Probability Theory & Mathematical Statistics

Random Processes

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Random Processes

(*Stochastic* processes)

- A **random process** is a family $\{X_t : t \in \mathcal{T}\}$ of random variables
- $\mathcal{T}$ is a set of indices, where each $t \in \mathcal{T}$ is usually interpreted as **time**
  - **discrete-time**: countable $\mathcal{T}$, usually $\mathcal{T} = \{0,1,2,\ldots\}$ or $\mathcal{T} = \{1,2,\ldots\}$
  - **continuous-time**: uncountable $\mathcal{T}$, usually $\mathcal{T} = [0,\infty)$
- X_t takes values in a **state space** $\mathcal{S}$
  - **discrete-space**: countable $\mathcal{S}$, e.g. $\mathcal{S} = \mathbb{Z}$
  - **continuous-space**: uncountable $\mathcal{S}$, e.g. $\mathcal{S} = \mathbb{R}$

Random Processes

(*Stochastic* processes)

- <u>Bernoulli process</u>: i.i.d. Bernoulli trials $X_0, X_1, X_2, \ldots \in \{0,1\}$

- <u>Branching (Galton-Watson) process</u>: $X_0 = 1$ and $X_{n+1} = \sum_{j=1}^{X_n} \xi_j^{(n)}$

  where $\{\xi_j^{(n)} : n, j \geq 0\}$ are i.i.d. non-negative integer-valued random variables

- <u>Poisson process</u>: continuous-time counting process $\{N(t) \mid t \geq 0\}$ such that

  $$N(t) = \max\{n \mid X_1 + \cdots + X_n \leq t\} \text{ for any } t \geq 0$$

  where $\{X_i\}$ are i.i.d. **exponential random variables** with parameter $\lambda > 0$

Martingales

Martingale (鞅)

- A sequence $\{Y_n : n \geq 0\}$ of random variables is a martingale with respect to another sequence $\{X_n : n \geq 0\}$ if, for all $n \geq 0$,
  - $\mathbb{E}\left[|Y_n|\right] < \infty$
  - $\mathbb{E}\left[Y_{n+1} \mid X_0, X_1, \ldots, X_n\right] = Y_n$ (martingale property)
- By definition: Y_n is a function of $X_0, X_1, \ldots, X_n$
- Current capital Y_n in a **fair gambling game** with outcomes $X_0, X_1, \ldots, X_n$
  - Super-martingale (上鞅): $\mathbb{E}\left[Y_{n+1} \mid X_0, X_1, \ldots, X_n\right] \leq Y_n$
  - Sub-martingale (下鞅): $\mathbb{E}\left[Y_{n+1} \mid X_0, X_1, \ldots, X_n\right] \geq Y_n$

