Combinatorics

Matching Theory

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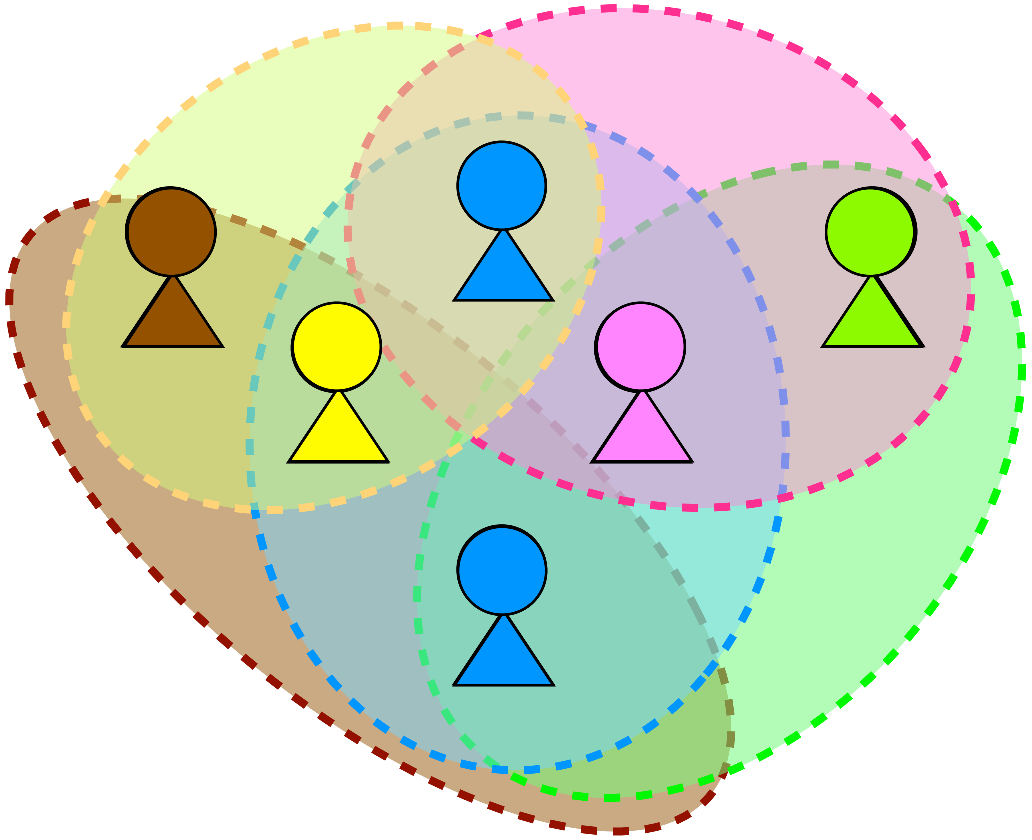
System of Distinct Representatives

(Transversal)

system of distinct representatives (SDR)

for sets $S_1, S_2, \ldots, S_m \subseteq [n]$



distinct representatives

$$x_1, x_2, \ldots, x_m \in [n]$$

$$x_i \in S_i$$

for $i = 1, 2, \ldots, m$

Marriage Problem

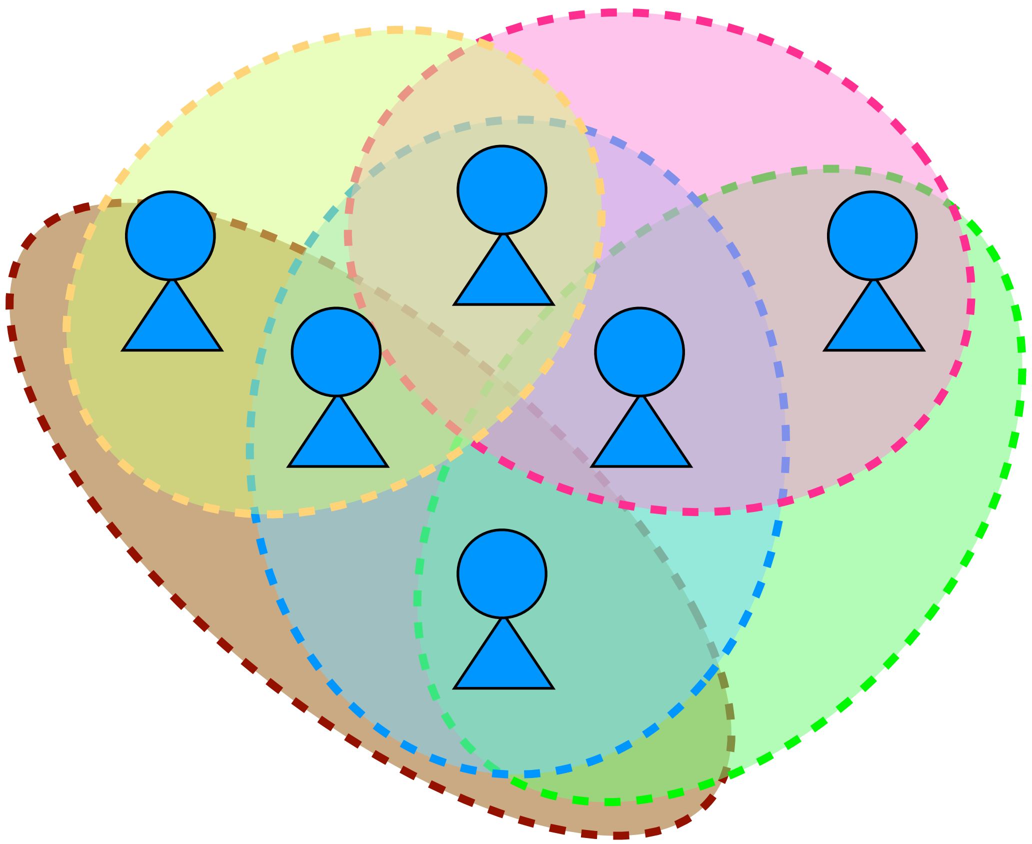
Does there exist an SDR for

$$S_1, S_2, \ldots, S_m \ ?$$

m girls

S_i: boys that girl i is OK to marry to

"Is there a way of marrying these m girls?"



$$S_1, S_2, \ldots, S_m \text{ have a SDR} \quad \Rightarrow$$

$$\exists \text{ dinstinct } x_1 \in S_1, x_2 \in S_2, \ldots, x_m \in S_m$$

$$\Downarrow$$

$$\forall I \subseteq \{1, 2, \ldots, m\},$$

$$\left|\bigcup_{i \in I} S_i\right| \geq |\{x_i \mid i \in I\}| \geq |I|.$$

distinct

$S_1, S_2, \ldots, S_m$ have a SDR $\Rightarrow$

$$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$$

Hall's Theorem (marriage theorem)

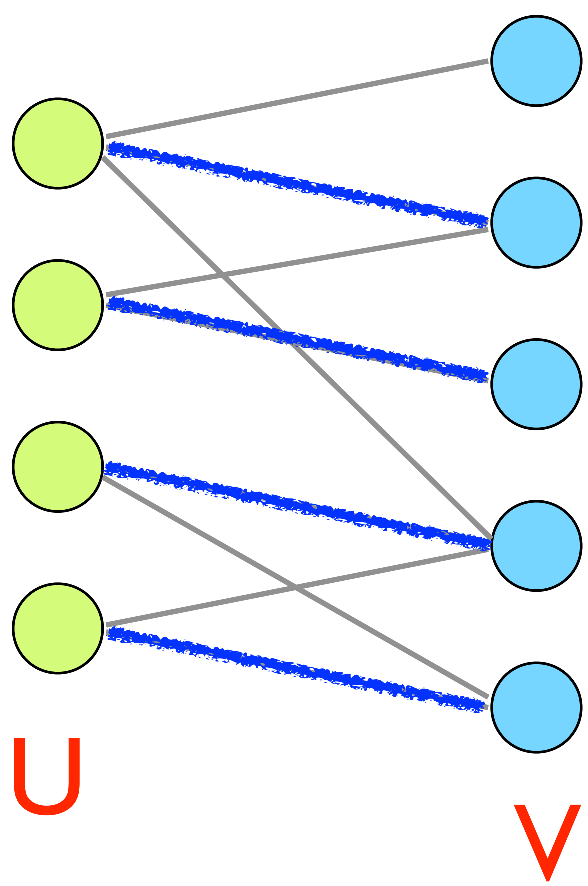
$$S_1, S_2, \ldots, S_m \text{ have a SDR} \quad \Longleftrightarrow$$

$$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$$

Hall's Theorem (graph theory form)

A bipartite graph $G(U, V, E)$ has a matching of U

$\Leftrightarrow$ $|N(S)| \geq |S|$ for all $S \subseteq U$



matching: edge independent set

$M \subseteq E$ with

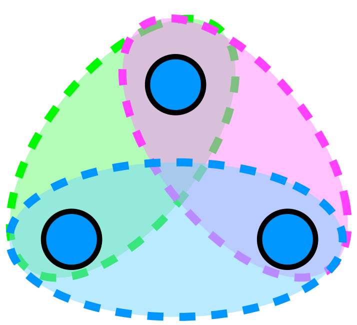
no $e_1, e_2 \in M$ share a vertex

$$N(S) = \{v \mid \exists u \in S \text{ s.t. } uv \in E\}$$

Hall's Theorem (marriage theorem)

$$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$$

$\Rightarrow$ $S_1, S_2, \ldots, S_m$ have a SDR



critical family: $S_1, S_2, \ldots, S_k$ $\quad k < m$

$$\left|\bigcup_{i=1}^{k} S_i\right| = k$$

Induction on m: $\quad m = 1,$ trivial

case.1: there is no critical family in $S_1, S_2, \ldots, S_m$

case.2: there is a critical family in $S_1, S_2, \ldots, S_m$

Hall's Theorem (marriage theorem)

$$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i\in I} S_i\right| \geq |I|.$$

$\Rightarrow$ $S_1, S_2, \ldots, S_m$ have a SDR

case.1: there is no critical family in $S_1, S_2, \ldots, S_m$

$$\forall I \subseteq \{1, 2, \ldots, m\} \text{ that } |I| < m, \quad \left|\bigcup_{i\in I} S_i\right| > |I|$$

take an arbitrary $x \in S_m$ as representative of S_m

remove S_m and x $\quad S_i' = S_i \setminus \{x\} \quad i = 1, 2, \ldots, m\text{-}1$

$$\forall I \subseteq \{1, 2, \ldots, m-1\}, \quad \left|\bigcup_{i\in I} S_i'\right| \geq |I|$$

due to I.H. $S_1', \ldots, S_{m-1}'$ have a SDR $\{x_1, \ldots, x_{m-1}\}$

$x_1, \ldots, x_{m-1}$ and x form a SDR for $S_1, S_2, \ldots, S_m$

Hall's Theorem (marriage theorem)

$$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$$

$\Rightarrow$ $S_1, S_2, \ldots, S_m$ have a SDR

case.2: there is a critical family in $S_1, S_2, \ldots, S_m$

say $|S_{m-k+1} \cup \cdots \cup S_m| = k$ $\quad k < m$

due to I.H. $S_{m-k+1}, \ldots, S_m$ have a SDR $X = \{x_1, \ldots, x_k\}$

$S_i' = S_i \setminus X$ $\quad i = 1, 2, \ldots, m-k$

$$\forall I \subseteq \{1, 2, \ldots, m-k\},$$

$$\left|\bigcup_{i=m-k+1}^{m} S_i \cup \bigcup_{i \in I} S_i\right| \geq k + |I|$$

$$\Rightarrow \quad \left|\bigcup_{i \in I} S_i'\right| \geq |I|$$

Hall's Theorem (marriage theorem)

$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$

⇒ $S_1, S_2, \ldots, S_m$ have a SDR

case.2: there is a critical family in $S_1, S_2, \ldots, S_m$

say $|S_{m-k+1} \cup \cdots \cup S_m| = k \qquad k < m$

due to I.H. $S_{m-k+1}, \ldots, S_m$ have a SDR $X = \{x_1, \ldots, x_k\}$

$S_i' = S_i \setminus X \qquad i = 1, 2, \ldots, m\text{-}k$

$\forall I \subseteq \{1, 2, \ldots, m-k\}, \; \left|\bigcup_{i \in I} S_i'\right| \geq |I|$

due to I.H.

$S_1', \ldots, S_{m-k}'$ have a SDR $Y = \{y_1, \ldots, y_{m-k}\}$

X and Y form a SDR for $S_1, S_2, \ldots, S_m$

Hall's Theorem (marriage theorem)

$$S_1, S_2, \ldots, S_m \text{ have a SDR} \quad \Longleftrightarrow$$

$$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$$

Min-Max Theorems

- **König-Egerváry theorem**: in bipartite graph

  min vertex cover = max matching

- **Dilworth's theorem**: in poset

  min chain-decomposition = max antichain

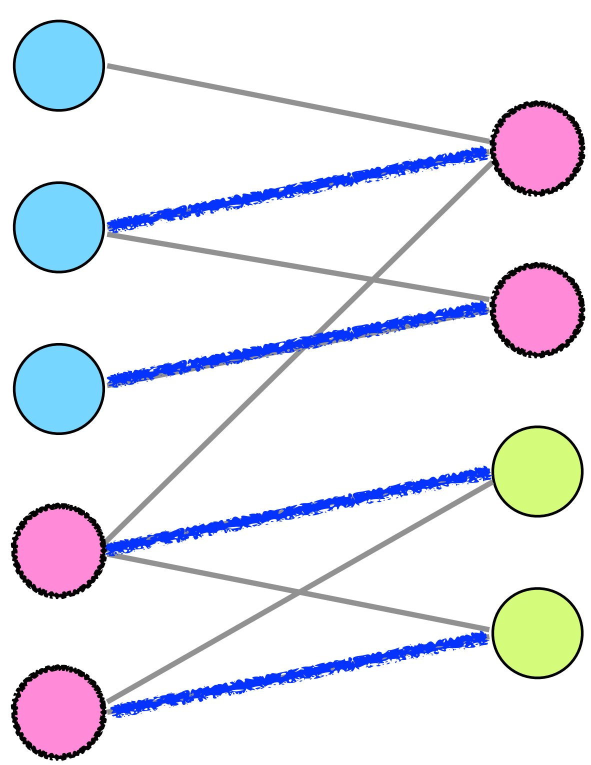
- **Menger's theorem**: in graph

  min vertex-cut = max vertex-disjoint paths

König-Egerváry theorem

Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.



matching: $M \subseteq E$

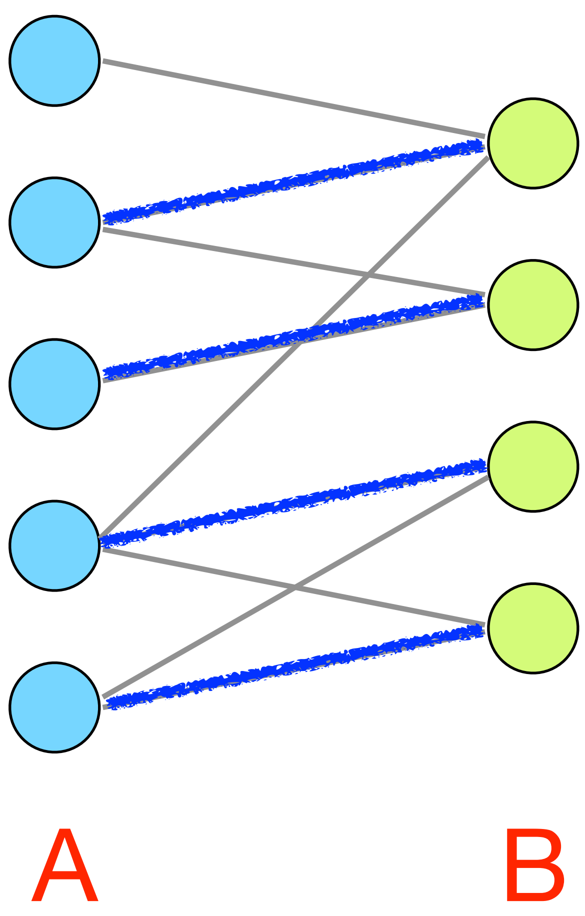
no $e_1, e_2 \in M$ share a vertex

vertex cover: $C \subseteq V$

all $e \in E$ adjacent to some $v \in C$

Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.



matching:

independent 1s

do not share row/column

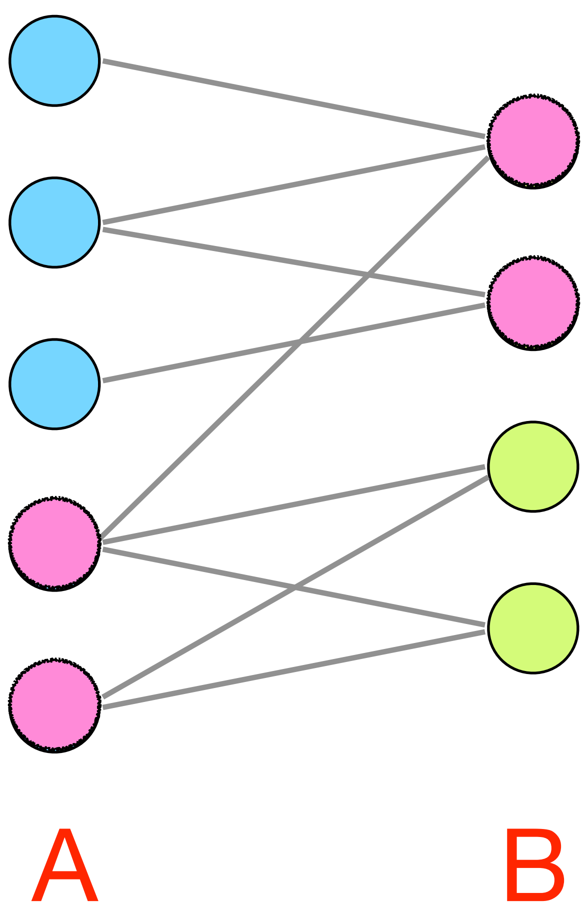
B

1	0	0	0
1	1	0	0
0	1	0	0
1	0	1	1
0	0	1	1

A

Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.



vertex cover:

rows/columns
covering all 1s

B

1	0	0	0
1	1	0	0
0	1	0	0
1	0	1	1
0	0	1	1

A

Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

König-Egerváry Theorem (matrix form)

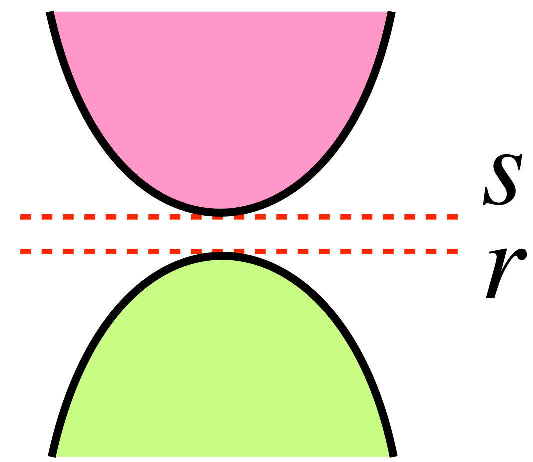
For any 0-1 matrix, the maximum number of independent 1's equals the minimum number of rows and columns required to cover all the 1's.

