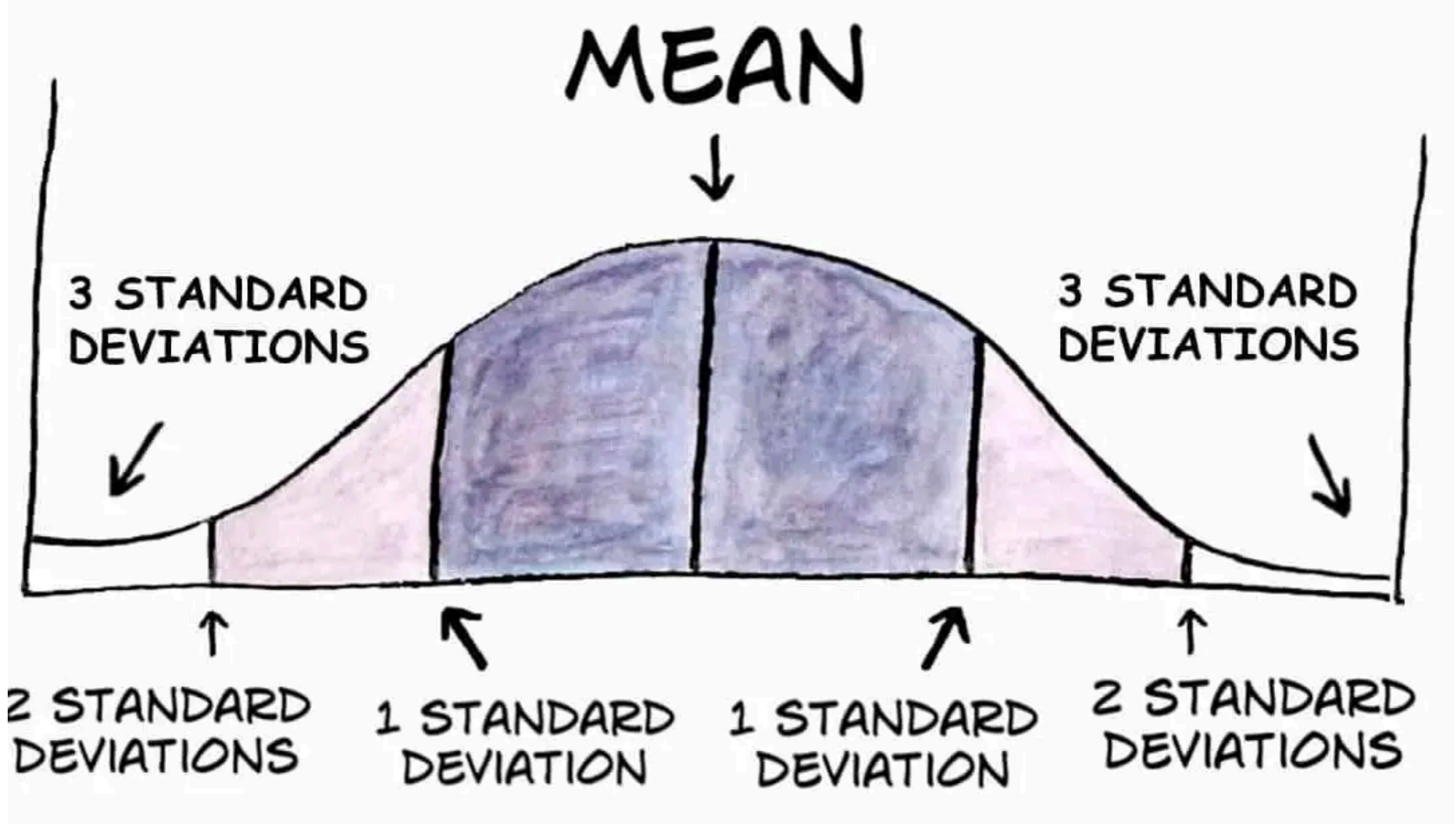
# Probability Theory & Mathematical Statistics

## Moment and Deviation

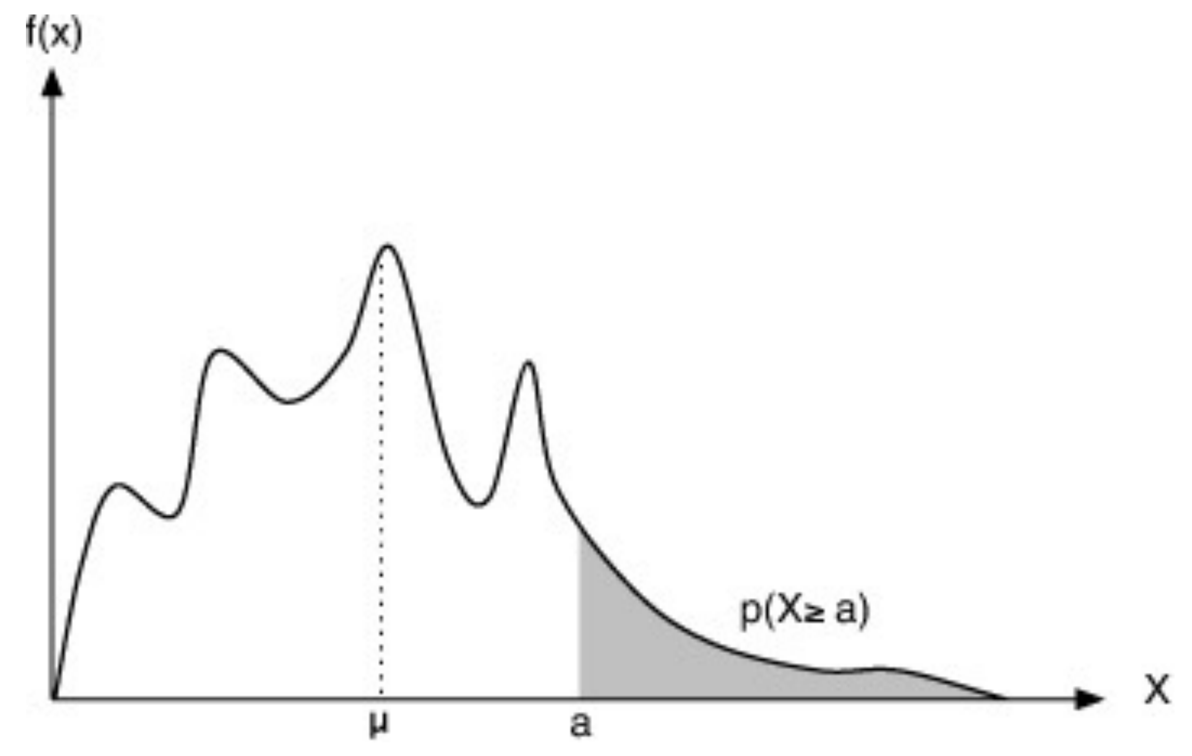
尹一通 Nanjing University, 2023 Spring

# Moments and Deviations



# Markov's Inequality

**(马尔可夫不等式)**



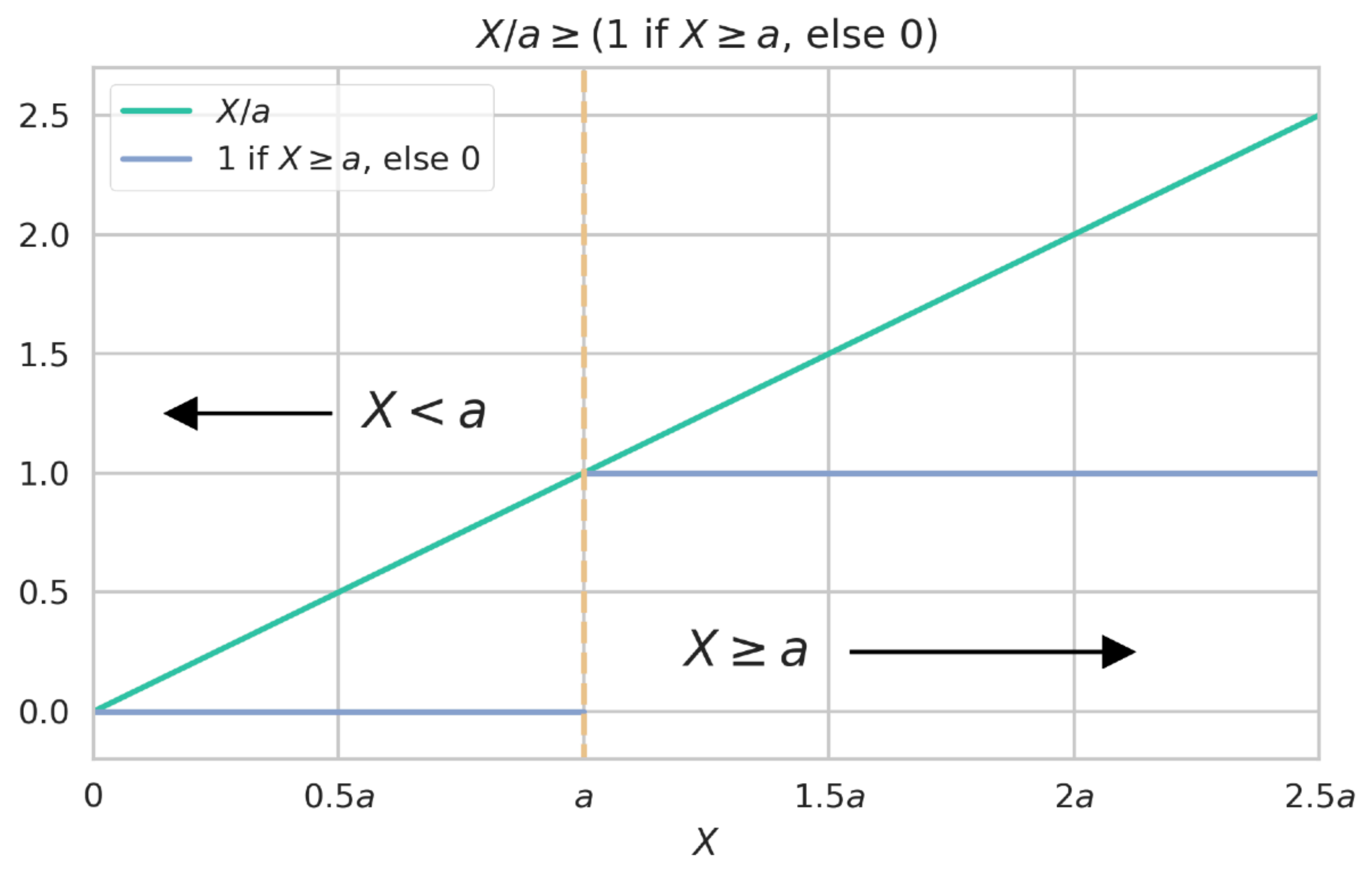

- <u>Markov's inequality</u>: Let $X$ be a *nonnegative-valued* random variable. Then,

$$\text{for any } a > 0, \qquad \Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