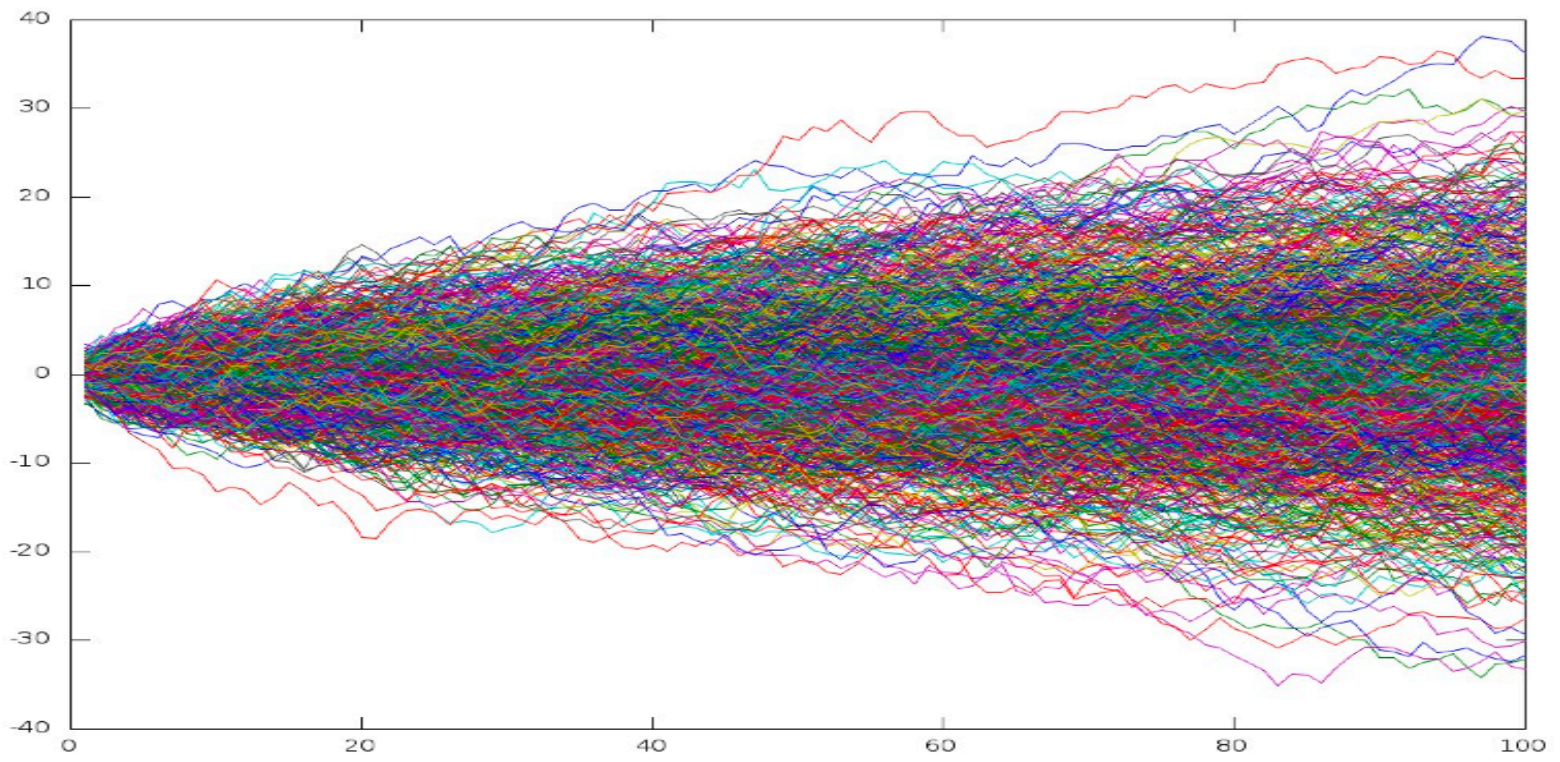
Martingale (鞅)

- A sequence $\{Y_n : n \geq 0\}$ of random variables is a <u>martingale</u> with respect to another sequence $\{X_n : n \geq 0\}$ if, for all $n \geq 0$,
  - $\mathbb{E}\left[\,|Y_n|\,\right] < \infty$
  - $\mathbb{E}\left[\, Y_{n+1} \mid X_0, X_1, \ldots, X_n \,\right] = Y_n$ (martingale property)
- $\{X_n : n \geq 0\}$ are defined on the probability space $(\Omega, \Sigma, \Pr)$
  - $(X_0, X_1, \ldots, X_n)$ defines a sub-σ-field $\Sigma_n \subseteq \Sigma$ (the smallest σ-field s.t. $(X_0, \ldots, X_n)$ is Σ_n-measurable)
  - $\{\Sigma_n : n \geq 0\}$ is a <u>filtration</u> of Σ, i.e. $\Sigma_0 \subseteq \Sigma_1 \subseteq \cdots \subseteq \Sigma$
  - The **martingale property** is expressed as $\mathbb{E}\left[\, Y_{n+1} \mid \Sigma_n \,\right] = Y_n$

Examples of Martingale

- **Doob martingale:** $Y_i = \mathbb{E}\left[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_i\right]$
  - **vertex/edge exposure martingale** for random graph
- Capital in a **fair gambling game** (arbitrary betting strategy)
- **Unbiased 1D random walk:** $Y_n = \sum_{i=1}^{n} X_i$ with *i.i.d.* uniform $X_i \in \{-1,1\}$
- **de Moivre's martingale:** $Y_n = (p/(1-p))^{X_n}$, where $X_n = \sum_{i=1}^{n} X_i$ and $X_i \in \{-1,1\}$ are independent with $\Pr(X_i = 1) = p$
- **Polya's urn**: The urn contains marbles with different colors. At each turn, a marble is selected *u.a.r.*, and replaced with k marbles of that same color.

Studies of Martingale



- For **martingale** $\{Y_n : n \geq 0\}$ with respect to $\{X_n : n \geq 0\}$:

$$\mathbb{E}\left[\, