# Combinatorics

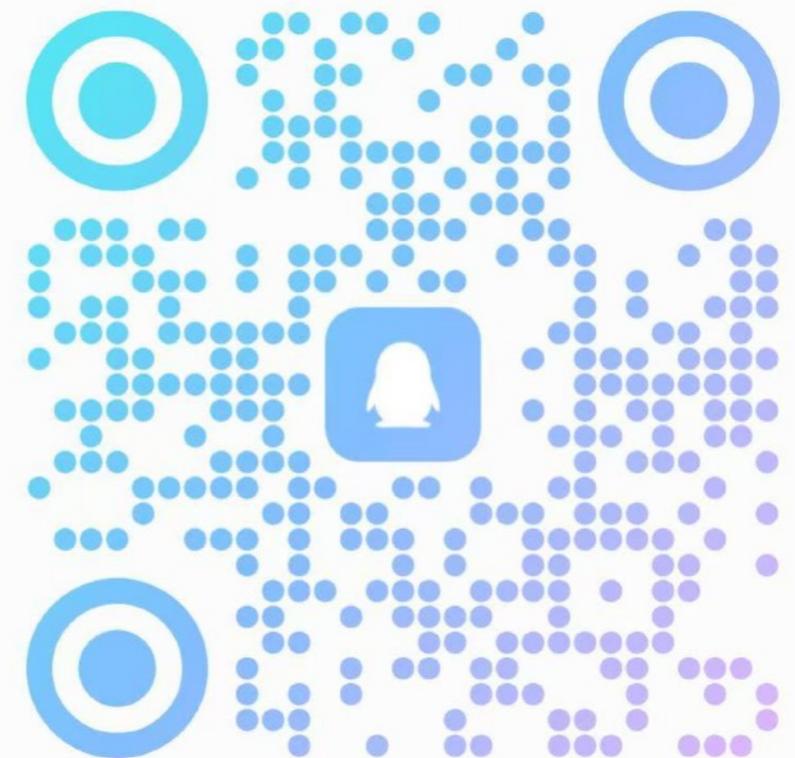**Basic Enumeration**

尹一通　**Nanjing University, 2026 Spring**

# Course Info

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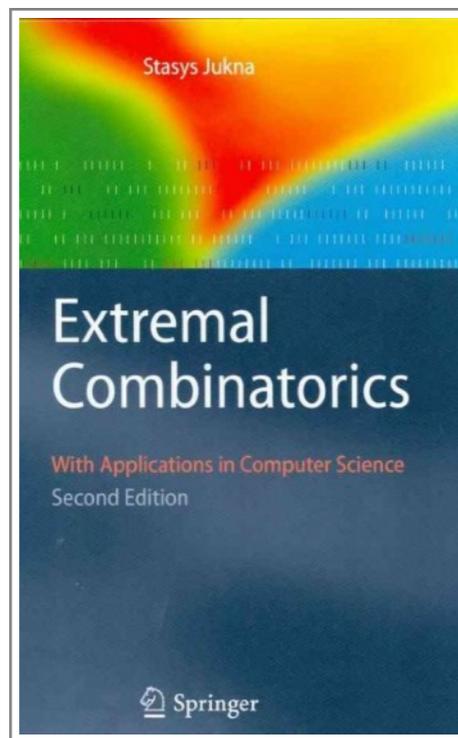组合数学（南京大学…
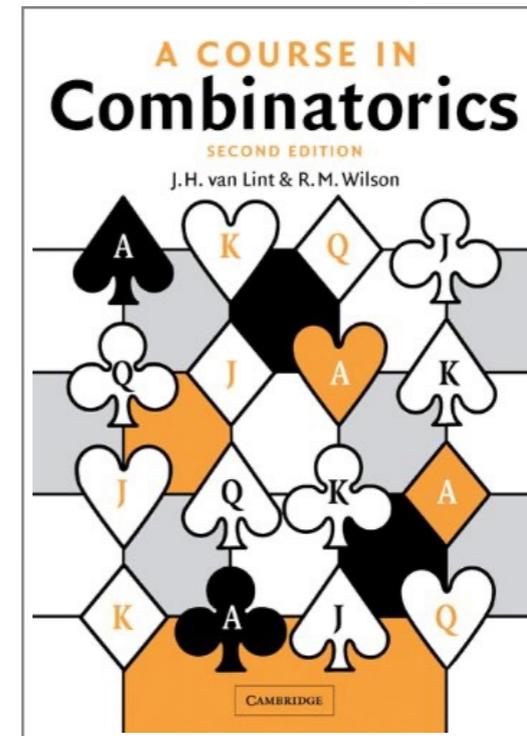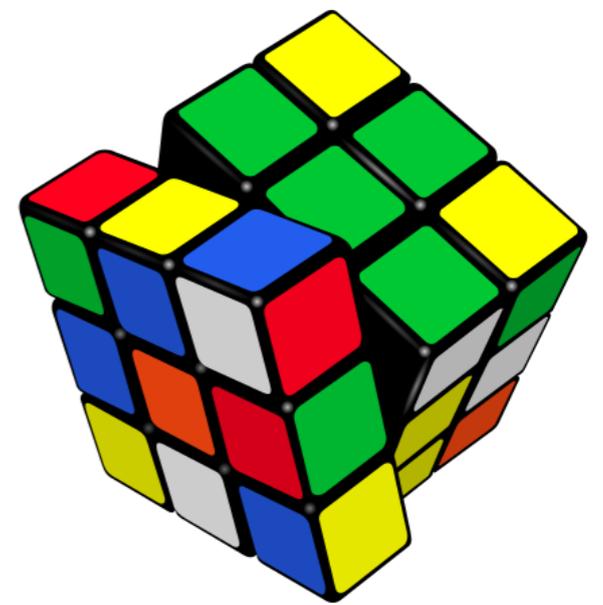群号: 1090691552

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QQ

# Textbooks

- van Lint and Wilson.
  ***A course in Combinatorics***,
  2nd Edition.

- Jukna.
  ***Extremal Combinatorics:***
  ***with applications in computer science***,
  2nd Edition.

# Combinatorics:
## *A math for finite structures* (*not really*)

- It is difficult to give rigorous definition of **Combinatorics**.

- According to *Wikipedia*, subfields of combinatorics include:

Problems
- *enumerative combinatorics, extremal combinatorics, combinatorial design*

Tools
- *algebraic combinatorics, analytic combinatorics, probabilistic combinatorics, geometric combinatorics, topological combinatorics, arithmetic combinatorics*

Objects
- *partition theory, graph theory, finite geometry, order theory, matroid theory, combinatorics on words, infinitary combinatorics*

- In *The Princeton Companion to Mathematics*, there are two branches of mathematics on the subject of combinatorics:

- *Enumerative and Algebraic Combinatorics*  (Counting)
- *Extremal and Probabilistic Combinatorics*  (Hungarian)
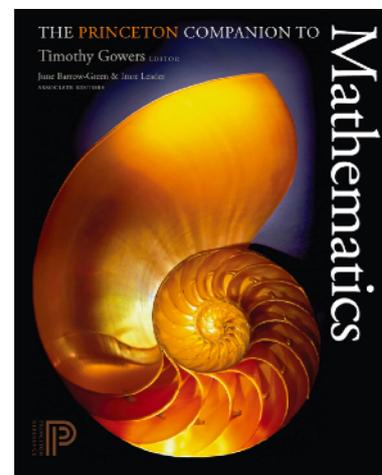
# Combinatorics:
## by the types of problems it studies
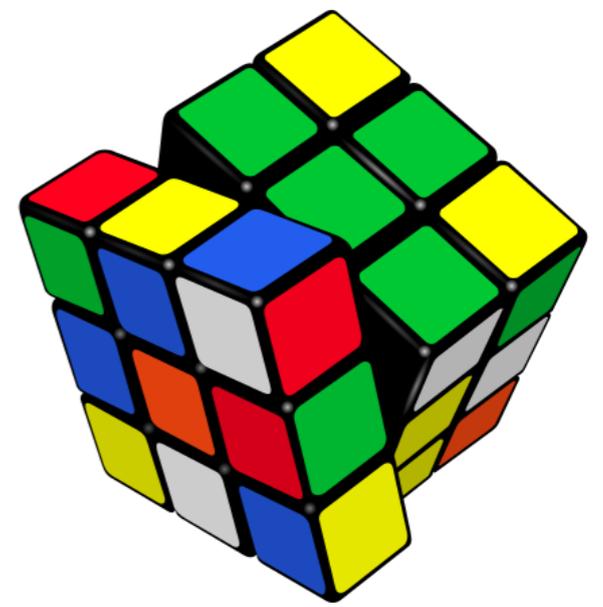
- Enumeration (counting) of finite structures (*e.g. solutions/ assignments/arrangements/configurations of finite systems*) satisfying certain given constraints.

- Existence of finite structures satisfying certain given constraints.

  - Extremal **problems**: How large/small a finite structure can be to satisfy certain given constraints?

  - Ramsey **problems**: When a finite structure becomes large enough, some regularity must show up somewhere.

- Construction (design) of such finite structures.

- Optimization: to find the best structure/solution in some sense.

# References
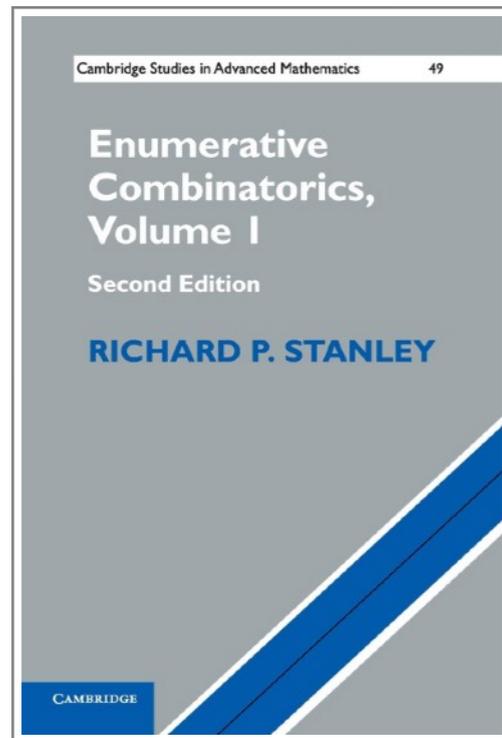
- Stanley.
  ***Enumerative Combinatorics***,
  Volume 1, 2nd Edition.

- Graham, Knuth, and Patashnik.
  ***Concrete Mathematics:
  A Foundation for Computer Science.***
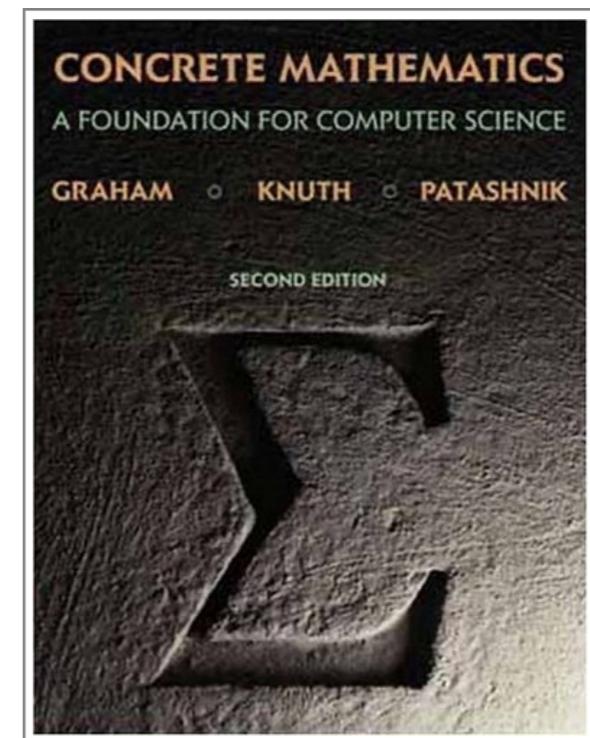
# References
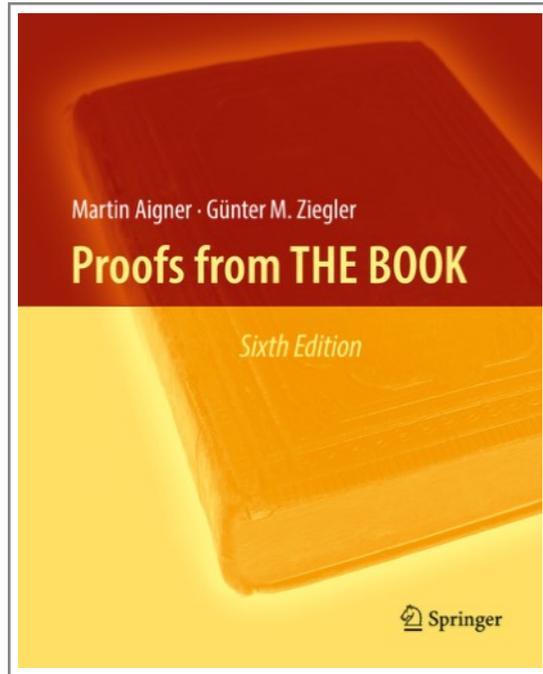
- Aigner and Ziegler.
  ***Proofs from THE BOOK***,
  6th Edition.
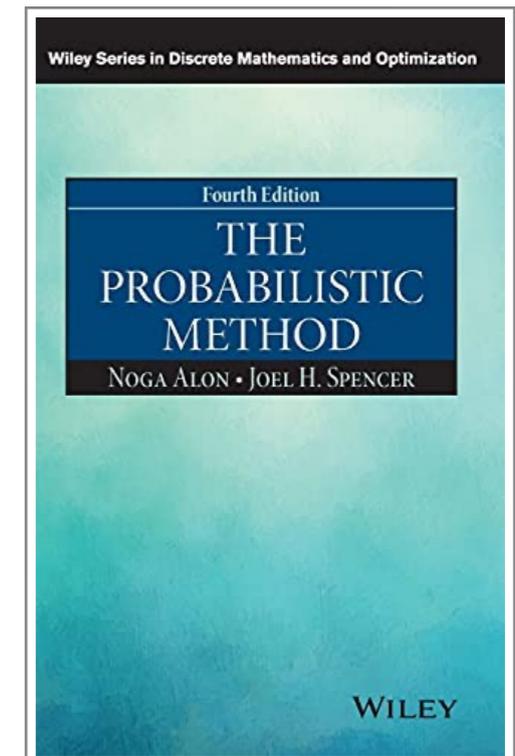
- Alon and Spencer.
  ***The Probabilistic Method***,
  4th Edition.
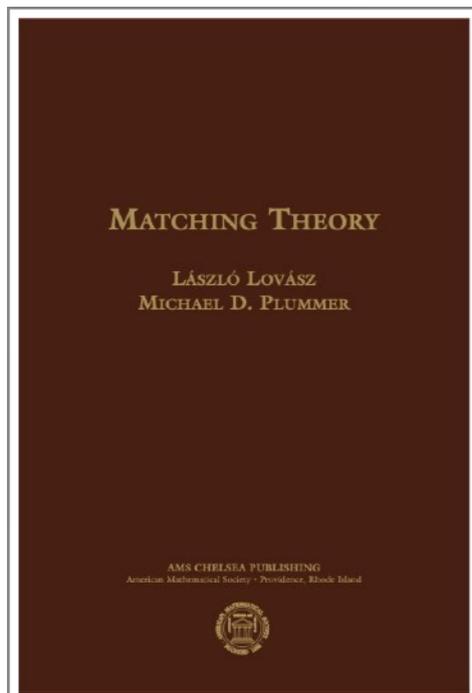
- Lovász and Plummer.
  ***Matching Theory***.

# Enumeration
## (Counting)

- How many ways are there:

  - to rank $n$ people?

  - to assign $m$ zodiac signs to $n$ people?

  - to choose $m$ people out of $n$ people?

  - to partition $n$ people into $m$ groups?

  - to distribute $m$ yuan to $n$ people?

  - to partition $m$ yuan to $n$ parts?

  - ... ...

# *Basic Enumeration*:
## The Twelvefold Way

Gian-Carlo Rota
(1932-1999)

# The Twelvefold Way

$$f : N \rightarrow M \qquad |N| = n, \ |M| = m$$

| elements of $N$ | elements of $M$ | any $f$ | 1-1 | on-to |
|---|---|---|---|---|
| *distinct* | *distinct* | | | |
| *identical* | *distinct* | | | |
| *distinct* | *identical* | | | |
| *identical* | *identical* | | | |

# Knuth's version (in *TAOCP* vol.4A)

$n$ balls are put into $m$ bins

| balls per bin: | unrestricted | $\leq 1$ | $\geq 1$ |
|---|---|---|---|
| $n$ *distinct balls,* $m$ *distinct bins* | $m^n$ | | |
| $n$ *identical balls,* $m$ *distinct bins* | | | |
| $n$ *distinct balls,* $m$ *identical bins* | | | |
| $n$ *identical balls,* $m$ *identical bins* | | | |

# Tuples
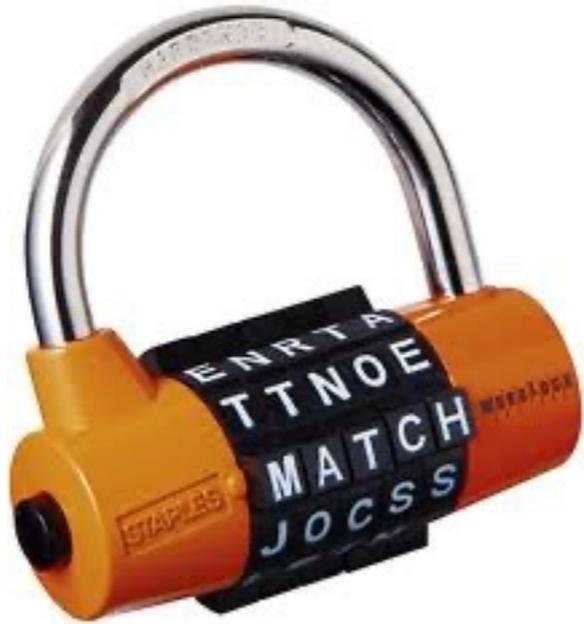


$$[m] = \frac{\{1, 2, \ldots, m\}}{\cancel{\{0, 1, \ldots, m-1\}}}$$

$$[m]^n = \underbrace{[m] \times \cdots \times [m]}_{n}$$

$$|[m]^n| = m^n$$

**Product rule:**

For finite sets $S$ and $T$
$$|S \times T| = |S| \cdot |T|$$

# Functions



count the # of functions

$$f : [n] \to [m]$$
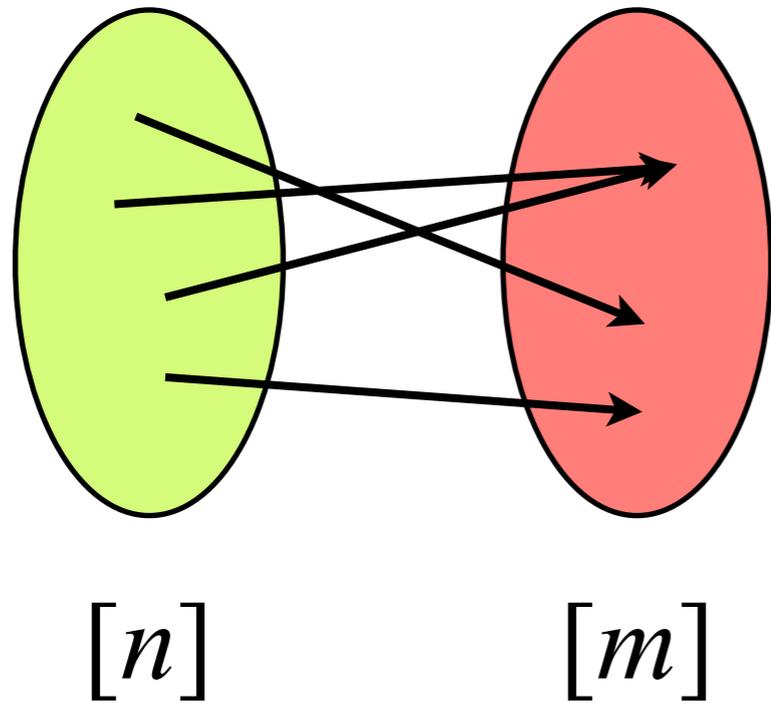
$[n]$    $[m]$

$$(f(1), f(2), \dots, f(n)) \in [m]^n$$

one-one correspondence

$$[n] \to [m] \Leftrightarrow [m]^n$$

# Functions



$[n]$       $[m]$

count the # of functions

$$f : [n] \rightarrow [m]$$
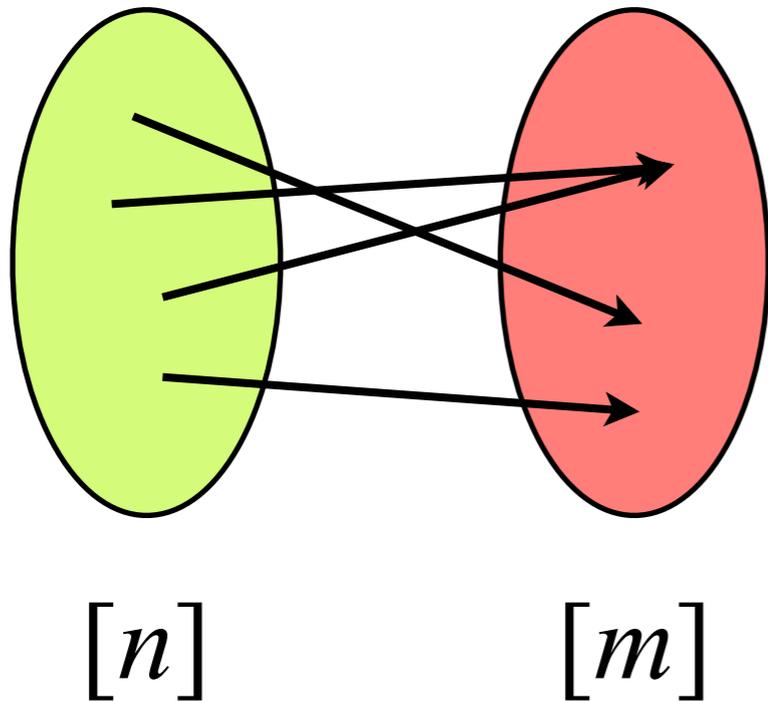
one-one correspondence

$$[n] \rightarrow [m] \Leftrightarrow [m]^n$$

**<u>Bijection rule:</u>**

For finite sets $S$ and $T$

$$\exists \phi : S \xrightarrow[\text{onto}]{1-1} T \implies |S| = |T|$$

# Functions



$[n]$      $[m]$

count the # of functions

$$f : [n] \longrightarrow [m]$$
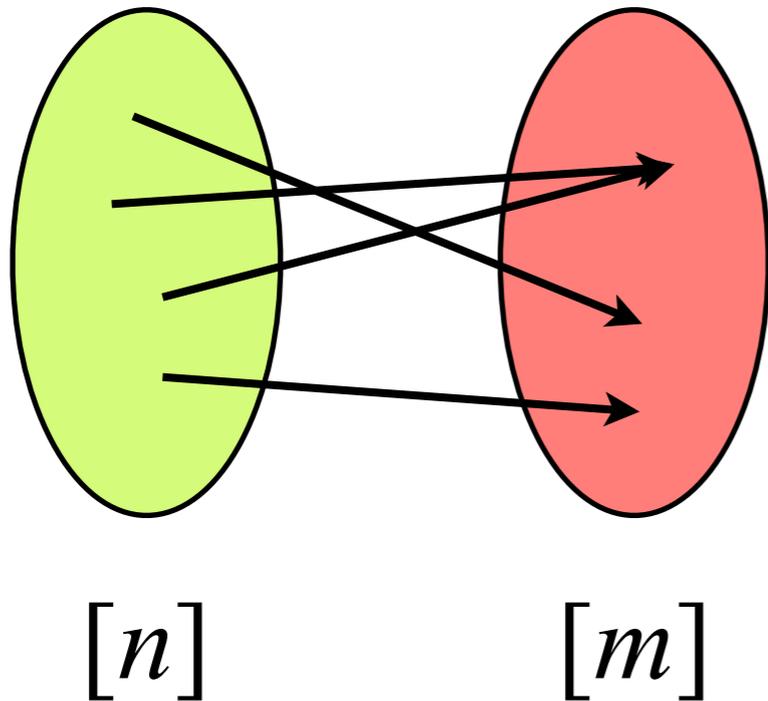
one-one correspondence

$$[n] \longrightarrow [m] \Longleftrightarrow [m]^n$$

$$|[n] \longrightarrow [m]| = |[m]^n| = m^n$$

"*Combinatorial* proof."

# Injections
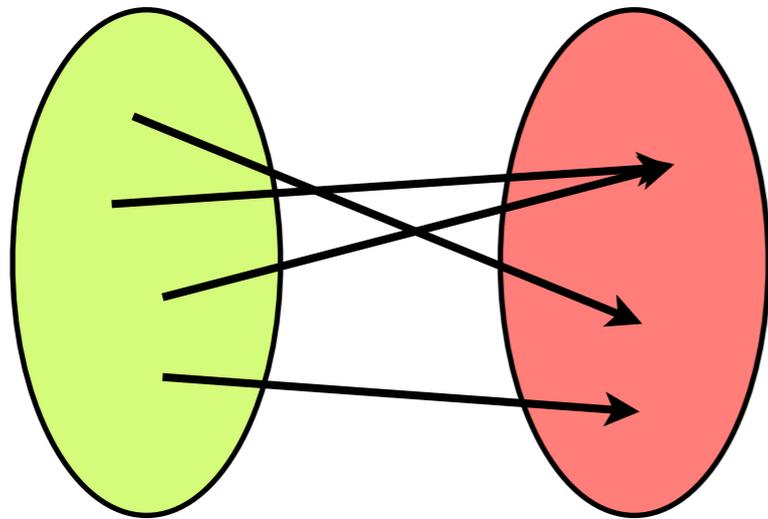
count the # of 1-1 functions

$$f : [n] \xrightarrow{\text{1-1}} [m]$$

one-to-one correspondence

$$\pi = (f(1), f(2), \ldots, f(n))$$

$[n]$  $[m]$

$n$-permutation:  $\pi \in [m]^n$ of distinct elements

$$(m)_n = m(m-1)\cdots(m-n+1) = \frac{m!}{(m-n)!}$$

"$m$ lower factorial $n$"

# Subsets

subsets of { 1, 2, 3 }:

$$\varnothing,$$

$$\{1\}, \{2\}, \{3\},$$

$$\{1, 2\}, \{1, 3\}, \{2, 3\},$$

$$\{1, 2, 3\}$$

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left| 2^{[n]} \right| =$$

# Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set:  $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left| 2^{[n]} \right| =$$

**Combinatorial proof**:

A subset $S \subseteq [n]$ corresponds to a string of $n$ bits, where the $i$-th bit indicates whether $i \in S$.

# Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set:  $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left| 2^{[n]} \right| =$$

**Combinatorial proof**:

$$S \subseteq [n] \Longleftrightarrow \chi_S \in \{0, 1\}^n \quad \chi_S(i) = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$$

one-to-one correspondence

# Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set:  $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left| 2^{[n]} \right| =$$

A not-so-combinatorial proof:

Let  $f(n) = \left| 2^{[n]} \right|$

$$f(n) = 2f(n-1)$$

$$f(n) = \left|2^{[n]}\right| \qquad \boxed{f(n) = 2f(n-1)}$$

$$2^{[n]} = \{S \subseteq [n] \mid n \notin S\} \cup \{S \subseteq [n] \mid n \in S\}$$

$$\left|2^{[n]}\right| = \left|2^{[n-1]}\right| + \left|2^{[n-1]}\right| = 2f(n-1)$$

**Sum rule:**

For **disjoint** finite sets $S$ and $T$
$$|S \cup T| = |S| + |T|$$

# Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set:  $\quad 2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left| 2^{[n]} \right| = 2^n$$

A not-so-combinatorial proof:

Let  $\quad f(n) = \left| 2^{[n]} \right|$

$$\boxed{f(n) = 2f(n-1)} \qquad f(0) = |2^{\emptyset}| = 1$$

# Basic Rules for Counting

**Sum rule:**

For **disjoint** finite sets $S$ and $T$

$$|S \cup T| = |S| + |T|$$

**Product rule:**

For finite sets $S$ and $T$

$$|S \times T| = |S| \cdot |T|$$

**Bijection rule:**

For finite sets $S$ and $T$

$$\exists \phi : S \xrightarrow[\text{onto}]{1-1} T \implies |S| = |T|$$

# Subsets of fixed size

2-subsets of $\{1, 2, 3\}$:   $\{1, 2\}, \{1, 3\}, \{2, 3\}$

$k$-uniform   $\binom{S}{k} = \{T \subseteq S \mid |T| = k\}$

$$\binom{n}{k} = \left| \binom{[n]}{k} \right|$$

"$n$ choose $k$"

# Subsets of fixed size

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{k!(n-k)!}$$

\# of <span style="color:red">ordered</span> $k$-subsets of $[n]$:

$$n(n-1)\cdots(n-k+1)$$

\# of permutations of a $k$-set:

$$k(k-1)\cdots 1$$

# Binomial Coefficients

**Binomial coefficient**: $\dbinom{n}{k}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

1. $\dbinom{n}{k} = \dbinom{n}{n-k}$

2. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$

choose a *k*-subset ⇔ choose its compliment

0-subsets + 1-subsets + ... + *n*-subsets = all subsets

# Binomial Theorem

**Binomial Theorem:**

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

**Proof:**

$$(1+x)^n = \underbrace{(1+x)(1+x)\cdots(1+x)}_{n}$$

\# of $x^k$:  choose $k$ factors out of $n$

**Binomial Theorem:**

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Let $x = 1$.

$$S = \{x_1, x_2, \ldots, x_n\}$$

$$\begin{aligned} & \text{\# of subsets of } S \text{ of odd sizes} \\ = \; & \text{\# of subsets of } S \text{ of even sizes} \end{aligned}$$
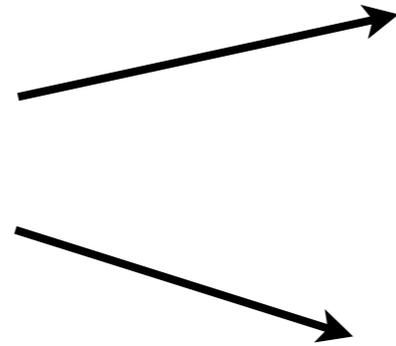
Let $x = -1$.

# The Twelvefold Way

$n$ balls are put into $m$ bins

| balls per bin: | unrestricted | $\leq 1$ | $\geq 1$ |
|---|---|---|---|
| *$n$ distinct balls, $m$ distinct bins* | $m^n$ | $(m)_n$ | |
| *$n$ identical balls, $m$ distinct bins* | | $\binom{m}{n}$ | |
| *$n$ distinct balls, $m$ identical bins* | | | |
| *$n$ identical balls, $m$ identical bins* | | | |

# Compositions of an Integer



$n$ beli

$k$ pirates

How many ways to assign $n$ beli to $k$ pirates?

How many ways to assign $n$ beli to $k$ pirates,
so that each pirate receives at least 1 beli?

# Compositions of an Integer

$n \in \mathbb{Z}^+$

*k*-composition of $n$:

an ordered sum of $k$ positive integers

a $k$-tuple $(x_1, x_2, \ldots, x_k)$ satisfying
$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$
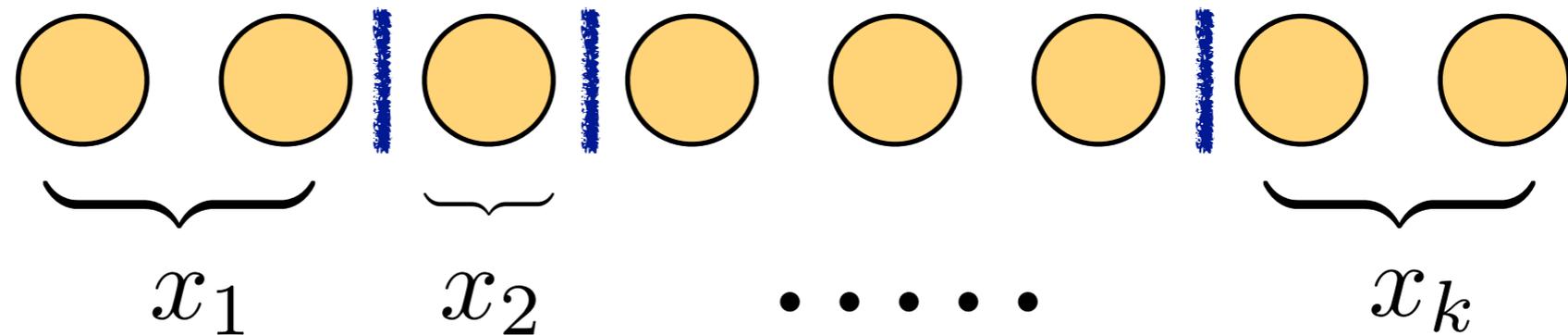
# Compositions of an Integer

$$n \in \mathbb{Z}^+$$

*k*-composition of *n*:

a $k$-tuple $(x_1, x_2, \ldots, x_k)$ satisfying

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

\# of *k*-compositions of *n*?    $\binom{n-1}{k-1}$

*n* identical balls



$x_1$          $x_2$          $\cdots\cdots$          $x_k$

# Compositions of an Integer

a $k$-tuple $(x_1, x_2, \ldots, x_k)$ satisfying

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

# of $k$-compositions of $n$?   $\dbinom{n-1}{k-1}$

$$\phi((x_1, x_2, \ldots, x_k)) = \{x_1, \ x_1 + x_2, \ x_1 + x_2 + x_3,$$
$$\ldots, \ x_1 + x_2 + \cdots + x_{k-1}\}$$

$\phi$ is a 1-1 correspondence between

{$k$-compositions of $n$} and $\dbinom{\{1,2,\ldots,n-1\}}{k-1}$

# Compositions of an Integer

weak $k$-composition of $n$:

an ordered sum of $k$ nonnegative integers

a $k$-tuple $(x_1, x_2, \ldots, x_k)$ satisfying
$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{N}$

# Compositions of an Integer

weak $k$-composition of $n$:

a $k$-tuple $(x_1, x_2, \ldots, x_k)$ satisfying

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{N}$

\# of weak $k$-compositions of $n$? $\quad \dbinom{n+k-1}{k-1}$

$(x_1 + 1) + (x_2 + 1) + \cdots + (x_k + 1) = n + k$

a $k$-composition of $n + k$

1-1 correspondence

# Multisets

$k$-subset of $S$

"$k$-combination of $S$
*without* repetition"

3-combinations of  { 1, 2, 3, 4 }

without repetition:

{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}

with repetition:

{1,1,1}, {1,1,2}, {1,1,3}, {1,1,4}, {1,2,2}, {1,3,3},
{1,4,4}, {2,2,2}, {2,2,3}, {2,2,4}, {2,3,3}, {2,4,4},
{3,3,3}, {3,3,4}, {3,4,4}, {4,4,4}

# **Multisets**

multiset $M$ on set $S$:

$$m : S \to \mathbb{N}$$

multiplicity of $\ x \in S$

$m(x)$: # of repetitions of $x$ in $M$

cardinality $\ |M| = \sum_{x \in S} m(x)$

"$k$-combination of $S$ with repetition" $\Longleftrightarrow$ $k$-multiset on $S$

$\left( \binom{n}{k} \right)$ : # of $k$-multisets on an $n$-set

# Multisets

$$\left(\binom{n}{k}\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

$k$-multiset on $S = \{x_1, x_2, \ldots, x_n\}$

$$m(x_1) + m(x_2) + \cdots + m(x_n) = k$$

$$m(x_i) \geq 0$$

a weak $n$-composition of $k$

# Multinomial Coefficients

permutations of a multiset on $k$ elements
of size $n$ with multiplicities $m_1, m_2, \ldots, m_k$

\# of reordering of "*multinomial*"
permutations of $\{a, i,i, l,l, m,m, n, o, t, u\}$

assignments of $n$ distinct balls to $k$ distinct bins
with the *i-th* bin receiving $m_i$ balls

multinomial
coefficient
$$\binom{n}{m_1, \ldots, m_k}$$

$$m_1 + m_2 + \cdots + m_k = n$$

# Multinomial Coefficients

# of permutations of a multiset on $k$ elements
  of size $n$ with multiplicities $m_1, m_2, \ldots, m_k$

$\|$

# of assignments of $n$ distinct balls to $k$ distinct bins
  with the *i-th* bin receiving $m_i$ balls

$$\binom{n}{m_1, \ldots, m_k} = \frac{n!}{m_1! m_2! \cdots m_k!}$$

$$\binom{n}{m, n-m} = \binom{n}{m}$$

# Multinomial Theorem

**Multinomial Theorem:**

$$(x_1 + x_2 + \cdots + x_k)^n$$

$$= \sum_{m_1 + \cdots + m_k = n} \binom{n}{m_1, \ldots, m_k} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$$

**Proof:** $(x_1 + x_2 + \cdots + x_k)^n$

$$= \underbrace{(x_1 + x_2 + \cdots + x_k) \cdots \cdots (x_1 + x_2 + \cdots + x_k)}_{n}$$

\# of $x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$ :

assign $n$ factors to $k$ groups of sizes $m_1, m_2, \ldots, m_k$

# The Twelvefold Way

$n$ balls are put into $m$ bins

| balls per bin: | unrestricted | $\leq 1$ | $\geq 1$ |
|---|---|---|---|
| $n$ distinct balls, $m$ distinct bins | $m^n$ | $(m)_n$ | |
| $n$ identical balls, $m$ distinct bins | $\left(\binom{m}{n}\right)$ | $\binom{m}{n}$ | $\binom{n-1}{m-1}$ |
| $n$ distinct balls, $m$ identical bins | | | |
| $n$ identical balls, $m$ identical bins | | | |

# The Twelvefold Way

$n$ balls are put into $m$ bins

| balls per bin: | unrestricted | $\leq 1$ | $\geq 1$ |
|---|---|---|---|
| $n$ distinct balls, $m$ distinct bins | $m^n$ | $(m)_n$ | |
| $n$ identical balls, $m$ distinct bins | $\binom{n+m-1}{m-1}$ | $\binom{m}{n}$ | $\binom{n-1}{m-1}$ |
| $n$ distinct balls, $m$ identical bins | | | |
| $n$ identical balls, $m$ identical bins | | | |

# Partitions of a Set



$n$ pirates
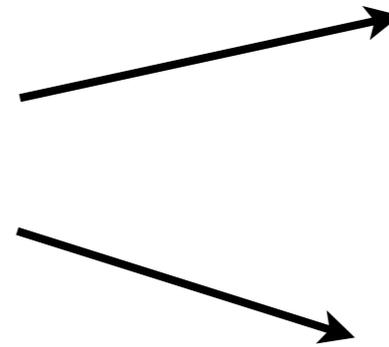
$k$ boats

$P = \{A_1, A_2, \ldots, A_k\}$ is a **partition** of $S$:
- $A_i \neq \varnothing$;
- $A_i \cap A_j = \varnothing$;
- $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

# Partitions of a Set

$P = \{A_1, A_2, \ldots, A_k\}$ is a **partition** of $S$:

- $A_i \neq \varnothing$;
- $A_i \cap A_j = \varnothing$;
- $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

$$\left\{ {n \atop k} \right\} : \quad \text{\# of } k\text{-partitions of an } n\text{-set}$$

"**Stirling number** of the second kind"

$$B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} : \quad \text{total \# of partitions of an } n\text{-set}$$

"Bell number"

# Stirling Number of the 2nd Kind

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} : \quad \text{# of } k\text{-partitions of an } n\text{-set}$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$$

**Case.1**: $\{n\}$ is not a partition block

$n$ is in one of the $k$ blocks in a $k$-partition of $[n-1]$

**Case.2**: $\{n\}$ is a partition block

the remaining $k-1$ blocks forms a $(k-1)$-partition of $[n-1]$

# The Twelvefold Way

$n$ balls are put into $m$ bins

| balls per bin: | unrestricted | $\leq 1$ | $\geq 1$ |
|---|---|---|---|
| $n$ distinct balls, $m$ distinct bins | $m^n$ | $(m)_n$ | |
| $n$ identical balls, $m$ distinct bins | $\left(\!\binom{m}{n}\!\right)$ | $\binom{m}{n}$ | $\binom{n-1}{m-1}$ |
| $n$ distinct balls, $m$ identical bins | $\sum_{k=1}^{m}\left\{\begin{matrix} n \\ k \end{matrix}\right\}$ | $\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$ | $\left\{\begin{matrix} n \\ m \end{matrix}\right\}$ |
| $n$ identical balls, $m$ identical bins | | | |

# Surjections

$$f : [n] \xrightarrow{\text{on-to}} [m]$$

$$\forall i \in [m]$$
$$f^{-1}(i) \neq \emptyset$$
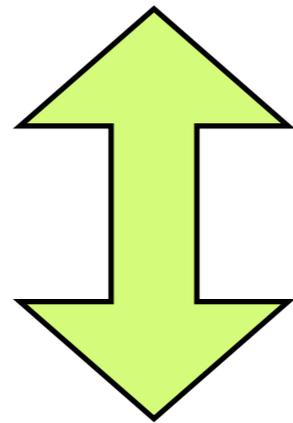
$$(f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(m))$$

ordered $m$-partition of $[n]$

$$m! \left\{ {n \atop m} \right\}$$

# The Twelvefold Way

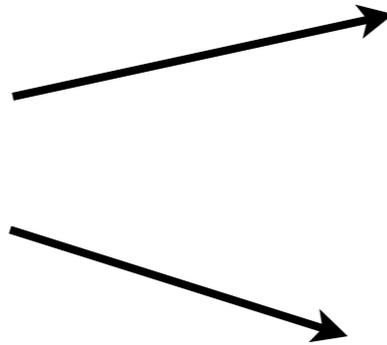$n$ balls are put into $m$ bins

| balls per bin: | unrestricted | $\leq 1$ | $\geq 1$ |
|---|---|---|---|
| *$n$ distinct balls, $m$ distinct bins* | $m^n$ | $(m)_n$ | $m! \left\{ {n \atop m} \right\}$ |
| *$n$ identical balls, $m$ distinct bins* | $\left(\!\!\binom{m}{n}\!\!\right)$ | $\binom{m}{n}$ | $\binom{n-1}{m-1}$ |
| *$n$ distinct balls, $m$ identical bins* | $\displaystyle\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$ | $\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$ | $\left\{ {n \atop m} \right\}$ |
| *$n$ identical balls, $m$ identical bins* | | | |

# Partitions of a Number



$n$ beli
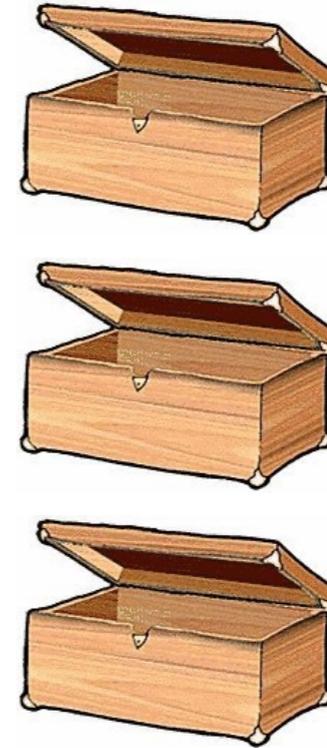
$k$ boxes

a **partition** of number $n$ into $k$ parts:

an unordered sum of $k$ positive integers

# Partitions of a Number

a **partition** of $n$ into $k$ parts:  "positive"

$n = 7$  "unordered"

$\{7\}$

$\{1,6\}, \{2,5\}, \{3,4\}$

$\{1,1,5\}, \{1,2,4\}, \{1,3,3\}, \{2,2,3\}$

$\{1,1,1,4\}, \{1,1,2,3\}, \{1,2,2,2\}$

$\{1,1,1,1,3\}, \{1,1,1,2,2\}$

$\{1,1,1,1,1,2\}$

$\{1,1,1,1,1,1,1\}$

$p_k(n)$ :  # of **partitions** of $n$ into $k$ parts

$p_k(n)$ : # of **partitions** of $n$ into $k$ parts

integral solutions to
$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

$$p_k(n) = ?$$

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k)$$

**Case.1**   $x_k = 1$

$(x_1, \ldots, x_{k-1})$ is a $(k-1)$-partition of $n-1$

**Case.2**   $x_k > 1$

$(x_1 - 1, \ldots, x_k - 1)$ is a $k$-partition of $n-k$

**partition** $\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$

**composition** $\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_i \geq 1 \end{cases}$

partition

$\{x_1, \cdots, x_k\}$

$\pi$
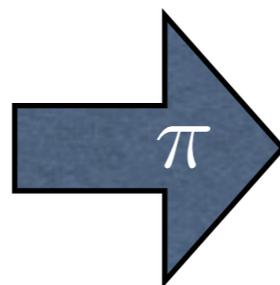
composition

$(x_1, \cdots, x_k)$

permutation

"on-to"

$$k! p_k(n) \geq \binom{n-1}{k-1}$$

partition $\{x_1, \cdots, x_k\}$    $y_i = x_i + k - i$

$$x_1 \geq x_2 \geq \cdots \geq x_{k-2} \geq x_{k-1} \geq x_k \geq 1$$
$+k-1$    $+k-2$    $+2$    $+1$

$$y_1 > y_2 > \cdots > y_{k-2} > y_{k-1} > y_k > 1$$
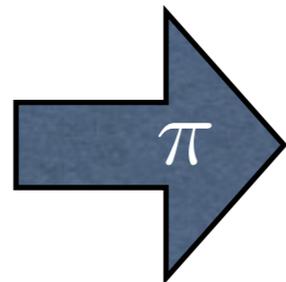
$\xrightarrow{\pi}$ composition of $n + \frac{k(k-1)}{2}$