A : $m \times n$ 0-1 matrix

s : min # of rows/columns covering all 1's

r : max # of independent 1's

$$r \le s$$



any r independent 1's

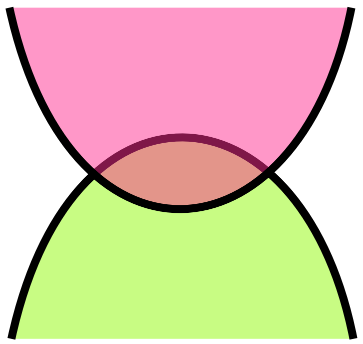
requires r rows/columns to cover

A : $m \times n$ 0-1 matrix

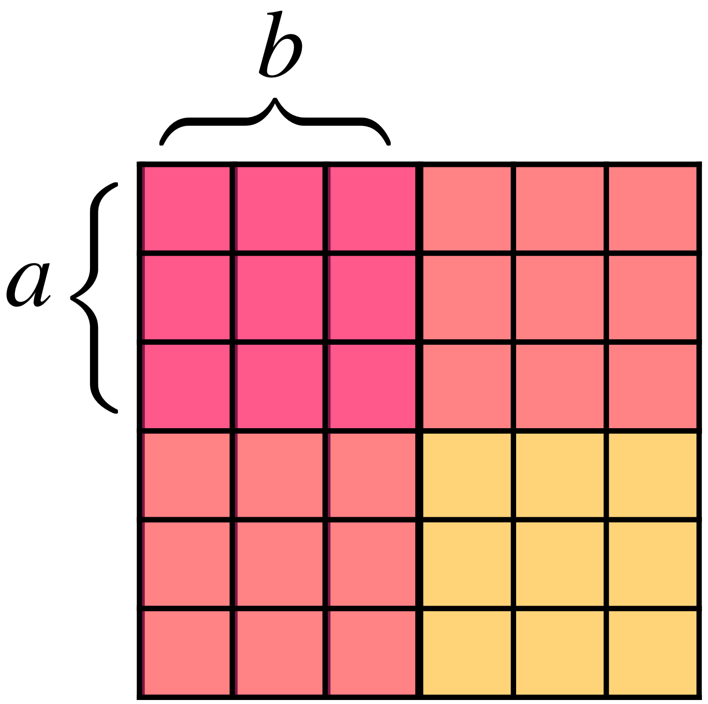
s : min # of rows/columns covering all 1's

r : max # of independent 1's

$$r \geq s$$



min covering: s = a rows + b columns



$$A = \begin{bmatrix} B_{a\times b} & C_{a\times(n-b)} \\ D_{(m-a)\times b} & \mathbf{0} \end{bmatrix}$$

C has a independent 1's

D has b independent 1's

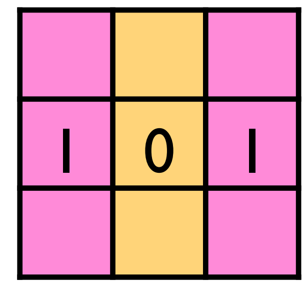
A has min covering: s = a rows + b columns

$$A = \begin{bmatrix} B_{a\times b} & C_{a\times(n-b)} \\ D_{(m-a)\times b} & \mathbf{0} \end{bmatrix}$$

C has a independent 1's

$$S_i = \{j \mid C_{ij} = 1\}$$

S_2



$S_1, S_2, \ldots, S_a$ have a SDR

otherwise $\exists 1 \le |I| \le a, \quad \left|\bigcup_{i\in I} S_i\right| < |I|$ (Hall)

C can be covered by $(a - |I|)$ rows $+ \left|\bigcup_{i\in I} S_i\right|$ columns

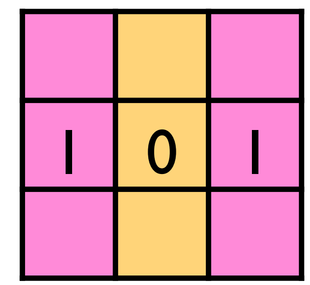
A has min covering: $s = a$ rows + b columns

$$A = \begin{bmatrix} B_{a\times b} & C_{a\times(n-b)} \\ D_{(m-a)\times b} & \mathbf{0} \end{bmatrix}$$

C has a independent 1's

$$S_i = \{j \mid C_{ij} = 1\}$$

S_2



$S_1, S_2, \ldots, S_a$ have a SDR

otherwise $\exists 1 \le |I| \le a, \quad \left|\bigcup_{i\in I} S_i\right| < |I|$ (Hall)

C can be covered by $< a$ rows&columns

A can be covered by $< a+b$ rows&columns

contradiction!

A has min covering: s = a rows + b columns

$$A = \begin{bmatrix} B_{a\times b} & C_{a\times(n-b)} \\ D_{(m-a)\times b} & \mathbf{0} \end{bmatrix}$$

C has a independent 1's

$$S_i = \{j \mid C_{ij} = 1\}$$

$S_1, S_2, \ldots, S_a$ have a SDR

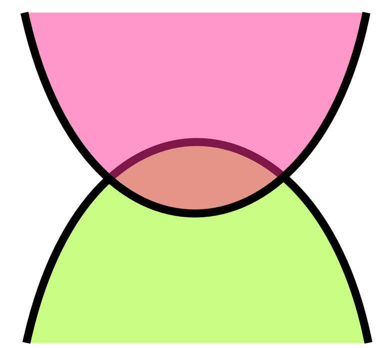
SDR: distinct $j_1, j_2, \ldots, j_a$ $\quad C(i, j_i) = 1$

A : $m \times n$ 0-1 matrix

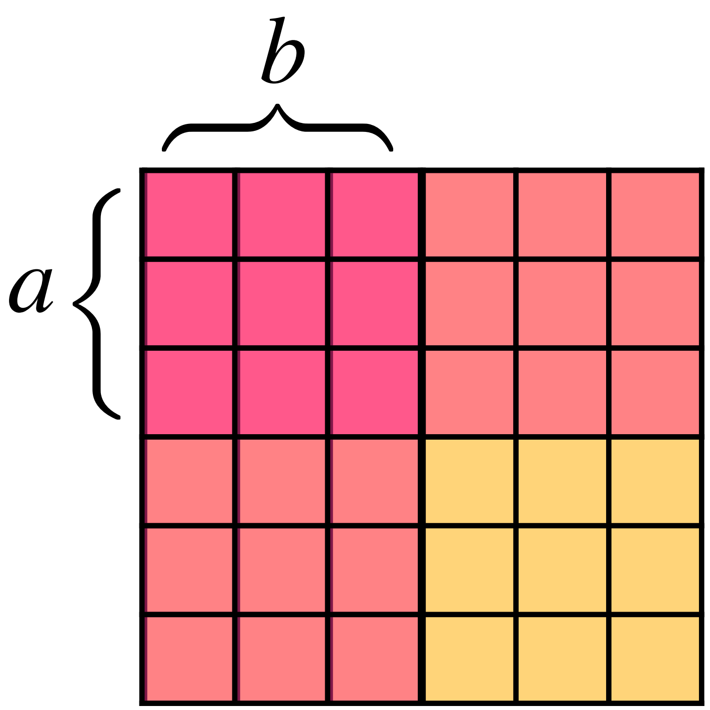
r : max # of independent 1's

s : min # of rows/columns covering all 1's

$$r \geq s$$



A has min covering: s = a rows + b columns



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C has a independent 1's

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König-Egerváry Theorem (matrix form)

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Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

Poset

$\mathcal{F} \subseteq 2^{[n]}$ with ⊆ define a

partially ordered set (poset)

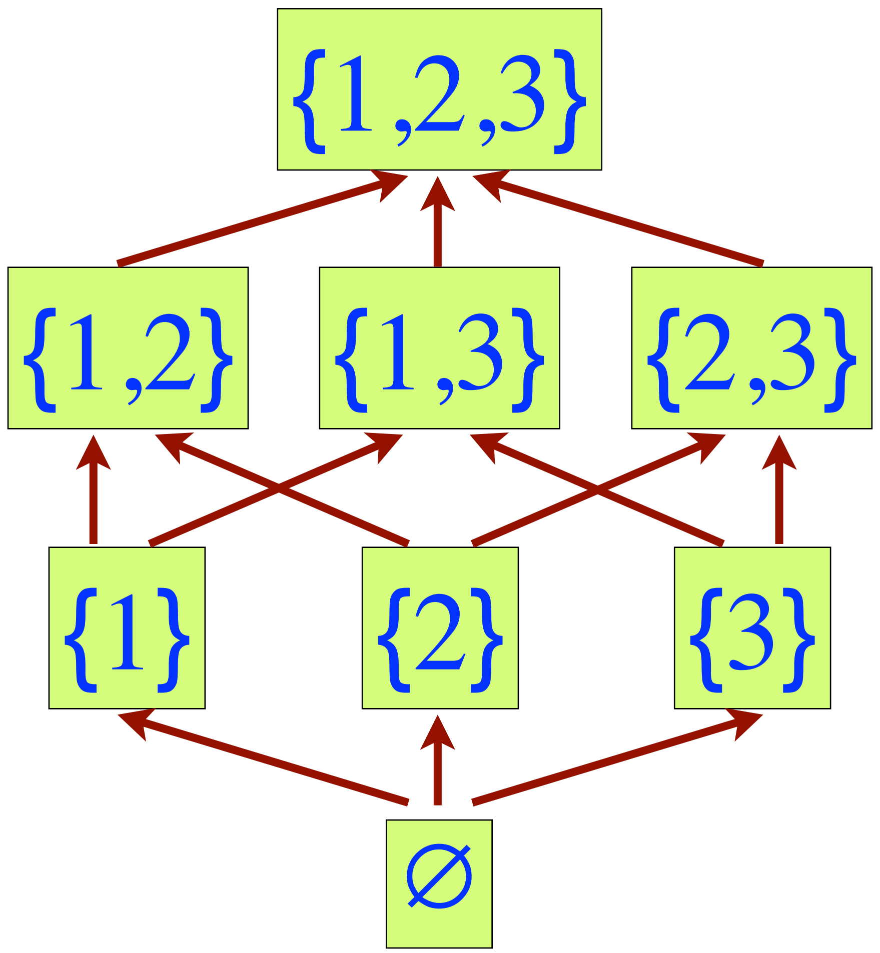
reflexivity: $A \subseteq A$

antisymmetry:

$A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

transitivity:

$A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

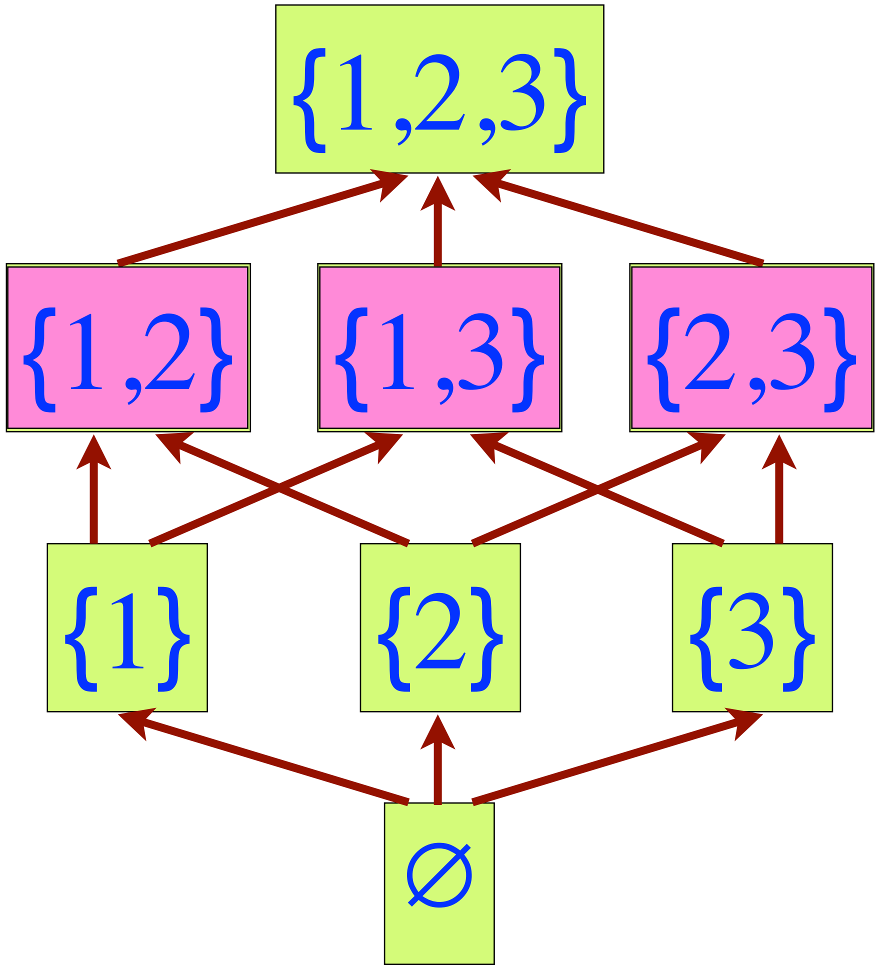


chain: $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r$

antichain: $A_1, A_2, \ldots, A_r$ that $\forall A_i, A_j, \quad A_i \not\subseteq A_j$

Dilworth's Theorem

Size of the largest **antichain** in the poset P =
size of the smallest **partition** of P into **chains**.



Dilworth's Theorem

Size of the largest antichain in the poset P = size of the smallest **partition** of P into chains.

Suppose: P has an antichain of size r.

P can be partitioned to s chains.

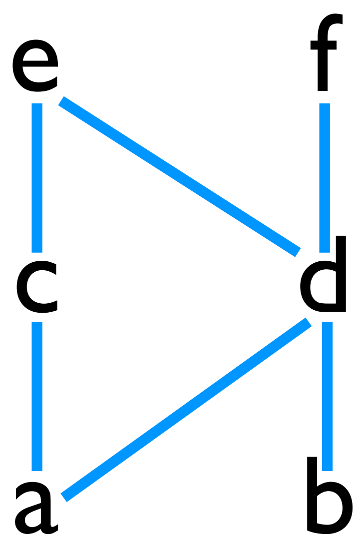
$r \leq s$

antichain A, chain C $|A \cap C| \leq 1$

We only need to prove:

There exist an **antichain** $A \subseteq P$ of size r and a **partition** of P into r **chains**.

poset P



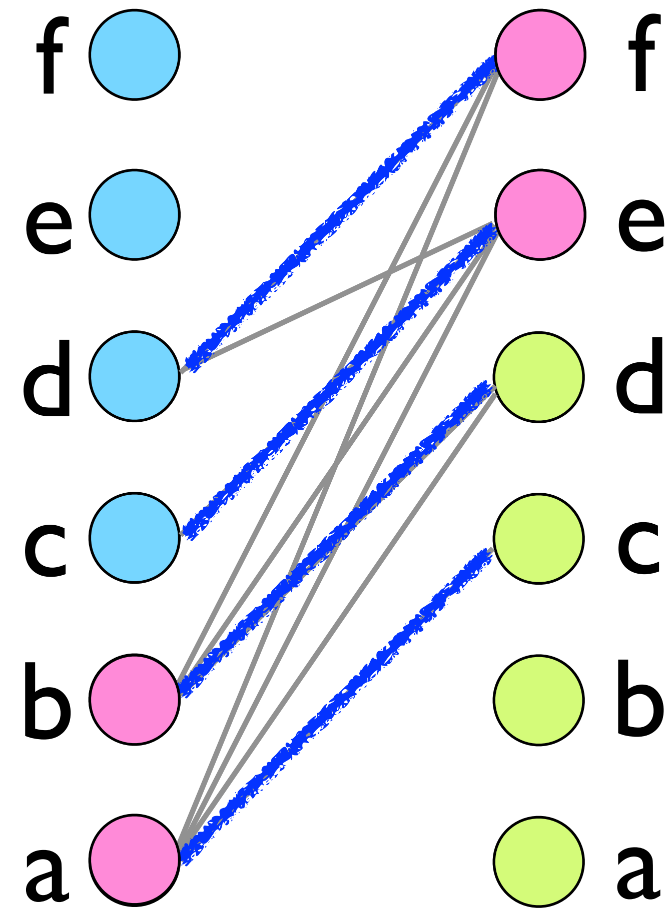
$|P| = n$

$G(U,V,E)$

$U=V=P$

$uv \in E$ if

$u<v$

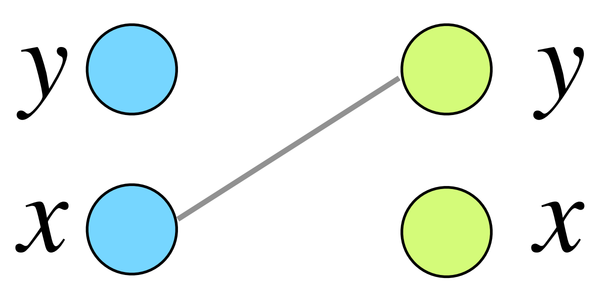


König-Egerváry Theorem:

$\exists$ matching M and vertex cover C, $|M| = |C| = k$

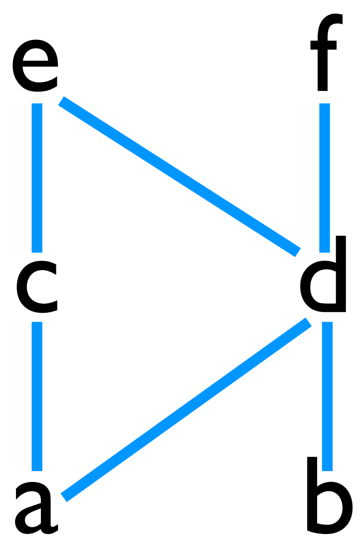
$x \in P$ uncovered by C ➡ antichain $\geq n\text{-}k$

otherwise



C is not a vertex cover

poset P

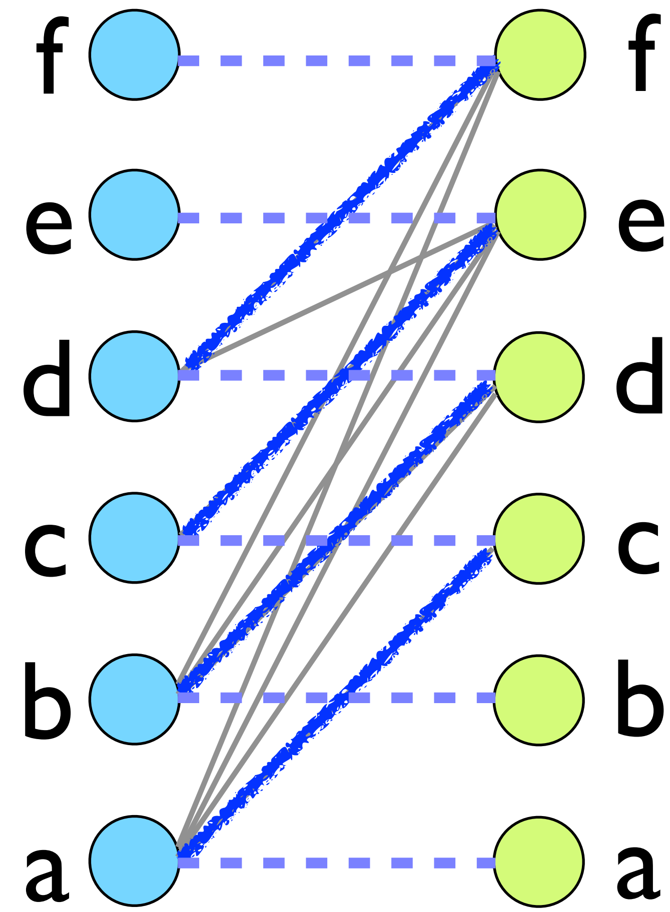


$|P| = n$

$G(U,V,E)$

$U=V=P$

$uv \in E$ if $u<v$



∃ matching M and vertex cover C, $|M| = |C| = k$

∃ antichain of size $\geq n\text{-}k$

decompose P into chains:

u,v in the same chain if $uv \in M$

chains = # unmatched vertices in U $= n\text{-}k$

Dilworth's Theorem

Size of the largest antichain in the poset P =
size of the smallest **partition** of P into chains.

$\exists$ antichain of size $\geq n\text{-}k$ = # chains

There exists an antichain $A \subseteq P$ and
a partition of P into r chains such that $|A| = r$.

