Combinatorics

Extremal Set Theory

尹一通 Nanjing University, 2026 Spring

Extremal Combinatorics

"how large or how small a collection of finite objects can be, if it has to satisfy certain restrictions"

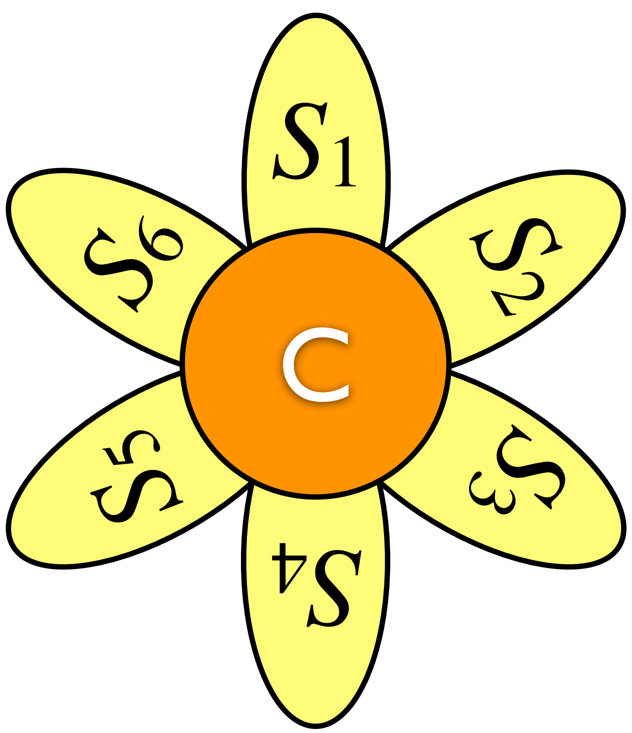
set system (family) $\mathscr{F} \subseteq 2^{[n]}$ with **ground set** $[n]$

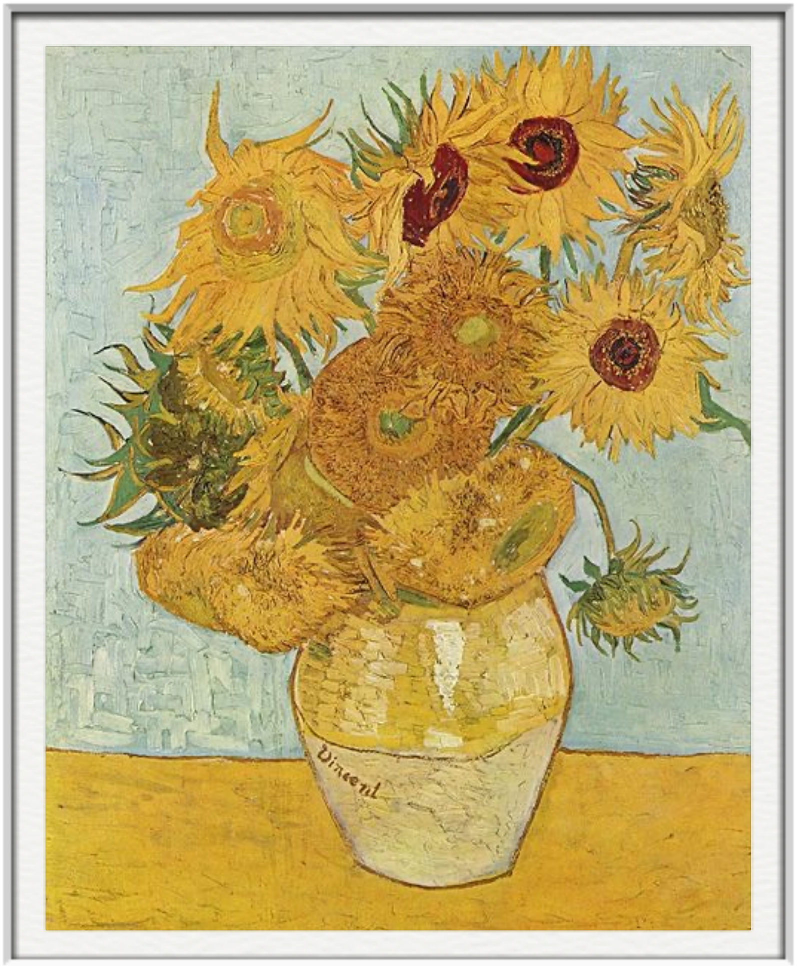
Sunflowers

$\mathscr{F} \subseteq 2^{[n]}$ is a sunflower of size r with center C:

$|\mathscr{F}| = r$ $\forall S, T \in \mathscr{F}$: $S \cap T = C$

a sunflower of size 6
with core C



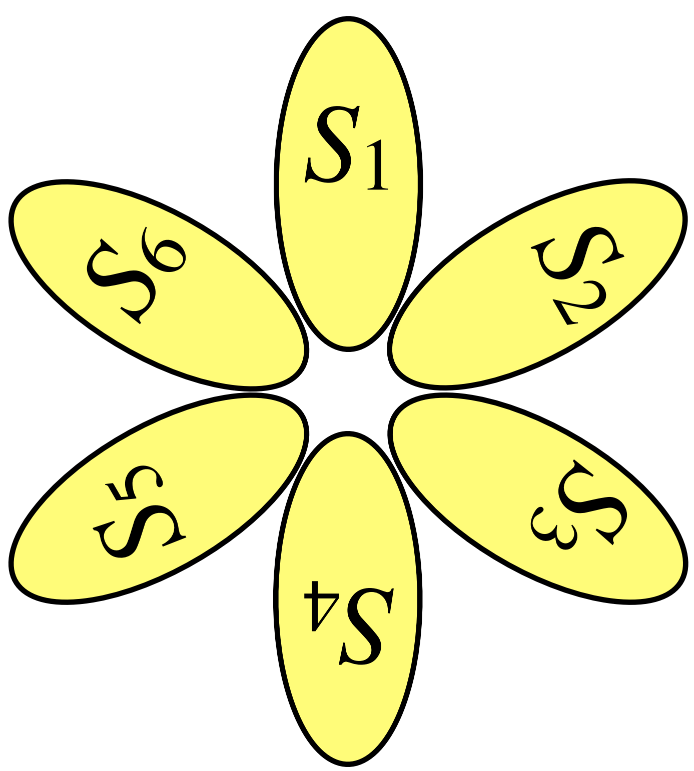


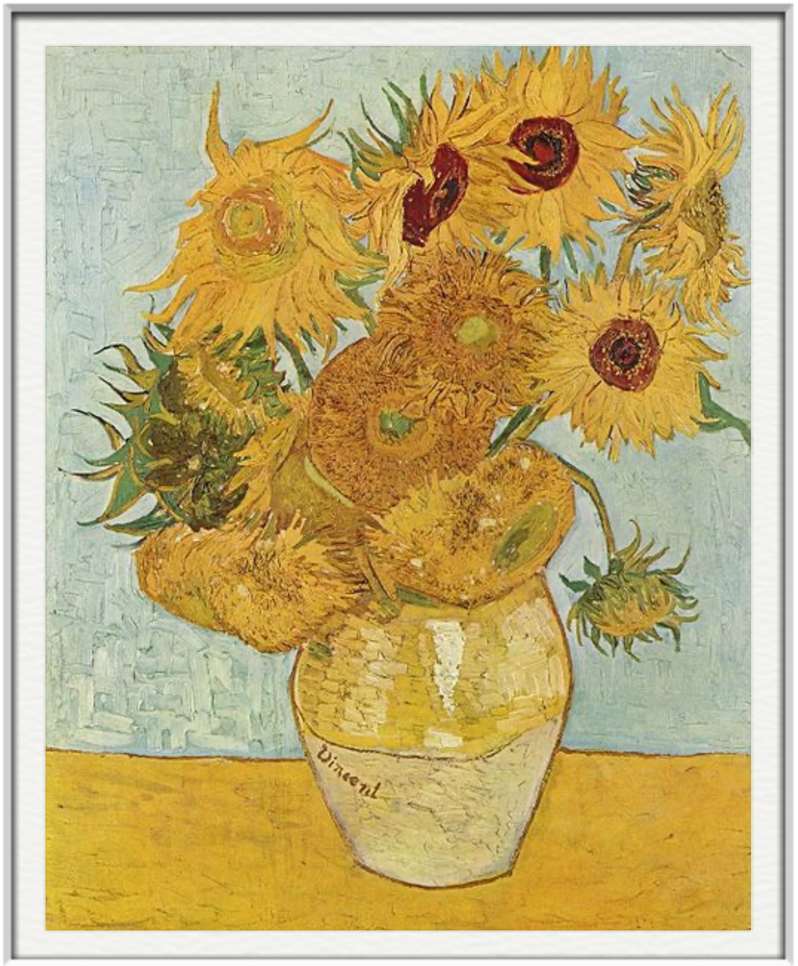
Sunflowers

$\mathscr{F} \subseteq 2^{[n]}$ is a sunflower of size r with center C:

$|\mathscr{F}| = r$ $\qquad$ $\forall S, T \in \mathscr{F}\colon\ S \cap T = C$

a sunflower of size 6
with core $\varnothing$





Sunflower Lemma (Erdős-Rado 1960)

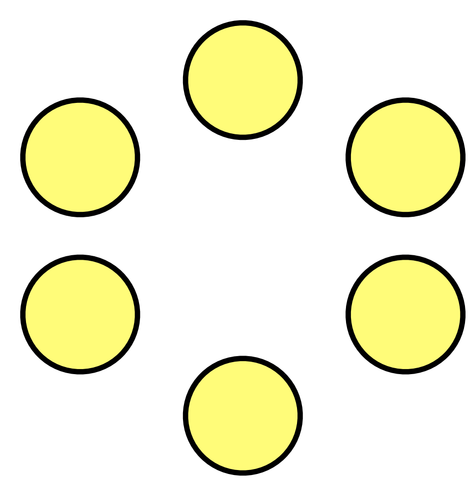
$$\mathcal{F} \subseteq \binom{[n]}{k}. \quad |\mathcal{F}| > k!(r-1)^k$$

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

Induction on k. **Basis:** $k = 1$

$$\mathcal{F} \subseteq \binom{[n]}{1} \qquad |\mathcal{F}| > r - 1$$

$\exists$ r singletons:



Sunflower Lemma (Erdős-Rado 1960)

$\mathcal{F} \subseteq \binom{[n]}{k}$. $|\mathcal{F}| > k!(r-1)^k$ ⇒

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

For $k \geq 2$,

take largest $\mathcal{G} \subseteq \mathcal{F}$ with disjoint members

$$\forall S, T \in \mathcal{G} \text{ that } S \neq T,\ S \cap T = \emptyset$$

Case.1: $|\mathcal{G}| \geq r$, $\mathcal{G}$ is a sunflower of size r

Case.2: $|\mathcal{G}| \leq r-1$,

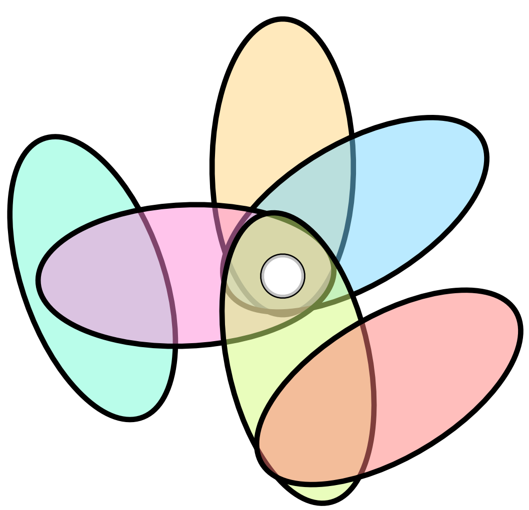
Goal: find a popular $x \in [n]$

Sunflower Lemma (Erdős-Rado 1960)

$\mathcal{F} \subseteq \binom{[n]}{k}$. $|\mathcal{F}| > k!(r-1)^k$ ⇒

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

$|\mathcal{G}| \le r - 1$, **Goal:** find a popular $x \in [n]$



consider

$$\{S \in \mathcal{F} \mid x \in S\}$$

remove x

$$\mathcal{H} = \{S \setminus \{x\} \mid S \in \mathcal{F} \land x \in S\}$$

$\mathcal{H} \subseteq \binom{[n]}{k-1}$ if $|\mathcal{H}| > (k-1)!(r-1)^{k-1}$ **I.H.**

$$\mathcal{F} \subseteq \binom{[n]}{k}. \qquad |\mathcal{F}| > k!(r-1)^k$$

take **maximal** $\mathcal{G} \subseteq \mathcal{F}$ with **disjoint** members

$$|\mathcal{G}| \leq r-1, \qquad \text{let } Y = \bigcup_{S\in\mathcal{G}} S \qquad |Y| \leq k(r-1)$$

claim: Y intersects all $S \in \mathcal{F}$

if otherwise: $\exists T \in \mathcal{F},\ T \cap Y = \emptyset$

T is disjoint with all $S \in \mathcal{G}$

contradiction!

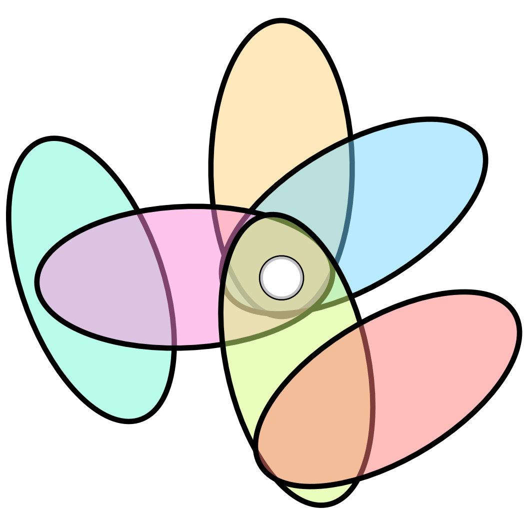
$$\mathcal{F} \subseteq \binom{[n]}{k}. \qquad |\mathcal{F}| > k!(r-1)^k$$

take **maximal** $\mathcal{G} \subseteq \mathcal{F}$ with **disjoint** members

$$|\mathcal{G}| \le r-1, \qquad \text{let } Y = \bigcup_{S \in \mathcal{G}} S \qquad |Y| \le k(r-1)$$

Y intersects all $S \in \mathcal{F}$

Pigeonhole: $\exists x \in Y$, # of $S \in \mathcal{F}$ containing x



$$|\{S \in \mathcal{F} \mid x \in S\}| \ge \frac{|\mathcal{F}|}{|Y|} > \frac{k!(r-1)^k}{k(r-1)}$$

$$= (k-1)!(r-1)^{k-1}$$

$$\mathcal{H} = \{S \setminus \{x\} \mid S \in \mathcal{F} \wedge x \in S\}$$

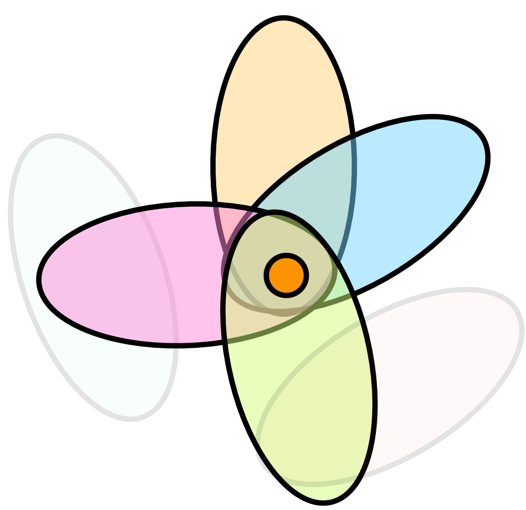
$$\mathcal{H} \subseteq \binom{[n]}{k-1} \qquad |\mathcal{H}| > (k-1)!(r-1)^{k-1}$$

Sunflower Lemma (Erdős-Rado 1960)

$\mathcal{F} \subseteq \binom{[n]}{k}$. $|\mathcal{F}| > k!(r-1)^k$ ⇒

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

$\exists x \in Y$, let $\mathcal{H} = \{S \setminus \{x\} \mid S \in \mathcal{F} \wedge x \in S\}$



$$\mathcal{H} \subseteq \binom{[n]}{k-1}$$

$$|\mathcal{H}| > (k-1)!(r-1)^{k-1}$$

I.H.: $\mathcal{H}$ contains a sunflower of size r

adding x back, it is a sunflower in $\mathcal{F}$

Sunflower Lemma (Erdős-Rado 1960)

$\mathcal{F} \subseteq \binom{[n]}{k}$. $|\mathcal{F}| > k!(r-1)^k$ ⇒

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

Induction on k. **Basis:** $k = 1$ *trivial*

For $k \geq 2$, take **maximal disjoint** $\mathcal{G} \subseteq \mathcal{F}$

case.1: $|\mathcal{G}| \geq r$, $\mathcal{G}$ is a sunflower of size r

case.2: $|\mathcal{G}| \leq r-1$, $\exists x \in Y$,

$$\mathcal{H} = \{S \setminus \{x\} \mid S \in \mathcal{F} \wedge x \in S\}$$

$\mathcal{H}$ contains a sunflower of size r

⇒ $\mathcal{F}$ contains a sunflower of size r

Sunflower Conjecture (Erdős-Rado 1960)

$\mathcal{F} \subseteq \binom{[n]}{k}$. $|\mathcal{F}| > c(r)^k$ ⇒

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

$c(r)$: constant depending only on r

Alweiss-Lovett-Wu-Zhang (2019):

$\mathcal{F} \subseteq \binom{[n]}{k}$. $|\mathcal{F}| > O(r \log(rk))^k$ ⇒

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

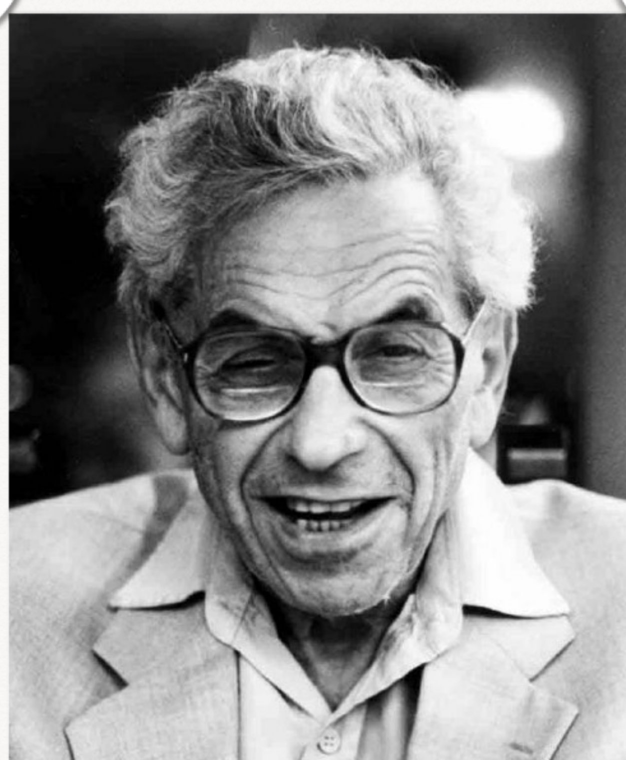
ABSTRACTIONS BLOG

Mathematicians Begin to Tame Wild 'Sunflower' Problem

6

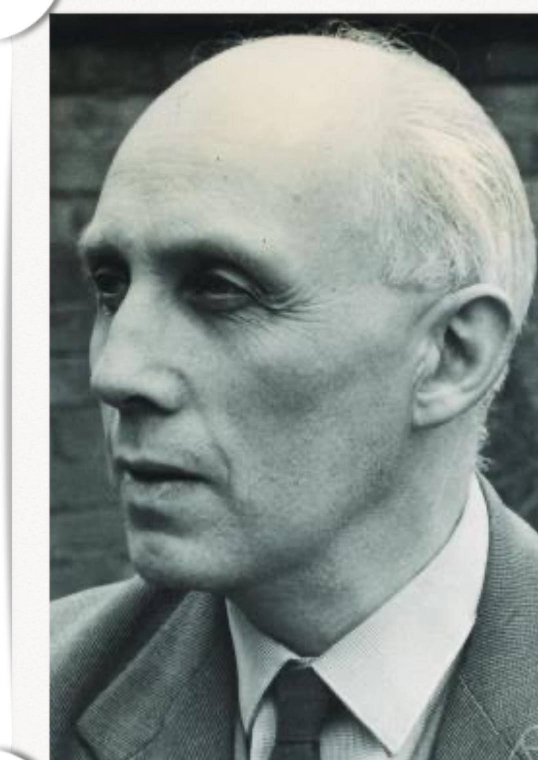
A major advance toward solving the 60-year-old sunflower conjecture is shedding light on how order begins to appear as random systems grow in size.

Erdős-Ko-Rado Theorem



Paul Erdős
(1913-1996)

柯召
(1910-2002)

