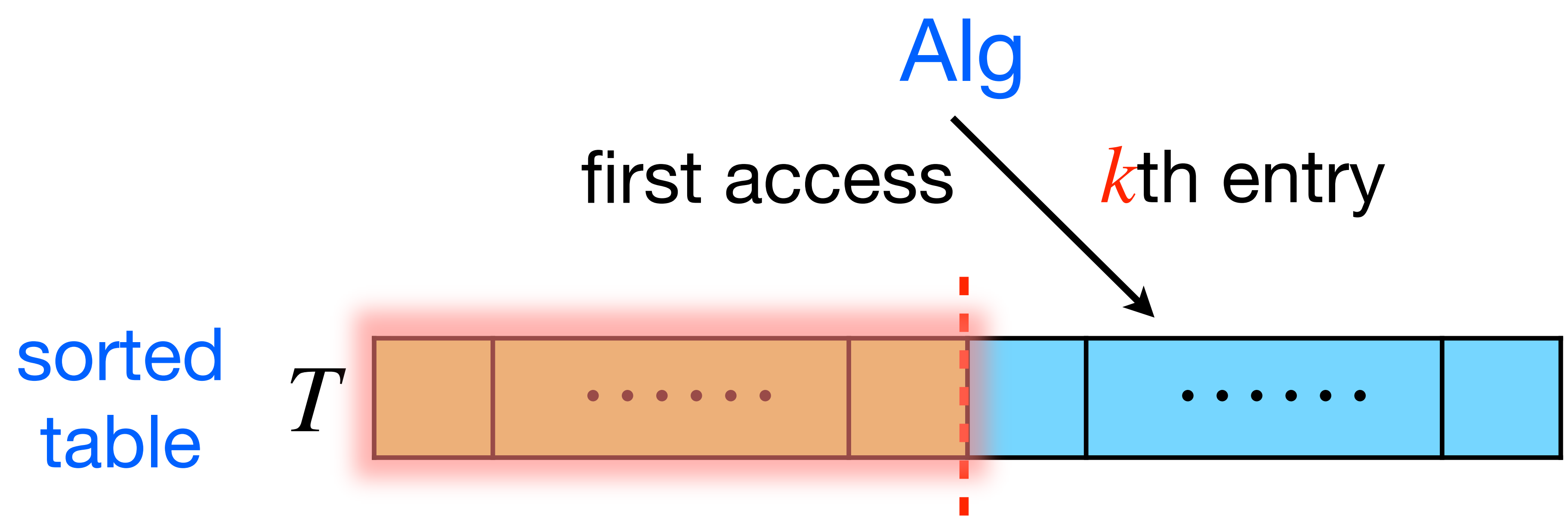


$$T[k] = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ N - (n-k) & \text{if } k > \frac{n}{2} \end{cases}$$

$n' = \frac{n}{2}$ “Is $n'\in S'$?” $\forall S' \in \binom{[N']}{n'}$ $\quad N' \geq 2n'$

I.H. require $\log \frac{n}{2}$ memory accesses

Problem: "Is $n \in S$?" $\quad \forall S \in \binom{[N]}{n} \quad N \geq 2n$



$$T[k] = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ N-(n-k) & \text{if } k > \frac{n}{2} \end{cases}$$

possible

$$\binom{\{\frac{n}{2},\ldots,N-\frac{n}{2}\}}{\frac{n}{2}} \subseteq \binom{\{1,\ldots,N-\frac{n}{2}\}}{\frac{n}{2}} \subseteq \{\, T[1],\ldots,T[\tfrac{n}{2}] \,\}$$

$$n' = \frac{n}{2} \qquad N' = |\{\tfrac{n}{2},\ldots,N-\tfrac{n}{2}\}| \geq 2n'$$

relative key in $[N']$: $\quad n - \frac{n}{2} = n'$

"Is $x \in S$?" $x \in [N]$ $S \in \binom{[N]}{n}$

Theorem (Yao 1981)
If $N \geq 2n$, on sorted table, any search Alg requires $\Omega(\log n)$ accesses in the worst-case.

implicit data structure:

each $S \in \binom{[N]}{n}$ is stored as a permutation of S

$$f : \binom{[N]}{n} \to [n!]$$

$f(S) = \pi$
dataset S: $x_1 < \cdots < x_n$
table: $(x_{\pi(1)}, \ldots, x_{\pi(n)})$

$\binom{[N]}{n}$ is mapped to the same π ⇒ same as sorted

$\exists X \subseteq [N], |X| \geq 2n$, $\binom{X}{n}$ monochromatic ⇒

"Is $x \in S$?" $x \in [N]$ $S \in \binom{[N]}{n}$

Theorem (Yao 1981)
For sufficiently large N, on any implicit data structure, any search Alg requires $\Omega(\log n)$ accesses in the worst-case.

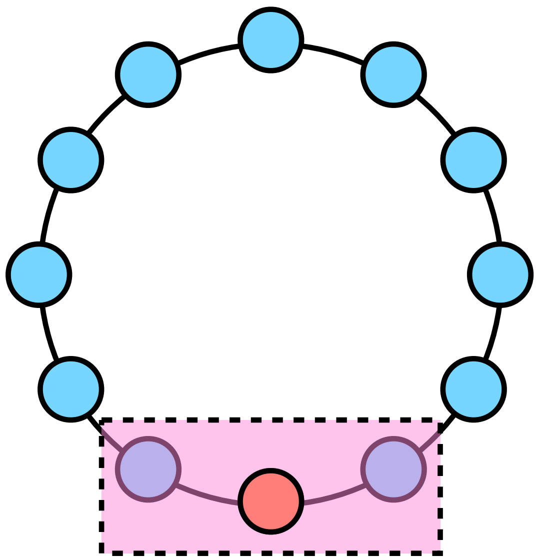
implicit data structure:

$$f : \binom{[N]}{n} \rightarrow [n!]$$

$N \geq R_n(n!; \underbrace{2n, \ldots, 2n}_{n!})$ or equivalently $N \rightarrow (\underbrace{2n, \ldots, 2n}_{n!})^n$

$\exists X \subseteq [N], |X| \geq 2n,$ $\binom{X}{n}$ monochromatic ⇨ $\geq \log n$ accesses

Local Computation



distributed computing in a ring

n nodes, ID from $[n]$

maximal independent set (MIS)

for any input ring,
locally compute the MIS

t-local algorithm:

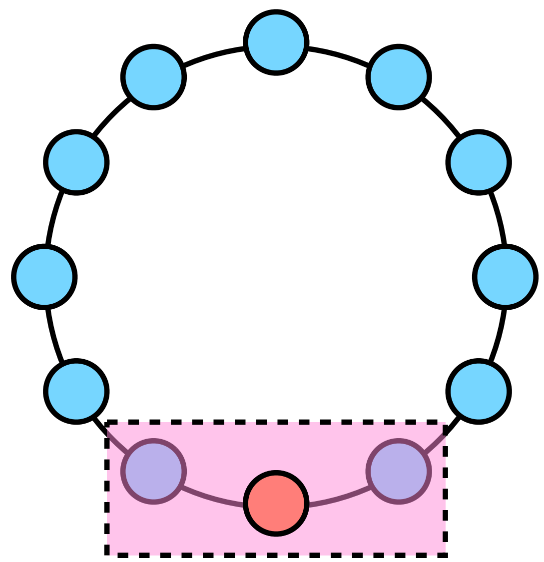
$$\mathcal{L} : \{t\text{-permutations of } [n]\} \to \{0, 1\}$$

$$\Rightarrow \quad f : \binom{[n]}{t} \to \{0, 1\}$$

$$f(\{a_1, \ldots, a_t\}) = \mathcal{L}(a_1, \ldots, a_t)$$

$$a_1 < a_2 < \ldots < a_t$$

Local Computation



distributed computing in a ring

n nodes, ID from $[n]$

maximal independent set (MIS)

t-local algorithm:

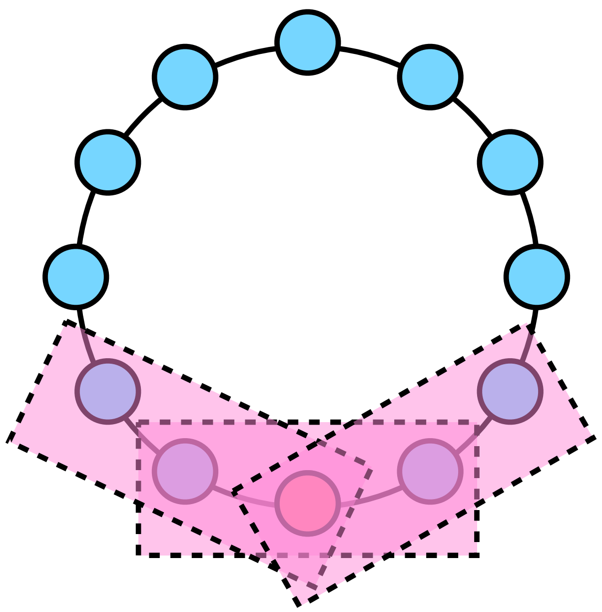
$$f : \binom{[n]}{t} \to \{0, 1\}$$

$$n \geq R_t(2; t+2, t+2)$$

∃ a monochromatic $\binom{S}{t}$ $\quad |S| = t + 2$

$$S = \{a_1, \ldots, a_t, a_{t+1}, a_{t+2}\} \qquad a_1 < \ldots < a_t < a_{t+1} < a_{t+2}$$

Local Computation



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t-local algorithm:

$$\mathcal{L} : \{t\text{-permutations of } [n]\} \to \{0, 1\}$$

$n \geq R_t(2; t+2, t+2)$ ⇒ $\exists S = \{a_1, \ldots, a_t, a_{t+1}, a_{t+2}\}$

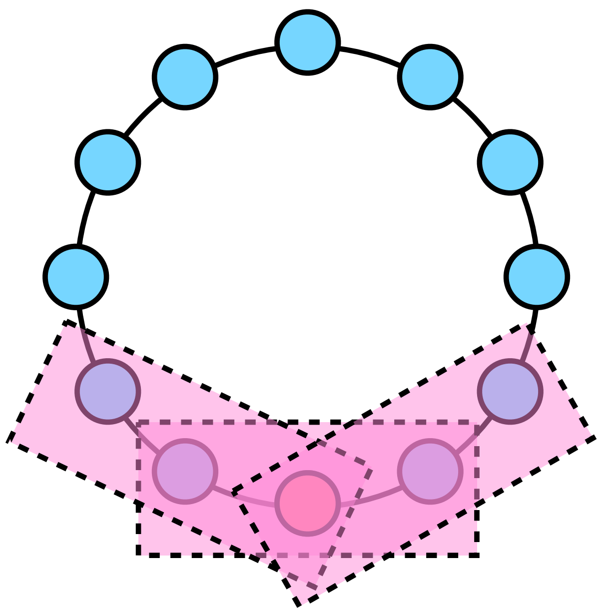
$$\mathcal{L}(a_1, \ldots, a_t) = \mathcal{L}(a_2, \ldots, a_{t+1}) = \mathcal{L}(a_3, \ldots, a_{t+2})$$

construct a bad ring starting with

$$(a_1, a_2, \ldots, a_t, a_{t+1}, a_{t+2})$$

Contradiction!

Local Computation



distributed computing in a ring

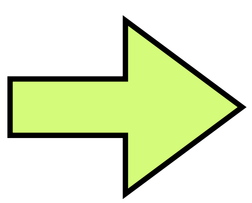
n nodes, ID from $[n]$

maximal independent set (MIS)

t-local algorithm:

$$\mathcal{L} : \{t\text{-permutations of } [n]\} \to \{0, 1\}$$

$$n < R_t(2; t+2, t+2) \le \underbrace{2^{2^{\cdot^{\cdot^{2^{ct}}}}}}_{t}$$



$$t = \Omega(\log^* n)$$

Ramsey Theory

Arithmetic Progression

2-coloring of 1 to 12

$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12$

monochromatic arithmetic progression

Can you give a 2-coloring of $\mathbb{N}$, no infinite monochromatic arithmetic progression?

$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ \cdots$

Van der Waerden Theorem

$W(r,k) \triangleq$ the smallest integer satisfying:

if $n \geq W(r,k)$, for any r-coloring of $[n]$, there exists a monochromatic arithmetic progression of length k

VdW Theorem
(Van der Waerden 1927)

$W(r,k)$ is finite.

Bartel Leendert van der Waerden
(1903-1996)

Van der Waerden Theorem

$W(r,k) \triangleq$ the smallest integer satisfying:

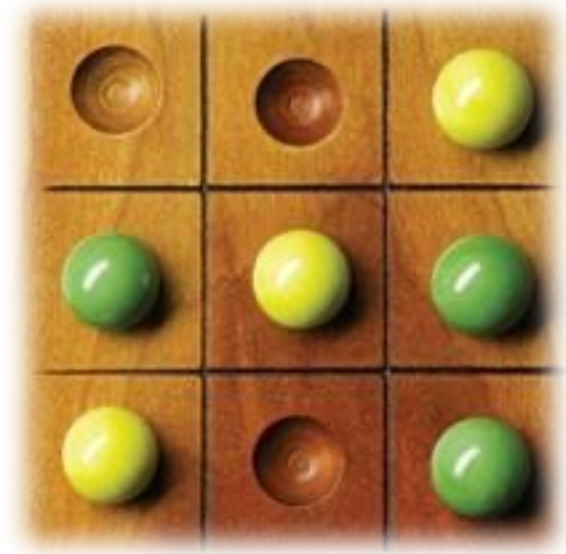
if $n \geq W(r,k)$, for any r-coloring of $[n]$, there exists a monochromatic arithmetic progression of length k

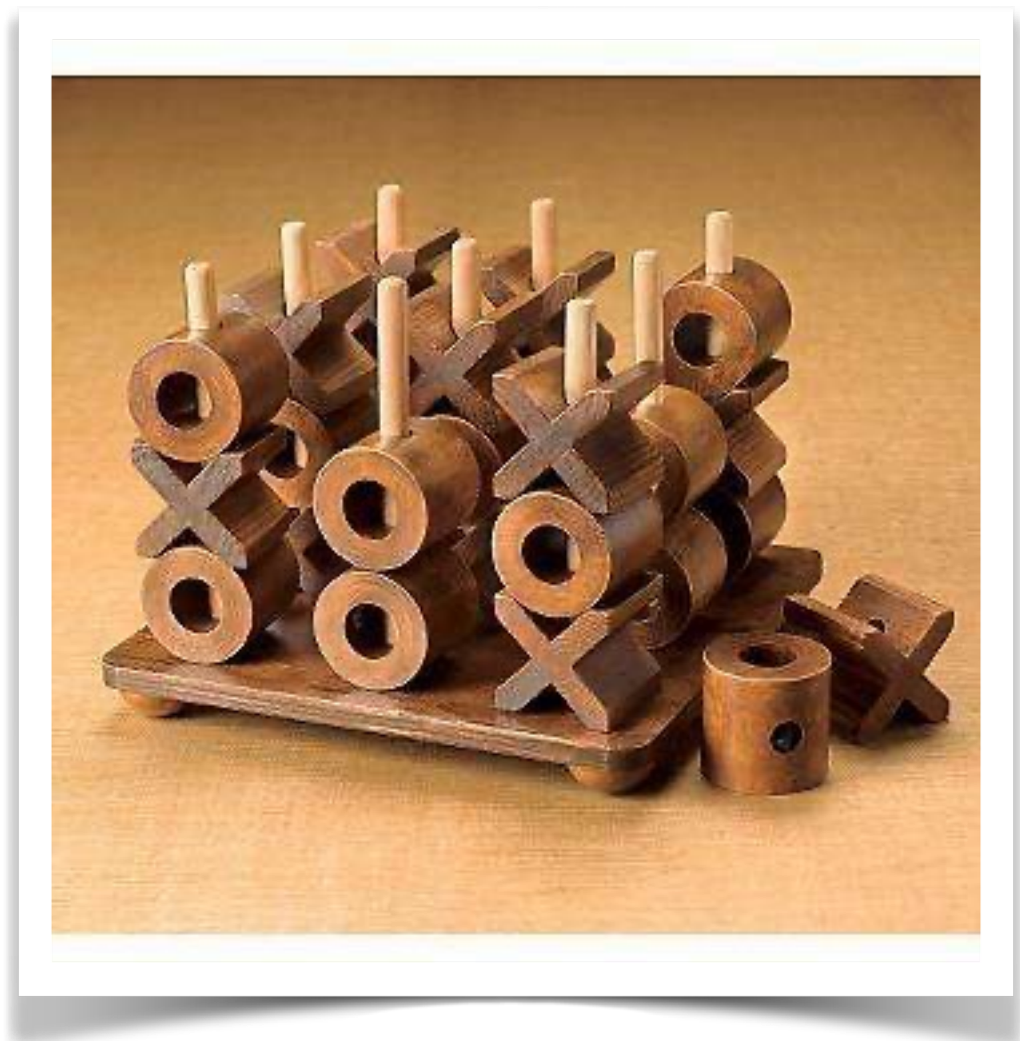
VdW Theorem $W(r,k)$ is finite.

$$\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$$

⇒ some C_i contains arbitrarily long arithmetic progression.

Hales-Jewett Theorem





Hales-Jewett Theorem

$[k] = \{1, \ldots, k\}$: an alphabet of k symbols

$[k]^n$: n-dimensional discrete cube

combinatorial line: $L_\tau = \{\tau(1), \tau(2), \ldots, \tau(k)\}$

$\tau \in ([k] \cup \{\star\})^n$ $\quad$ τ contains "$\star$"

$\forall i \in [k], \quad \tau(i) =$ replacing $\star$ by i in τ

$\tau = 12\star3\star2$

$k = 4$

$$L_\tau = \{121312, 122322, 123332, 124342\}$$

Hales-Jewett Theorem

$[k] = \{1, \ldots, k\}$: an alphabet of k symbols

$[k]^n$: n-dimensional discrete cube

combinatorial line:

$HJ(r,k) \triangleq$ the smallest integer satisfying:

If $n \geq HJ(r,k)$, for every r-coloring of the cube $[k]^n$, there exists a monochromatic combinatorial line.

Hales-Jewett Theorem $HJ(r,k)$ is finite.

HJT $\Longrightarrow$ VdW

If $n \geq HJ(r,k)$, for every r-coloring of the cube $[k]^n$, there exists a monochromatic combinatorial line.

reduction ϕ: $[k]^n \longrightarrow [N]$

$$\forall x \in [k]^n \qquad \phi(x) = x_1 + x_2 + \ldots + x_n$$

combinatorial line $\quad L_\tau = \{\tau(1), \ldots, \tau(k)\}$

arithmetic progression $\quad \{\phi(\tau(1)), \ldots, \phi(\tau(k))\}$

$\tau = 12 \star 3 \star 2 \qquad L_\tau = \{ 121312, 122322, 123332, 124342 \}$

if $N \geq W(r,k)$, for any r-coloring of $[N]$, there exists a monochromatic arithmetic progression of length k

HJT $\Longrightarrow$ VdW

If $n \geq HJ(r,k)$, for every r-coloring of the cube $[k]^n$, there exists a monochromatic combinatorial line.

reduction $\phi: [k]^n \to [N]$

$\forall x \in [k]^n \qquad \phi(x) = x_1+x_2+\ldots+x_n$

combinatorial line $\qquad L_\tau = \{\tau(1), \ldots, \tau(k)\}$

arithmetic progression $\qquad \{\phi(\tau(1)), \ldots, \phi(\tau(k))\}$

$f : [N] \to [r]$ $\Rightarrow$ $f' : [k]^n \to [r]$

$f'(x) = f(\phi(x))$

if $N \geq W(r,k)$, for any r-coloring of $[N]$, there exists a monochromatic arithmetic progression of length k