- **Proof** (by indicator): Let $I = I(X \geq a)$. Since $X \geq 0$ and $a > 0$, we have

$$I = I(X \geq a) \leq \left\lfloor \frac{X}{a} \right\rfloor \leq \frac{X}{a}.$$

Therefore, $\Pr(X \geq a) = \mathbb{E}[I] \leq \mathbb{E}\left[\frac{X}{a}\right] = \frac{\mathbb{E}[X]}{a}$



# Markov's Inequality

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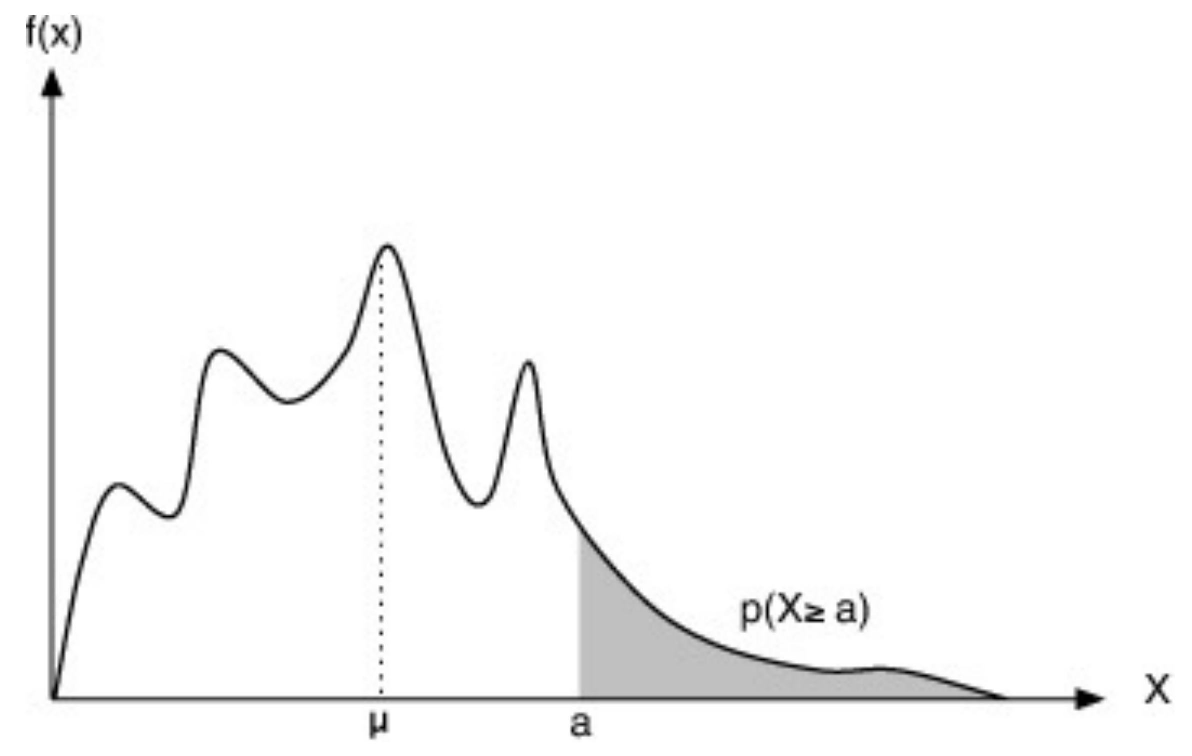


- <u>Markov's inequality</u>: Let $X$ be a *nonnegative-valued* random variable. Then,

$$\text{for any } a > 0, \qquad \Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

- **Proof** (by total expectation):

($X \geq a$ is possible) ($X$ is nonnegative)

$$\mathbb{E}[X] = \mathbb{E}[X \mid X \geq a] \cdot \Pr(X \geq a) + \mathbb{E}[X \mid X < a] \cdot \Pr(X < a)$$

$$\geq a \cdot \Pr(X \geq a) + 0 \cdot \Pr(X < a) \qquad = a \cdot \Pr(X \geq a)$$

$$\implies \Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

# Markov’s Inequality

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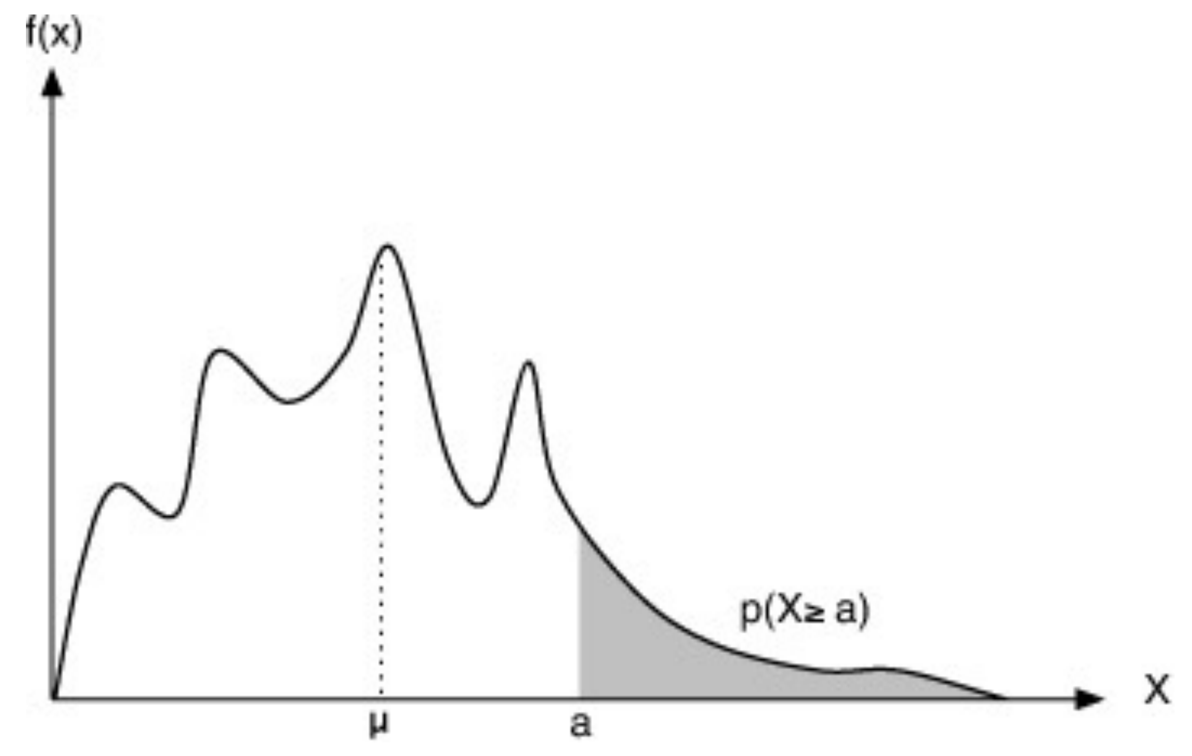


- <u>Markov’s inequality</u>: Let $X$ be a *nonnegative-valued* random variable. Then,

$$\text{for any } a > 0, \qquad \Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

- **Corollary**: for any $c > 1, \qquad \Pr(X \geq c\mathbb{E}[X]) \leq 1/c$

- **Tight in the worst case:** $\forall c > 1, \forall \mu \in \mathbb{R}$, $\exists$ nonnegative $X$ with $\mathbb{E}[X] = \mu$, such that $\Pr(X \geq c\mu) = 1/c$

- **Lower tail variant** (sometimes called <u>*reverse Markov’s inequality*</u>):

  $\Pr(X \leq a) \leq (u - \mathbb{E}[X])/(u - a)$ requires $X$ to have bounded range $X \leq u$

# From Las Vegas to Monte Carlo

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**Monte Carlo algorithm**: randomized algorithms that are correct by chance
- 

**Las Vegas algorithm**: randomized algorithms that always give correct result upon termination (but may run for a random period of time before termination)

- If there is a Las Vegas algorithm $\mathscr{A}$ with expected running time at most $t(n)$ for any input of size $n$ ($\mathscr{A}$ has worst-case expected time complexity $t(n)$):

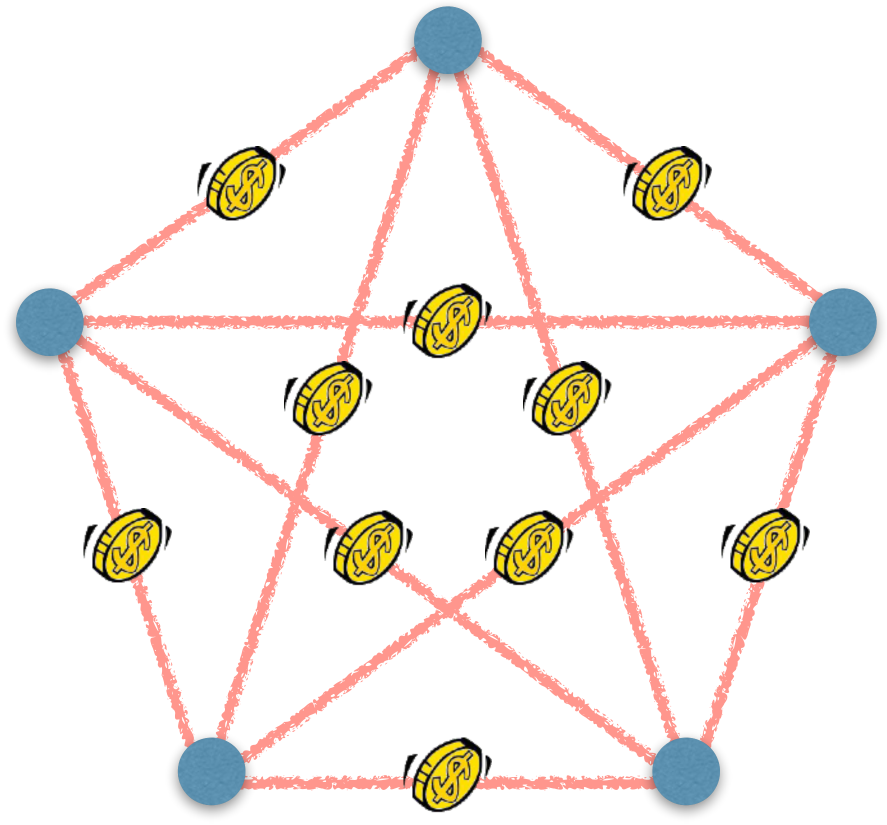
**Algorithm $\mathscr{B}$:**

simulate algorithm $\mathscr{A}$ up to $\lceil t(n)/\epsilon \rceil$ steps;
if algorithm $\mathscr{A}$ terminates
  return the output of $\mathscr{A}$;
else return an arbitrary answer;

- Algorithm $\mathscr{B}$ is a Monte Carlo algorithm s.t.
  - $\mathscr{B}$ has worst-case running time $\leq \lceil t(n)/\epsilon \rceil$
  - $\mathscr{B}$ is correct with probability at least $1 - \epsilon$
    (by Markov inequality)

# Cliques in Random Graph

- $G(n, p)$: between every pair $u, v$ among $n$ vertices, an edge is added i.i.d. with prob. $p$
- Fix a constant integer $k \geq 3$. Let $X$ be the number of $k$-**cliques** ($K_k$) in $G \sim G(n, p)$.
- For every distinct $S \subseteq \in [n]$ of size $|S| = k$, let $I_S = I(K_S \subseteq G)$. Then:
  - $\mathbb{E}[I_S] = \Pr(K_S \subseteq G) = p^{\binom{k}{2}}$
  - $X = \sum_{S \in \binom{[n]}{k}} I_S$



- **Linearity of expectation:** $\mathbb{E}[X] = \binom{n}{k} p^{\binom{k}{2}} \leq n^k p^{k(k-1)/2} = o(1)$ for $p = o\left(n^{-2/(k-1)}\right)$
- **Markov's inequality**: $\Pr(X \geq 1) \leq \mathbb{E}[X] = o(1) \Longrightarrow \Pr(X = 0) = 1 - o(1)$

  $\Longrightarrow$ If $p = o\left(n^{-2/(k-1)}\right)$, then $G(n, p)$ is $K_k$-free **a.a.s.** (asymptotically almost surely)

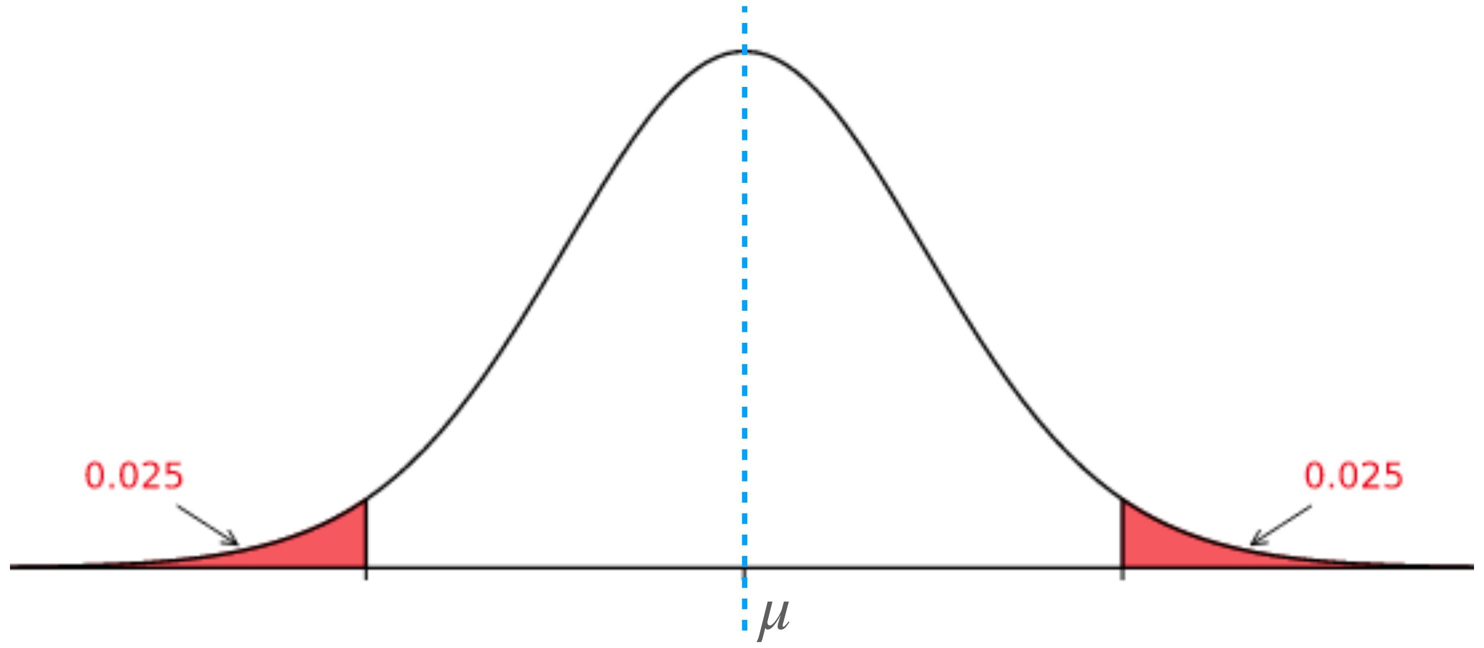
# Generalized Markov's Inequality

- Let $X$ be a random variable and $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ a nonnegative-valued function.

$$\text{For any } a > 0, \qquad \Pr(f(X) \geq a) \leq \frac{\mathbb{E}[f(X)]}{a}$$

- **Proof**: Apply the Markov's inequality to the random variable $Y = f(X)$.
- **Applications**: useful if $f(X)$ can "*extract*" useful information about $X$
    - **Chebyshev's inequality, $k$th moment method:** $f(X)$ extracts the $k$th moment
    - **Chernoff-Hoeffding bounds, Bernstein inequalities:** $f(X)$ extracts all moments

# Deviation Inequality




- Let $X$ be a random variable with **mean** $\mu = \mathbb{E}[X]$. For $a > 0$

$$\Pr(|X - \mu| \geq a) \leq ?$$

- Applying **Markov's inequality** to $Y = |X - \mu|$ gives us

$$\Pr(|X - \mu| \geq a) \leq \frac{\mathbb{E}[|X - \mu|]}{a}$$

difficult to calculate

- Alternatively, we may apply **Markov's inequality** to $Y = (X - \mu)^2$

$$\Pr(|X - \mu| \geq a) = \Pr((X - \mu)^2 \geq a^2) \leq \frac{\mathbb{E}[(X - \mu)^2]}{a^2}$$

**Variance**

(2nd central moment)

# Variance (方差) and Moments (矩)

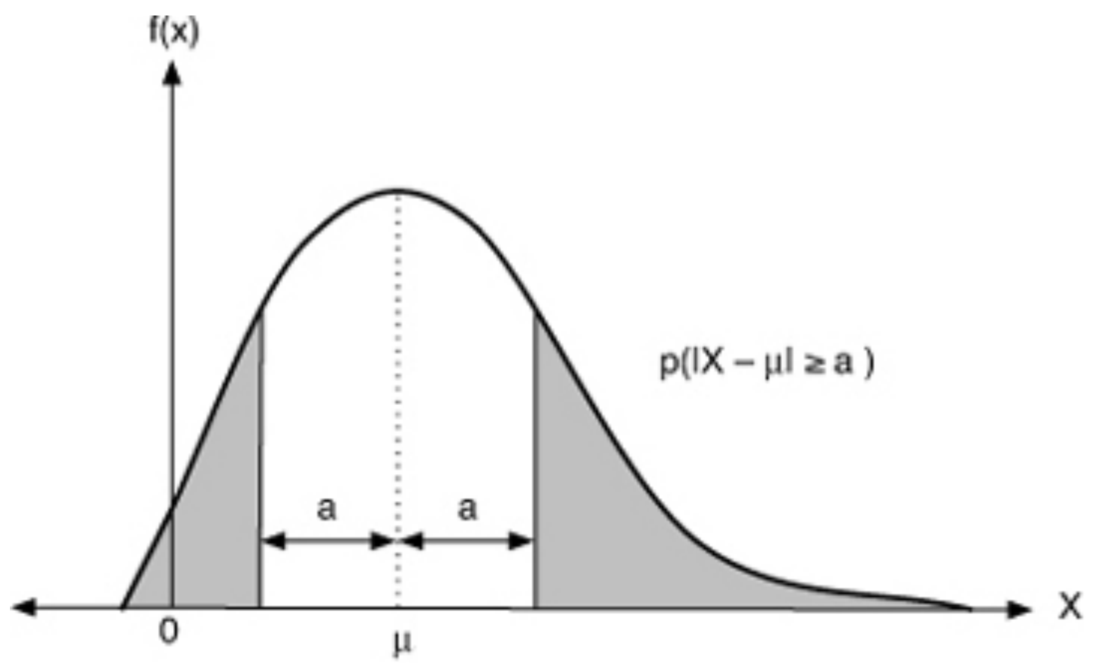
- For integer $k > 0$, the $k$th moment ($k$阶矩) of a random variable $X$ is $\mathbb{E}[X^k]$, and the $k$th central moment ($k$阶中心矩) of $X$ is $\mathbb{E}[(X - \mathbb{E}[X])^k]$.
- Sometimes, a random variable $X$ is called **centralized** (中心化的) if $\mathbb{E}[X] = 0$. A random variable $X$ can be centralized by $Y = X - \mathbb{E}[X]$.
- The variance (方差) of a random variable $X$ is its 2nd central moment:

$$\mathbf{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$

and the standard deviation (标准差) of $X$ is $\sigma = \sigma[X] = \sqrt{\mathbf{Var}[X]}$

# Chebyshev's Inequality

**(切比雪夫不等式)**




- <u>Chebyshev's inequality</u>: Let $X$ be a random variable. For any $a > 0$,

$$\Pr(|X - \mathbb{E}[X]| \geq a) \leq \frac{\mathbf{Var}[X]}{a^2}$$

- **Proof**: Apply Markov's inequality to $Y = (X - \mathbb{E}[X])^2$.

- **Corollary**: For standard deviation $\sigma = \sqrt{\mathbf{Var}[X]}$, for any $k \geq 1$,

$$\Pr(|X - \mathbb{E}[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

# Median and Mean

- The <u>median</u> (中位数) of random variable $X$ is defined to be any value $m$ s.t.:

$$\Pr(X \leq m) \geq 1/2 \quad \text{and} \quad \Pr(X \geq m) \geq 1/2$$

- The expectation $\mu = \mathbb{E}[X]$ is the value that minimizes

$$\mathbb{E}[(X-\mu)^2]$$

- **Proof**: $f(x) = \mathbb{E}[(X-x)^2] = \mathbb{E}[X^2] - 2x\mathbb{E}[X] + x^2$ is convex and has $f'(\mu) = 0$

- The median $m$ is the value that minimizes

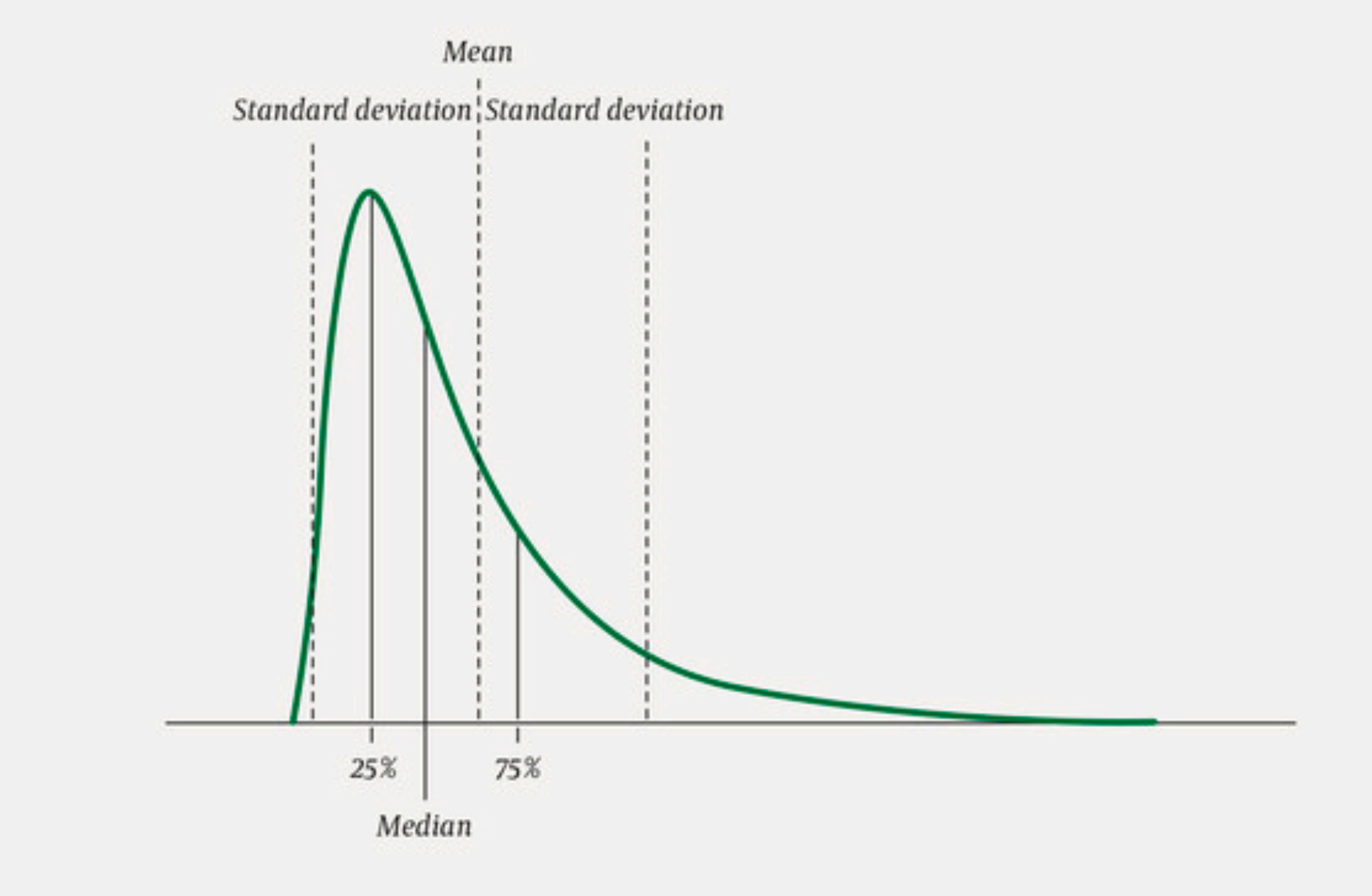
$$\mathbb{E}[\,|X-m|\,]$$

- **Proof**: By symmetry, suppose non-median $y > m$ so that $\Pr(X \geq y) < 1/2$.

$$\mathbb{E}[\,|X-y| - |X-m|\,] = (m-y)\Pr(X \geq y) + \sum_{m<x<y} (m+y-2x)\Pr(X=x) + (y-m)\Pr(X \leq m)$$

$$> (m-y)/2 + (y-m)/2 = 0$$

# Median and Mean




- If $X$ is a random variable with finite expectation $\mu$, median $m$, and standard deviation $\sigma$, then

$$|\mu - m| \leq \sigma$$

- **Proof**: $|\mu - m| = |\mathbb{E}[X] - m| = |\mathbb{E}[X - m]|$

$$\leq \mathbb{E}[|X - m|] \quad \text{(Jensen's inequality / triangle inequality)}$$

$$\leq \mathbb{E}[|X - \mu|] \quad \text{(the median } m \text{ minimizes } \mathbb{E}[|X - m|])$$

$$= \mathbb{E}\left[\sqrt{(X-\mu)^2}\right] \leq \sqrt{\mathbb{E}\left[(X-\mu)^2\right]} = \sigma \quad \text{(Jensen's inequality)}$$

# Variance

# Calculation of Variance

$$\mathbf{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

- **Proof**: $\mathbf{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$

$$= \mathbb{E}\left[X^2 - 2\mathbb{E}[X]X + \mathbb{E}[X]^2\right]$$

$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

- $X$ is constant ***a.s.*** $(\Pr(X = \mathbb{E}[X]) = 1) \iff \mathbb{E}[X^2] = \mathbb{E}[X]^2 \iff \mathbf{Var}[X] = 0$

# Variance of Linear Function

- For random variables $X, Y$ and real number $a \in \mathbb{R}$:
  - $\mathbf{Var}[a] = 0$
  - $\mathbf{Var}[X + a] = \mathbf{Var}[X]$ (variance is a central moment)
  - $\mathbf{Var}[aX] = a^2\mathbf{Var}[X]$ (variance is quadratic)
  - $\mathbf{Var}[X + Y] = \mathbf{Var}[X] + \mathbf{Var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y])$
- **Proof**: All can be verified through $\mathbf{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

# Covariance (协方差)

- The covariance (协方差) of two random variables $X$ and $Y$ is

$$\mathbf{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

- **Properties**: $\mathbf{Var}[X] = \mathbf{Cov}(X, X)$
  - *Symmetric*: $\mathbf{Cov}(X, Y) = \mathbf{Cov}(Y, X)$
  - *Distributive*: $\mathbf{Cov}(X + Y, Z) = \mathbf{Cov}(X, Z) + \mathbf{Cov}(Y, Z)$

$$\mathbf{Cov}(aX, Y) = a\mathbf{Cov}(X, Y)$$

- If $X$ and $Y$ are independent then

$$\mathbf{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$$

# Covariance of Independent Variables

- If random variables $X$ and $Y$ are **independent**, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

- If random variables $X_1, X_2, \ldots, X_n$ are **mutually independent**, then

$$\mathbb{E}\left[\prod_{i=1}^{n} X_i\right] = \mathbb{E}\left[\prod_{i=1}^{n-1} X_i\right] \cdot \mathbb{E}[X_n] = \prod_{i=1}^{n} \mathbb{E}[X_i]$$

**Proof**: By change of variable

$$\mathbb{E}[XY] = \sum_{x,y} xy \Pr(X = x \cap Y = y) = \sum_{x,y} xy \Pr(X = x) \Pr(Y = y)$$

$$= \left(\sum_{x} x \Pr(X = x)\right)\left(\sum_{y} y \Pr(Y = y)\right) = \mathbb{E}[X]\mathbb{E}[Y]$$

# Expectation of Product

- For random variables $X$ and $Y$:

if $X$ and $Y$ independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

- (**Cauchy-Schwarz**)

$$\mathbb{E}[XY]^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2]$$

- (**Hölder**) for any $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$

$$\mathbb{E}[XY] \leq \mathbb{E}[\,|X|^p\,]^{1/p}\mathbb{E}[\,|Y|^q\,]^{1/q}$$

# Correlation (相关性)

- The covariance (协方差) of two random variables $X$ and $Y$ is

$$\mathbf{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

- The correlation coefficient (相关性系数) of $X$ and $Y$ is

$$\rho(X, Y) = \frac{\mathbf{Cov}(X, Y)}{\sqrt{\mathbf{Var}[X] \cdot \mathbf{Var}[Y]}}$$

- Two random variables $X$ and $Y$ are called uncorrelated if $\mathbf{Cov}(X, Y) = 0$

- $X$ and $Y$ are **uncorrelated** means:
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
  - $\mathbf{Var}[X + Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$

# Variance of Sum

- For random variables $X, Y$:

$$\mathbf{Var}[X+Y] = \mathbf{Var}[X] + \mathbf{Var}[Y] + 2\mathbf{Cov}(X, Y)$$

- For random variables $X_1, X_2, \ldots, X_n$:

$$\mathbf{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{Var}[X_i] + \sum_{i \neq j} \mathbf{Cov}(X_i, X_j)$$

- For pairwise independent $X_1, X_2, \ldots, X_n$:

$$\mathbf{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{Var}[X_i]$$

# Variance of Indicator

$p$ $\quad$ $1 - p$

- For **Bernoulli random variable** $X \in \{0,1\}$ with parameter $p$

$$X^2 = X \implies \mathbb{E}[X^2] = \mathbb{E}[X] = p$$

$$\mathbf{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = p(1-p)$$

- For the **indicator** random variable $X = I(A)$ of event $A$:

$$\mathbf{Var}[X] = \Pr(A)(1 - \Pr(A)) = \Pr(A)\Pr(A^c)$$

# Variance of Discrete Uniform Distribution

- For integers $a \leq b$, let $X$ be chosen from $[a, b] = \{a, a+1, \ldots, b\}$ **u.a.r.**
- $$\mathbb{E}[X] = \sum_{k=a}^{b} \frac{k}{b-a+1} = \frac{a+b}{2}$$
- $$\mathbb{E}[X^2] = \sum_{k=a}^{b} \frac{k^2}{b-a+1} = \frac{2b^2 + 2ab + 2a^2 + b - a}{6}$$
- $$\mathbf{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{(a-b)(a-b-2)}{12}$$

# Poisson Distribution

- For **Poisson random variable** $X \sim \text{Pois}(\lambda)$, recall $\mathbb{E}[X] = \lambda$, and

$$\mathbb{E}[X^2] = \sum_{k\geq 0} k^2 \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k\geq 1} k \frac{e^{-\lambda}\lambda^k}{(k-1)!}$$

$$= \sum_{k\geq 0} (k+1) \frac{e^{-\lambda}\lambda^{k+1}}{k!} = \lambda \sum_{k\geq 0} (k+1) \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \lambda\mathbb{E}[X+1] = \lambda(\mathbb{E}[X]+1) = \lambda(\lambda+1)$$

$$\textbf{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$$

# Geometric Distribution (几何分布)

- For geometric random variable $X \sim \text{Geo}(p)$, recall $\mathbb{E}[X] = 1/p$, and

$$\mathbb{E}[X^2] = \sum_{k \geq 1} k^2 (1-p)^{k-1} p = (2-p)p^{-2}$$

$$\mathbf{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = (2-p)p^{-2} - p^{-2} = (1-p)/p^2$$

- Total expectation: $\mathbb{E}[X^2] = \mathbb{E}[X^2 \mid X > 1] \cdot (1-p) + \mathbb{E}[X^2 \mid X = 1] \cdot p$

$$= \mathbb{E}[((X-1)+1)^2 \mid X > 1] \cdot (1-p) + p$$

(memoryless) $= \mathbb{E}[(X+1)^2] \cdot (1-p) + p$

$$= (1-p)\mathbb{E}[X^2] + 2(1-p)/p + 1$$

$$\implies \mathbb{E}[X^2] = (2-p)/p^2 \implies \mathbf{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = (1-p)/p^2$$

# Binomial Distribution (二项分布)

- For **binomial random variable** $X \sim \text{Bin}(n, p)$, recall $\mathbb{E}[X] = np$, and

$$\mathbf{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} - (np)^2$$

- **Observation**: $X \sim \text{Bin}(n, p)$ can be expressed as $X = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are i.i.d. Bernoulli random variables with parameter $p$

- For **mutually independent** $X_1, \ldots, X_n$:

$$\mathbf{Var}\left[X\right] = \sum_{i=1}^{n} \mathbf{Var}[X_i] = np(1-p)$$

# Negative Binomial Distribution (负二项分布)

- For negative binomial random variable $X$ with parameters $r, p$

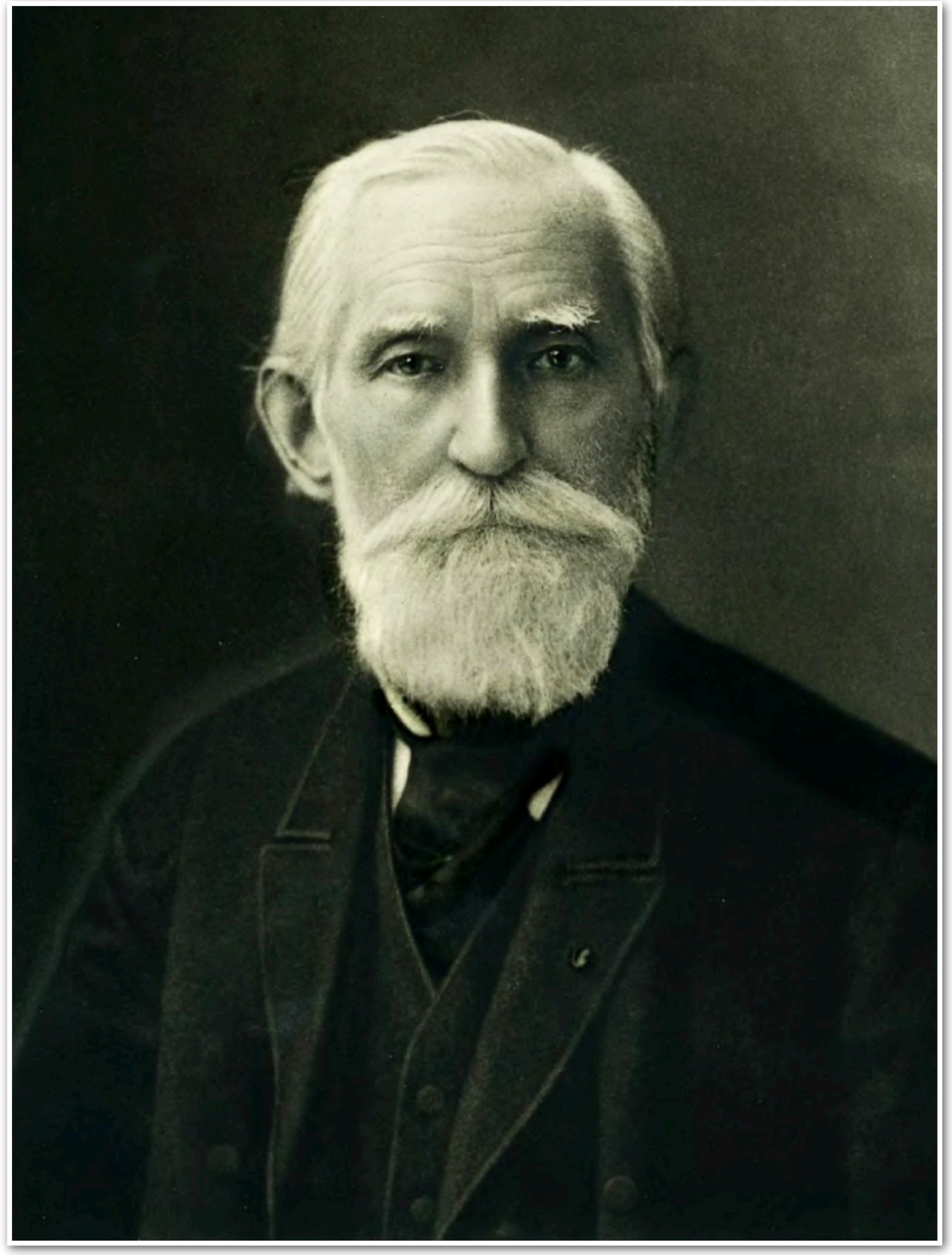
$$\mathbf{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sum_{k \geq 1} k^2 \binom{k+r-1}{k} (1-p)^k p^r - r^2(1-p)^2/p^2$$

- **Observation**: $X$ can be expressed as $X = (X_1 - 1) + \cdots + (X_r - 1)$, where $X_1, \ldots, X_r$ are i.i.d. geometric random variables with parameter $p$

- For mutually independent $X_1, \ldots, X_r$:

$$\mathbf{Var}[X] = \sum_{i=1}^{r} \mathbf{Var}[X_i - 1] = \sum_{i=1}^{r} \mathbf{Var}[X_i] = \frac{r(1-p)}{p^2}$$

# Chebyshev (Чебышёв)’s Inequality



# Chebyshev's Inequality

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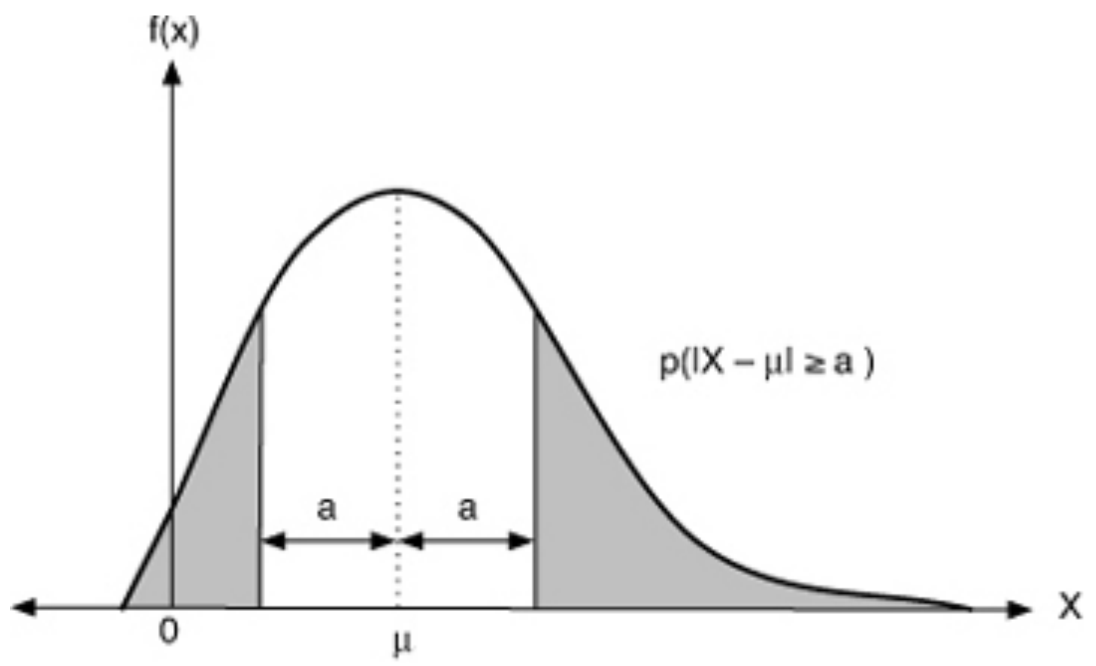


- <u>Chebyshev's inequality</u>: Let $X$ be a random variable. For any $a > 0$,

$$\Pr(|X - \mathbb{E}[X]| \geq a) \leq \frac{\mathbf{Var}[X]}{a^2}$$

- **Corollary**: For standard deviation $\sigma = \sqrt{\mathbf{Var}[X]}$, for any $k \geq 1$,

$$\Pr(|X - \mathbb{E}[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

- **Tight in the worst case:** $\forall k \geq 1$, $\forall \mu \in \mathbb{R}$ and $\forall \sigma > 0$, $\exists X$ with $\mathbb{E}[X] = \mu$ and $\mathbf{Var}[X] = \sigma^2$ such that $\Pr(|X - \mu| \geq k\sigma) = 1/k^2$

# Unbiased Estimator

- Let $X_1, \ldots, X_n$ be *i.i.d.* random variables with $\mathbb{E}[X_i] = \mu$ and $\mathbf{Var}[X_i] = \sigma^2$.
- Empirical mean: $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$\mathbb{E}[\overline{X}] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \mu \text{ and } \mathbf{Var}[\overline{X}] = \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{Var}[X_i] = \frac{\sigma^2}{n}$$

- **Chebyshev's inequality:**

$$\Pr(|\overline{X} - \mu| \geq \epsilon\mu) \leq \frac{\mathbf{Var}[\overline{X}]}{\epsilon^2\mu^2} = \frac{\sigma^2}{\epsilon^2\mu^2 n} \leq \delta \text{ if } n \geq \frac{\sigma^2}{\epsilon^2\mu^2\delta}$$

# (one-sided) Error Reduction

- Decision problem $f : \{0,1\}^* \to \{0,1\}$.
- Monte Carlo randomized algorithm $\mathscr{A}$ with ***one-sided*** error:

  for any input $x$ and uniform ***random seed*** $r \in [p]$ for some prime number $p$
  - $f(x) = 1 \implies \Pr\left(\mathscr{A}(x, r) = 1\right) \geq \epsilon$
  - $f(x) = 0 \implies \mathscr{A}(x, r) = 0$ for all $r \in [p]$
- $\mathscr{A}^k(x, r_1, \ldots, r_k) = \vee_{i=1}^{k} \mathscr{A}(x, r_i)$: for mutually independent $r_1, \ldots, r_k \in [p]$
  - $f(x) = 1 \implies \Pr\left(\mathscr{A}^k(x, r_1, \ldots, r_k) = 0\right) \leq (1 - \epsilon)^k$

# Two-Point Sampling (2-Universal Hashing)

- Let $p > 1$ be a prime number and $[p] = \{0,1,\ldots,p-1\} = \mathbb{Z}_p$.
- Pick $\boldsymbol{a}, \boldsymbol{b} \in [p]$ *u.a.r.* and let $r_i = (\boldsymbol{a} \cdot i + \boldsymbol{b}) \bmod p$ for $i = 1,2,\ldots,p$
  - $r_1, \ldots, r_p \in [p]$ are <u>pairwise independent</u>
  - each $r_i$ is <u>uniformly distributed</u> over $[p]$
- **Proof**: For any $i \neq j$, $\forall c, d \in [p]$, $\Pr(r_i = c \cap r_j = d) = 1/p^2$ because

$$\begin{cases} \boldsymbol{a} \cdot i + \boldsymbol{b} \equiv c \pmod p \\ \boldsymbol{a} \cdot j + \boldsymbol{b} \equiv d \pmod p \end{cases}$$ has a unique solution $(a, b) \in [p]^2$

$$\Pr(r_i = c) = \Pr(\boldsymbol{a} \cdot i + \boldsymbol{b} \equiv c \pmod p) = \frac{1}{p} \sum_{a \in [p]} \Pr(\boldsymbol{b} \equiv c - ai \pmod p) = \frac{1}{p}$$

# *Derandomization* with Two-Point Sampling

- $\mathscr{A}$: for any input $x$ and uniform ***random seed*** $r \in [p]$ for prime number $p$
  - $f(x) = 1 \implies \Pr\left(\mathscr{A}(x, r) = 1\right) \geq \epsilon$
  - $f(x) = 0 \implies \mathscr{A}(x, r) = 0$ for all $r \in [p]$
- $\mathscr{A}^k(x, r_1, \ldots, r_k) = \vee_{i=1}^k \mathscr{A}(x, r_i)$: $k \leq p$ for $r_i = (\boldsymbol{a} \cdot i + \boldsymbol{b}) \bmod p$ with uniform $\boldsymbol{a}, \boldsymbol{b} \in [p]$
  - If $f(x) = 0 \implies \mathscr{A}^k(x, r_1, \ldots, r_k) = \vee_{i=1}^k \mathscr{A}(x, r_i) = 0$
  - If $f(x) = 1 \implies \Pr\left(\mathscr{A}(x, r_i) = 1\right) \geq \epsilon$ because each $r_i$ is uniform over $[p]$
  - Let $X_i = \mathscr{A}(x, r_i)$ and let $X = \sum_{i=1}^k X_i$.
    - $X_1, \ldots, X_k$ are **pairwise independent** Bernoulli random variables with $\Pr(X_i = 1) \geq \epsilon$
    - $\Pr\left(\mathscr{A}^k(x, r_1, \ldots, r_k) = 0\right) = \Pr(X = 0) \leq \Pr\left(|X - \mathbb{E}[X]| \geq \mathbb{E}[X]\right) \leq \dfrac{\mathbf{Var}[X]}{\mathbb{E}[X]^2}$

(Chebyshev's inequality)

# ***Derandomization*** **with Two-Point Sampling**

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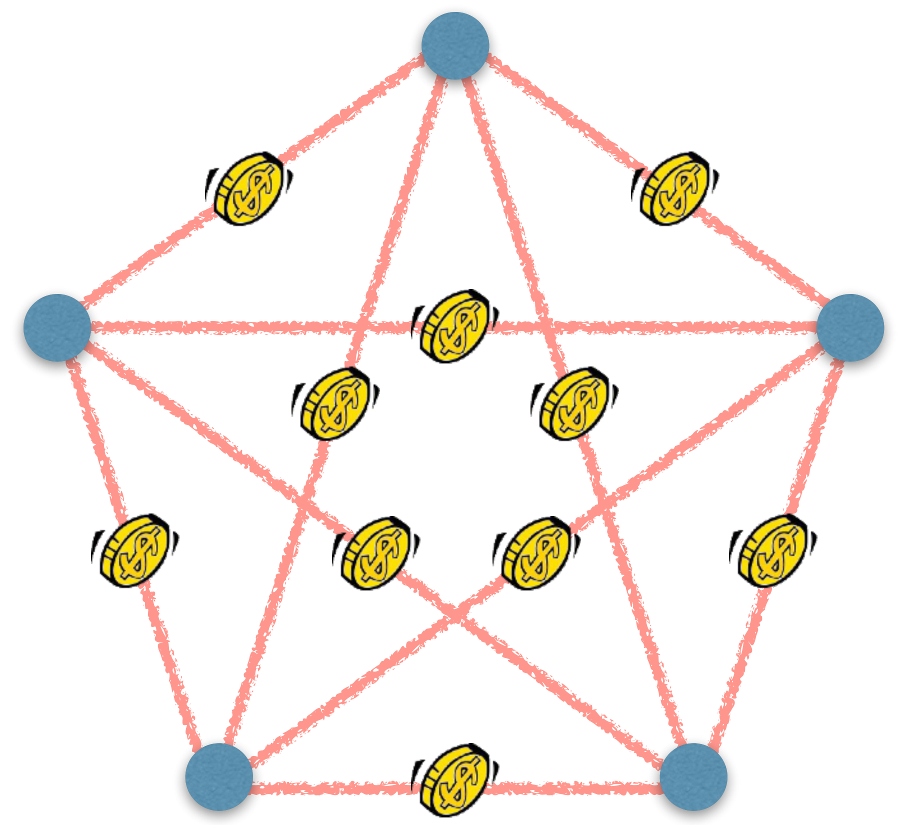
  - Let $X_i = \mathscr{A}(x, r_i)$ and let $X = \sum_{i=1}^k X_i$.
    - $X_1, \ldots, X_k$ are **pairwise independent** Bernoulli random variables with $\Pr(X_i = 1) \geq \epsilon$
    - $\Pr\left(\mathscr{A}^k(x, r_1, \ldots, r_k) = 0\right) = \Pr(X = 0) \leq \Pr\left(|X - \mathbb{E}[X]| \geq \mathbb{E}[X]\right) \leq \dfrac{\mathbf{Var}[X]}{\mathbb{E}[X]^2} \leq \dfrac{1}{\epsilon k}$
      - **Linearity of expectation:** $\mathbb{E}[X] = \sum_{i=1}^k \mathbb{E}[X_i] \geq \epsilon k$
      - **Pairwise independence:** $\mathbf{Var}[X] = \sum_{i=1}^{k} \mathbf{Var}[X_i] \leq \sum_{i=1}^{k} \mathbb{E}[X_i^2] = \sum_{i=1}^{k} \mathbb{E}[X_i] = \mathbb{E}[X]$
- Reduce any 1-sided error $1 - \epsilon$ to $1/(\epsilon k)$ with $k \leq p$ runs of the algorithm using only **2 random seeds** in total.

# Cliques in Random Graph (revisited)

- Fix a constant integer $k \geq 3$. Let $X$ be the number of $k$-**cliques** ($K_k$) in $G \sim G(n,p)$.
- For every distinct $S \subseteq \in [n]$ of size $|S| = k$, let $I_S = I(K_S \subseteq G)$. Then:

$$X = \sum_{S \in \binom{[n]}{k}} I_S \text{ and } \mathbb{E}[I_S] = \Pr(K_S \subseteq G) = p^{\binom{k}{2}}$$



- **Linearity of expectation:** $\mathbb{E}[X] = \binom{n}{k} p^{\binom{k}{2}} = \Theta\left(n^k p^{\binom{k}{2}}\right)$

$$\mathbb{E}[X] = \Theta\left(n^k p^{\binom{k}{2}}\right) = \begin{cases} o(1) & \text{if } p = o\left(n^{-2/(k-1)}\right) \\ \omega(1) & \text{if } p = \omega\left(n^{-2/(k-1)}\right) \end{cases}$$

(Markov) $\Longrightarrow \Pr(X \geq 1) = o(1)$

$\overset{?}{\Longrightarrow} \Pr(X \geq 1) = 1 - o(1)$

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- Chebyshev: $\Pr(X=0) \leq \Pr(|X - \mathbb{E}[X]| \geq \mathbb{E}[X]) \leq \dfrac{\mathbf{Var}[X]}{\mathbb{E}[X]^2} \leq \dfrac{1}{\mathbb{E}[X]} + \dfrac{\sum_{S \neq T} \mathbb{E}[I_S I_T]}{\mathbb{E}[X]^2}$

$$\mathbf{Var}[X] = \sum_{S \in \binom{[n]}{k}} \mathbf{Var}[I_S] + \sum_{\substack{S \neq T \\ S,T \in \binom{[n]}{k}}} \mathbf{Cov}(I_S, I_T) = \sum_{S \in \binom{[n]}{k}} (\mathbb{E}[I_S^2] - \mathbb{E}[I_S]^2) + \sum_{\substack{S \neq T \\ S,T \in \binom{[n]}{k}}} (\mathbb{E}[I_S I_T] - \mathbb{E}[I_S]\mathbb{E}[I_T])$$

$$= \sum_S (\mathbb{E}[I_S] - \mathbb{E}[I_S]^2) + \sum_{S \neq T} (\mathbb{E}[I_S I_T] - \mathbb{E}[I_S]\mathbb{E}[I_T])$$

$$\leq \mathbb{E}[X] + \sum_{S \neq T} \mathbb{E}[I_S I_T]$$

# Cliques in Random Graph (revisited)

- Fix a constant integer $k \geq 3$. Let $X$ be the number of $k$-**cliques** ($K_k$) in $G \sim G(n, p)$.
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- **Chebyshev**: $\Pr(X = 0) \leq \Pr(|X - \mathbb{E}[X]| \geq \mathbb{E}[X]) \leq \dfrac{\mathbf{Var}[X]}{\mathbb{E}[X]^2} \leq \dfrac{1}{\mathbb{E}[X]} + \dfrac{\sum_{S \neq T} \mathbb{E}[I_S I_T]}{\mathbb{E}[X]^2}$

$$\mathbb{E}[I_S I_T] = \Pr((K_S \cup K_T) \subseteq G) = p^{2\binom{k}{2} - \binom{|S \cap T|}{2}}$$

$$\sum_{\substack{S \neq T \\ S, T \in \binom{[n]}{k}}} \mathbb{E}[I_S I_T] = \sum_{\ell=2}^{k-1} \sum_{\substack{|S \cap T| = \ell \\ S, T \in \binom{[n]}{k}}} \mathbb{E}[I_S I_T] = \sum_{\ell=2}^{k-1} \binom{n}{2k - \ell} \cdot O(1) \cdot p^{2\binom{k}{2} - \binom{\ell}{2}} = O\left(n^{2k} p^{2\binom{k}{2}} \sum_{\ell=2}^{k-1} n^{-\ell} p^{-\binom{\ell}{2}}\right)$$

# Cliques in Random Graph (revisited)

- Fix a constant integer $k \geq 3$. Let $X$ be the number of $k$-**cliques** ($K_k$) in $G \sim G(n,p)$.
- For every distinct $S \subseteq \in [n]$ of size $|S| = k$, let $I_S = I(K_S \subseteq G)$. Then:

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- **Chebyshev:** $\Pr(X = 0) \leq \Pr(|X - \mathbb{E}[X]| \geq \mathbb{E}[X]) \leq \dfrac{\mathbf{Var}[X]}{\mathbb{E}[X]^2} \leq \dfrac{1}{\mathbb{E}[X]} + \dfrac{\sum_{S \neq T} \mathbb{E}[I_S I_T]}{\mathbb{E}[X]^2}$

$$= O\left(n^{-k} p^{-\binom{k}{2}}\right) + O\left(\sum_{\ell=2}^{k-1} n^{-\ell} p^{-\binom{\ell}{2}}\right) = O\left(\sum_{\ell=2}^{k} n^{-\ell} p^{-\binom{\ell}{2}}\right)$$

$$= o(1) \text{ if } p = \omega\left(n^{2/(1-k)}\right)$$

- $\implies \Pr(X \geq 1) \geq 1 - o(1)$

# A "Threshold Behavior" in Random Graphs

## (Erdős–Rényi 1960)

- Fix a constant integer $k \geq 3$.
- Let $G \sim G(n, p)$, as $n \to \infty$:

$$\Pr(G \text{ contains a } K_k) = \begin{cases} o(1) & \text{if } p = o\left(n^{-2/(k-1)}\right) \\ 1 - o(1) & \text{if } p = \omega\left(n^{-2/(k-1)}\right) \end{cases}$$

- For $H(V, E)$ with $k = |V|, m = |E|$ s.t. every subgraph of $H$ has density $\leq m/k$:

$$\Pr(G \text{ contains a subgraph } H) = \begin{cases} o(1) & \text{if } p = o\left(n^{-k/m}\right) \\ 1 - o(1) & \text{if } p = \omega\left(n^{-k/m}\right) \end{cases}$$

# Weierstrass Approximation Theorem

**(魏尔施特拉斯逼近定理)**

- Weierstrass Approximation Theorem: Let $f : [0,1] \to [0,1]$ be a continuous function. For any $\epsilon > 0$, there exists a polynomial $p$ such that

$$\sup_{x \in [0,1]} |p(x) - f(x)| \leq \epsilon$$

- **Proof**: Let integer $n$ be sufficiently large (to be fixed later).

  For $x \in [0,1]$, let $Y_x \sim \text{Bin}(n, x)$. Define polynomial $p$ on $x \in [0,1]$ to be:

$$p(x) = \mathbb{E}\left[f\left(\frac{Y_x}{n}\right)\right] = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$$

Let $f : [0,1] \to [0,1]$ be continuous. For $x \in [0,1]$, let $Y_x \sim \mathrm{Bin}(n, x)$, and:

$$p(x) = \mathbb{E}\left[f\left(\frac{Y_x}{n}\right)\right] = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

$$|p(x) - f(x)| = \left|\mathbb{E}\left[f(Y_x/n) - f(x)\right]\right| \le \mathbb{E}\left[\left|f(Y_x/n) - f(x)\right|\right]$$

( $f$ is continuous on $[0,1] \implies \exists \delta > 0$ s.t. $|f(x) - f(y)| \le \epsilon/2$ for all $|x - y| \le \delta$ )

$$= \mathbb{E}\left[\left|f\left(Y_x/n\right) - f(x)\right| \mid \left|Y_x/n - x\right| \le \delta\right] \cdot \Pr\left(\left|Y_x/n - x\right| \le \delta\right)$$

$$+\mathbb{E}\left[\left|f\left(Y_x/n\right) - f(x)\right| \mid \left|Y_x/n - x\right| > \delta\right] \cdot \Pr\left(\left|Y_x/n - x\right| > \delta\right)$$

$$\le \mathbb{E}\left[\epsilon/2\right] + \left|1 - 0\right| \cdot \Pr\left(\left|Y_x - nx\right| > n\delta\right) \le \frac{\epsilon}{2} + \frac{x(1-x)}{n\delta^2}$$ (Chebyshev)

$$\le \frac{\epsilon}{2} + \frac{1}{4n\delta^2} \le \epsilon$$ if we choose $n \ge \dfrac{1}{2\epsilon\delta^2}$