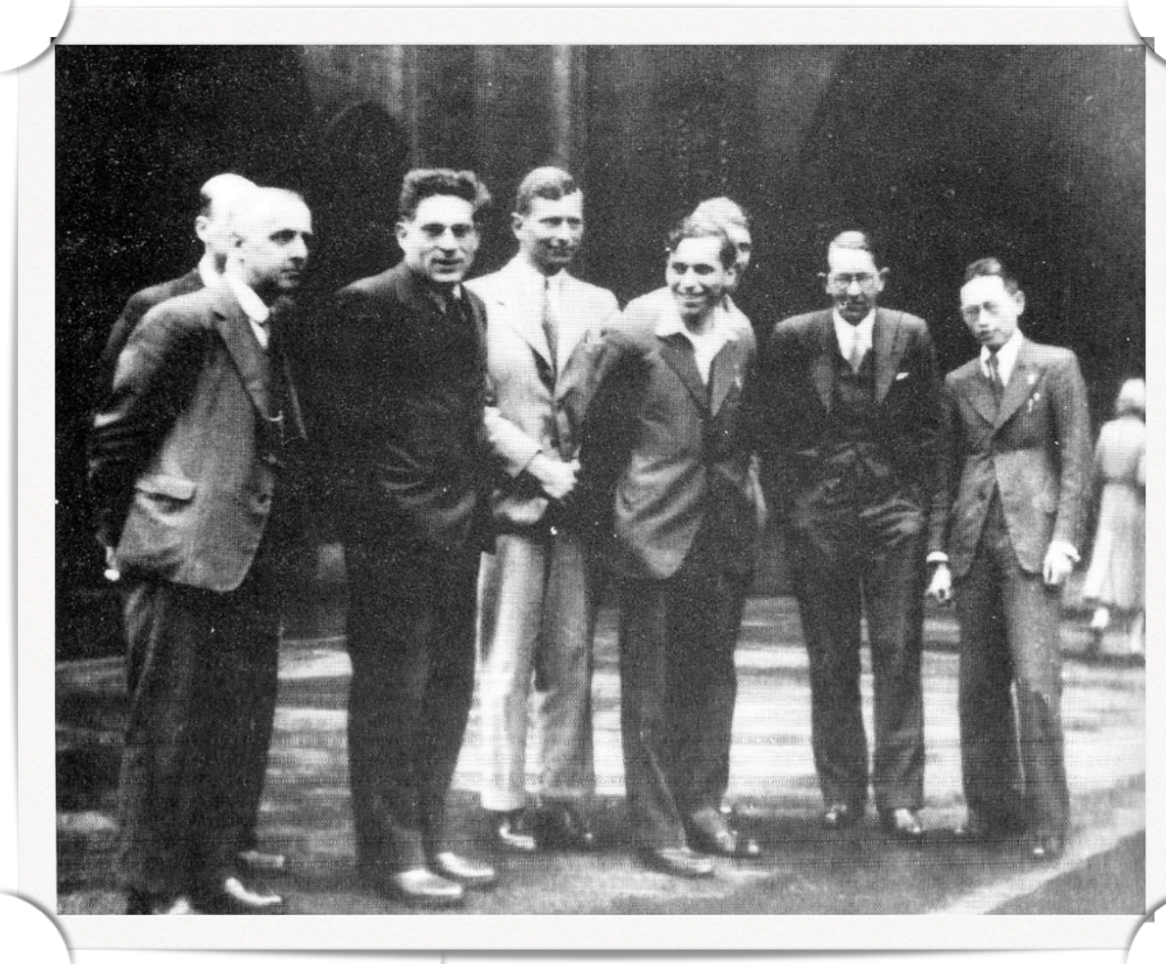


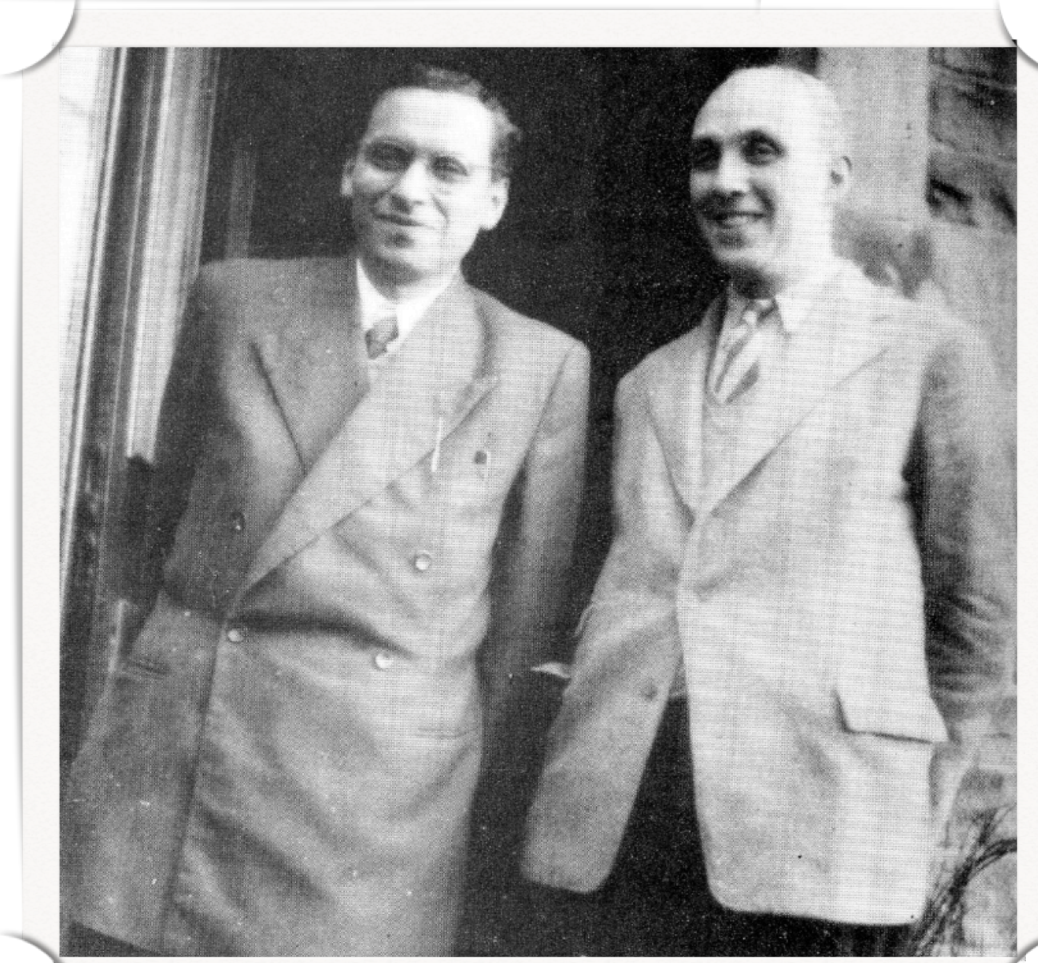
Richard Rado
(1906-1989)

柯召

Paul Erdős

Harold Davenport





Paul Erdős Richard Rado

Intersecting Families

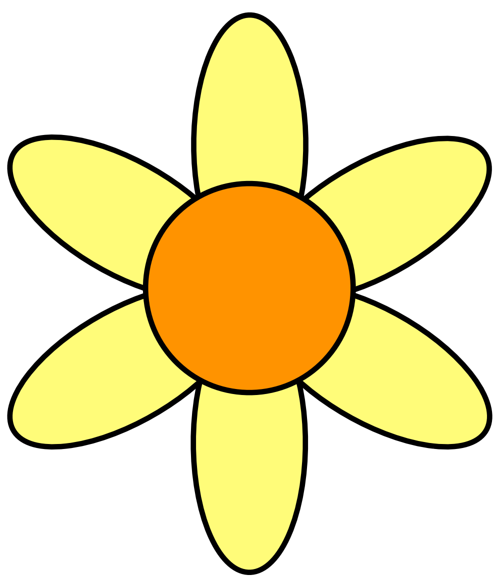
$\mathcal{F} \subseteq \binom{[n]}{k}$

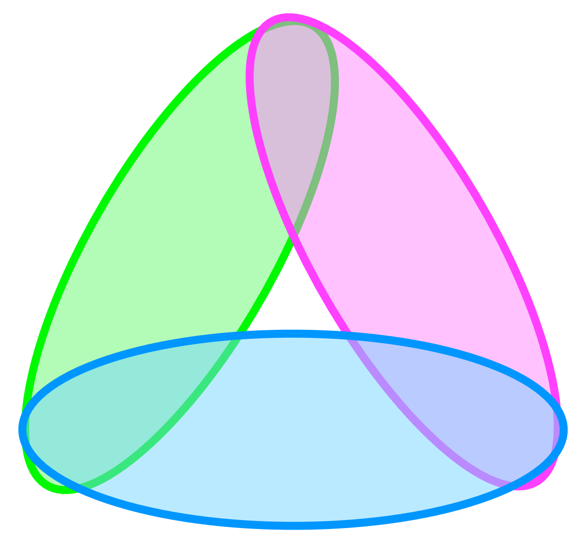
intersecting:

$\forall S, T \in \mathcal{F}, \quad S \cap T \neq \emptyset$

trivial case: $n < 2k$

nontrivial examples:





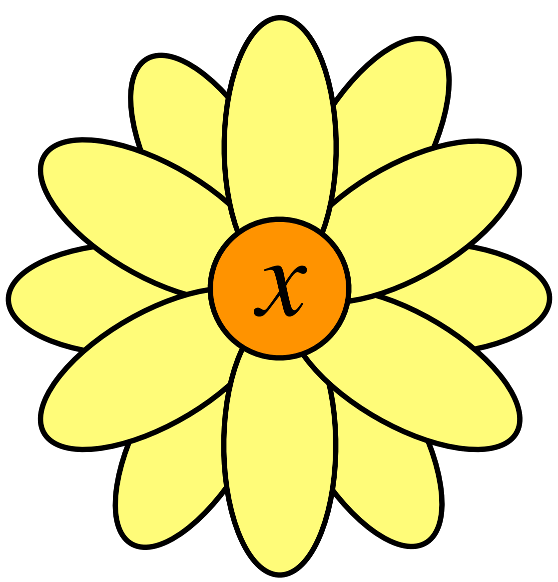
"How large can a nontrivial intersecting family be?"

Erdős-Ko-Rado Theorem

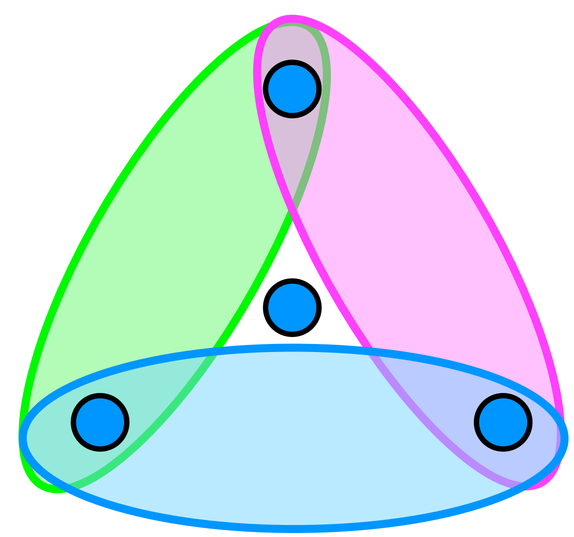
Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\forall S,T \in \mathcal{F}, S \cap T \neq \emptyset$ / $\mathcal{F}$ is intersecting $\Rightarrow$ $|\mathcal{F}| \leq \binom{n-1}{k-1}$

proved in 1938;   published in 1961;



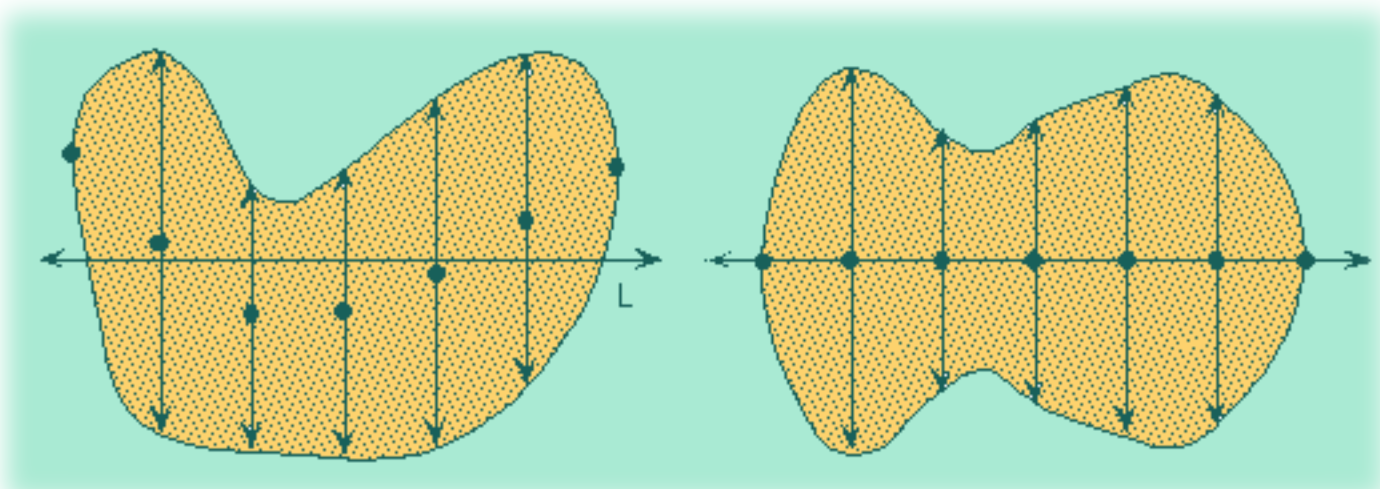
all $S \ni x$



Shifting (*Symmetrization*)

Isoperimetric problem:

With fixed perimeter,
what plane figure has the largest area?

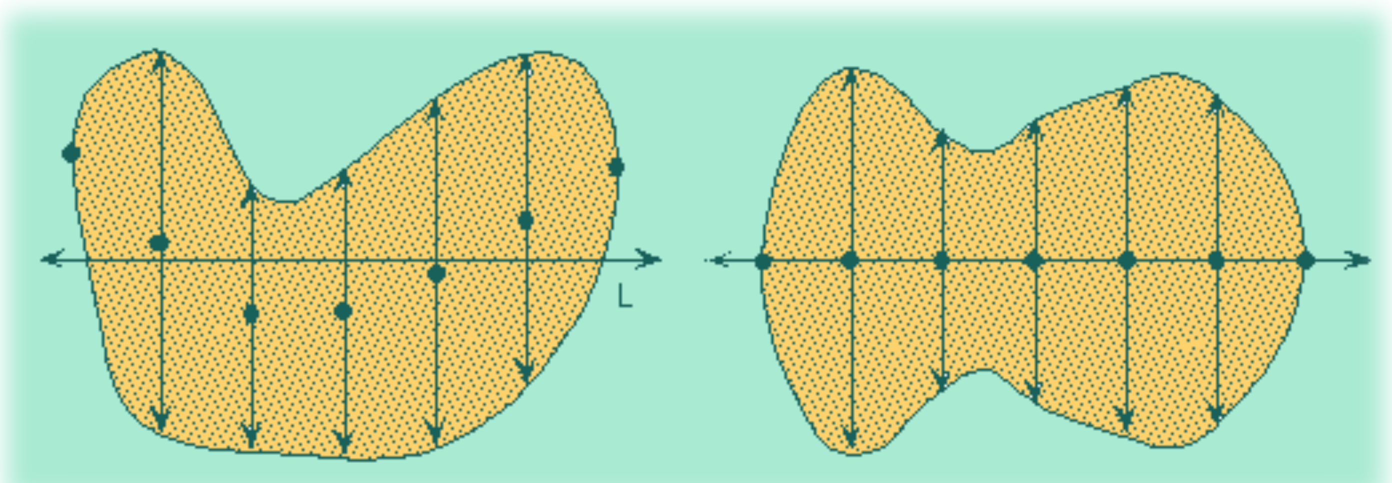


Steiner's symmetrization

Shifting (*Symmetrization*)

Isoperimetric problem:

With fixed area,
what plane figure has the smallest perimeter?



Steiner's symmetrization

Erdős-Ko-Rado Theorem

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

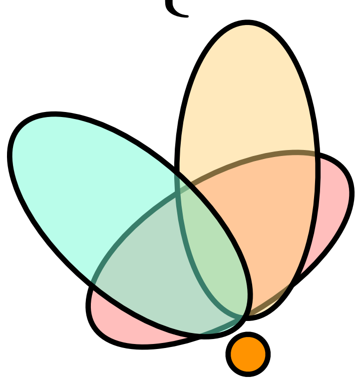
$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$

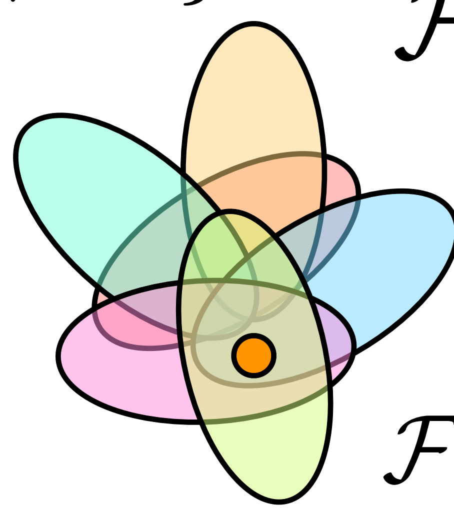
induction on n and k

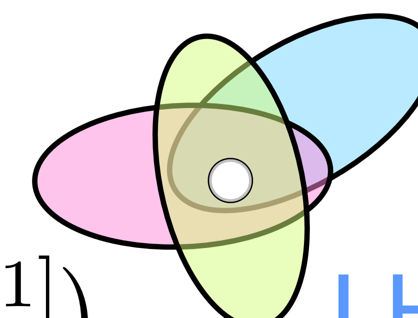
$\mathcal{F}_0 = \{S \in \mathcal{F} \mid n \notin S\}$

$\mathcal{F}_1 = \{S \in \mathcal{F} \mid n \in S\}$

$\mathcal{F}_1' = \{S \setminus \{n\} \mid S \in \mathcal{F}_1\}$







$\mathcal{F}_0 \subseteq \binom{[n-1]}{k}$

intersecting

I.H.

$|\mathcal{F}_0| \leq \binom{n-2}{k-1}$

$\mathcal{F}_1' \subseteq \binom{[n-1]}{k-1}$

intersecting?

I.H.

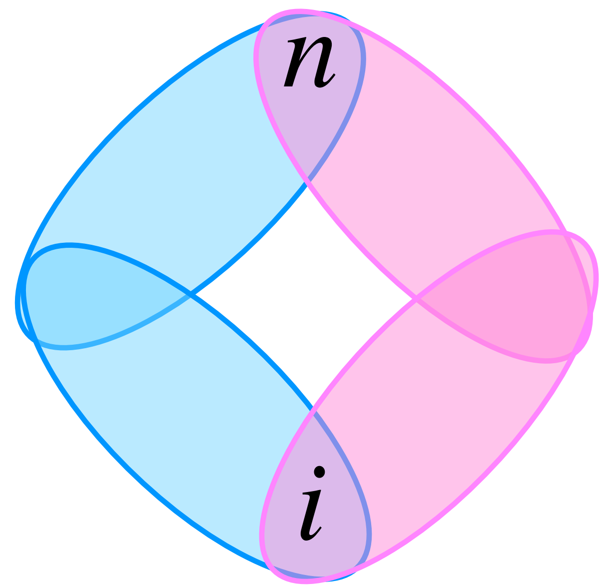
$|\mathcal{F}_1'| \leq \binom{n-2}{k-2}$

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{F}_0| + |\mathcal{F}_1'| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$$

Shifting (*compression*)

special $\mathcal{F} \subseteq \binom{[n]}{k}$

$\mathcal{F}$ remains intersecting after deleting n



Shifting (*compression*)

$\mathcal{F} \subseteq 2^{[n]}$ for $1 \leq i < j \leq n$

$\forall T \in \mathcal{F},\;$ write $T_{ij} = (T \setminus \{j\}) \cup \{i\}$

(i, j)-shift: $S_{ij}(\cdot)$

$\forall T \in \mathcal{F},$

$$S_{ij}(T) = \begin{cases} T_{ij} & \text{if } j \in T, i \notin T, \text{ and } T_{ij} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(T) \mid T \in \mathcal{F}\}$$

$$1 \le i < j \le n \quad \forall T \in \mathcal{F}, \ \text{write } T_{ij} = (T \setminus \{j\}) \cup \{i\}$$

$$S_{ij}(T) = \begin{cases} T_{ij} & \text{if } j \in T, i \notin T, \text{ and } T_{ij} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(T) \mid T \in \mathcal{F}\}$$

1. $|S_{ij}(T)| = |T|$ and $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$
2. $\mathcal{F}$ intersecting ⇨ $S_{ij}(\mathcal{F})$ intersecting

(2) the only bad case: $A, B \in \mathcal{F} \quad A \cap B = \{j\}$

$$A_{ij} = A \setminus \{j\} \cup \{i\} \in \mathcal{F} \quad B_{ij} = B \setminus \{j\} \cup \{i\} \notin \mathcal{F} \quad i \notin B$$

⇨ $A_{ij} \cap B = \emptyset$ contradiction!

$$1 \le i < j \le n \quad \forall T \in \mathcal{F}, \ \text{write } T_{ij} = (T \setminus \{j\}) \cup \{i\}$$

$$S_{ij}(T) = \begin{cases} T_{ij} & \text{if } j \in T, i \notin T, \text{ and } T_{ij} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(T) \mid T \in \mathcal{F}\}$$

1. $|S_{ij}(T)| = |T|$ and $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$
2. $\mathcal{F}$ intersecting $\Rightarrow$ $S_{ij}(\mathcal{F})$ intersecting

repeat applying (i, j)-shifting $S_{ij}(\mathcal{F})$ **for** $1 \le i < j \le n$

eventually, $\mathcal{F}$ is unchanged by any $S_{ij}(\mathcal{F})$

called: $\mathcal{F}$ is shifted

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\forall S, T \in \mathcal{F}, S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$

Erdős-Ko-Rado's proof:

true for $k = 1$;

when $n = 2k$,

$\forall S \in \binom{[n]}{k}$ at most one of S and $\bar{S}$ is in $\mathcal{F}$

$$|\mathcal{F}| \leq \frac{1}{2}\binom{n}{k} = \frac{n!}{2 \cdot k!(n-k)!}$$

$$= \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1}$$

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \implies |\mathcal{F}| \leq \binom{n-1}{k-1}$$

arbitrary intersecting $\mathcal{F}$ → shifted $\mathcal{F}'$

$|\mathcal{F}| = |\mathcal{F}'|$

keep intersecting

$$|\mathcal{F}'| \leq \binom{n-1}{k-1}$$

←

$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\forall S, T \in \mathcal{F}, S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$

when $n > 2k$, induction on n WLOG: $\mathcal{F}$ is shifted

$\mathcal{F}_1 = \{S \in \mathcal{F} \mid n \in S\}$ $\mathcal{F}_1' = \{S \setminus \{n\} \mid S \in \mathcal{F}_1\}$

$\mathcal{F}_1'$ is intersecting

otherwise, $\exists A, B \in \mathcal{F}$ $A \cap B = \{n\}$

$|A \cup B| \leq 2k - 1 < n - 1 \Rightarrow \exists i < n,\ i \notin A \cup B$

$C = A \setminus \{n\} \cup \{i\} \in \mathcal{F} \Leftarrow \mathcal{F}$ is shifted

$C \cap B = \emptyset$ contradiction!

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\forall S, T \in \mathcal{F}, S \cap T \neq \emptyset$ ⇨ $|\mathcal{F}| \leq \binom{n-1}{k-1}$

when $n > 2k$, induction on n WLOG: $\mathcal{F}$ is shifted

$\mathcal{F}_0 = \{S \in \mathcal{F} \mid n \notin S\}$ $\mathcal{F}_1 = \{S \in \mathcal{F} \mid n \in S\}$

$\mathcal{F}_0 \subseteq \binom{[n-1]}{k}$ and intersecting ⇨ (I.H.) $|\mathcal{F}_0| \leq \binom{n-2}{k-1}$

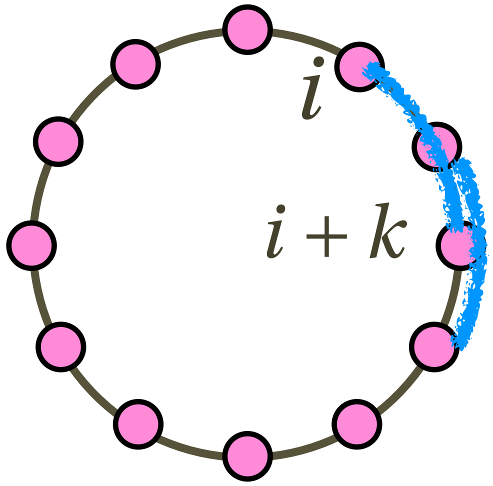
$\mathcal{F}_1' = \{S \setminus \{n\} \mid S \in \mathcal{F}_1\}$

$\mathcal{F}_1' \subseteq \binom{[n-1]}{k-1}$ and intersecting ⇨ (I.H.) $|\mathcal{F}_1'| \leq \binom{n-2}{k-2}$

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{F}_0| + |\mathcal{F}_1'| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$$

Katona's proof (1972)

n-cycle:



k-arc: length k path on cycle

intersecting arcs: share edges

Lemma
If $n \geq 2k$ and $A_1, A_2, \ldots, A_t$ are distinct pairwise intersecting k-arcs, then $t \leq k$.

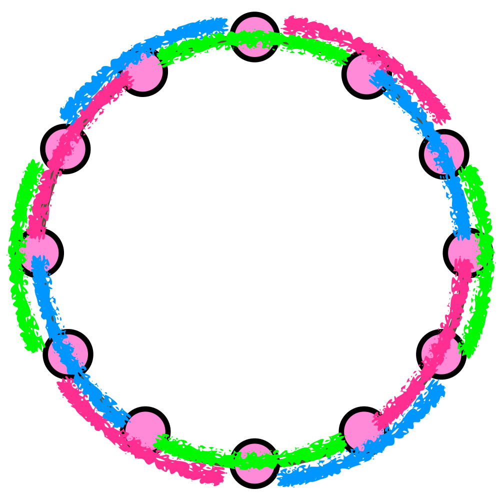
every node can be endpoint of at most 1 arc

take A_1: A_1 has $k+1$ nodes

2 endpoints of itself

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\forall S, T \in \mathcal{F}, S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$



take an n-cycle π of $[n]$

family of all k-arcs in π

$$\mathcal{G}_\pi = \{\{\pi_{(i+j) \bmod n} \mid j \in [k]\} \mid i \in [n]\}$$

double counting: $X = \{(S, \pi) \mid S \in \mathcal{F} \cap \mathcal{G}_\pi\}$

each n-cycle π an n-cycle has $\leq k$ intersecting k-arcs

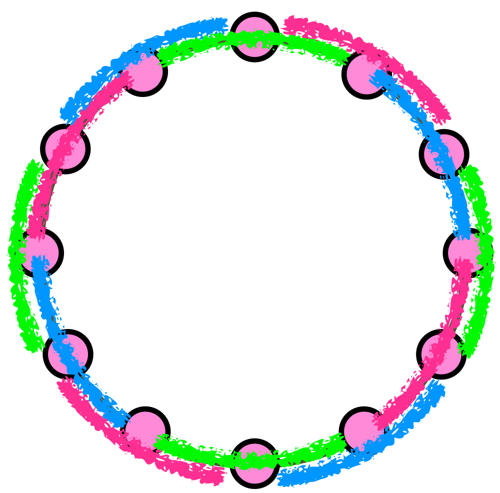
$$|\mathcal{F} \cap \mathcal{G}_\pi| \leq k$$

of n-cycles: $(n-1)!$

$$|X| = \sum_{n\text{-cycle } \pi} |\mathcal{F} \cap \mathcal{G}_\pi| \leq k(n-1)!$$

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\forall S, T \in \mathcal{F}, S \cap T \neq \emptyset$ ⇒ $|\mathcal{F}| \leq \binom{n-1}{k-1}$



take an n-cycle π of $[n]$

family of all k-arcs in π

$$\mathcal{G}_\pi = \{\{\pi_{(i+j) \bmod n} \mid j \in [k]\} \mid i \in [n]\}$$

double counting: $X = \{(S, \pi) \mid S \in \mathcal{F} \cap \mathcal{G}_\pi\}$

$|X| \leq k(n-1)!$

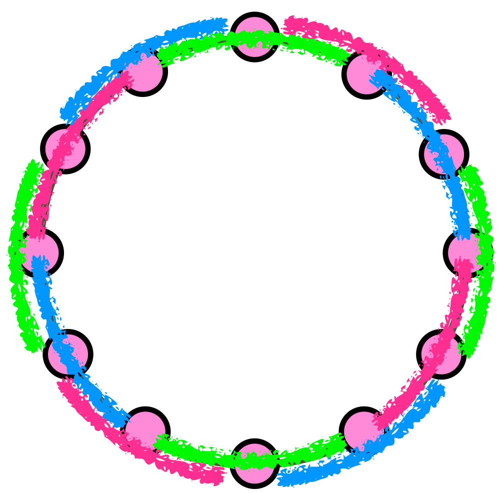
each S is a k-arc in

$k!(n-k)!$ cycles

$$|X| = \sum_{S \in \mathcal{F}} |\{\pi \mid S \in \mathcal{G}_\pi\}| = |\mathcal{F}|k!(n-k)!$$

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$$



take an n-cycle π of $[n]$

family of all k-arcs in π

$$\mathcal{G}_\pi = \{\{\pi_{(i+j) \bmod n} \mid j \in [k]\} \mid i \in [n]\}$$

double counting: $X = \{(S, \pi) \mid S \in \mathcal{F} \cap \mathcal{G}_\pi\}$

$|X| \leq k(n-1)!$

$|X| = |\mathcal{F}| k!(n-k)!$

$$|\mathcal{F}| \leq \frac{k(n-1)!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1}$$

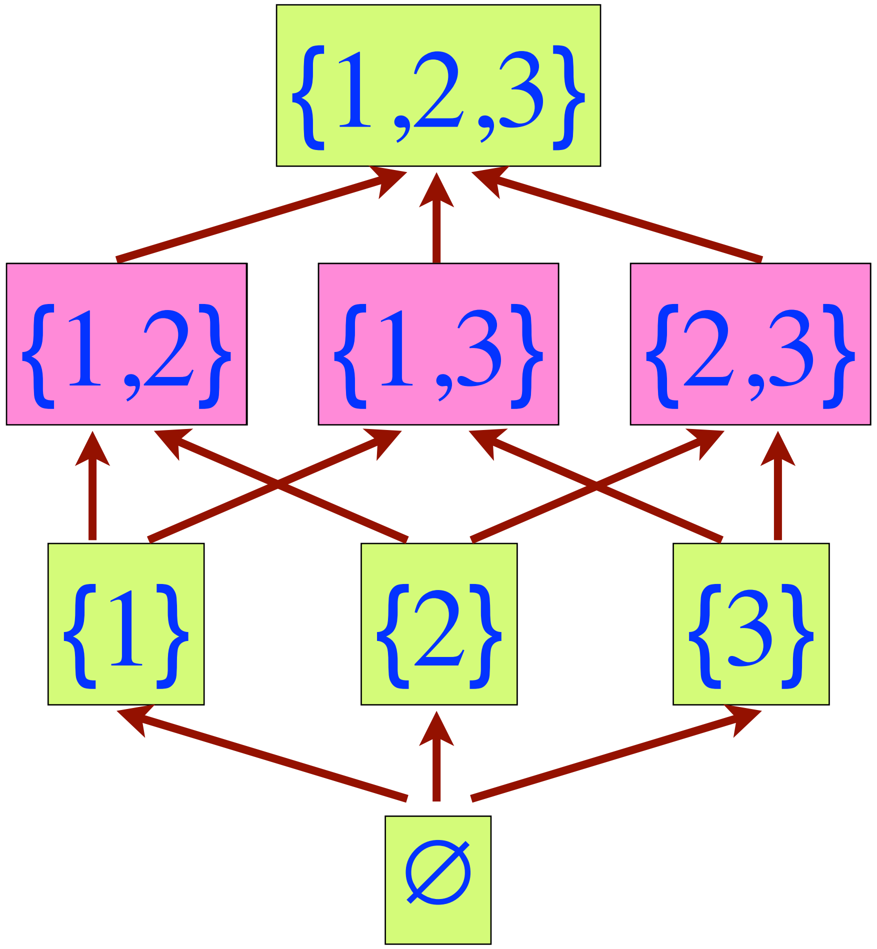
Antichains

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain

$$\forall A, B \in \mathcal{F}, \quad A \not\subseteq B$$

$\binom{[n]}{k}$ is antichain

largest size: $\binom{n}{\lfloor n/2 \rfloor}$



"Is this the largest size for all antichains?"

Sperner's Theorem

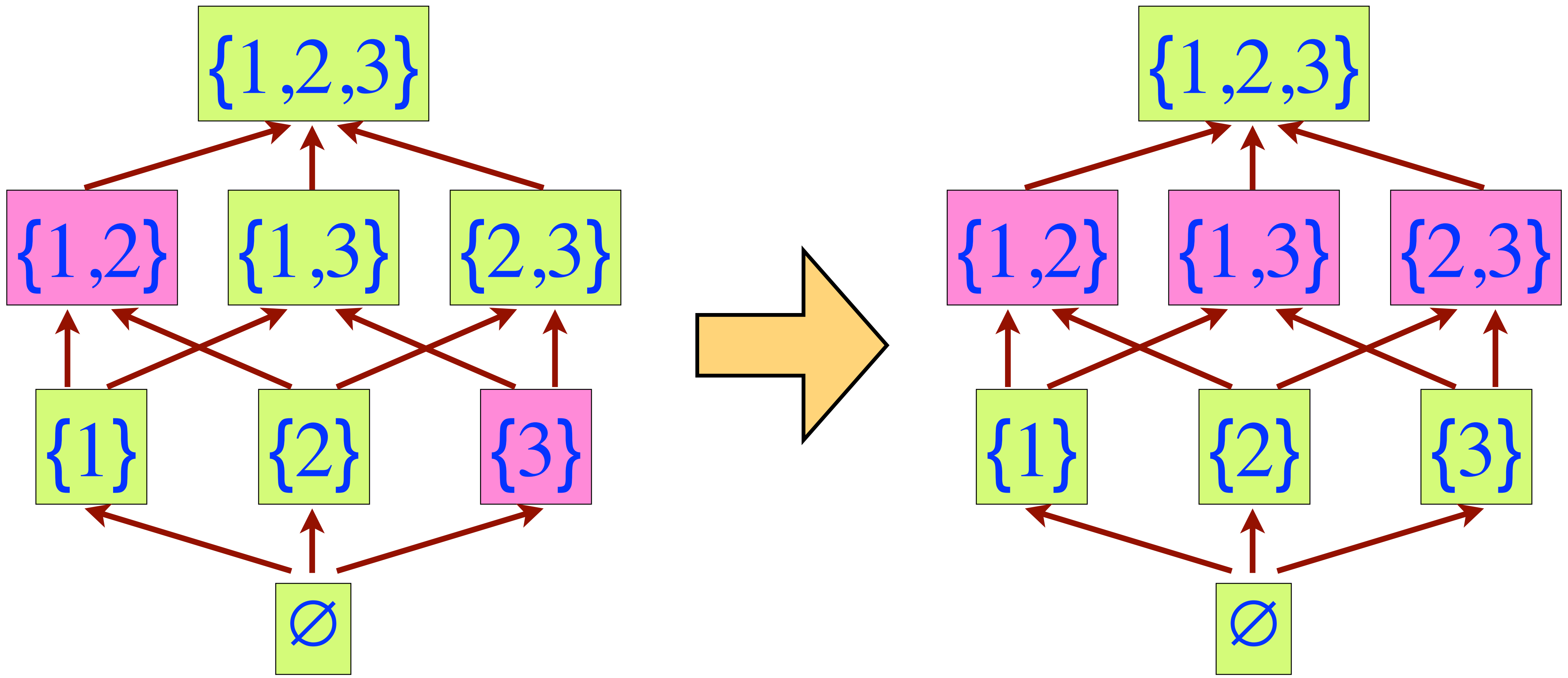
Theorem (Sperner 1928)

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain.

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Emanuel Sperner
(1905 - 1980)

Sperner's proof



$$\mathcal{F} \subseteq \binom{[n]}{k}$$

shade: $$\nabla\mathcal{F} = \left\{T \in \binom{[n]}{k+1} \mid \exists S \in \mathcal{F}, S \subset T\right\}$$

shadow: $$\Delta\mathcal{F} = \left\{T \in \binom{[n]}{k-1} \mid \exists S \in \mathcal{F}, T \subset S\right\}$$

$$[n] = \{1,2,3,4,5\}$$

$$\mathcal{F} = \{\{1,2,3\}, \{1,3,4\}, \{2,3,5\}\}$$

$$\nabla\mathcal{F} = \{\{1,2,3,4\}, \{1,2,3,5\}, \{1,3,4,5\}, \{2,3,4,5\}\}$$

$$\Delta\mathcal{F} = \{\{1,2\}, \{2,3\}, \{1,3\}, \{3,4\}, \{1,4\}, \{2,5\}, \{3,5\}\}$$

Lemma (Sperner)

Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Then

$$|\nabla \mathcal{F}| \geq \frac{n-k}{k+1}|\mathcal{F}| \qquad \text{(for } k < n\text{)}$$

$$|\Delta \mathcal{F}| \geq \frac{k}{n-k+1}|\mathcal{F}| \qquad \text{(for } k > 0\text{)}$$

double counting

$$\mathcal{R} = \{(S,T) \mid S \in \mathcal{F}, T \in \nabla\mathcal{F}, S \subset T\}$$

$$\forall S \in \mathcal{F}, \qquad n-k \; T \in \binom{[n]}{k+1} \text{ have } T \supset S$$

$$|\mathcal{R}| = (n-k)|\mathcal{F}|$$

$$\forall T \in \nabla\mathcal{F}, \quad T \text{ has } \binom{k+1}{k} = k+1 \text{ many } k\text{-subsets}$$

$$|\mathcal{R}| \leq (k+1)|\nabla\mathcal{F}|$$

Lemma (Sperner)

Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Then

$$|\nabla \mathcal{F}| \geq \frac{n-k}{k+1}|\mathcal{F}| \qquad \text{(for } k < n\text{)}$$

$$|\Delta \mathcal{F}| \geq \frac{k}{n-k+1}|\mathcal{F}| \qquad \text{(for } k > 0\text{)}$$

Corollary:

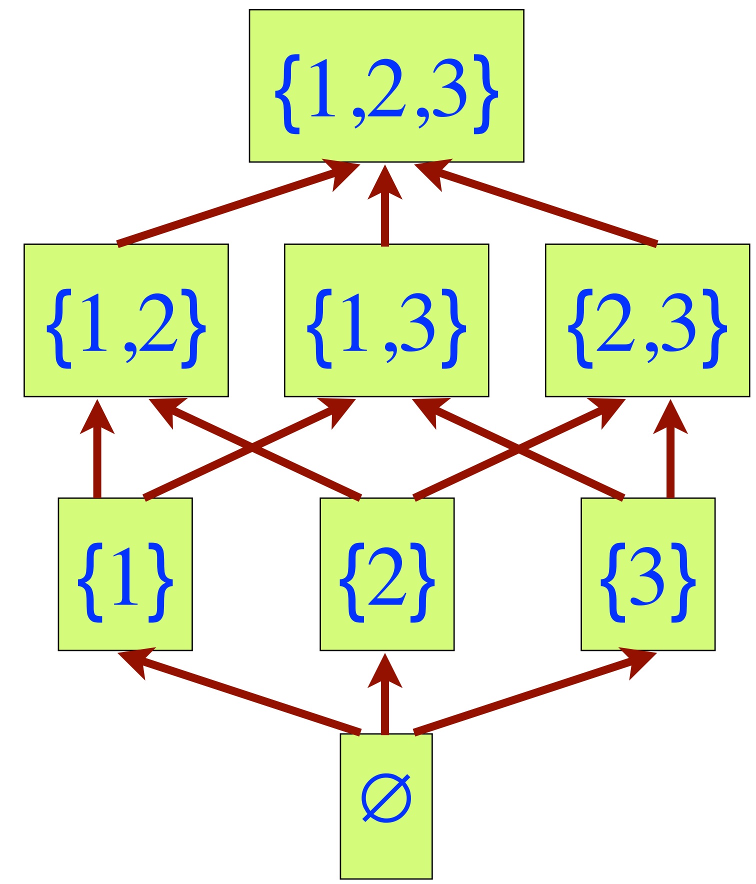
If $k \leq \frac{1}{2}(n-1)$, then $|\nabla \mathcal{F}| \geq |\mathcal{F}|$.

If $k \geq \frac{1}{2}(n+1)$, then $|\Delta \mathcal{F}| \geq |\mathcal{F}|$.

Sperner's Theorem

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Then $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$.

let $\mathcal{F}_k = \mathcal{F} \cap \binom{[n]}{k}$



If $k \le \frac{1}{2}(n-1)$, then $|\nabla\mathcal{F}| \ge |\mathcal{F}|$.

If $k \ge \frac{1}{2}(n+1)$, then $|\Delta\mathcal{F}| \ge |\mathcal{F}|$.

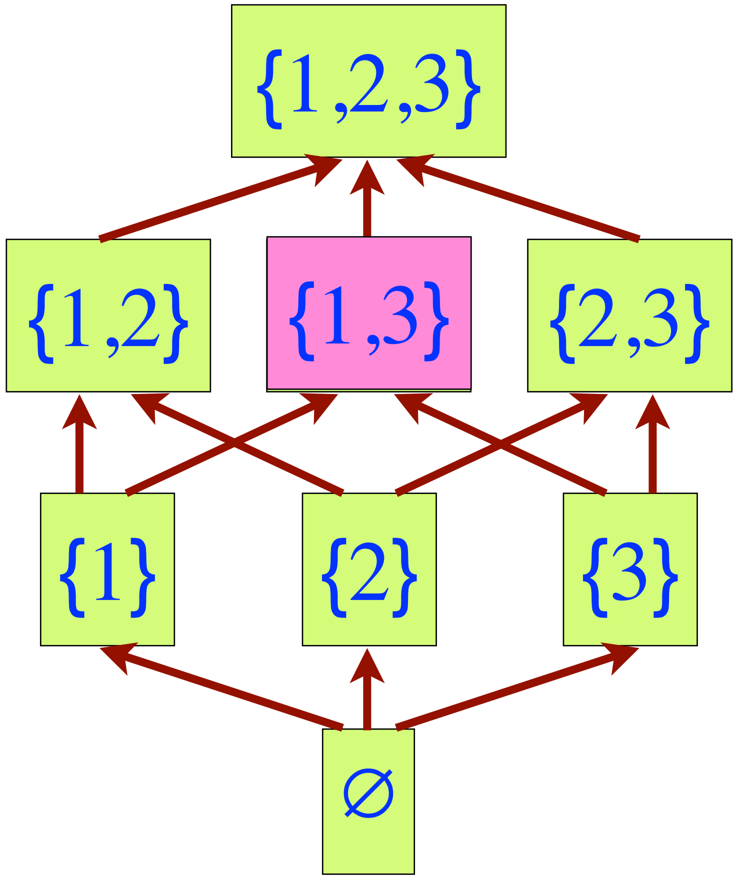
replace $\mathcal{F}_k$ by $\begin{cases} \nabla\mathcal{F}_k & \text{if } k < \frac{1}{2}(n-1) \\ \Delta\mathcal{F}_k & \text{if } k \ge \frac{1}{2}(n+1) \end{cases}$ still antichain!

repeat until $\mathcal{F} \subseteq \binom{[n]}{\lfloor n/2 \rfloor}$ with no decreasing of $|\mathcal{F}|$

Sperner's Theorem

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Then $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$.

Lubell's proof (double counting)



maximal chain:

$$\emptyset \subset S_1 \subset \cdots \subset S_{n-1} \subset [n]$$

# of maximal chains in $2^{[n]}$: $n!$

$\forall S \subseteq [n]$,

# of maximal chains containing S: $|S|!(n-|S|)!$

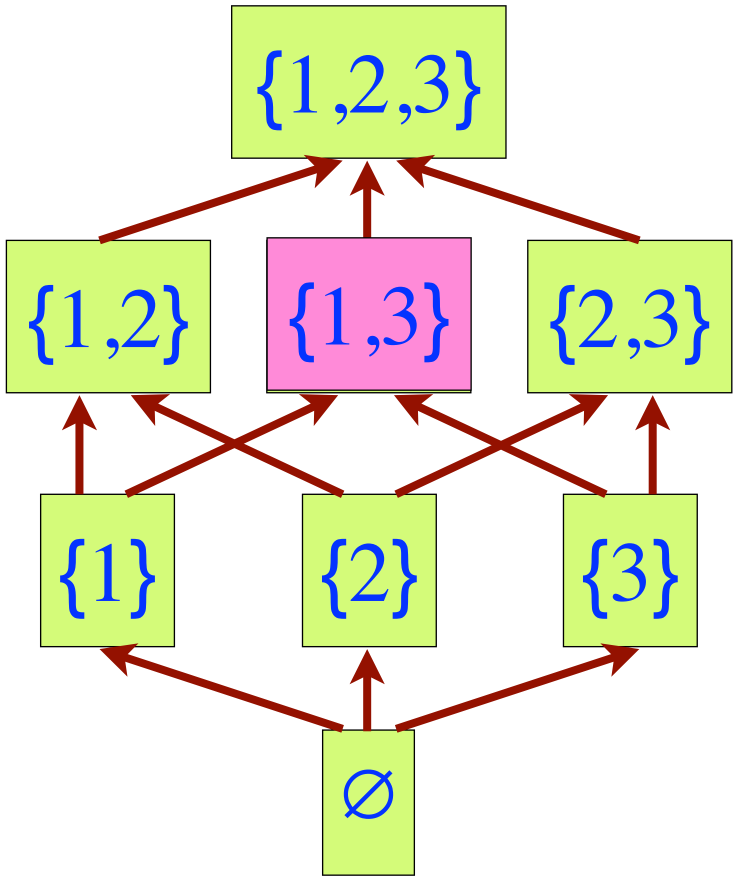
$\mathcal{F}$ is antichain ⇒ $\forall$ chain C, $|\mathcal{F} \cap C| \le 1$

maximal chains crossing $\mathcal{F}$ ≤ # all maximal chains

Sperner's Theorem

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Lubell's proof (double counting)



maximal chain:

$$\emptyset \subset S_1 \subset \cdots \subset S_{n-1} \subset [n]$$

# of maximal chains in $2^{[n]}$: $n!$

$\forall S \subseteq [n]$,

# of maximal chains containing S: $|S|!(n-|S|)!$

$\mathcal{F}$ is antichain ⇒ $\forall$ chain C, $|\mathcal{F} \cap C| \leq 1$

$$\sum_{S \in \mathcal{F}} |S|!(n-|S|)! \leq n!$$

Sperner's Theorem

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Lubell's proof (double counting)

$$\sum_{S \in \mathcal{F}} |S|!(n-|S|)! \leq n!$$

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} = \sum_{S \in \mathcal{F}} \frac{|S|!(n-|S|)!}{n!} \leq 1$$

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$$

LYM Inequality

(Lubell-Yamamoto 1954, Meschalkin 1963)

LYM inequality

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain.

$$\sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1$$

$$\mathcal{F} \subseteq 2^{[n]} \text{ is an antichain. } \sum_{S\in\mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1$$

Alon's proof (the probabilistic method)

let π be a random permutation $[n]$

$$\mathcal{C}_\pi = \{\{\pi_1\}, \{\pi_1, \pi_2\}, \ldots, \{\pi_1, \ldots, \pi_n\}\}$$

$$\forall S \in \mathcal{F}, \qquad X_S = \begin{cases} 1 & S \in \mathcal{C}_\pi \\ 0 & \text{otherwise} \end{cases}$$

let $X = \sum_{S\in\mathcal{F}} X_S = |\mathcal{F} \cap \mathcal{C}_\pi|$

$$\mathbf{E}[X_S] = \Pr[S \in \mathcal{C}_\pi] = \frac{1}{\binom{n}{|S|}}$$

$\mathcal{C}_\pi$ contains precisely 1 $|S|$-set

uniform over all $|S|$-sets

$$\mathcal{F} \subseteq 2^{[n]} \text{ is an antichain.} \quad \sum_{S\in\mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1$$

Alon's proof (the probabilistic method)

let π be a random permutation $[n]$

$$\mathcal{C}_\pi = \{\{\pi_1\}, \{\pi_1, \pi_2\}, \ldots, \{\pi_1, \ldots, \pi_n\}\}$$

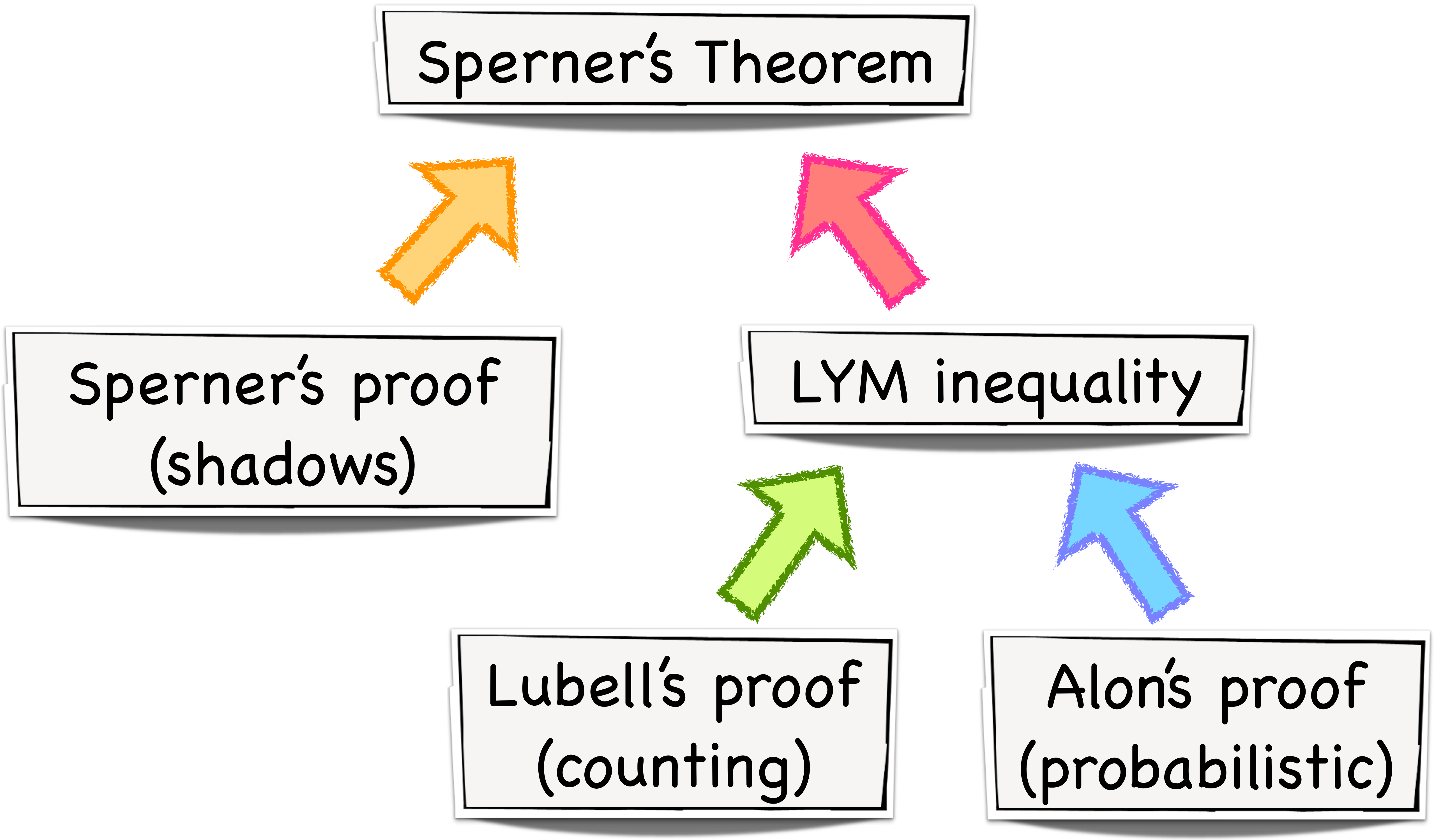
$$X = \sum_{S\in\mathcal{F}} X_S = |\mathcal{F} \cap \mathcal{C}_\pi| \leq 1$$

$\mathcal{F}$ is antichain

$\mathcal{C}_\pi$ is chain

$$\mathbf{E}[X_S] = \frac{1}{\binom{n}{|S|}}$$

$$1 \geq \mathbf{E}[X] = \sum_{S\in\mathcal{F}} \mathbf{E}[X_S] = \sum_{S\in\mathcal{F}} \frac{1}{\binom{n}{|S|}}$$