$$(y_1, y_2, \ldots, y_k)$$

permutation

"1-1"

$$k! \, p_k(n) \leq \binom{n + \frac{k(k-1)}{2} - 1}{k - 1}$$

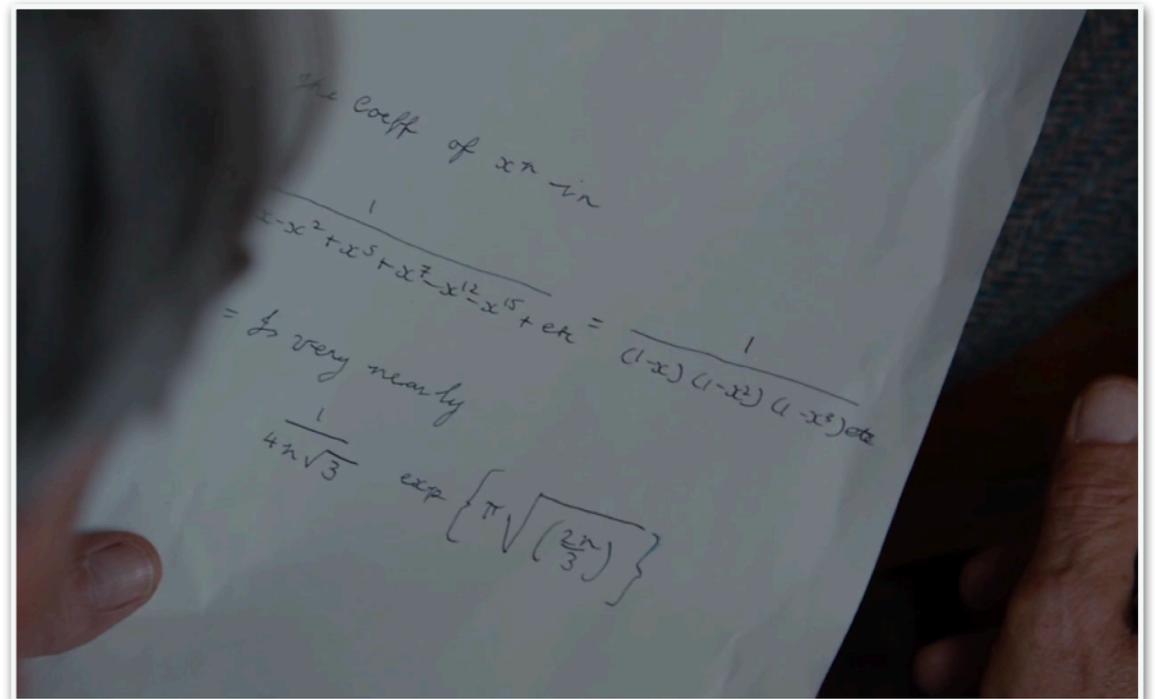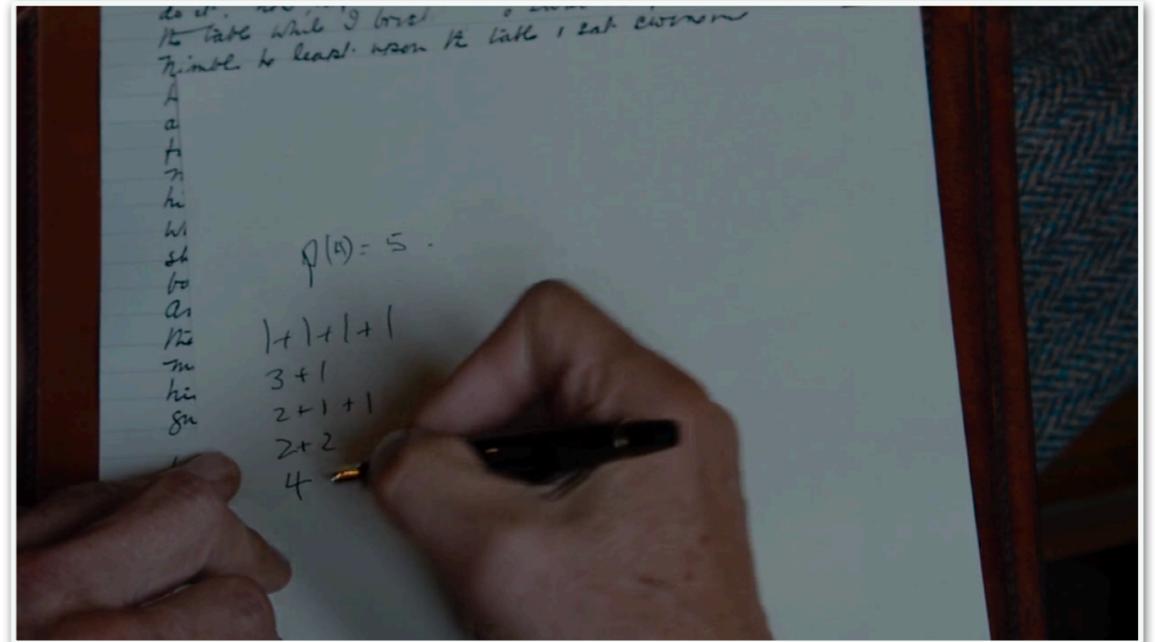$$\frac{\binom{n-1}{k-1}}{k!} \le p_k(n) \le \frac{\binom{n+\frac{k(k-1)}{2}-1}{k-1}}{k!}$$

If $k$ is fixed,

$$p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!} \qquad \text{as} \quad n \to \infty$$

The Man Who Knew Infinity
(2015 film)
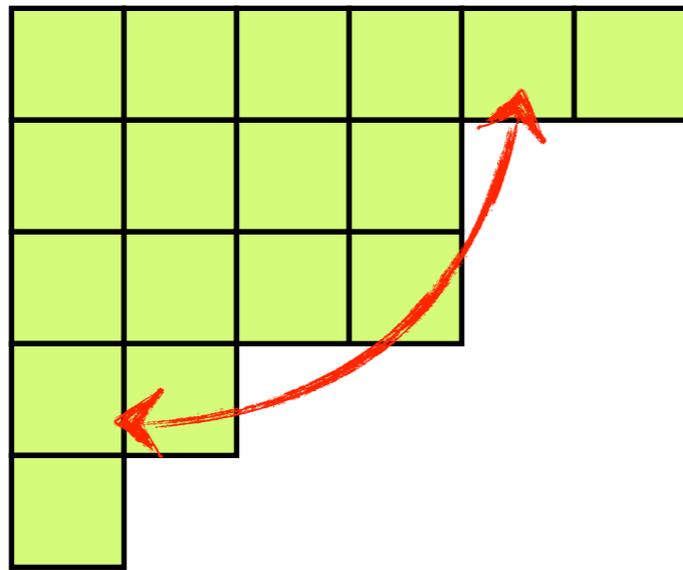
G. H. Hardy
(1877-1947)

Srinivasa
Ramanujan
(1887-1920)

$$p(n) = \sum_{k=1}^{n} p_k(n)$$

$$\approx \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}$$
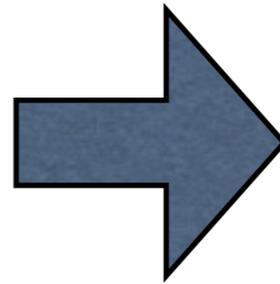
# Ferrers diagram
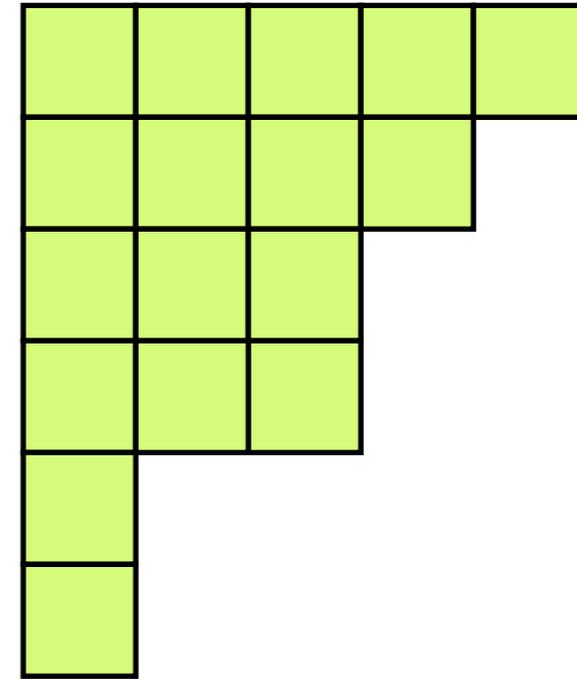## (Young diagram)

$x_1$
$x_2$
$x_3$
$x_4$
$x_5$
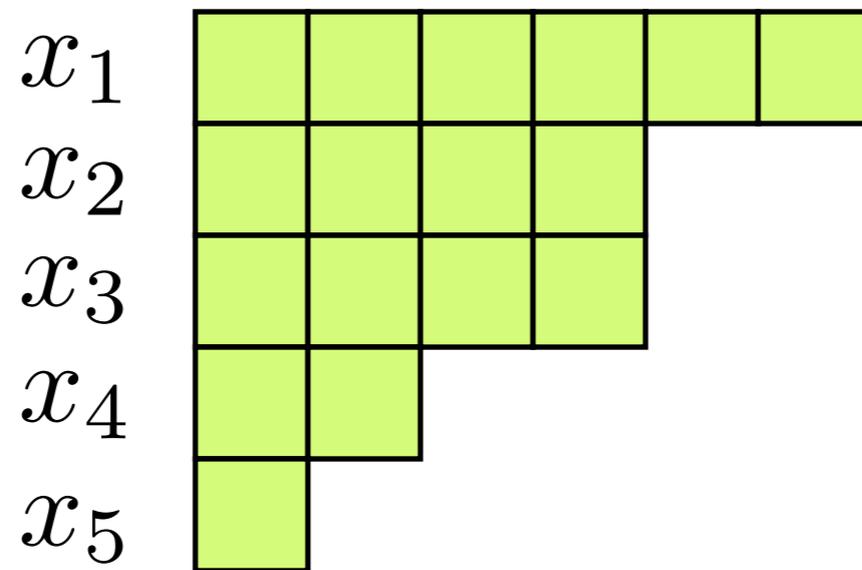
partition $\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$
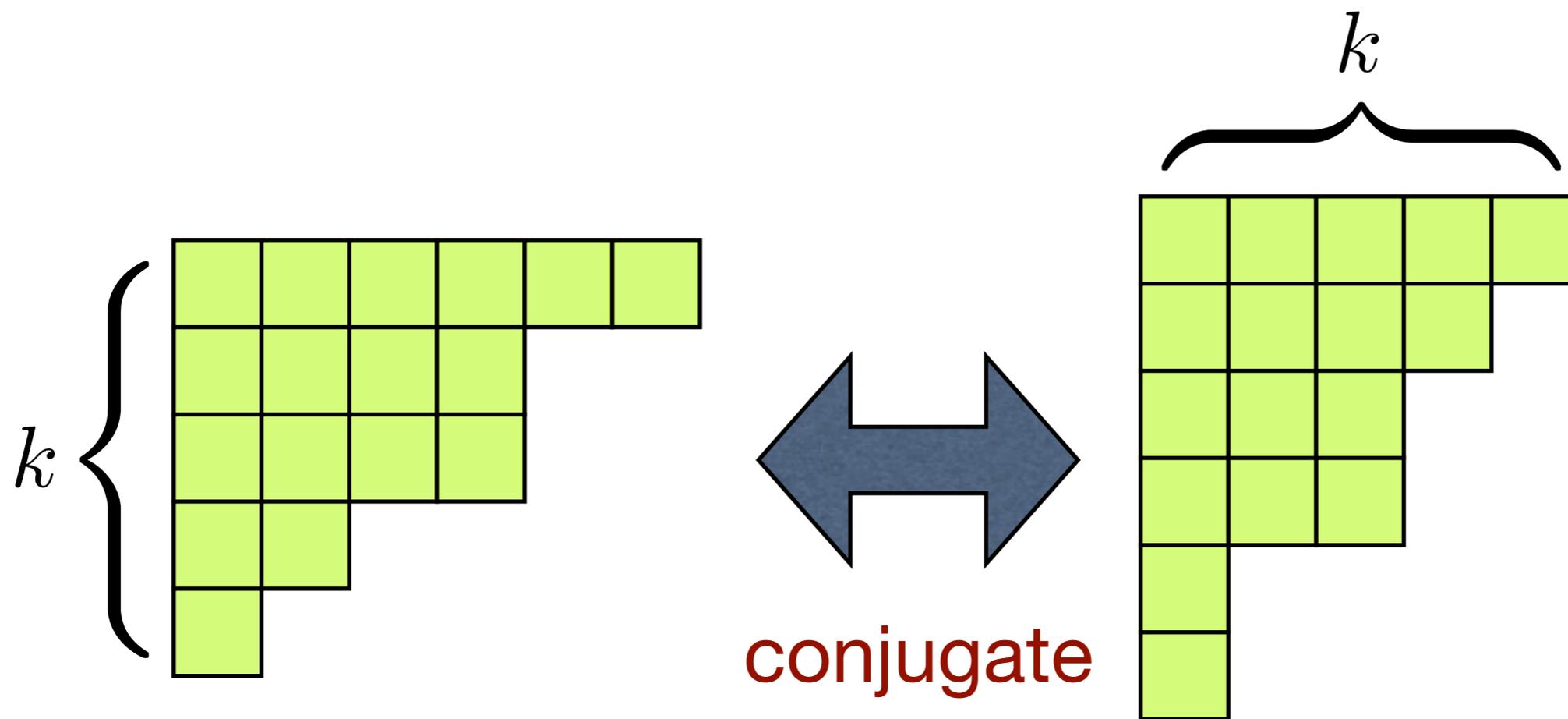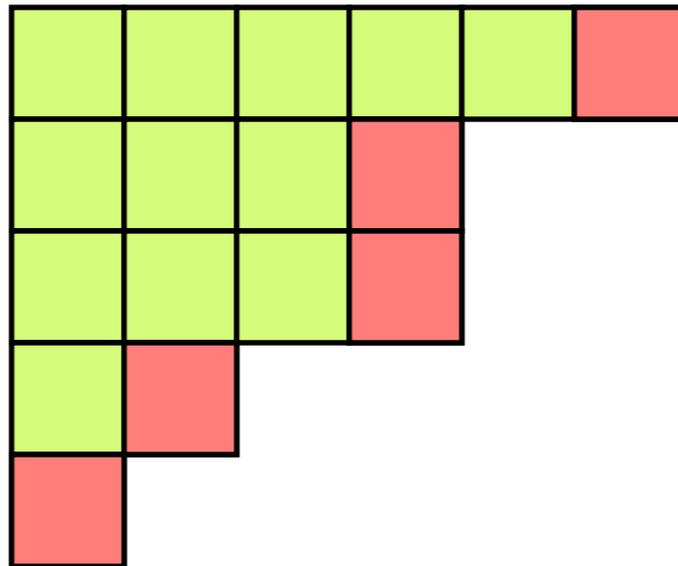
(6,4,4,2,1) → conjugate → (5,4,3,3,1,1)

one-to-one
correspondence

conjugate

# of partitions of $n$ into $k$ parts = # of partitions of $n$ with largest part $k$

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