Hall's Theorem (marriage theorem)

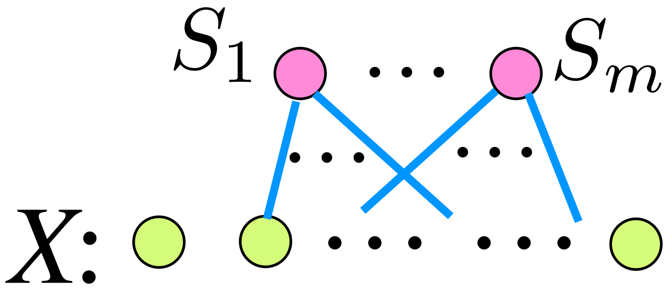
$\forall I \subseteq \{1,2,\ldots,m\}, \quad \left|\bigcup_{i\in I} S_i\right| \geq |I|.$

$\Rightarrow$ $S_1, S_2, \ldots, S_m$ have a SDR

let $X = S_1 \cup \cdots \cup S_m$

poset P: $X \cup \{S_1, \ldots, S_m\}$

$x < S_i$ if $x \in S_i$



X is the largest antichain in P.

$A \subseteq P$ is an antichain $\quad I = \{\, i \mid S_i \in A\}$ $\quad S_I = \bigcup_{i\in I} S_i$

$A \cap S_I = \emptyset$ $\Rightarrow$ $|A| \leq |I| + |X| - |S_I| \quad \leq |X|$

Hall condition

Hall's Theorem (marriage theorem)

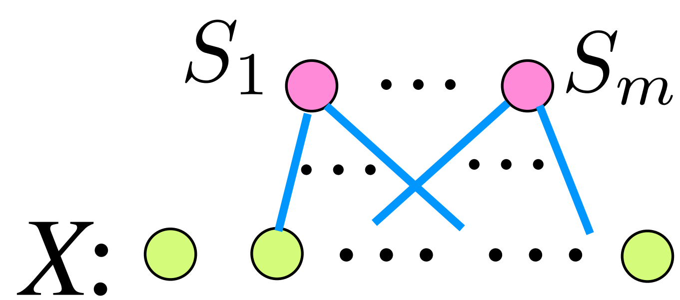
$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i\in I} S_i\right| \geq |I|.$

⇒ $S_1, S_2, \ldots, S_m$ have a SDR

let $X = S_1 \cup \cdots \cup S_m$

poset P: $X \cup \{S_1, \ldots, S_m\}$

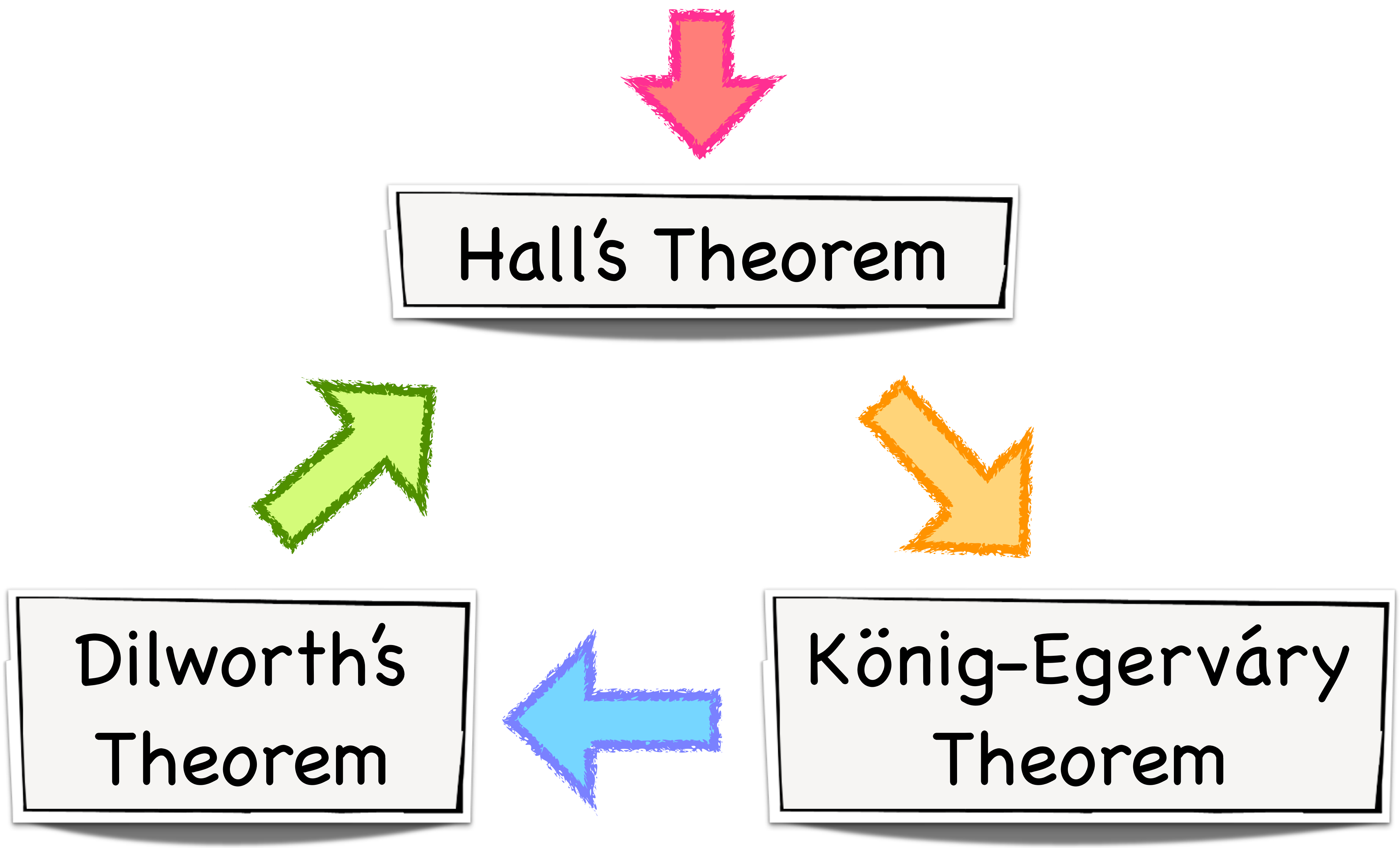
$x < S_i$ if $x \in S_i$



X is the largest antichain in P.

Dilworth: P can be partitioned into $n=|X|$ chains

$\{S_1, x_1\}, \{S_2, x_2\}, \ldots, \{S_m, x_m\}, \{x_{m+1}\}, \ldots, \{x_n\}$


Hall's Theorem
Dilworth's Theorem
König-Egerváry Theorem

Erdős-Szekeres Theorem

Any sequence of $N > mn$ distinct numbers must contain at least one of the followings:

- an increasing subsequence of length $m+1$
- a decreasing subsequence of length $n+1$

$(a_1, \ldots, a_N)$ of N different numbers $\qquad N > mn$

poset P: $\{(i, a_i) \mid i = 1, 2, \ldots, N\}$

$$(i, a_i) \le (j, a_j) \text{ if } a_i \le a_j \text{ and } i \le j$$

chain: increasing subseq

antichain: decreasing subseq

Use Dilworth!

Birkhoff - von Neumann Theorem

Every doubly stochastic matrix is a convex combination of permutation matrix.

doubly stochastic matrix A: $n \times n$ $A_{ij} \geq 0$

$$\forall j, \ \sum_i A_{ij} = 1 \quad \text{and} \quad \forall i, \ \sum_j A_{ij} = 1$$

permutation matrix P: $P_{ij} \in \{0, 1\}$

every row/column has precisely one 1

convex combination:

$$A = \sum_{i=1}^{m} \lambda_i P_i \qquad \lambda_i \geq 0 \quad \sum_{i=1}^{m} \lambda_i = 1$$

$n \times n$ nonnegative matrix A:

$$\forall j, \ \sum_i A_{ij} = \gamma \qquad \forall i, \ \sum_j A_{ij} = \gamma \qquad \gamma > 0$$

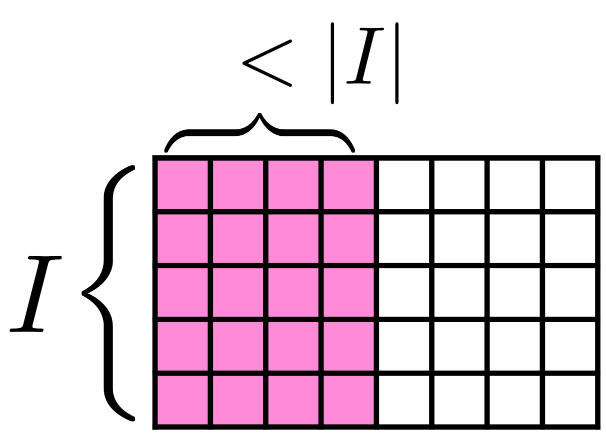
$$A = \sum_{i=1}^{m} \lambda_i P_i \qquad \lambda_i \geq 0 \quad \sum_{i=1}^{m} \lambda_i = \gamma$$

induction on # of non-zeros in A denoted m

$\gamma > 0$ $\Rightarrow$ $m \geq n$ Basis: $m = n$

$$S_i = \{j \mid A_{ij} > 0\} \qquad i = 1, 2, \ldots, n$$

$$\text{If } \exists I \subseteq \{1, \ldots, n\}, \ \left|\bigcup_{i \in I} S_i\right| < |I|$$



sum by columns $< |I|\gamma$

sum by rows $= |I|\gamma$

contradiction!

$n \times n$ nonnegative matrix A:

$$\forall j, \ \sum_i A_{ij} = \gamma \quad \forall i, \ \sum_j A_{ij} = \gamma \quad \gamma > 0$$

$$A = \sum_{i=1}^{m} \lambda_i P_i \qquad \lambda_i \geq 0 \quad \sum_{i=1}^{m} \lambda_i = \gamma$$

induction on # of non-zeros in A denoted m

$$S_i = \{j \mid A_{ij} > 0\} \qquad i = 1, 2, \ldots, n$$

$$\forall I \subseteq \{1, \ldots, n\}, \left|\bigcup_{i \in I} S_i\right| \geq |I|$$

Hall's Thm: ∃ SDR $j_1 \in S_1, \ldots, j_n \in S_n$

permutation matrix $P_m(i, j_i) = 1$ otherwise = 0

$$\lambda_m = \min_{1 \leq i \leq n} A(i, j_i) \qquad A' = A - \lambda_m P_m$$

$$\gamma' = \gamma - \lambda_m \quad m' \leq m - 1$$ I.H.

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doubly stochastic matrix A: $n \times n$ $A_{ij} \geq 0$

$$\forall j, \sum_i A_{ij} = 1 \quad \text{and} \quad \forall i, \sum_j A_{ij} = 1$$

permutation matrix P: $P_{ij} \in \{0, 1\}$

every row/column has precisely one 1

convex combination:

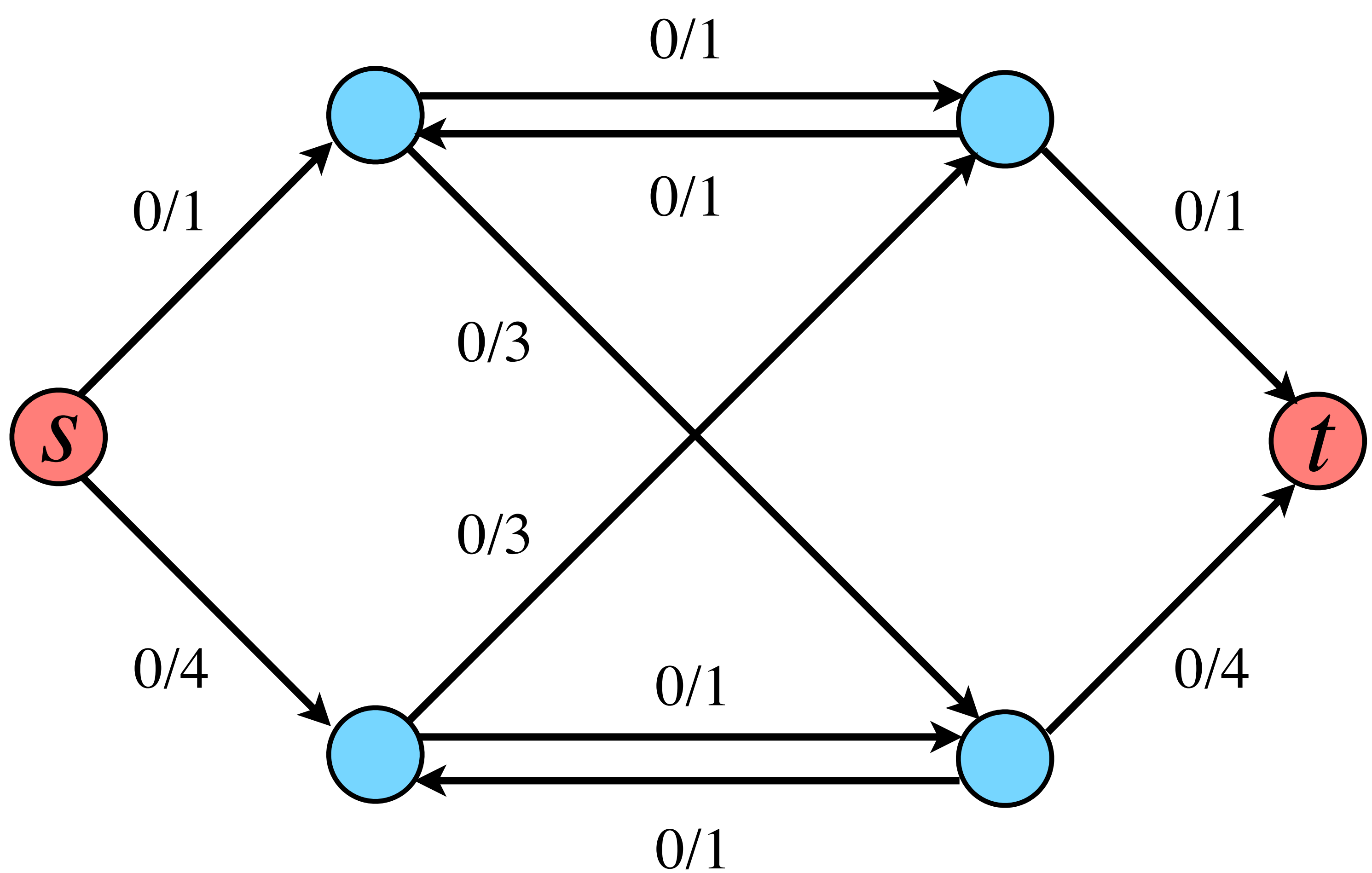
$$A = \sum_{i=1}^{m} \lambda_i P_i \qquad \lambda_i \geq 0 \qquad \sum_{i=1}^{m} \lambda_i = 1$$

Flow and Cut

Flow

digraph: $D(V, E)$ **source:** $s \in V$ **sink:** $t \in V$

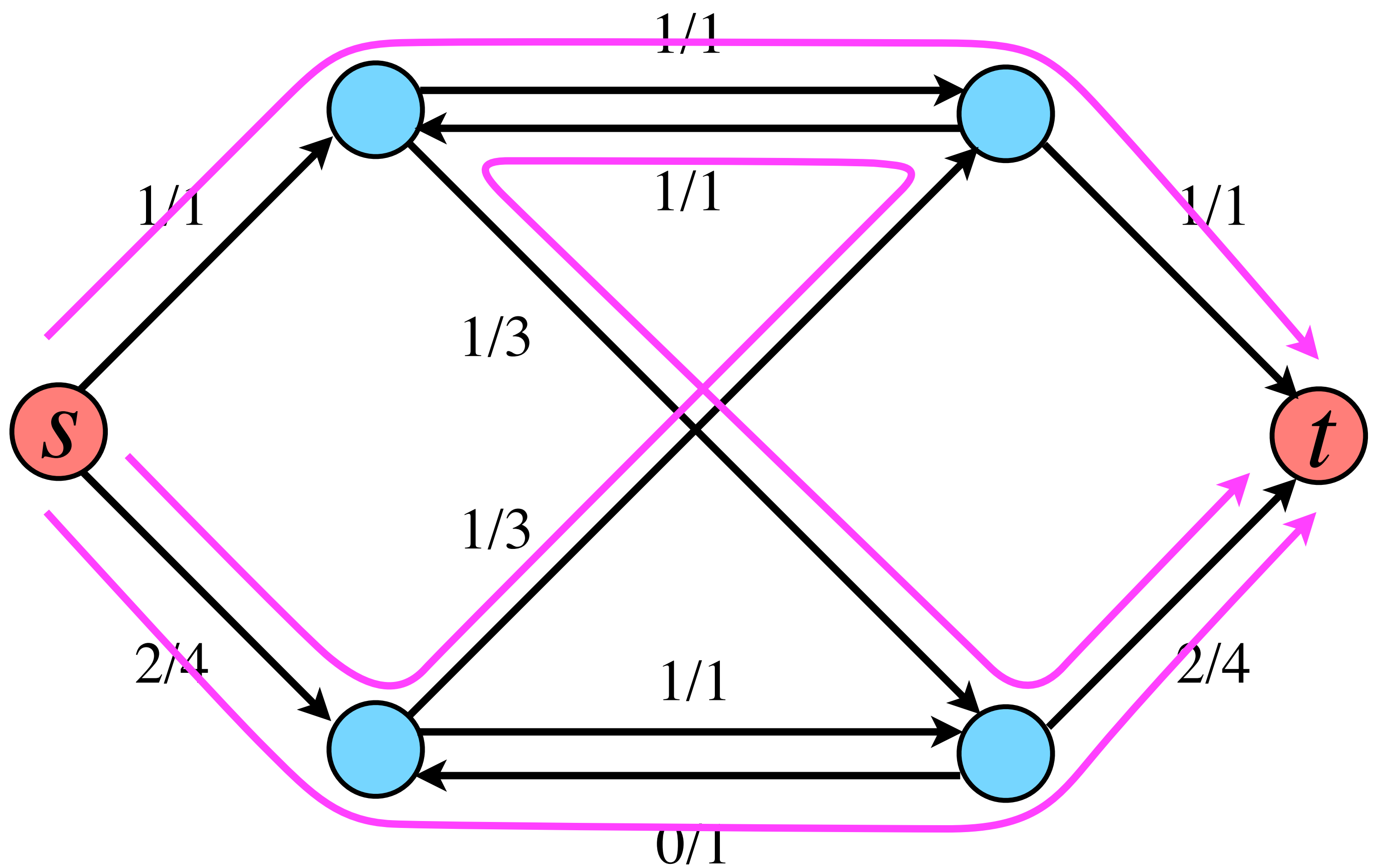
capacity $c : E \to \mathbb{R}^+$



Flow

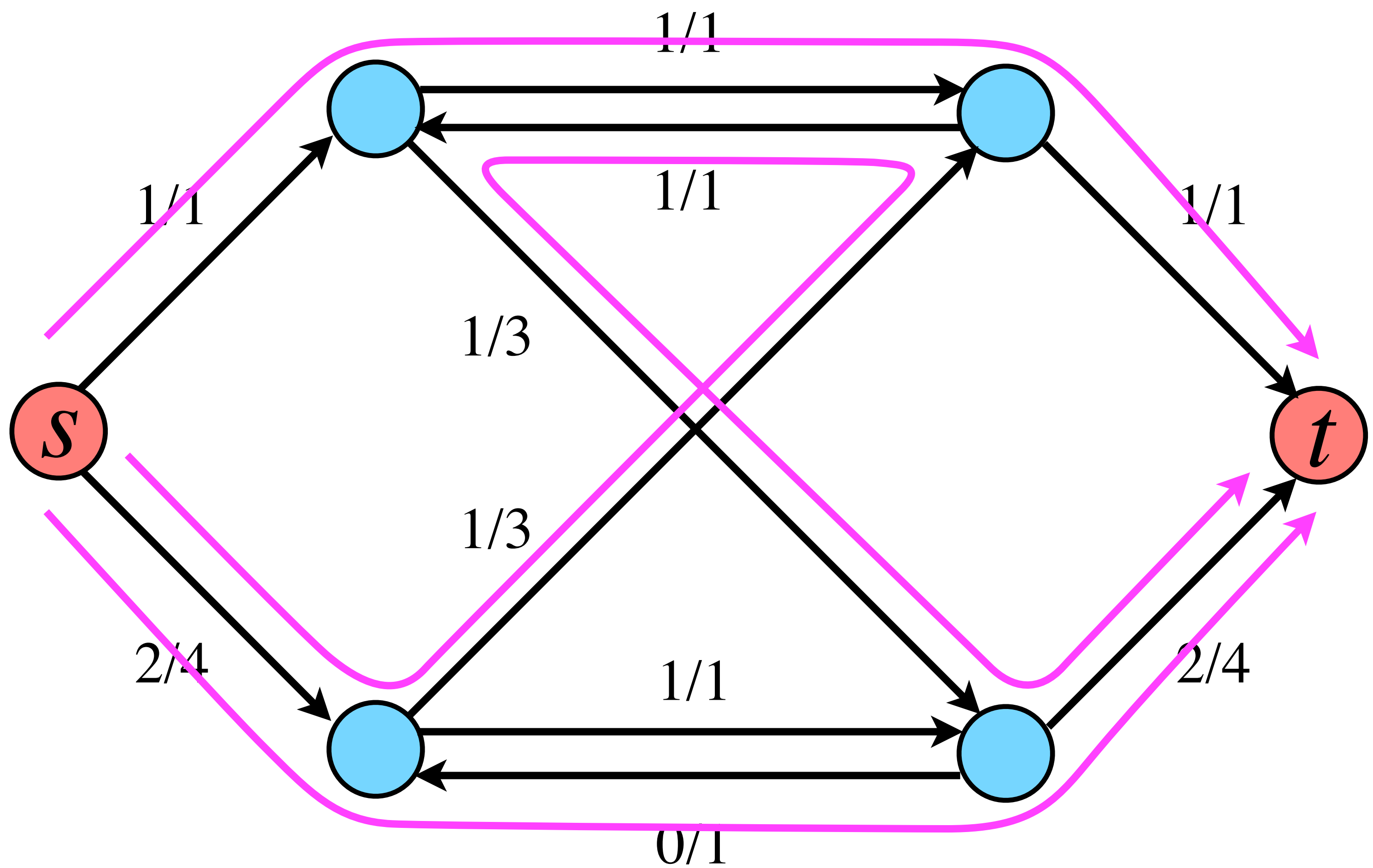
digraph: $D(V, E)$ **source:** $s \in V$ **sink:** $t \in V$

capacity $c : E \to \mathbb{R}^+$ **flow** $f : E \to \mathbb{R}^+$



capacity: $\forall (u, v) \in E,\ f_{uv} \leq c_{uv}$

conservation: $\forall u \in V \setminus \{s, t\},\ \sum_{(w,u)\in E} f_{wu} = \sum_{(u,v)\in E} f_{uv}$

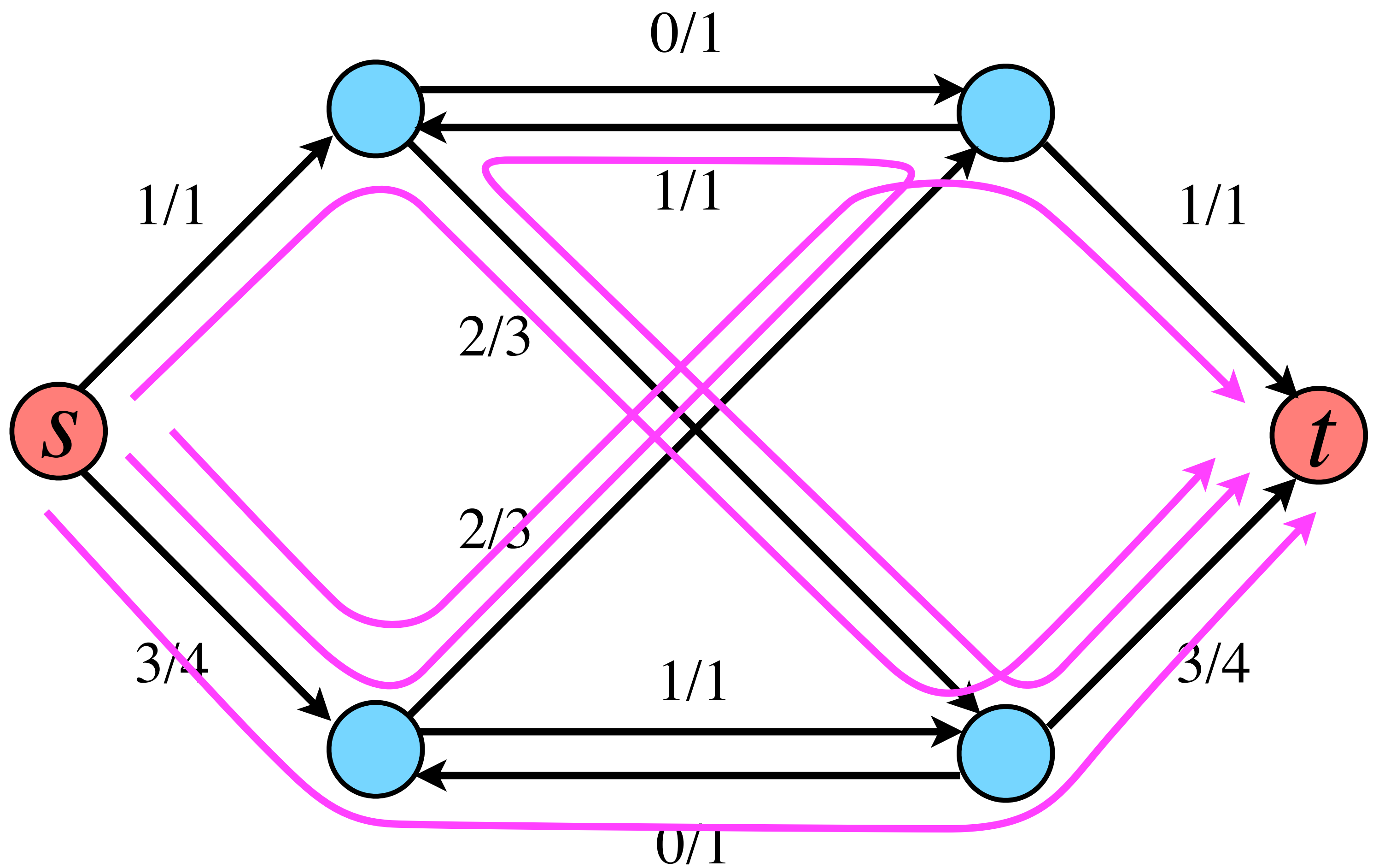


capacity: $\forall (u,v) \in E,\ f_{uv} \le c_{uv}$

conservation: $\forall u \in V \setminus \{s,t\},\ \sum_{(w,u)\in E} f_{wu} = \sum_{(u,v)\in E} f_{uv}$

value of flow: $$\sum_{(s,u)\in E} f_{su} = \sum_{(v,t)\in E} f_{vt}$$

maximum flow



capacity: $\forall (u,v) \in E,\ \ f_{uv} \le c_{uv}$

conservation: $\forall u \in V \setminus \{s,t\},\ \ \sum_{(w,u)\in E} f_{wu} = \sum_{(u,v)\in E} f_{uv}$

value of flow: $$\sum_{(s,u)\in E} f_{su} = \sum_{(v,t)\in E} f_{vt}$$

maximum flow

Maximum Flow

digraph: $D(V, E)$ **source:** $s \in V$ **sink:** $t \in V$

capacity $c : E \to \mathbb{R}^+$

max $$\sum_{u:(s,u)\in E} f_{su}$$

s.t. $$0 \le f_{uv} \le c_{uv} \qquad \forall (u,v) \in E$$

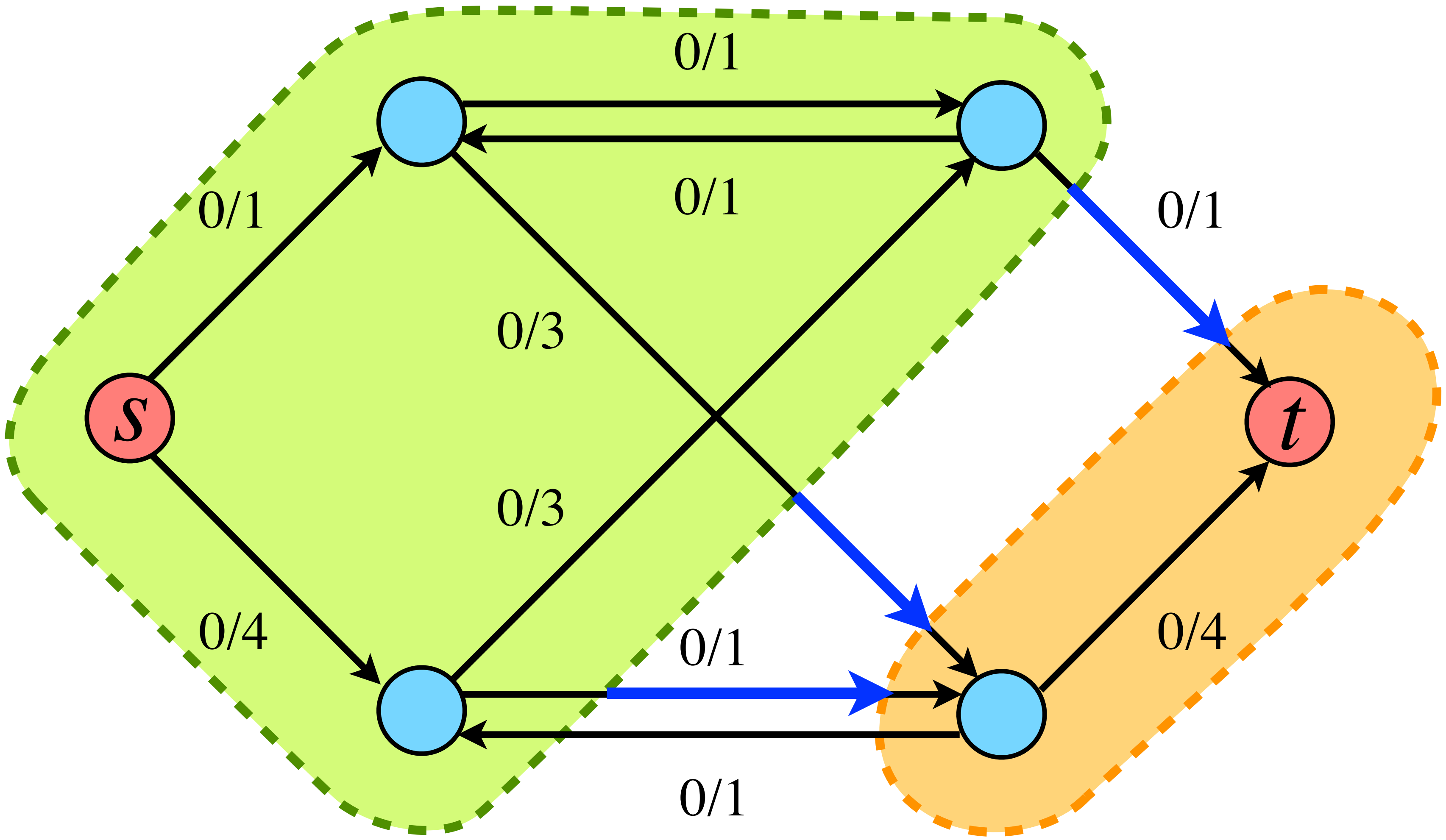
$$\sum_{w:(w,u)\in E} f_{wu} - \sum_{v:(u,v)\in E} f_{uv} = 0 \qquad \forall u \in V \setminus \{s,t\}$$

integral flow: $$f_{uv} \in \mathbb{Z} \qquad \forall (u,v) \in E$$

Cut

digraph: $D(V, E)$ **source:** $s \in V$ **sink:** $t \in V$

capacity $c : E \to \mathbb{R}^+$



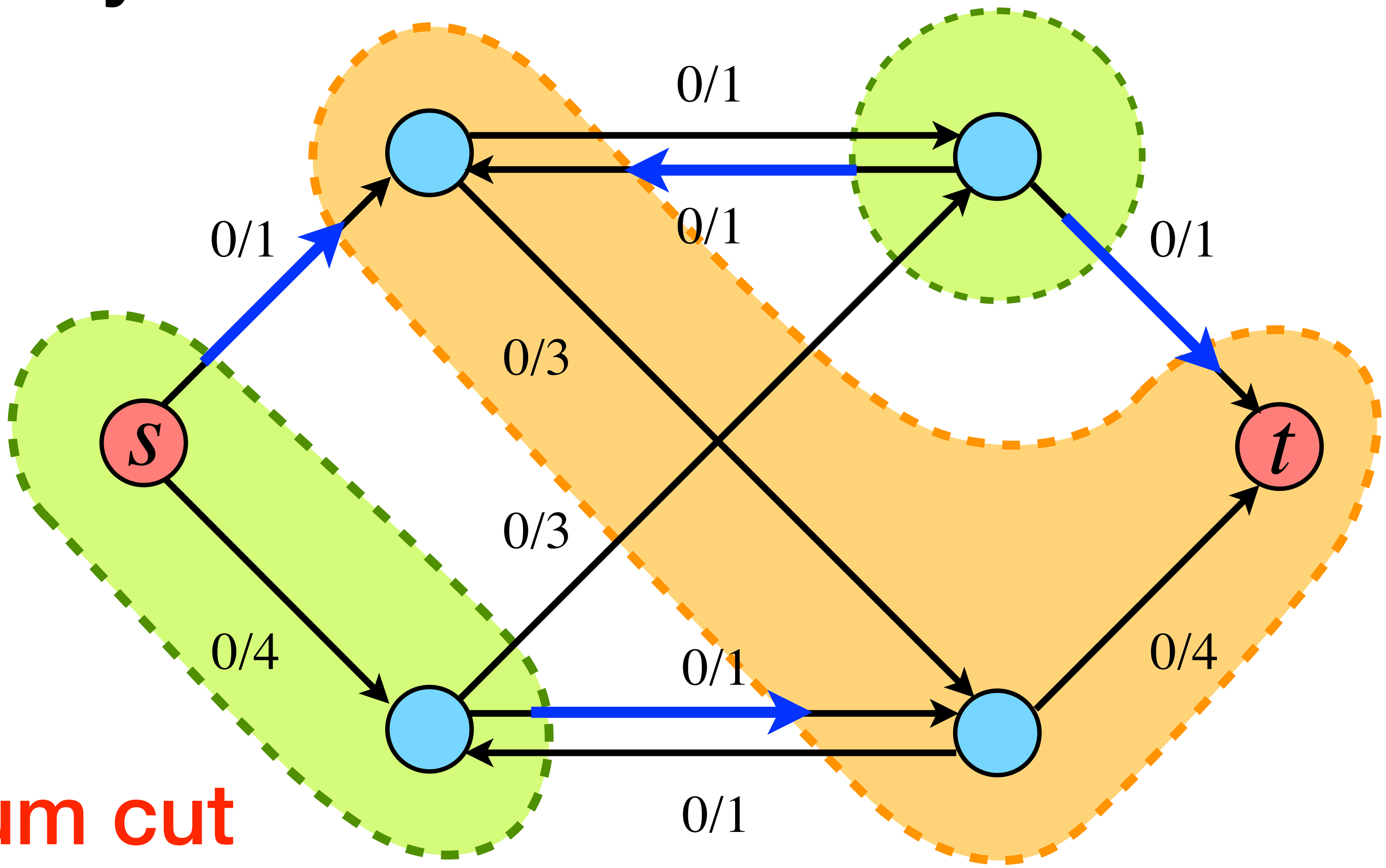
s-t **cut:** $S \subset V$ $\sum_{u \in S, v \notin S, (u,v) \in E} c_{uv}$

$s \in S, t \notin S$

Cut

digraph: $D(V, E)$ **source:** $s \in V$ **sink:** $t \in V$

capacity $c : E \to \mathbb{R}^+$



minimum cut

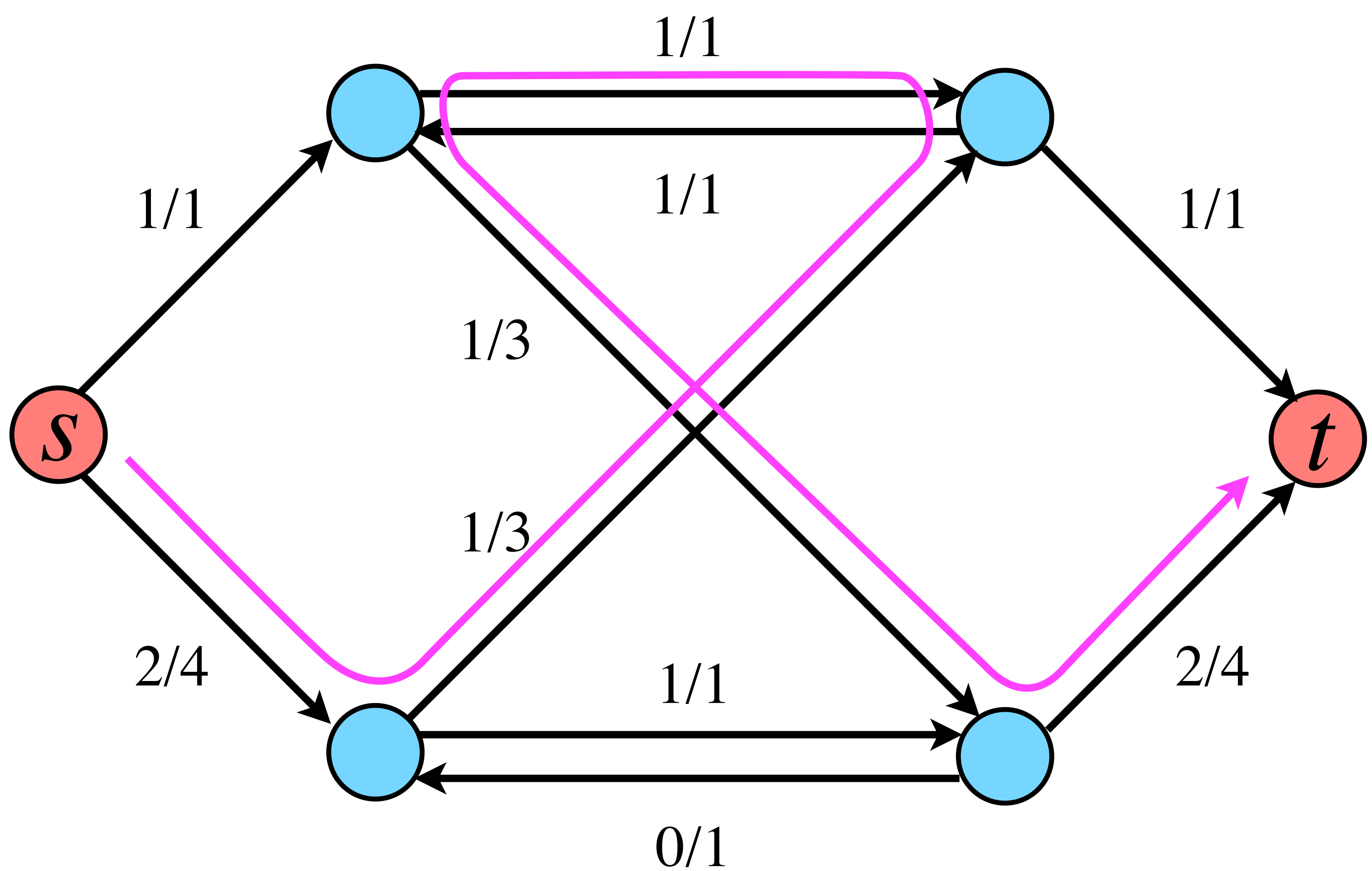
s-t **cut:** $S \subset V$ $\sum_{u \in S, v \notin S, (u,v) \in E} c_{uv}$

$s \in S, t \notin S$

Fundamental Theorems of Flow

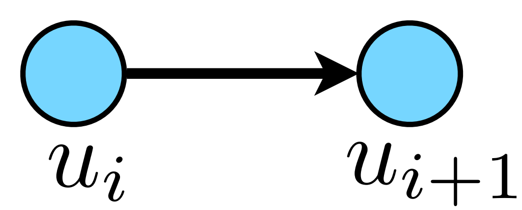
- **max integral flow = max flow**
  (assuming integral capacities)
- **max flow = min cut**

Augmenting Path

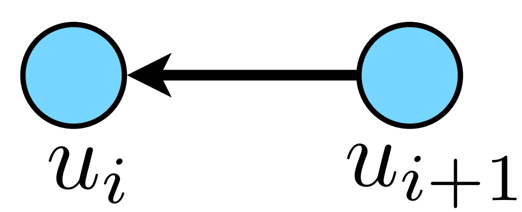


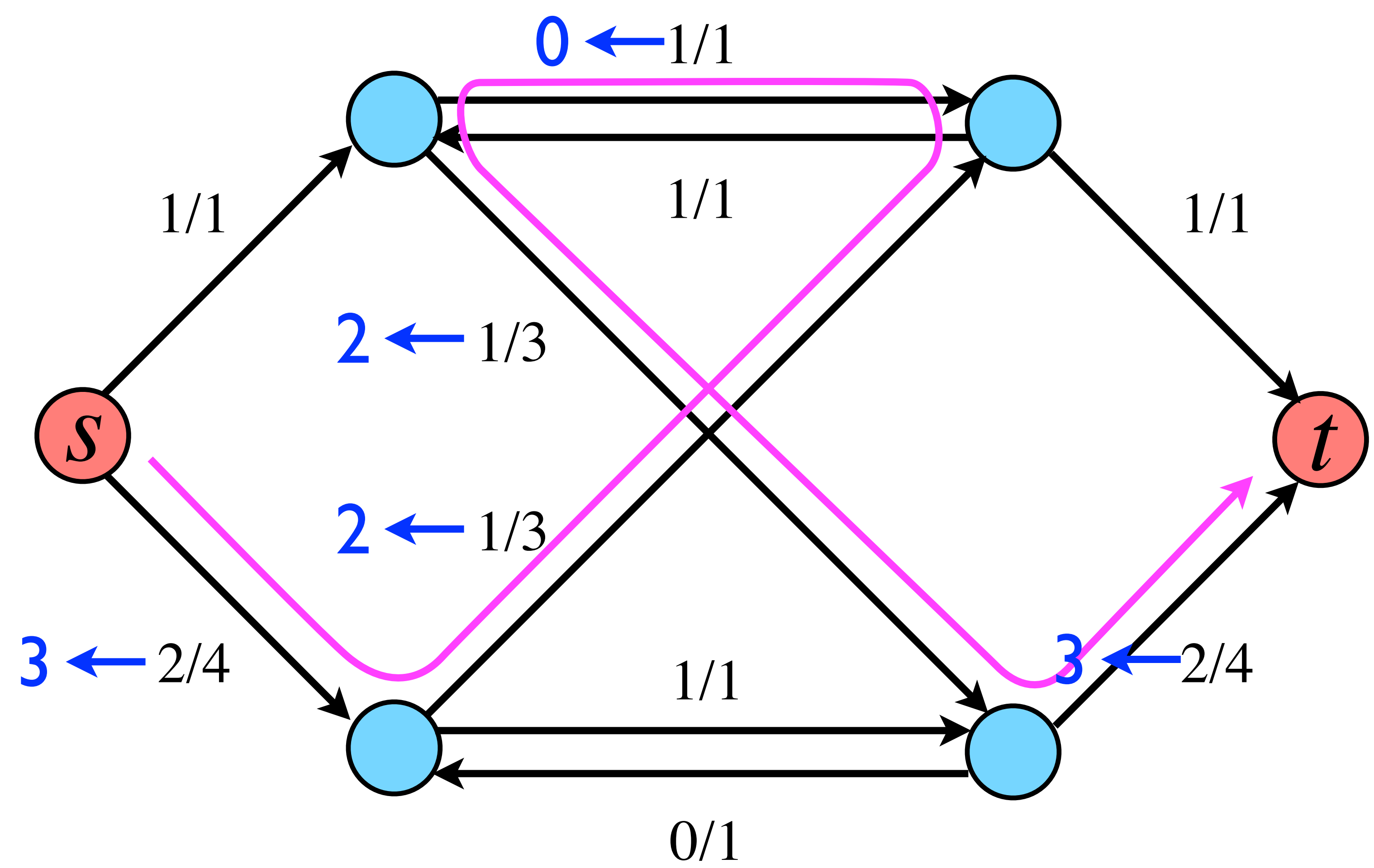
augmenting path: $s = u_0 u_1 \cdots u_k = t$

$$f(u_i u_{i+1}) < c(u_i u_{i+1})$$ if



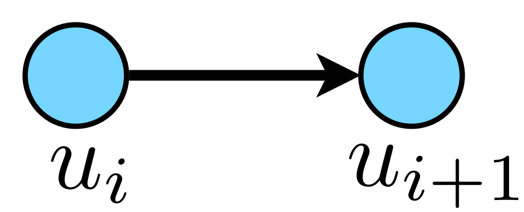
$$f(u_{i+1} u_i) > 0$$ if





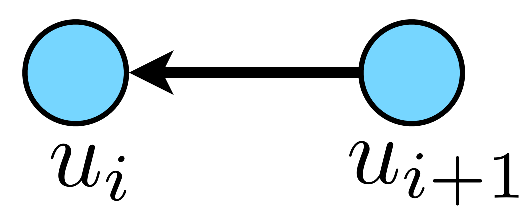
augmenting path: $s = u_0 u_1 \cdots u_k = t$ flow increased

$f(u_i u_{i+1}) < c(u_i u_{i+1})$ if



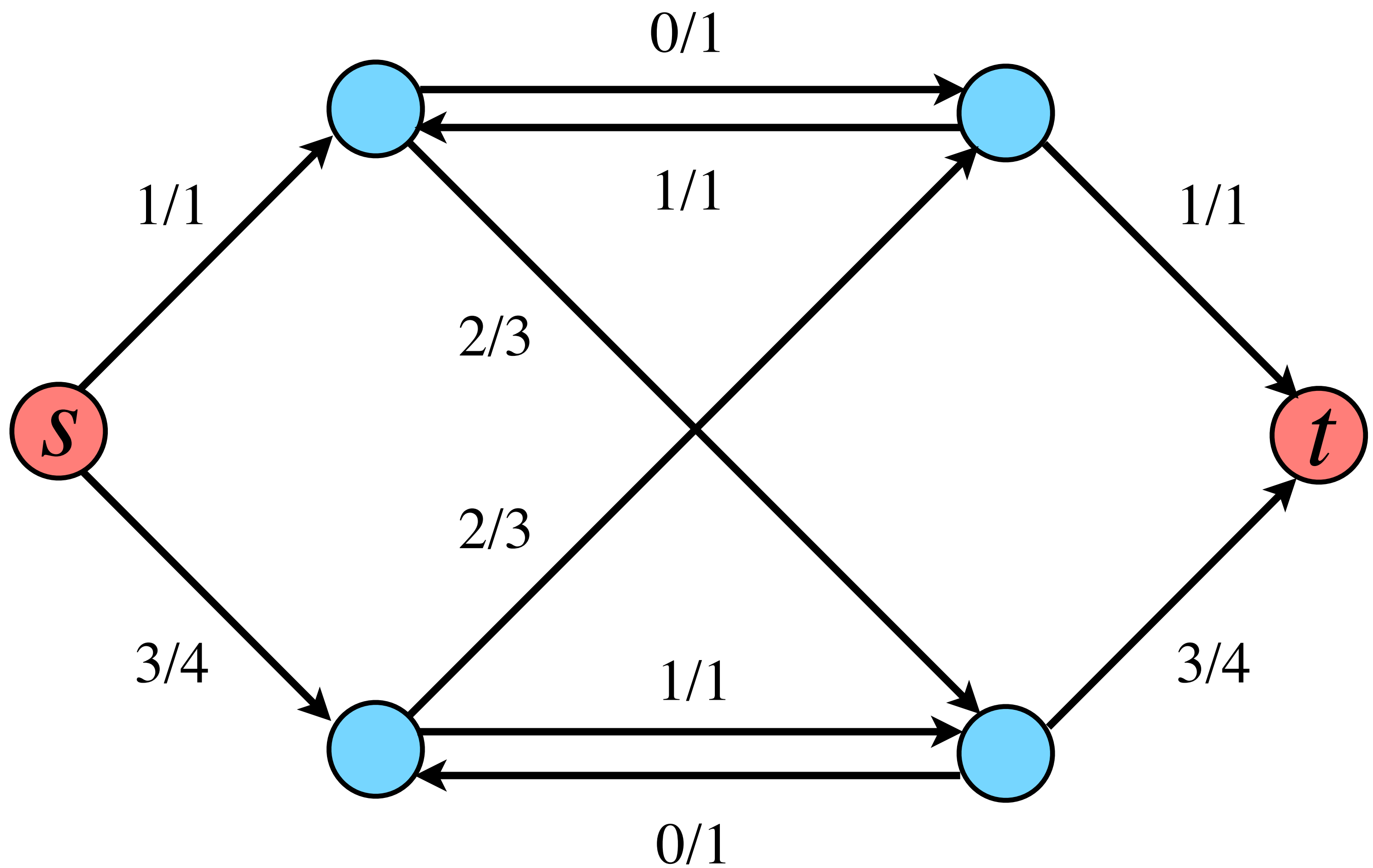
$f(u_i u_{i+1}) + \epsilon$

$f(u_{i+1} u_i) > 0$ if



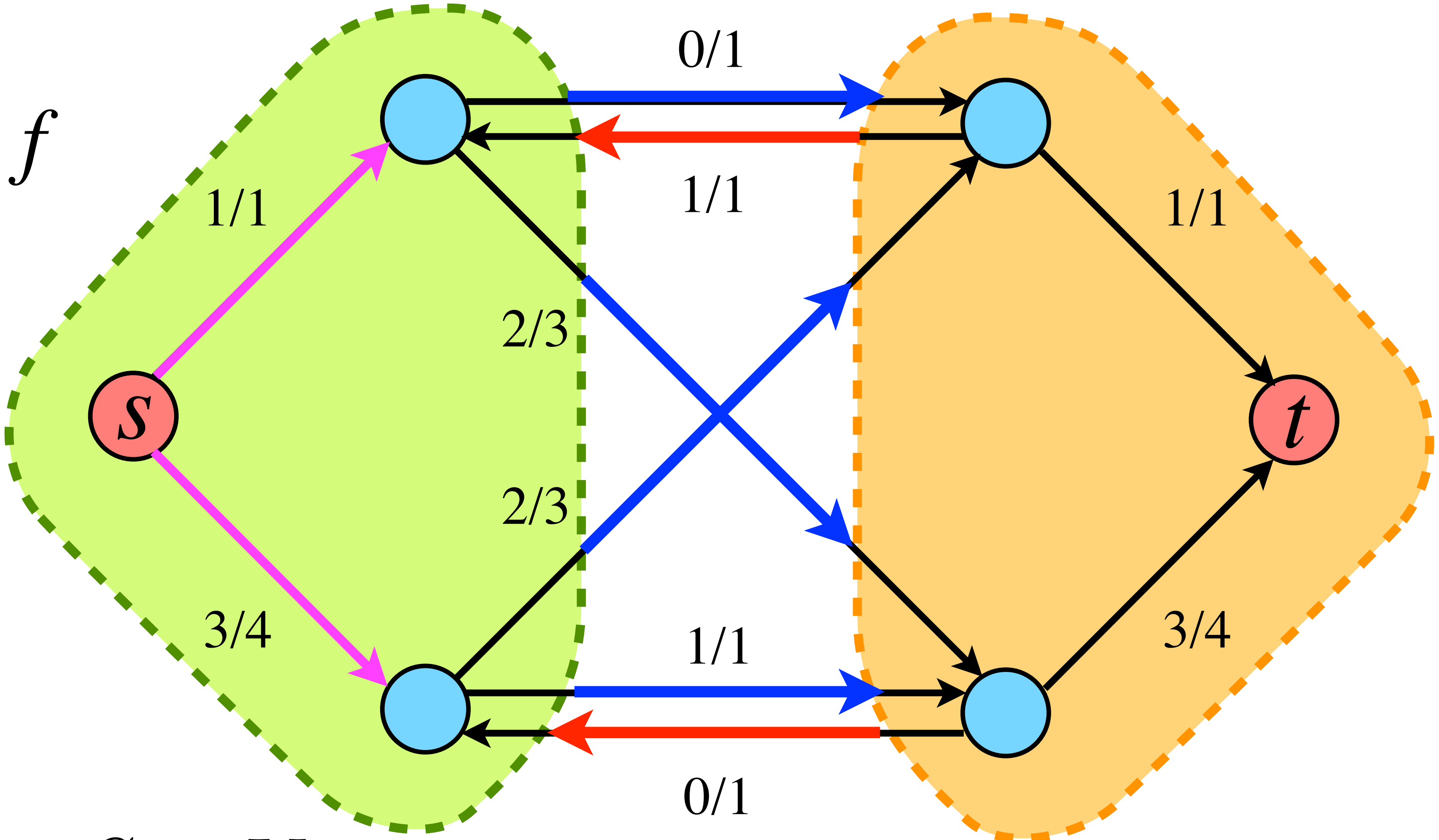
$f(u_{i+1} u_i) - \epsilon$

maximum flow ⇒ no augmenting path



maximum flow ⟺ no augmenting path

$\forall$ flow f



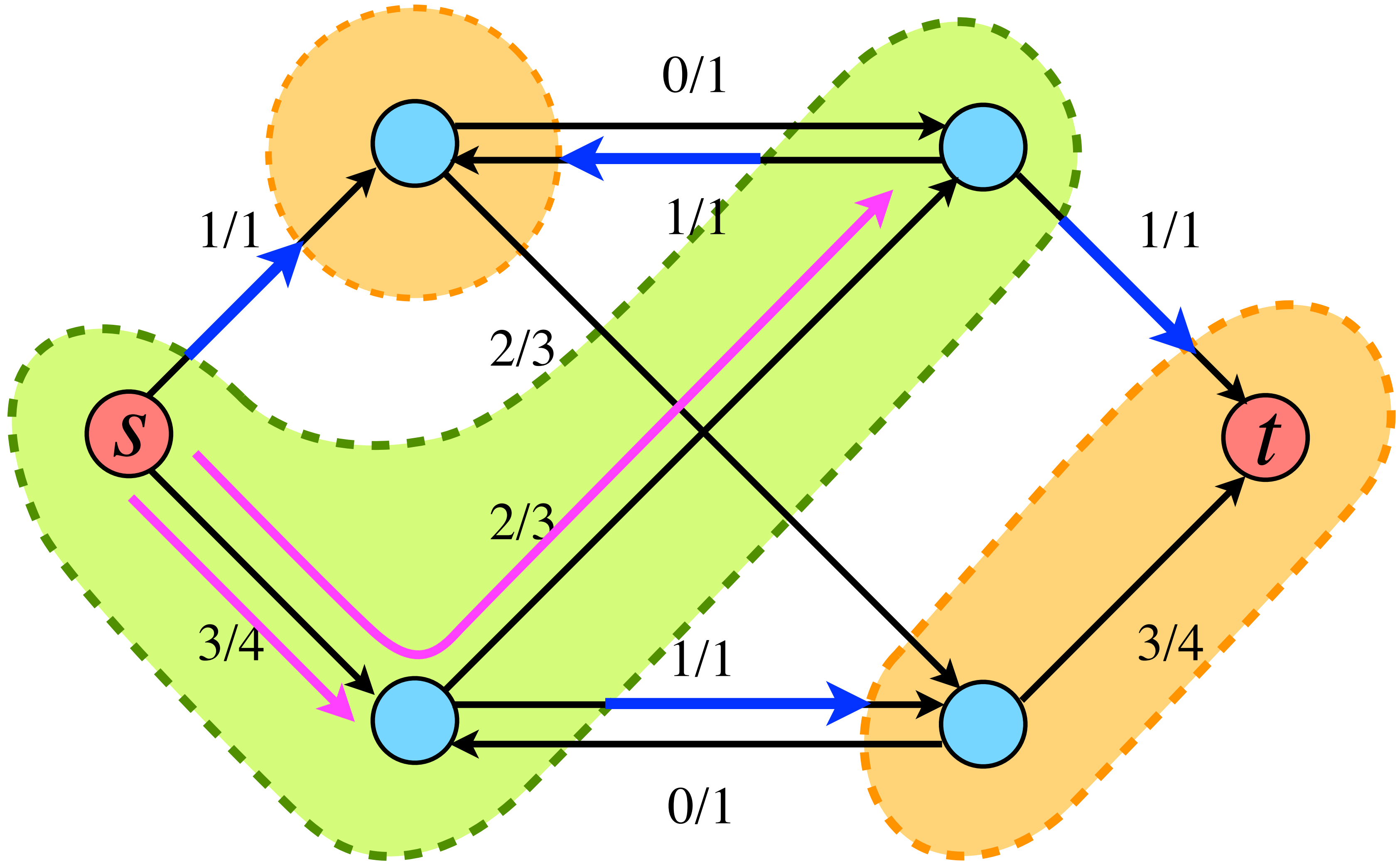
$\forall s$-t cut $S \subset V$

$$\sum_{u:(s,u)\in E} f_{su} = \sum_{\substack{u\in S, v\notin S\\(u,v)\in E}} f_{uv} - \sum_{\substack{u\in S, v\notin S\\(v,u)\in E}} f_{vu}$$

$$\sum_{u:(s,u)\in E} f_{su} \leq \sum_{\substack{u\in S, v\notin S\\(u,v)\in E}} f_{uv} \leq \sum_{\substack{u\in S, v\notin S\\(u,v)\in E}} c_{uv}$$

max-flow

min-cut



$$S = \{u \mid \exists \text{augmenting path from } s \text{ to } u\}$$

no augmenting path $\Rightarrow$ $s \in S, t \notin S$ $\quad s$-t cut

$$\forall u \in S, v \notin S, (u,v) \in E \quad \begin{cases} f_{uv} = c_{uv} \\ f_{vu} = 0 \end{cases}$$

flow $\displaystyle\sum_{u:(s,u)\in E} f_{su} = \sum_{\substack{u\in S, v\notin S \\ (u,v)\in E}} f_{uv} - \sum_{\substack{u\in S, v\in S \\ (v,u)\in E}} f_{vu} = \sum_{\substack{u\in S, v\notin S \\ (u,v)\in E}} c_{uv}$ cut maximum flow ⟺ no augmenting path

Max-Flow Min-Cut Theorem

(Ford-Fulkerson 1956; Kotzig 1956)

max-flow = min-cut

Flow Integrality Theorem

If capacities are integers, then

max integral flow = max-flow

in an integral flow f: ∃ augmenting path ⇒

∃ integral augmenting path ⇒ ∃ larger integral flow

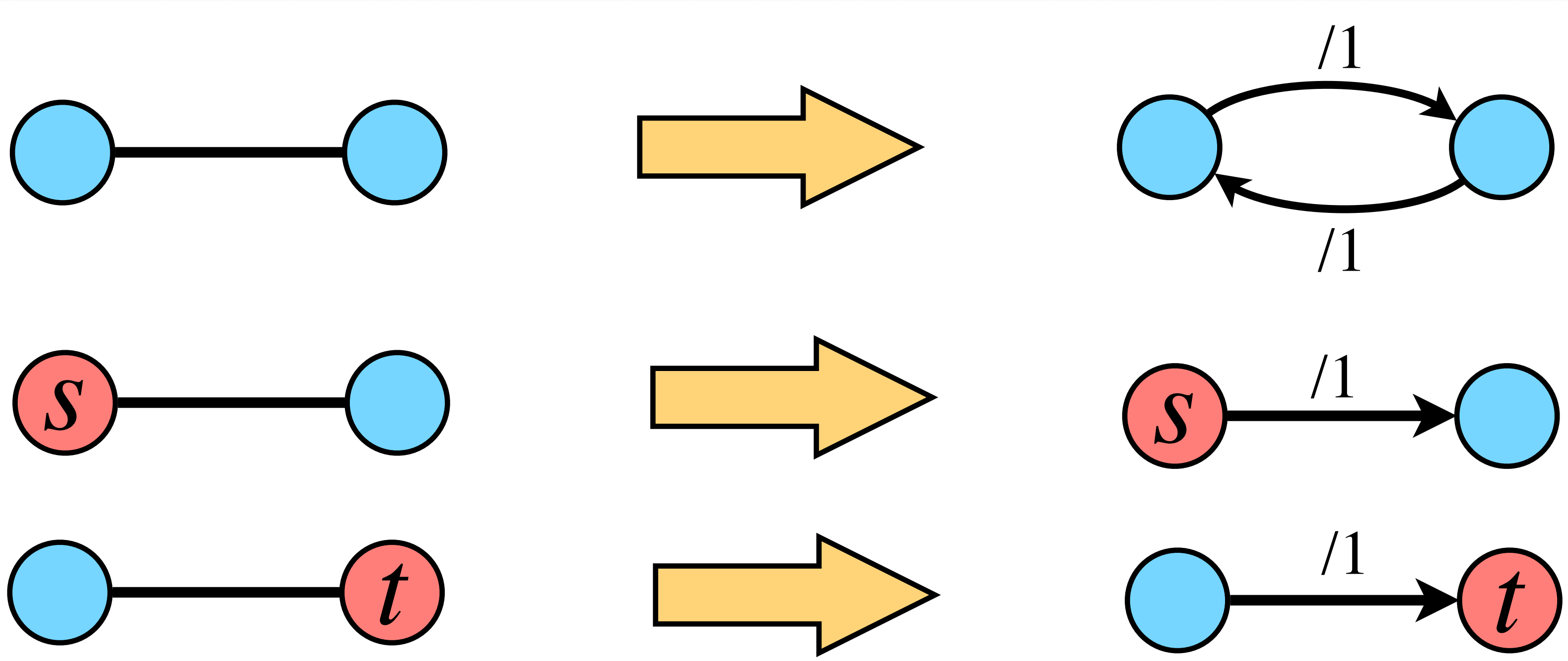
Menger's Theorem

undirected graph: $G(V, E)$ $\forall s, t \in V$

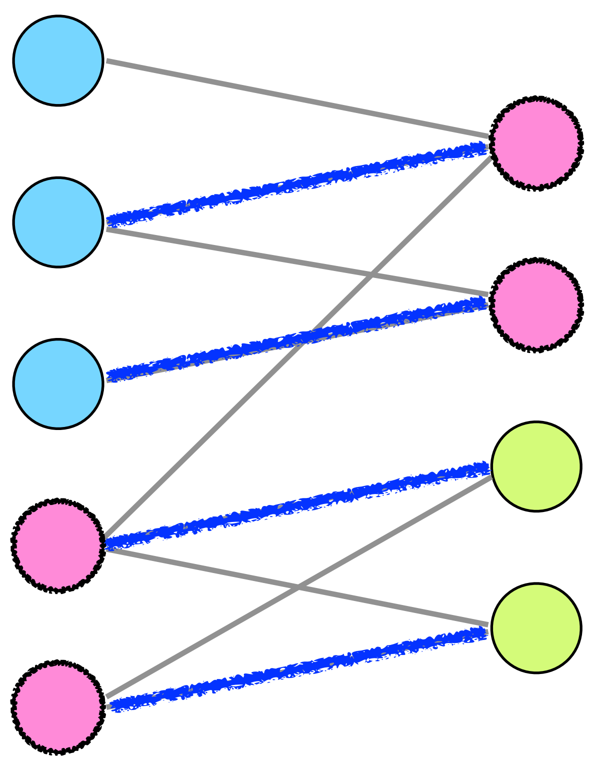
s-t cut $C \subset E$ removing C disconnects s, t

Theorem (Menger 1927)

min s-t cut = max # of disjoint s-t paths



Bipartite Matching



matching: $M \subseteq E$

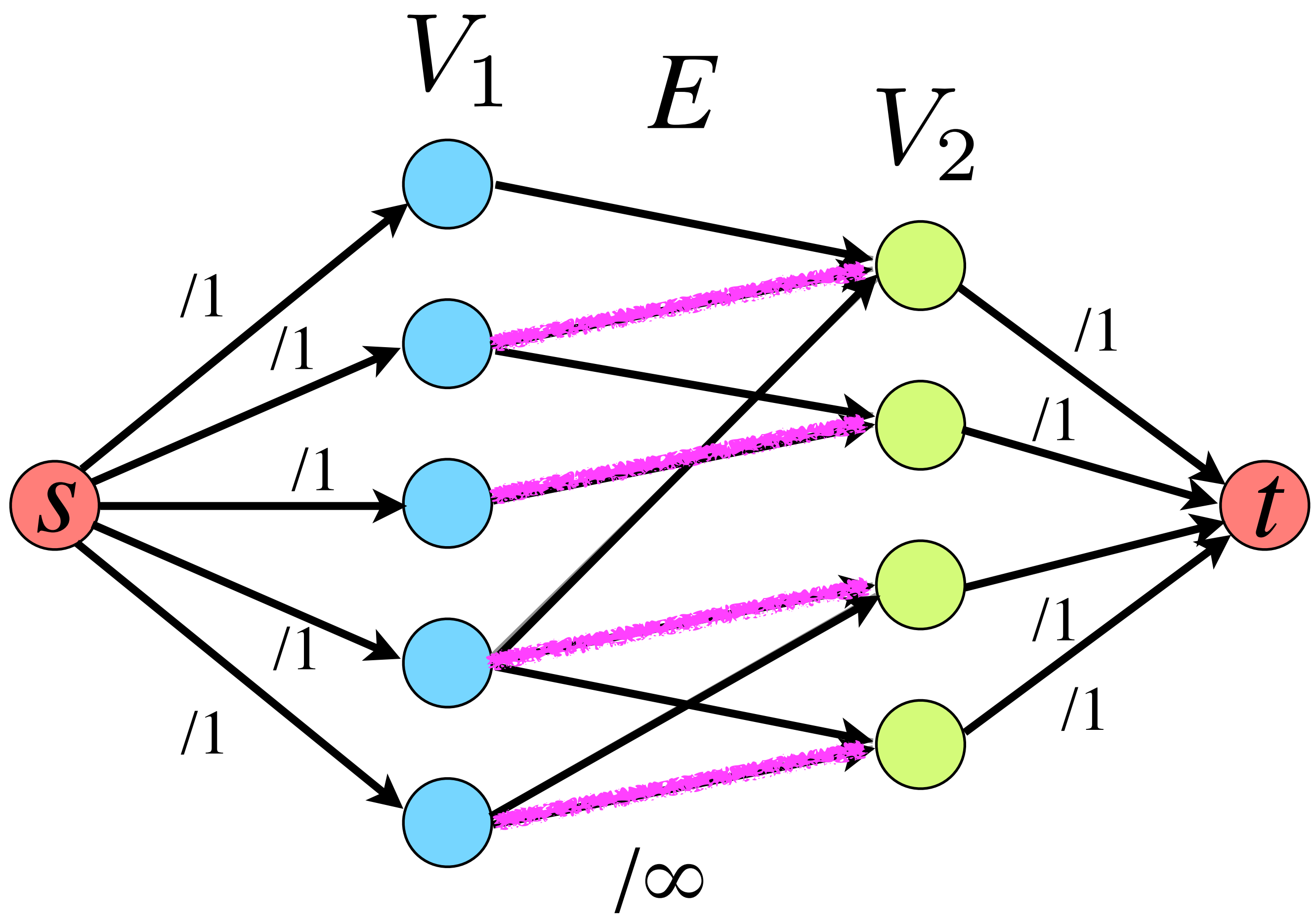
no $e_1, e_2 \in M$ share a vertex

vertex cover: $C \subseteq V$

all $e \in E$ adjacent to some $v \in C$

Theorem (König 1931, Egerváry 1931)

In a bipartite graph,
max matching = min vertex cover.

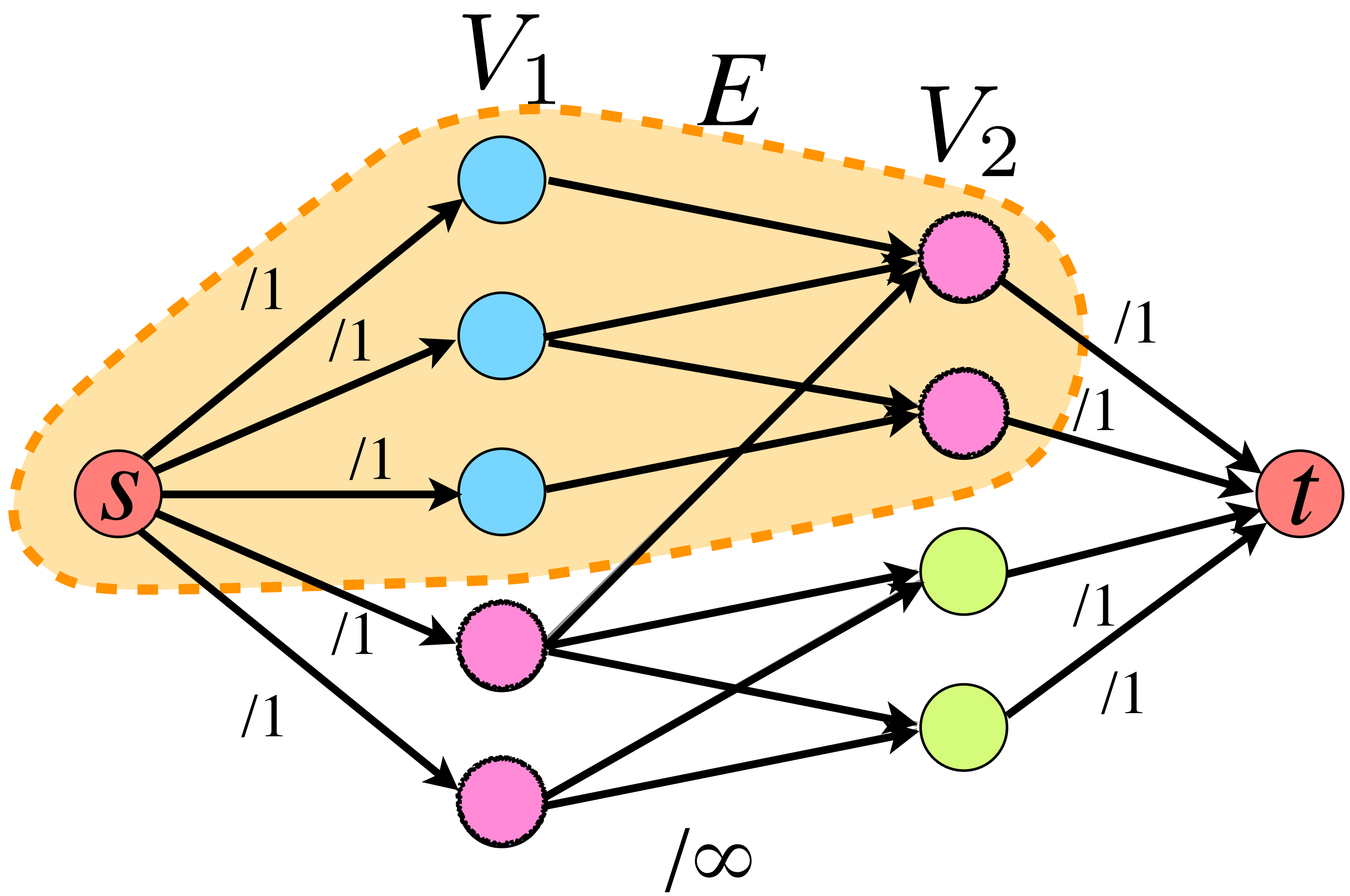


max integral flow = max matching

$$\forall (u,v) \in E \qquad f_{uv} \in \{0,1\}$$

$$\forall u \in V_1, \quad \sum_{v:(u,v)\in E} f_{uv} \leq 1$$

$$\forall v \in V_2, \quad \sum_{u:(u,v)\in E} f_{uv} \leq 1$$



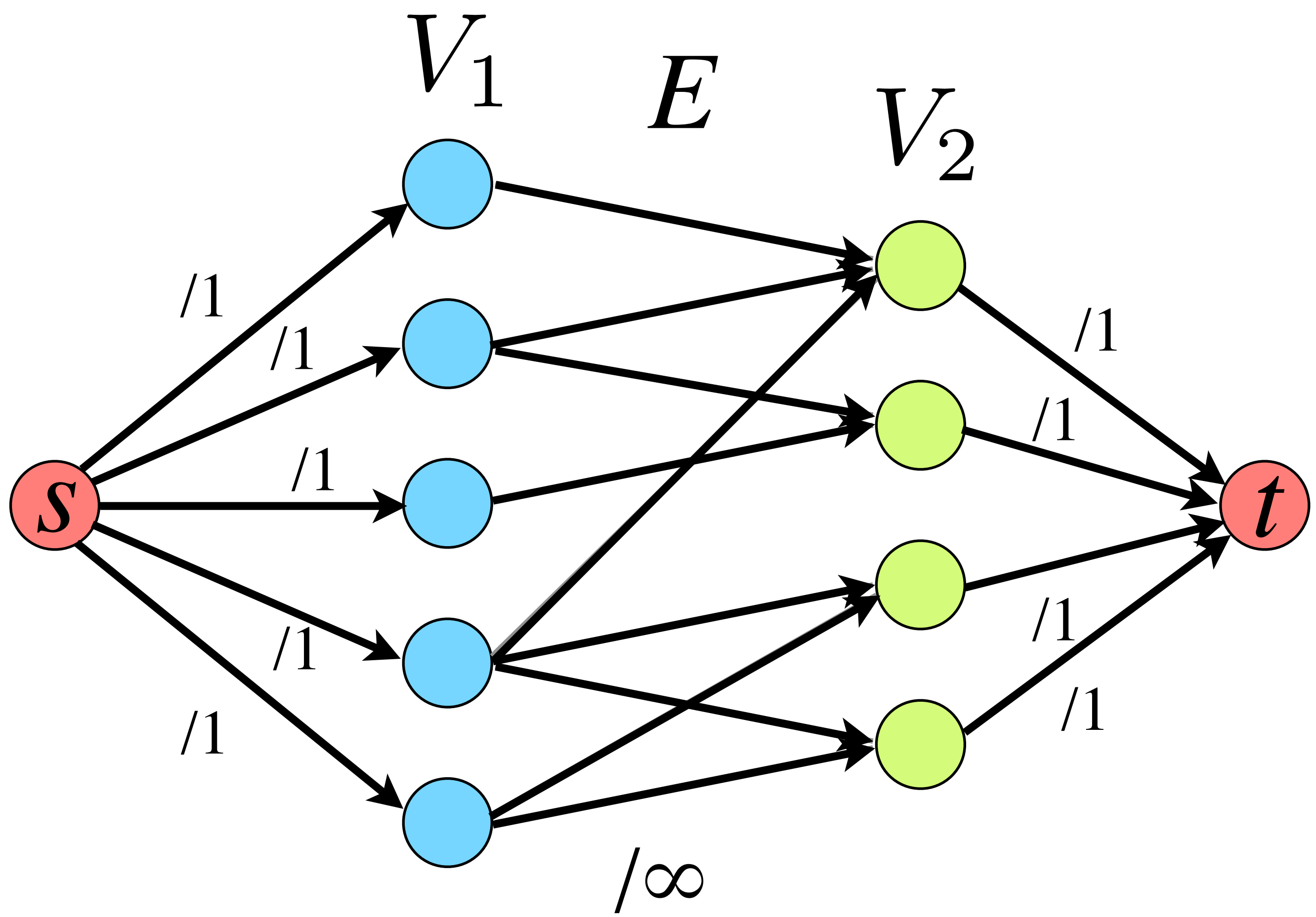
min *s*-*t* cut = vertex cover

min-cut
$s \in S, t \notin S$
⇒ $\sum_{\substack{u \in S, v \notin S \\ (u,v) \in E}} c_{uv} < \infty$ ⇒

no edge $(u, v) \in E$ has $u \in V_1 \cap S, v \in V_2 \setminus S$

⇒ $(V_1 \setminus S) \cup (V_2 \cap S)$ is a vertex cover

$$|V_1 \setminus S| + |V_2 \cap S| = \sum_{v \in V_1 \setminus S} c_{sv} + \sum_{u \in V_2 \cap S} c_{ut} = \sum_{\substack{u \in S, v \notin S \\ (u,v) \in E}} c_{uv}$$



max integral flow = max matching

min s-t cut = vertex cover

Theorem (König 1931, Egerváry 1931)

In a bipartite graph,
max matching = min vertex cover.

Duality and Integrality

Maximum Flow

digraph: $D(V, E)$ **source:** $s \in V$ **sink:** $t \in V$

capacity $c : E \to \mathbb{R}^+$

max $\displaystyle\sum_{u:(s,u)\in E} f_{su}$

s.t. $0 \leq f_{uv} \leq c_{uv}$ $\forall (u,v) \in E$

$$\sum_{w:(w,u)\in E} f_{wu} - \sum_{v:(u,v)\in E} f_{uv} = 0 \qquad \forall u \in V \setminus \{s,t\}$$

Linear Programming (LP)

general form: matrix $A = \{a_{ij}\}_{m \times n}$

sets $M \subseteq [m]$ $N \subseteq [n]$

$$
\begin{array}{llr}
\text{min} & \boldsymbol{c}^T \boldsymbol{x} & \\
\text{s.t.} & \boldsymbol{a}_i^T \boldsymbol{x} = b_i & i \in M \\
& \boldsymbol{a}_i^T \boldsymbol{x} \geq b_i & i \in \overline{M} \\
& x_j \geq 0 & i \in N \\
& x_j \text{ unconstrained} & i \in \overline{N}
\end{array}
$$

Canonical Form for LP

general form:

$$\begin{array}{lll} \min & \boldsymbol{c}^T\boldsymbol{x} & \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} = b_i & i \in M \\ & \boldsymbol{a}_i^T\boldsymbol{x} \geq b_i & i \in \overline{M} \\ & x_j \geq 0 & i \in N \\ & x_j \text{ unconstrained} & i \in \overline{N} \end{array}$$

canonical form:

$m \times n$ matrix A

$$\begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

$$\boldsymbol{a}_i^T\boldsymbol{x} = b_i \Rightarrow \begin{cases} \boldsymbol{a}_i^T\boldsymbol{x} \geq b_i \\ -\boldsymbol{a}_i^T\boldsymbol{x} \geq -b_i \end{cases}$$

$$x_j \text{ unconstrained} \Rightarrow \begin{array}{l} x_j^+ \geq 0 \\ x_j^- \geq 0 \end{array} \quad x_j = x_j^+ - x_j^-$$

Standard Form for LP

canonical form:

$m \times n$ matrix A

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

standard form:

$m \times n$ matrix A

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \Rightarrow \quad \begin{cases} \sum_{j=1}^{n} a_{ij} x_j - s_i = b_i \\ s_i \geq 0 \end{cases}$$

slack variable

Convex Polytopes

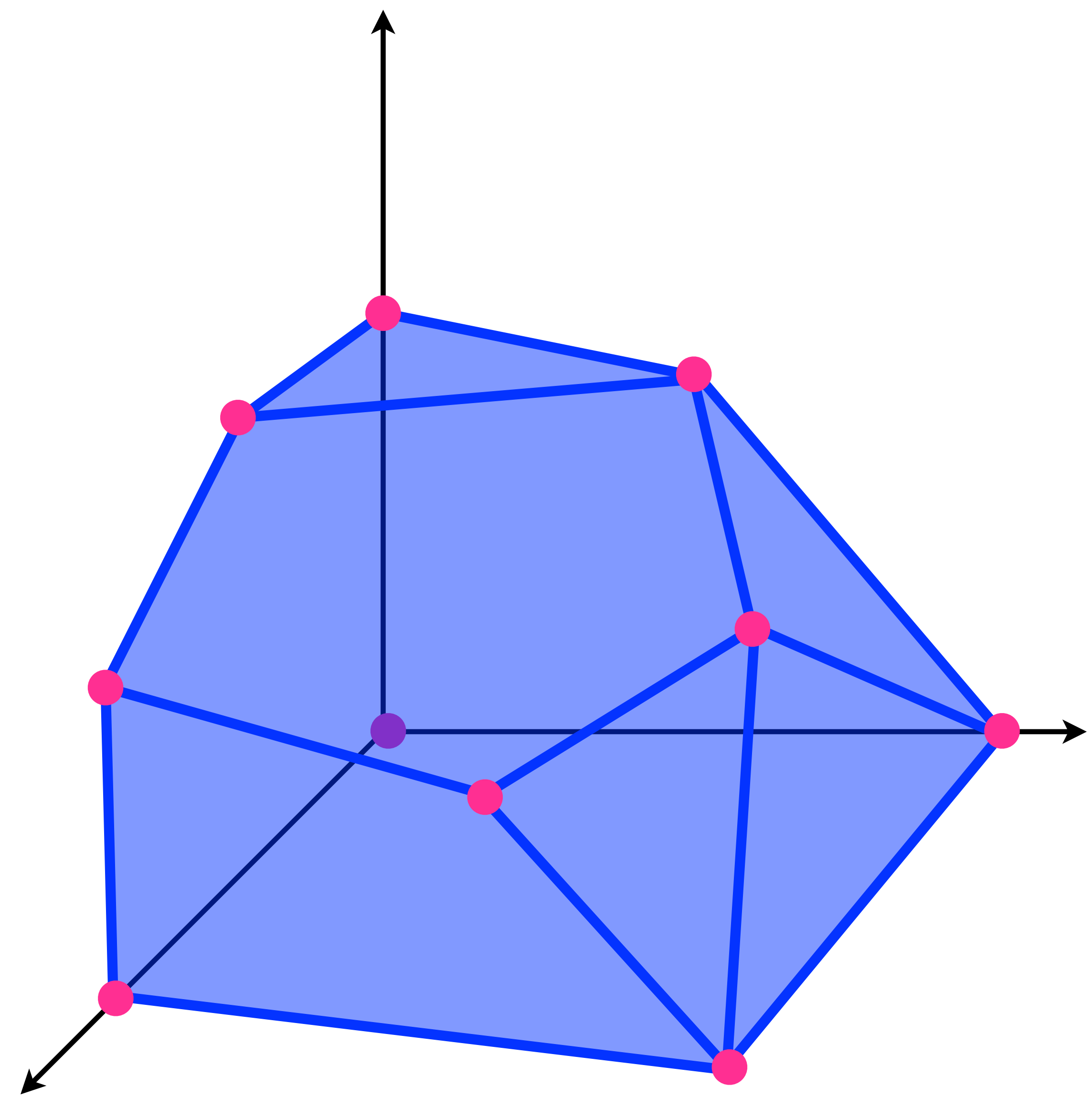
hyperplane:

subspace of dimension n-1

$$\sum_{j=1}^{n} a_j x_j = b$$

(closed) halfspace:

$$\sum_{j=1}^{n} a_j x_j \geq b$$



convex polyhedron:

intersection of halfspaces

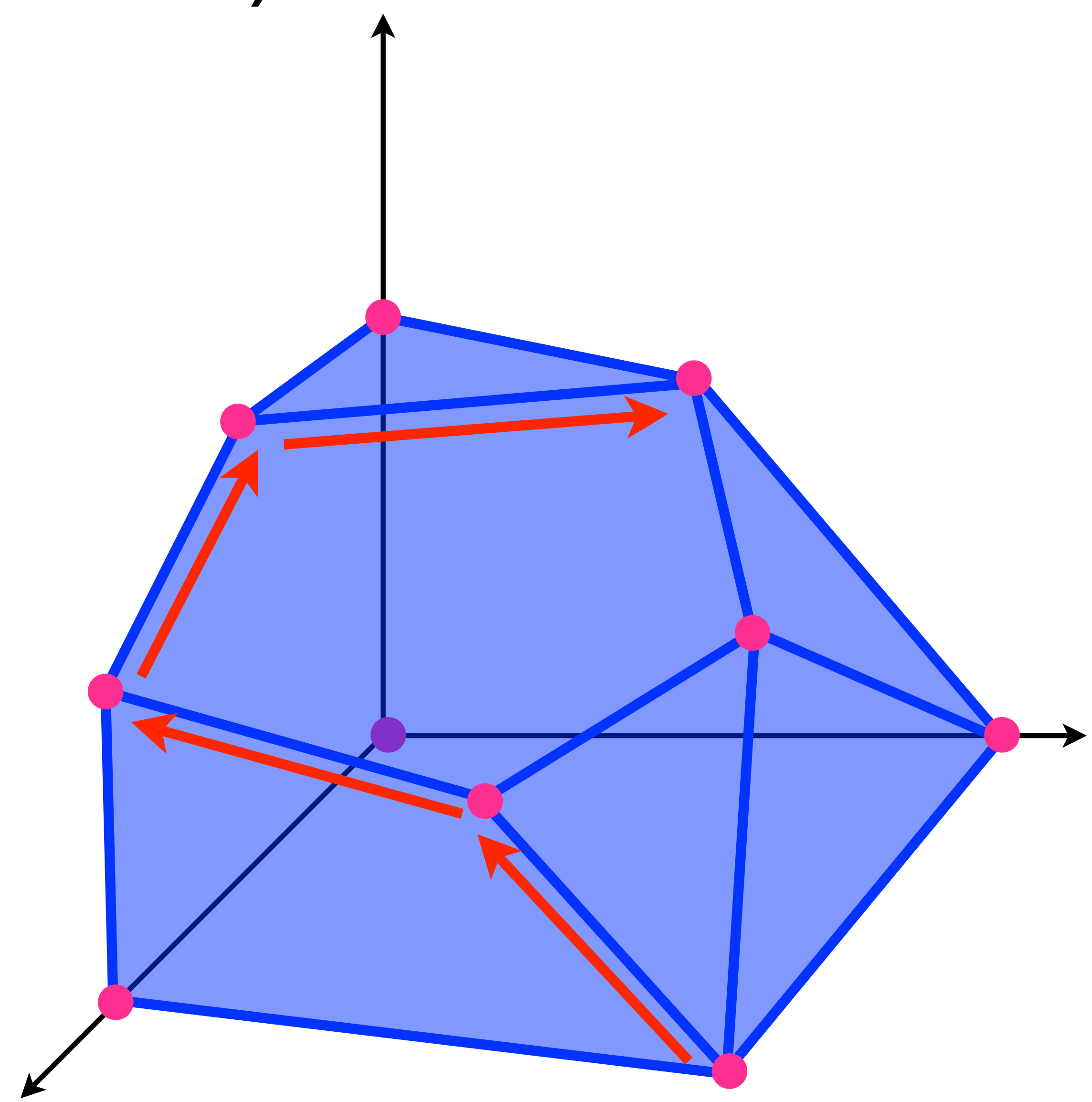
convex polytope: bounded convex polyhedron

The Simplex Algorithm

(Dentzig, 1947)

y is a neighbor of ***x*** if ∃ an edge between ***x*** and ***y***

find a vertex ***x***;
repeat:
  pick a neighbor ***y***
  with $\mathbf{c}^T\mathbf{y} < \mathbf{c}^T\mathbf{x}$;
  x ← ***y***;
until no such ***y***.



Estimate the Optima

$$\begin{array}{llcccccccc}
\text{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & & & \text{VI} & & & & \\
\text{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & & & + & & & & + \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & & & \text{II} \\
 & x_1,\ x_2,\ x_3 \geq 0 & & & & & & 16
\end{array}$$

16 ? ≤ OPT ≤ 30 (any feasible solution)

$$\boldsymbol{x} = (2, 1, 3)$$

Estimate the Optima

minimize $\quad 7x_1 + x_2 + 5x_3$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \geq$ (VI)

subject to

$$y_1 \quad x_1 - x_2 + 3x_3 \geq 10 \; y_1$$

$$+ \qquad\qquad +$$

$$y_2 \quad 5x_1 + 2x_2 - x_3 \geq 6 \; y_2$$

$$x_1,\, x_2,\, x_3 \geq 0$$

$$10y_1 + 6y_2 \leq \text{OPT}$$

for any

$$\begin{aligned} y_1 + 5y_2 &\leq 7 \\ -y_1 + 2y_2 &\leq 1 \\ 3y_1 - y_2 &\leq 5 \end{aligned} \qquad y_1,\, y_2 \geq 0$$

Primal-Dual

Primal

$$\begin{array}{llcrcrcl} \text{min} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \text{s.t.} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & x_1, x_2, x_3 \geq 0 & & & & & & \end{array}$$

Dual

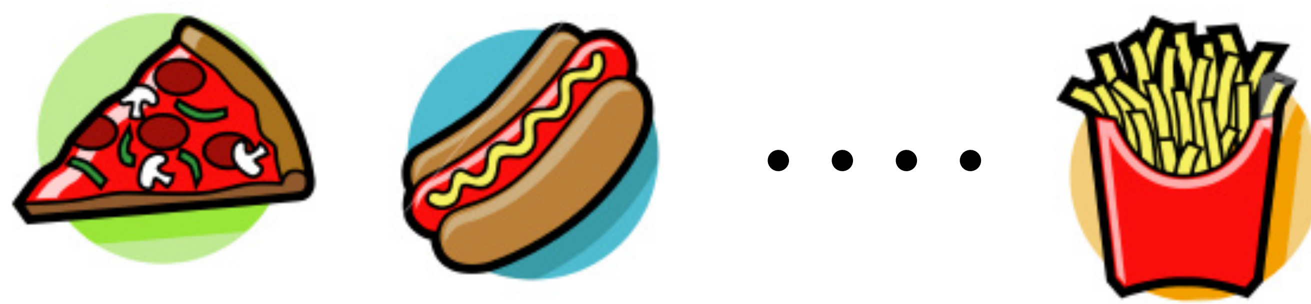
$$\begin{array}{llcrcl} \text{max} & 10y_1 + 6y_2 & & & & \\ \text{s.t.} & y_1 & + & 5y_2 & \leq & 7 \\ & -y_1 & + & 2y_2 & \leq & 1 \\ & 3y_1 & - & y_2 & \leq & 5 \\ & y_1, y_2 \geq 0 & & & & \end{array}$$

dual feasible solution

≤

primal OPT

Diet Problem



calories	c_1	c_2	$\cdots\cdots$	c_n	healthy
vitamin 1	a_{11}	a_{12}	$\cdots\cdots$	a_{1n}	$\geq b_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
vitamin m	a_{m1}	a_{m2}	$\cdots\cdots$	a_{mn}	$\geq b_m$
solution:	x_1	x_2	$\cdots\cdots$	x_n	

minimize the calories while keeping healthy

Diet Problem

minimize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

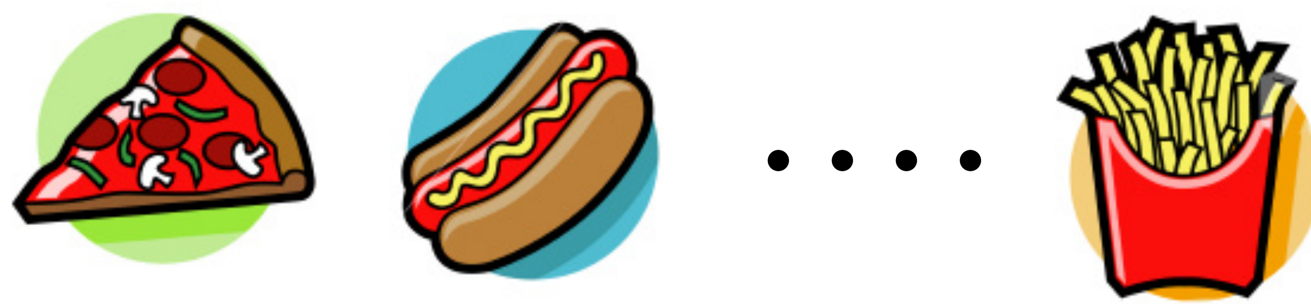
subject to $A\boldsymbol{x} \geq \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$

calories	c_1	c_2	$\cdots\cdots$	c_n	healthy
vitamin 1	a_{11}	a_{12}	$\cdots\cdots$	a_{1n}	$\geq b_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
vitamin m	a_{m1}	a_{m2}	$\cdots\cdots$	a_{mn}	$\geq b_m$
solution:	x_1	x_2	$\cdots\cdots$	x_n	

minimize the calories while keeping healthy

Surviving Problem



					healthy
price	c_1	c_2	$\cdots\cdots$	c_n	
vitamin 1	a_{11}	a_{12}	$\cdots\cdots$	a_{1n}	$\geq b_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
vitamin m	a_{m1}	a_{m2}	$\cdots\cdots$	a_{mn}	$\geq b_m$
solution:	x_1	x_2	$\cdots\cdots$	x_n	

minimize the total price while keeping healthy

Surviving Problem

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

price	c_1	c_2	$\cdots\cdots$	c_n	healthy
vitamin 1	a_{11}	a_{12}	$\cdots\cdots$	a_{1n}	$\geq b_1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
vitamin m	a_{m1}	a_{m2}	$\cdots\cdots$	a_{mn}	$\geq b_m$
solution:	x_1	x_2	$\cdots\cdots$	x_n	

minimize the total price while keeping healthy

Dual LP

Primal:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

$$\text{s.t.} \quad \boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

dual solution:	price	c_1	c_2	$\cdots\cdots$	c_n	healthy
y_1	vitamin 1	a_{11}	a_{12}	$\cdots\cdots$	a_{1n}	b_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
y_m	vitamin m	a_{m1}	a_{m2}	$\cdots\cdots$	a_{mn}	b_m

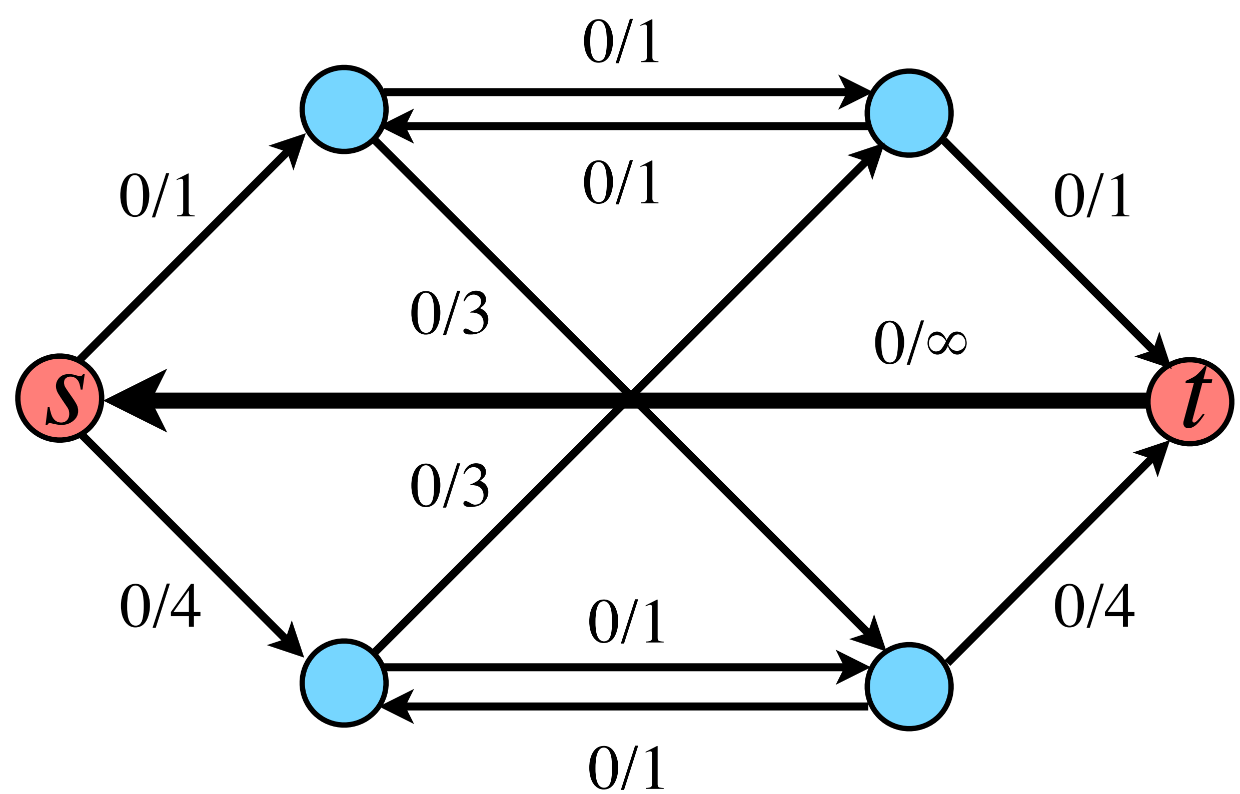
m types of vitamin pills, design a pricing system

competitive to n natural foods, max the total price

Max-Flow

digraph: $D(V,E)$ **source:** $s \in V$ **sink:** $t \in V$

capacity $c : E \to \mathbb{R}^+$



max f_{ts} $\sum_{u:(s,u)\in E} f_{su}$

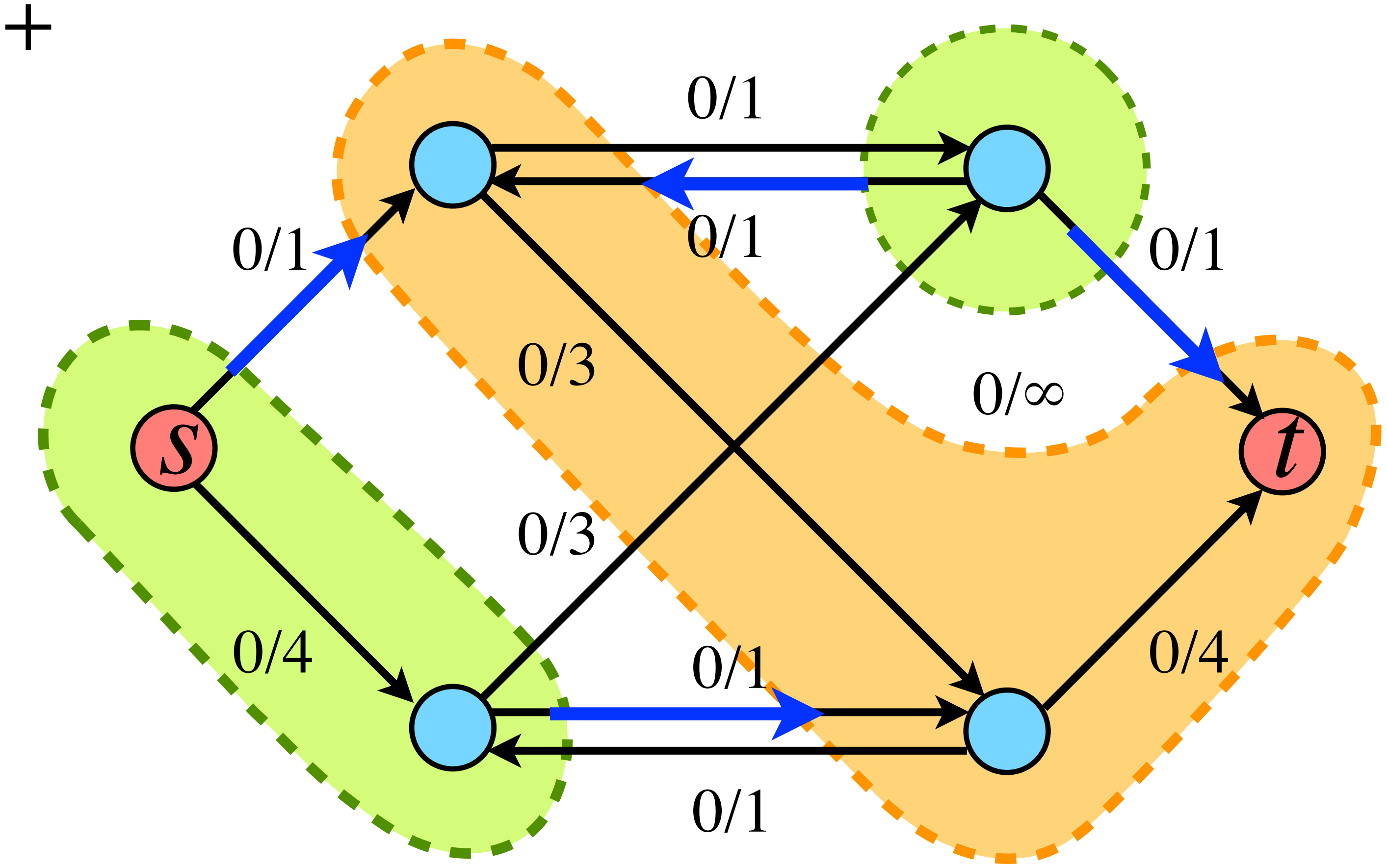
d_{uv} s.t. $0 \le f_{uv} \le c_{uv}$ $\forall (u,v) \in E$

p_u $\sum_{w:(w,u)\in E} f_{wu} - \sum_{v:(u,v)\in E} f_{uv} \le 0$ $\forall u \in V \setminus \{s,t\}$