# Weierstrass Approximation Theorem

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- Weierstrass Approximation Theorem: Let $f : [0,1] \to [0,1]$ be a continuous function. For any $\epsilon > 0$, there exists a polynomial $p$ such that

$$\sup_{x \in [0,1]} |p(x) - f(x)| \le \epsilon$$

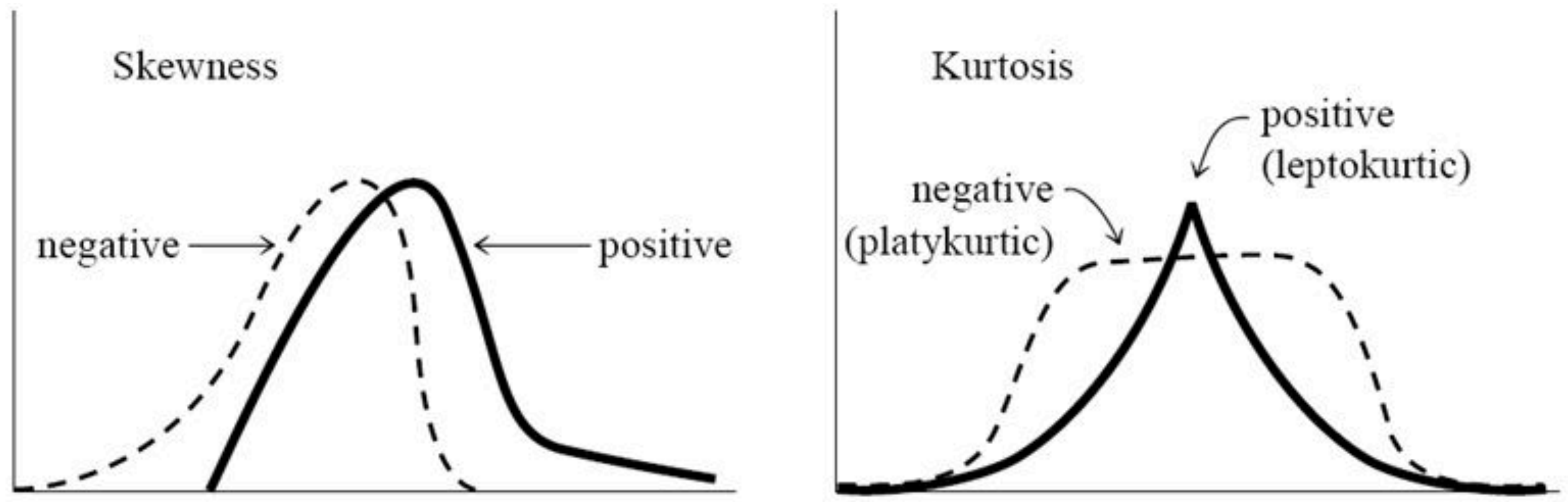
- **Proof**: By continuity, $\exists \delta > 0$ s.t. $|f(x) - f(y)| \le \epsilon/2$ if $|x - y| \le \delta$.

  Let $n \ge 1/(2\epsilon\delta^2)$ be any integer. For $x \in [0,1]$, let $Y_x \sim \mathrm{Bin}(n, x)$, and:

$$p(x) = \mathbb{E}\left[f\left(\frac{Y_x}{n}\right)\right] = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

  For any $x \in [0,1]$, it holds that $|p(x) - f(x)| \le \epsilon$.

# Higher Moments

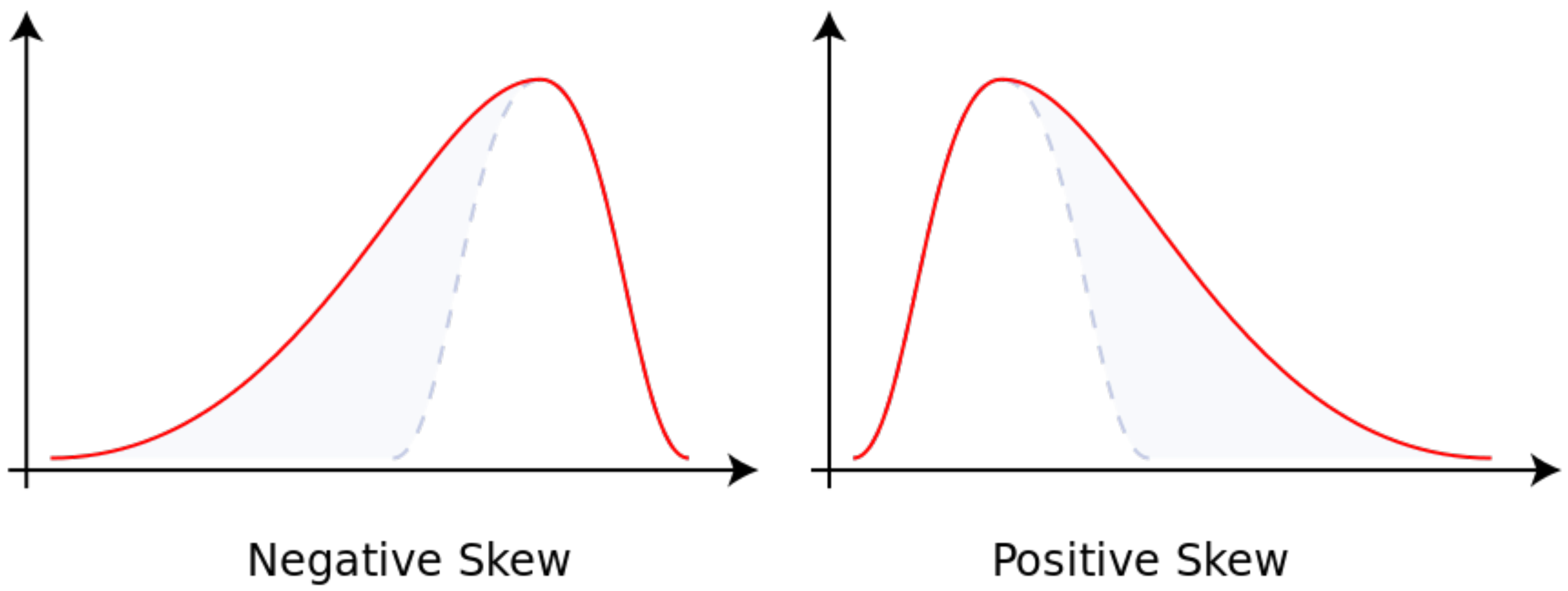


# Skewness (偏度)

- The <u>**skewness**</u> (偏度) of a random variable $X$ with expectation $\mu = \mathbb{E}[X]$ and standard deviation $\sigma = \sqrt{\mathbf{Var}[X]}$ is defined by

$$\mathrm{Skew}[X] = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3}$$

standardized moment (of degree 3)

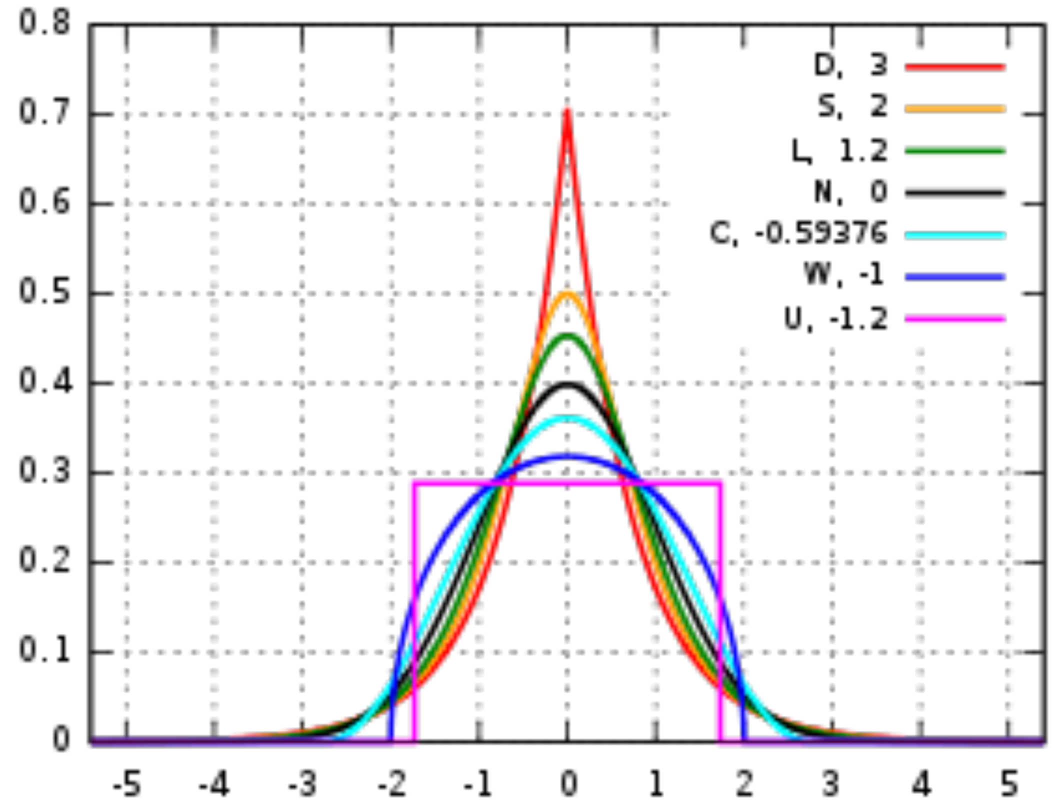


# Kurtosis (峰度)

- The **kurtosis** (峰度) of a random variable $X$ with expectation $\mu = \mathbb{E}[X]$ and standard deviation $\sigma = \sqrt{\mathbf{Var}[X]}$ is defined by

$$\mathrm{Kurt}[X] = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mathbb{E}[(X-\mu)^4]}{\sigma^4}$$

standardized moment (of degree 4)



# The $k$th Moment Method

- Let $X$ be a random variable with $\mathbb{E}[X] = \mu$. For any $C > 1$ and integer $k \geq 1$

$$\Pr\left( |X - \mu| \geq C \cdot \mathbb{E}\left[ |X - \mu|^k \right]^{\frac{1}{k}} \right) \leq \frac{1}{C^k}$$

- **Proof**: Apply Markov's inequality to $Z = |X - \mu|^k$.

# The Moment Problem

- Do moments $m_k = \mathbb{E}[X^k]$, $\forall k \geq 1$, uniquely identify the distribution of $X$?
- If $X$ takes values from a **finite** set $\{x_1, \ldots, x_n\}$,

  then solving the Vandermonde system:

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n & x_2^n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}$$

  can recover the *pmf* $p_i = p_X(x_i)$

# The Moment Problem

- Do moments $m_k = \mathbb{E}[X^k]$, $\forall k \geq 1$, uniquely identify the distribution of $X$?
  - If $\mathbb{E}[X^k] = \mathbb{E}[Y^k]$ for all $k \geq 1$, are $X$ and $Y$ always identically distributed?
- If $X$ and $Y$ have the same **moment generating function (MGF)**

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k \geq 0} \frac{t^k \mathbb{E}[X^k]}{k!}$$

  then $X$ and $Y$ are identically distributed.

- The MGF $M_X(t)$ is convergent if the sequence $\mathbb{E}[X^k]$ does not grow too fast.