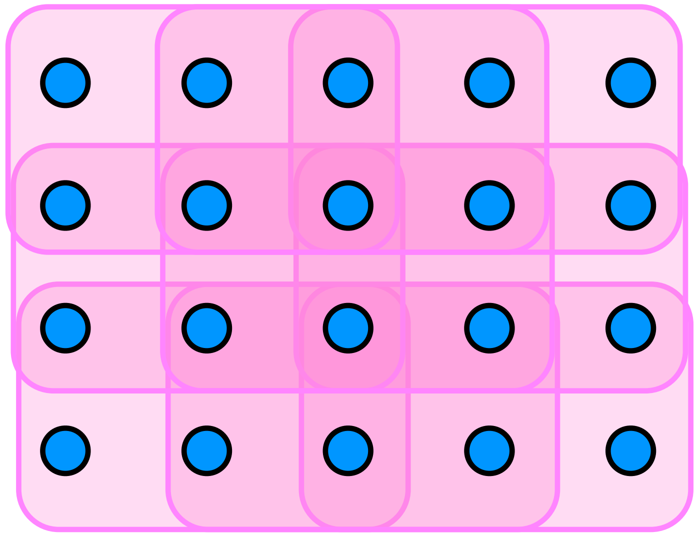


Sperner's Theorem
Sperner's proof (shadows)
LYM inequality
Lubell's proof (counting)
Alon's proof (probabilistic)

Shattering

$\mathcal{F} \subseteq 2^{[n]}$ $\qquad R \subseteq [n]$

trace $\mathcal{F}|_R$: $\qquad \mathcal{F}|_R = \{S \cap R \mid S \in \mathcal{F}\}$



$\mathcal{F}$ shatters R $\qquad \mathcal{F}|_R = 2^R$

Sauer's Lemma

$$|\mathcal{F}| > \sum_{0 \leq i < k} \binom{n}{i} \Rightarrow \exists R \in \binom{[n]}{k},\ \mathcal{F} \text{ shatters } R$$

Sauer; Shelah-Perles; Vapnik-Cervonenkis;

VC-dimension of $\mathcal{F}$

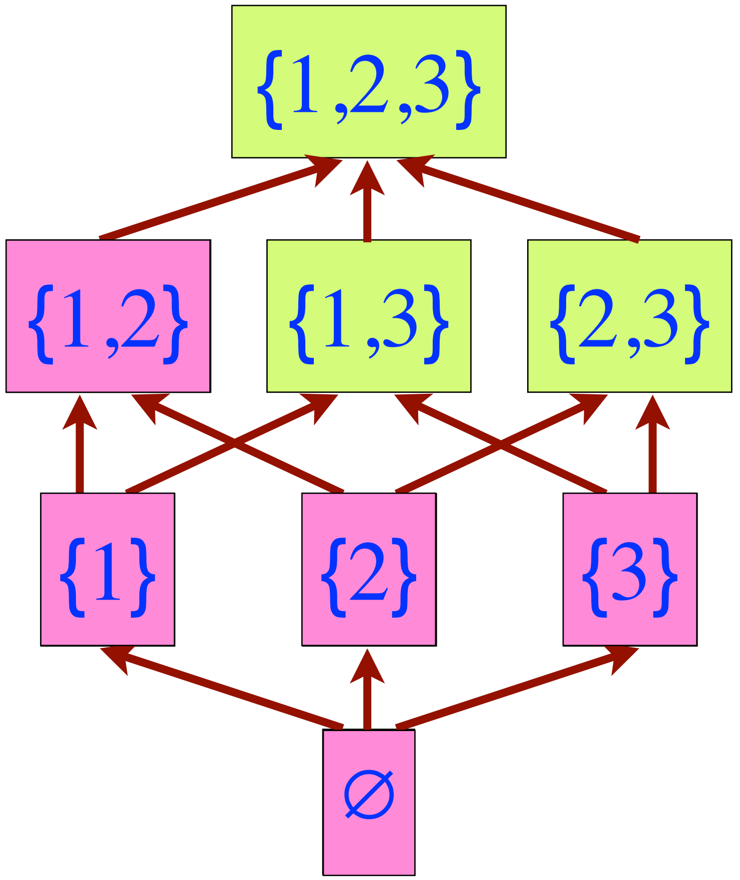
size of the largest R shattered by $\mathcal{F}$

$$\mathcal{F} \subseteq 2^{[n]} \qquad \mathcal{F}|_R = \{S \cap R \mid S \in \mathcal{F}\}$$

$$\text{VC-dim}(\mathcal{F}) = \max\left\{|R| \mid R \subseteq [n], \mathcal{F}|_R = 2^R\right\}$$

Heredity (*ideal, simplicial complex*)

$\mathcal{F}$ is hereditary if $\forall B \subseteq A \in \mathcal{F}, \quad B \in \mathcal{F}$



Heredity (*ideal, simplicial complex*)

Sauer's Lemma

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \Rightarrow \exists R \in \binom{[n]}{k}, \mathcal{F} \text{ shatters } R$$

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \Rightarrow \exists R \in \mathcal{F}, |R| \ge k$$

for hereditary $\mathcal{F}$: $\quad \forall B \subseteq A \in \mathcal{F}, \quad B \in \mathcal{F}$

$$R \in \mathcal{F} \Rightarrow \mathcal{F} \text{ shatters } R$$

Sauer's Lemma

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \Rightarrow \exists R \in \binom{[n]}{k}, \mathcal{F} \text{ shatters } R$$

$$|\mathcal{F}| \le |\mathcal{F}'|$$

arbitrary $\mathcal{F}$ → hereditary $\mathcal{F}'$

$$\text{VC-dim}(\mathcal{F}) \ge \text{VC-dim}(\mathcal{F}')$$

↓

$\mathcal{F}$ shatters a k-set ⇐ $\mathcal{F}'$ shatters a k-set

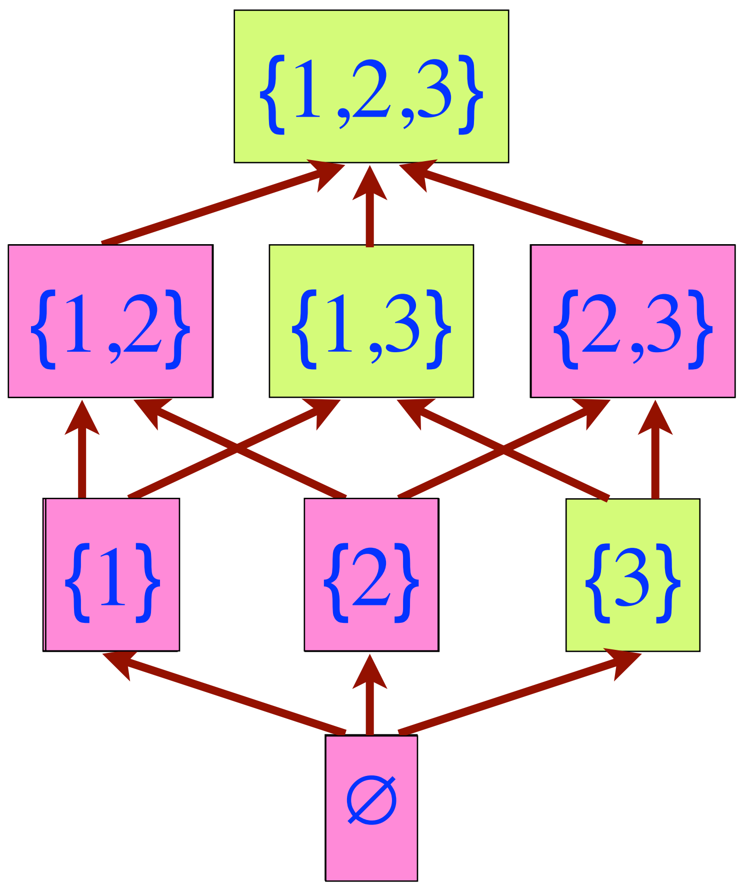
Down Shift

$\mathcal{F} \subseteq 2^{[n]}$ **for** $i \in [n]$

down-shift: $S_i(\cdot)$

$$S_i(T) = \begin{cases} T \setminus \{i\} & \text{if } i \in T \in \mathcal{F}, \text{ and } T \setminus \{i\} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_i(\mathcal{F}) = \{S_i(T) \mid T \in \mathcal{F}\}$$



$\mathcal{F} \subseteq 2^{[n]}$ $\quad \mathcal{F}|_R = \{S \cap R \mid S \in \mathcal{F}\}$ $\quad$ for $i \in [n]$

$$S_i(T) = \begin{cases} T \setminus \{i\} & \text{if } i \in T \in \mathcal{F}, \text{ and } T \setminus \{i\} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_i(\mathcal{F}) = \{S_i(T) \mid T \in \mathcal{F}\}$$

1. $|S_i(\mathcal{F})| = |\mathcal{F}|$ ✓
2. $|S_i(\mathcal{F})|_R| \leq |\mathcal{F}|_R|$ for all $R \subseteq [n]$

$$S_i(\mathcal{F})|_R \subseteq S_i(\mathcal{F}|_R)$$

by case analysis

$$A \in S_i(\mathcal{F}) \Rightarrow \left\{ \begin{array}{l} A = S_i(A \cup \{i\}) \\ \\ A = S_i(A) \end{array} \right\} \Rightarrow A \cap R \in S_i(\mathcal{F}|_R)$$

$\mathcal{F} \subseteq 2^{[n]} \quad \mathcal{F}|_R = \{S \cap R \mid S \in \mathcal{F}\}$ **for** $i \in [n]$

$$S_i(T) = \begin{cases} T \setminus \{i\} & \text{if } i \in T \in \mathcal{F}, \text{ and } T \setminus \{i\} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_i(\mathcal{F}) = \{S_i(T) \mid T \in \mathcal{F}\}$$

1. $|S_i(\mathcal{F})| = |\mathcal{F}|$
2. $|S_i(\mathcal{F})|_R| \leq |\mathcal{F}|_R|$ for all $R \subseteq [n]$

repeat applying down-shifting $S_i(\mathcal{F})$ **for** $i \in [n]$

eventually, $\mathcal{F}$ is unchanged by any $S_i(\mathcal{F})$

$\forall A \in \mathcal{F} \quad$ if $B \subseteq A \Rightarrow B \in \mathcal{F}$

$\mathcal{F}$ is hereditary

Sauer's Lemma

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \Rightarrow \exists R \in \binom{[n]}{k}, \ \mathcal{F} \text{ shatters } R$$

repeat down-shift $\mathcal{F}$ until unchanged

$\mathcal{F}$ is hereditary

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i}$$

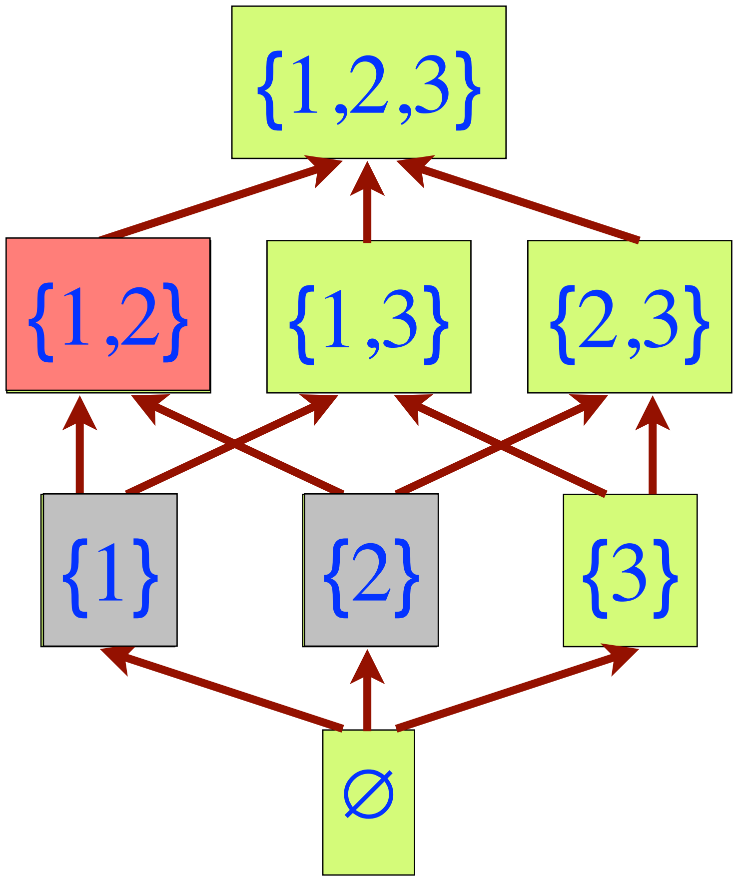
$\Rightarrow$ $\exists S \in \binom{[n]}{\ell}$ with $\ell \ge k$

$$2^S \subseteq \mathcal{F}$$

take any $R \in \binom{S}{k}$ $\qquad$ $\mathcal{F}$ shatters R

Kruskal-Katona Theorem

$$\mathcal{F} \subseteq \binom{[n]}{k}$$



shadow: $\Delta\mathcal{F} = \left\{ T \in \binom{[n]}{k-1} \mid \exists S \in \mathcal{F}, T \subseteq S \right\}$

$|\mathcal{F}| = m$ How small can the shadow $\Delta\mathcal{F}$ be?