Dual-LP

digraph: $D(V, E)$ **source:** $s \in V$ **sink:** $t \in V$

capacity $c : E \to \mathbb{R}^+$



min $$\sum_{(u,v)\in E} c_{uv} d_{uv}$$

s.t.
$$d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \in E$$
$$p_s - p_t \geq 1$$
$$d_{uv} \geq 0, p_u \geq 0 \qquad \forall (u,v) \in E \quad \forall u \in V$$
$$\boldsymbol{d_{uv}}, \boldsymbol{p_u} \in \{0,1\}$$

Flow-Cut Duality by Another LP

Primal:

$$\max \sum_{s\text{-}t \text{ path } p} x_p$$

$$\text{s.t.} \quad \sum_{p: e \in p} x_p \le c_e \qquad \forall e \in E$$

$$x_p \ge 0 \qquad \forall s\text{-}t \text{ path } p$$

Dual:

$$\min \sum_{e \in E} y_e c_e$$

$$\text{s.t.} \quad \sum_{e: e \in p} y_e \ge 1 \qquad \forall s\text{-}t \text{ path } p$$

$$y_e \ge 0 \qquad \forall e \in E$$

Duality

Primal:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

$\geq$

Dual:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

$$\text{s.t.} \quad \boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

dual of dual is primal

∀ feasible $\boldsymbol{x}$ and $\boldsymbol{y}$

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{x} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

Duality

Primal:

$$\min \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

$$\text{s.t.} \quad A\boldsymbol{x} \geq \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

Dual:

$$\max \quad \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

$$\text{s.t.} \quad \boldsymbol{y}^{\mathrm{T}}A \leq \boldsymbol{c}^{\mathrm{T}}$$

$$\boldsymbol{y} \geq \boldsymbol{0}$$

Weak Duality Theorem

Strong Duality Theorem

$$\mathrm{OPT}_{\mathrm{primal}} \geqq \mathrm{OPT}_{\mathrm{dual}}$$

Maximum Integral Flow

digraph: $D(V,E)$ **source:** $s \in V$ **sink:** $t \in V$

capacity $c : E \to \mathbb{Z}^+$

$$\max \sum_{u:(s,u)\in E} f_{su}$$

s.t. $$0 \le f_{uv} \le c_{uv} \qquad \forall (u,v) \in E$$

$$\sum_{w:(w,u)\in E} f_{wu} - \sum_{v:(u,v)\in E} f_{uv} = 0 \qquad \forall u \in V \setminus \{s,t\}$$

integral flow: $$f_{uv} \in \mathbb{Z} \qquad \forall (u,v) \in E$$

Integer Programming

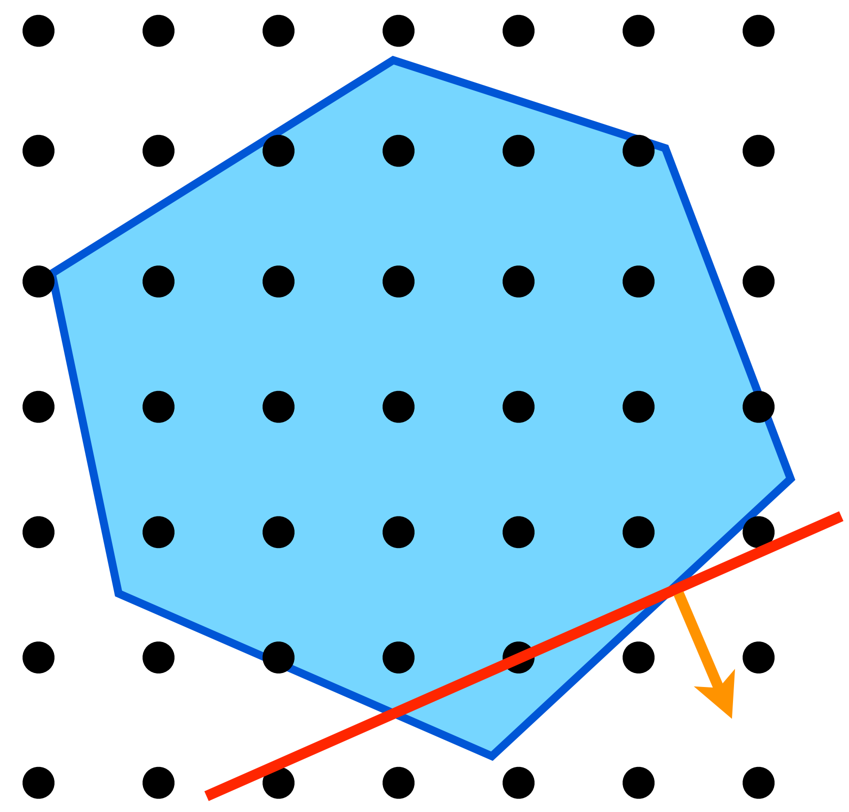
canonical form:

$m \times n$ matrix A

min $\quad \boldsymbol{c}^T \boldsymbol{x}$

s.t. $\quad A\boldsymbol{x} \geq \boldsymbol{b}$

~~$\boldsymbol{x} \in \mathbb{Z}^n$~~



LP-relaxation

Integrality gap

3-SAT: $$\bigwedge_{i=1}^{m} (l_{i_1} \vee l_{i_2} \vee l_{i_1})$$

literal $l_{i_j} \in \{x_{i_j}, \neg x_{i_j}\}$

Boolean variable $x_i \in \{\texttt{true}, \texttt{false}\}$

max $$\sum_{i=1}^{m} z_i$$

s.t.
$$z_i \leq y_{i_1} + y_{i_2} + y_{i_3} \qquad \forall\, 1 \leq i \leq m$$
$$y_{i_j} = x_{i_j} \qquad \text{if } l_{i_j} = x_{i_j}$$
$$y_{i_j} = 1 - x_{i_j} \qquad \text{if } l_{i_j} = \neg x_{i_j}$$
$$x_j, z_i \in \{0, 1\} \qquad \forall\, 1 \leq i \leq m, \quad 1 \leq j \leq n$$

ILP (Integer Linear Program) is **NP**-hard

Integral Polytope

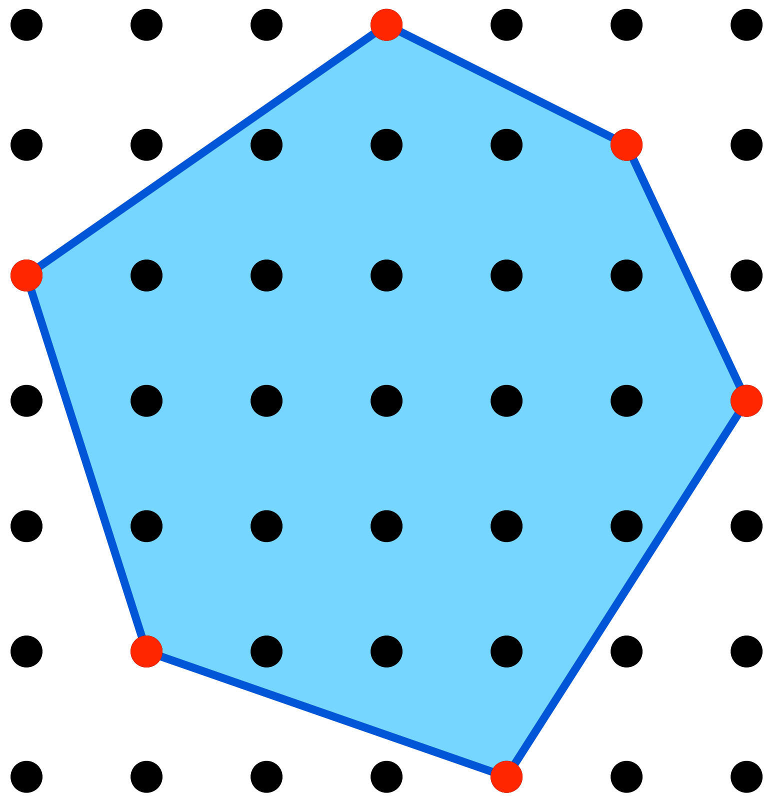
Integral polyhedron:

all vertices are integral

OPT for **ILP** =

OPT for LP-relaxation

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^n \end{aligned}$$



How to tell whether $A\boldsymbol{x} \geq \boldsymbol{b}$ is an integral polyhedron?

Unimodularity

$m \times m$ integer matrix B

B is unimodular if $\det(B) = \pm 1$.

$m \times n$ integer matrix A

A is totally unimodular if

$\forall$ square submatrix B, $\det(B) \in \{0, 1, -1\}$.

> A is totally unimodular $\Rightarrow$ $\forall \boldsymbol{b} \in \mathbb{Z}^m$ polyhedron $A\boldsymbol{x}=\boldsymbol{b}$ is integral

$\forall$ basis B: a set of m linearly independent columns of A

basic solution $\boldsymbol{x} = B^{-1}\boldsymbol{b} = B^{*}\boldsymbol{b}/\det(B)$ integral

Total Unimodularity

A is totally unimodular if
$\forall$ square submatrix B, $\det(B) \in \{0, 1, -1\}$.

Theorem (Hoffman-Kruskal, 1956)

A is totally unimodular ⇒ polyhedron $A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ is integral for $\forall \boldsymbol{b} \in \mathbb{Z}^m$

A is totally unimodular ⇒ so is $[\ A \quad I\ ]$

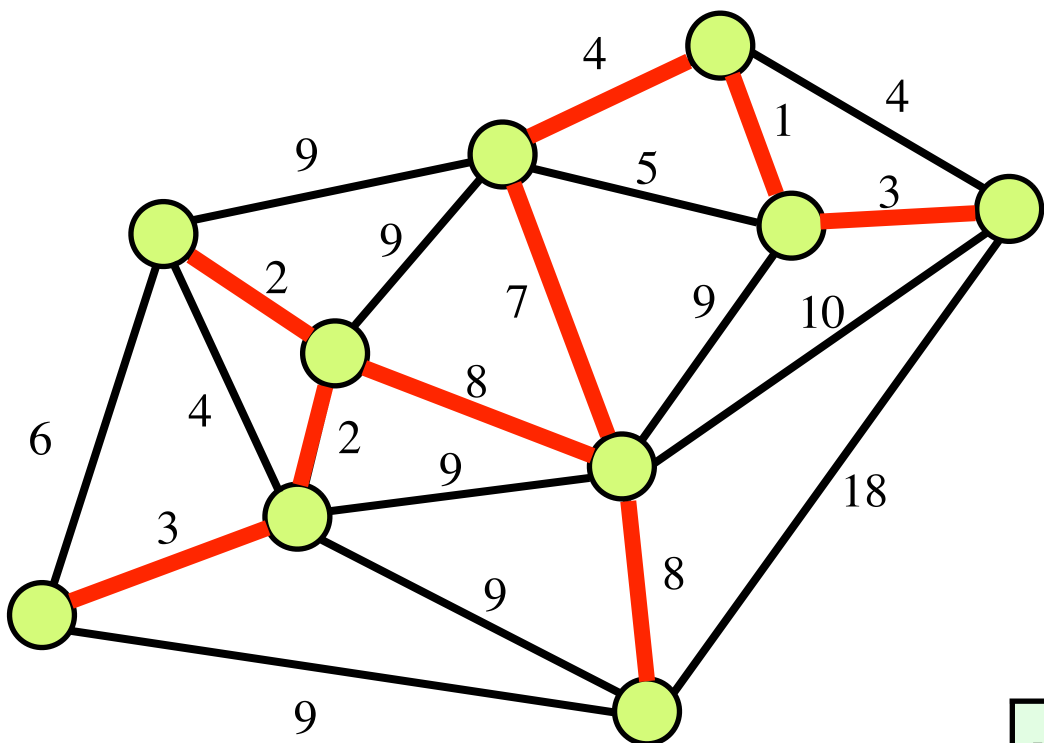
canonical to standard by adding slack variables

Totally Unimodular LP

- Max-Flow
- Maximum Bipartite Matching
- Shortest Paths

Matroid

Minimum Spanning Tree (MST)



Find the

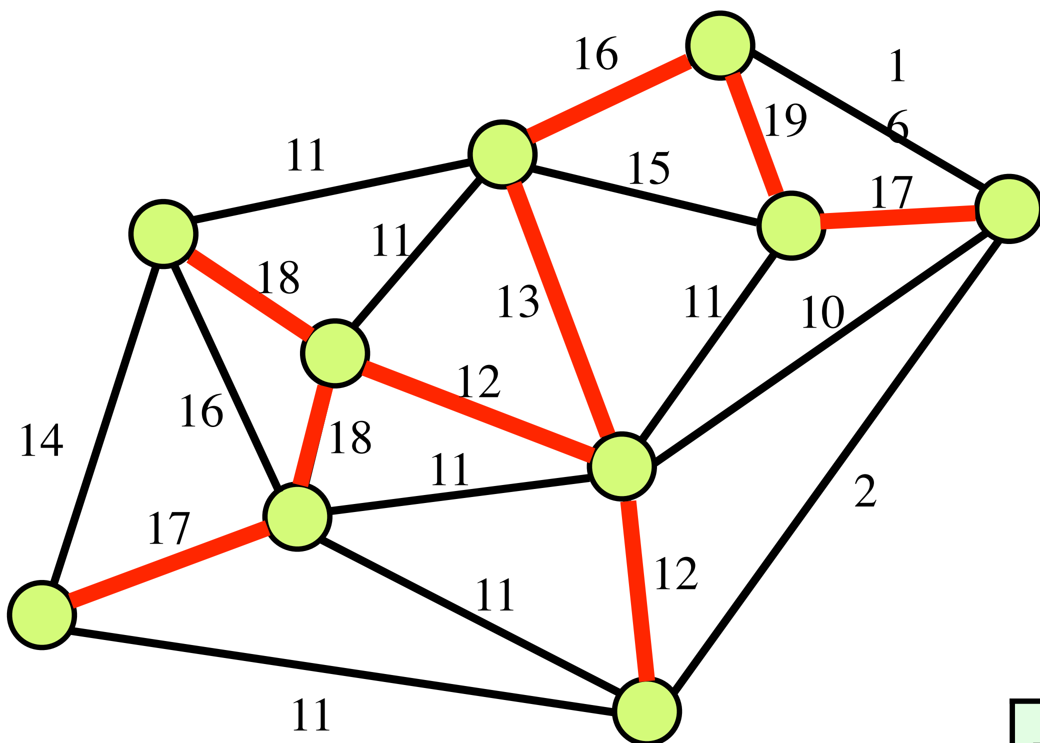
minimum spanning tree

undirected graph $G(V,E)$

weight $c : E \to \mathbb{R}^+$

Kruskal's Algorithm: Greedy!
$S = \emptyset$;   while $\exists e \in E$ that $S \cup \{e\}$ is a forest: pick such e with min c_e; $S = S \cup \{e\}$;

Maximum Weight Spanning Tree



Find the

maximum weight

spanning tree

undirected graph

$G(V,E)$

weight $c : E \to \mathbb{R}^+$

Kruskal's Algorithm:
$S = \varnothing$;
while $\exists e \in E$ that $S \cup \{e\}$ is a forest:
pick such e with max c_e;
$S = S \cup \{e\}$;

Matroid

set system $\mathcal{F} \subseteq 2^X$

each $S \in \mathcal{F}$ is called an independent set

hereditary: $S \in \mathcal{F}, T \subset S \Rightarrow T \in \mathcal{F}$

matroid property:

$$\forall Y \subseteq X, \quad \left.\begin{array}{r} S, T \in \mathcal{F} \\ S, T \subseteq Y \\ S, T \text{ maximal} \end{array}\right\} \Rightarrow |S| = |T|$$

$\forall Y \subseteq X,$ basis: maximal $S \in \mathcal{F}, \ S \subseteq Y$

rank: $r(Y) = |S|$ $\quad S$ is a basis of Y

Graph Matroid

undirected graph $G(V,E)$

set system $\mathcal{F} \subseteq 2^E$

$$\mathcal{F} = \{ \text{ all forests in } G \}$$

hereditary: subgraphs of a forest are forests

matroid property:

$\forall$ subgraph of G with k components

all spanning forests of G have the same size

$|V| - k$

Linear Matroid

$m \times n$ matrix A

set system $\mathcal{F} \subseteq 2^{[n]}$

$\mathcal{F} = \{S \mid \text{columns } A_{\cdot j}, j \in S \text{ are linearly independent}\}$

hereditary: subsets of a linearly independent set are linearly independent

matroid property:

$\forall$ subset of columns of A sub-matrix B

all basis of B have the same size

Greedy Algorithm

matroid $\mathcal{F} \subseteq 2^X$

weight $c : X \to \mathbb{R}^+$

find the $S \in \mathcal{F}$ with the

maximum weight $c(S) = \sum_{i \in S} c_i$

Greedy Algorithm:
$S = \emptyset$;
while $\exists i \in X$ that $S \cup \{i\} \in \mathcal{F}$:
pick such i with max c_i;
$S = S \cup \{i\}$;

Greedy Algorithm

matroid $\mathcal{F} \subseteq 2^X$

weight $c : X \to \mathbb{R}^+$

find the $S \in \mathcal{F}$ with the

maximum weight $c(S) = \sum_{i \in S} c_i$

Theorem (Rado 1957; Edmonds 1970)

The Greedy Algorithm finds the maximum weight independent set in a matroid.

Proof: Same as Kruskal's Algorithm.