Colex order of sets

lexicographic order

$\binom{[5]}{3}$

{1,2,3}
{1,2,4}
{1,2,5}
{1,3,4}
{1,3,5}
{1,4,5}
{2,3,4}
{2,3,5}
{2,4,5}
{3,4,5}

elements in increasing order
sets in lexicographic order

co-lexicographic(colex) order
(reversed lexicographic order)

{3,2,1} → $\binom{[3]}{3}$
{4,2,1}
{4,3,1}
{4,3,2}
{5,2,1}
{5,3,1}
{5,3,2}
{5,4,1}
{5,4,2}
{5,4,3}

$\binom{[4]}{3}$ (first four sets)

$\binom{[5]}{3}$ (all ten sets)

elements in decreasing order
sets in lexicographic order

Colex order of sets

$\mathcal{R}(m, k)$:

first m members of $\binom{\mathbb{N}}{k}$ in colex order

$$\mathcal{R}\left(\binom{n}{k}, k\right) = \binom{[n]}{k}$$

co-lexicographic(colex) order

(reversed lexicographic order)

$\binom{[4]}{3}$ $\binom{[3]}{3}$ $\binom{[5]}{3}$

{3,2,1}
{4,2,1}
{4,3,1}
{4,3,2}
{5,2,1}
{5,3,1}
{5,3,2}
{5,4,1}
{5,4,2}
{5,4,3}

elements in decreasing order

sets in lexicographic order

k-cascade Representation

$\forall$ positive integers m and k

m can be uniquely represented as

$$m = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_t}{t}$$

with $m_k > m_{k-1} > \cdots > m_t \geq t \geq 1$

k-cascade Representation

$\forall$ positive integers m and k

m can be uniquely represented as

$$m = \sum_{\ell=t}^{k} \binom{m_\ell}{\ell}$$

with $m_k > m_{k-1} > \cdots > m_t \geq t \geq 1$

greedy algorithm:

for $\ell = k, k-1, k-2, \ldots$

take the max m_ℓ with $\binom{m_\ell}{\ell} \leq m$

$m \leftarrow m - \binom{m_\ell}{\ell}$

until m=0

Colex order of sets

$\mathcal{R}(m,k)$:

first m members of $\binom{\mathbb{N}}{k}$ in colex order

k-cascade

$$m = \sum_{\ell=t}^{k} \binom{m_\ell}{\ell}$$

$\mathcal{R}(m,k)$:

$\binom{[m_\ell]}{\ell}$ adjoining $\{m_r + 1 \mid \ell < r \leq k\}$

$$|\Delta\mathcal{R}(m,k)| = \sum_{\ell=t}^{k} \binom{m_\ell}{\ell - 1}$$

colex order of $\binom{\mathbb{N}}{k}$

$\binom{[4]}{3}$:
{3,2,1}
{4,2,1}
{4,3,1}
{4,3,2}

$\binom{[3]}{2}$:
{5,2,1}
{5,3,1}
{5,3,2}

$\binom{[1]}{1}$:
{5,4,1}

{5,4,2}
{5,4,3}
{6,2,1}
⋮

Kruskal-Katona Theorem

$\mathcal{F} \subseteq \binom{[n]}{k}$, $|\mathcal{F}| = m$, where

$$m = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_t}{t},$$

for $m_k > m_{k-1} > \cdots > m_t \geq t \geq 1$. Then

$$|\Delta\mathcal{F}| \geq \binom{m_k}{k-1} + \binom{m_{k-1}}{k-2} + \cdots + \binom{m_t}{t-1}.$$

(Frankl 1984) induction on m, and for given m, on k

$$\mathcal{F}_0 = \{A \in \mathcal{F} \mid 1 \notin A\} \quad \mathcal{F}_1 = \{A \in \mathcal{F} \mid 1 \in A\}$$

$$\mathcal{F}_1' = \{A \setminus \{1\} \mid A \in \mathcal{F}_1\} \quad \mathcal{F}_1' \subseteq \binom{[n-1]}{k-1}$$

$$|\Delta\mathcal{F}| \geq |\Delta\mathcal{F}_1'| + |\mathcal{F}_1'|$$

can apply I.H. if we know

$$|\mathcal{F}_1'| \geq ?$$

$$\mathcal{F} \subseteq \binom{[n]}{k},\ |\mathcal{F}| = m,$$

$$m = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_t}{t},$$

$$\Rightarrow |\Delta\mathcal{F}| \geq \binom{m_k}{k-1} + \binom{m_{k-1}}{k-2} + \cdots + \binom{m_t}{t-1}.$$

$\mathcal{F}_0 = \{A \in \mathcal{F} \mid 1 \notin A\}$ $\mathcal{F}_1 = \{A \in \mathcal{F} \mid 1 \in A\}$

$\mathcal{F}_1' = \{A \setminus \{1\} \mid A \in \mathcal{F}_1\}$ $\mathcal{F}_1' \subseteq \binom{[n]}{k-1}$

$$|\Delta\mathcal{F}| \geq |\Delta\mathcal{F}_1'| + |\mathcal{F}_1'|$$

$\mathcal{F}$ shifted

$$|\mathcal{F}_1'| \geq \binom{m_k - 1}{k-1} + \binom{m_{k-1} - 1}{k-2} + \cdots + \binom{m_t - 1}{t-1}$$

I.H. $|\Delta\mathcal{F}_1'| \geq \binom{m_k - 1}{k-2} + \binom{m_{k-1} - 1}{k-3} + \cdots + \binom{m_t - 1}{t-2}$

$\mathcal{F} \subseteq 2^{[n]}$ **for** $1 \le i < j \le n$

$\forall T \in \mathcal{F}$, write $T_{ij} = (T \setminus \{j\}) \cup \{i\}$

(i, j)-shift: $S_{ij}(\cdot)$

$\forall T \in \mathcal{F}$,

$$S_{ij}(T) = \begin{cases} T_{ij} & \text{if } j \in T, i \notin T, \text{ and } T_{ij} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(T) \mid T \in \mathcal{F}\}$$

1. $|S_{ij}(T)| = |T|$ **and** $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$
2. $|\Delta S_{ij}(\mathcal{F})| \le |\Delta \mathcal{F}|$ by-case analysis

$\mathcal{F} \subseteq \binom{[n]}{k}$, $|\mathcal{F}| = m$,

$$m = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_t}{t},$$

$\mathcal{F}_0 = \{A \in \mathcal{F} \mid 1 \notin A\}$ $\qquad$ $\mathcal{F}_1 = \{A \in \mathcal{F} \mid 1 \in A\}$

$\mathcal{F}_1' = \{A \setminus \{1\} \mid A \in \mathcal{F}_1\}$ $\qquad$ $\mathcal{F}_1' \subseteq \binom{[n-1]}{k-1}$

Lemma 1: $|\Delta\mathcal{F}| \geq |\Delta\mathcal{F}_1'| + |\mathcal{F}_1'|$

Lemma 1.5: $\mathcal{F}$ is shifted ⇒ $\Delta\mathcal{F}_0 \subseteq \mathcal{F}_1'$

Lemma 2:
$\mathcal{F}$ is shifted ⇒ $|\mathcal{F}_1'| \geq \sum_{\ell=t}^{k} \binom{m_\ell - 1}{\ell - 1}$

$$\mathcal{F} \subseteq \binom{[n]}{k},\ |\mathcal{F}| = m,$$

$$m = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_t}{t},$$

$$\Rightarrow |\Delta\mathcal{F}| \geq \binom{m_k}{k-1} + \binom{m_{k-1}}{k-2} + \cdots + \binom{m_t}{t-1}.$$

$$\mathcal{F}_0 = \{A \in \mathcal{F} \mid 1 \notin A\} \qquad \mathcal{F}_1 = \{A \in \mathcal{F} \mid 1 \in A\}$$

$$\mathcal{F}_1' = \{A \setminus \{1\} \mid A \in \mathcal{F}_1\} \qquad \mathcal{F}_1' \subseteq \binom{[n]}{k-1}$$

$$|\Delta\mathcal{F}| \geq |\Delta\mathcal{F}_1'| + |\mathcal{F}_1'| \geq \sum_{\ell=t}^{k} \left\{ \binom{m_\ell - 1}{\ell - 1} + \binom{m_\ell - 1}{\ell - 2} \right\}$$

I.H.

$$|\mathcal{F}_1'| \geq \sum_{\ell=t}^{k} \binom{m_\ell - 1}{\ell - 1} \qquad |\Delta\mathcal{F}_1'| \geq \sum_{\ell=t}^{k} \binom{m_\ell - 1}{\ell - 2}$$

$$\mathcal{F} \subseteq \binom{[n]}{k},\ |\mathcal{F}| = m,$$

$$m = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_t}{t},$$

$$\Rightarrow |\Delta\mathcal{F}| \geq \binom{m_k}{k-1} + \binom{m_{k-1}}{k-2} + \cdots + \binom{m_t}{t-1}.$$

$\mathcal{F}_0 = \{A \in \mathcal{F} \mid 1 \notin A\}$ $\mathcal{F}_1 = \{A \in \mathcal{F} \mid 1 \in A\}$

$\mathcal{F}_1' = \{A \setminus \{1\} \mid A \in \mathcal{F}_1\}$ $\mathcal{F}_1' \subseteq \binom{[n]}{k-1}$

$$|\Delta\mathcal{F}| \geq |\Delta\mathcal{F}_1'| + |\mathcal{F}_1'| \geq \sum_{\ell=t}^{k} \binom{m_\ell}{\ell - 1}$$

I.H.

$$|\mathcal{F}_1'| \geq \sum_{\ell=t}^{k} \binom{m_\ell - 1}{\ell - 1}$$

$$|\Delta\mathcal{F}_1'| \geq \sum_{\ell=t}^{k} \binom{m_\ell - 1}{\ell - 2}$$

Kruskal-Katona Theorem

$\mathcal{F} \subseteq \binom{[n]}{k}$, $|\mathcal{F}| = m$, the k-cascade of m is

$$m = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_t}{t}.$$

Then $|\Delta \mathcal{F}| \geq \binom{m_k}{k-1} + \binom{m_{k-1}}{k-2} + \cdots + \binom{m_t}{t-1}$.

The first m k-sets in colex order have the smallest shadow.

$\mathcal{R}(m, k)$: first m k-sets in colex order

K-K Theorem: $|\Delta \mathcal{F}| \geq |\Delta \mathcal{R}(|\mathcal{F}|, k)|$

Kruskal-Katona Theorem

$\mathcal{F} \subseteq \binom{[n]}{k}$, $|\mathcal{F}| = m$, the k-cascade of m is

$$m = \sum_{\ell=t}^{k} \binom{m_\ell}{\ell}.$$

Then $|\Delta_r \mathcal{F}| \geq \sum_{\ell=t-k+r}^{r} \binom{m_\ell}{\ell}.$

r-shadow:

$$\Delta_r \mathcal{F} = \left\{ S \in \binom{[n]}{r} \;\middle|\; \exists T \in \mathcal{F}, S \subset T \right\}$$

$$\Delta_r \mathcal{F} = \underbrace{\Delta \cdots \Delta}_{k-r} \mathcal{F}$$

Erdős-Ko-Rado Theorem

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset$ ⇒ $|\mathcal{F}| \leq \binom{n-1}{k-1}$

Suppose $|\mathcal{F}| > \binom{n-1}{k-1}$ let $\mathcal{G} = \{\bar{S} \mid S \in \mathcal{F}\}$

$|\mathcal{G}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$ K-K ⇒ $|\Delta_k \mathcal{G}| > \binom{n-1}{k}$

$S \cap T \neq \emptyset$ ⇒ $S \not\subseteq \bar{T}$ ⇒ $\mathcal{F}$ and $\Delta_k \mathcal{G}$ are disjoint

⇓

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} < |\mathcal{F}| + |\Delta_k \mathcal{G}| \leq \binom{n}{k}$$

Contradiction!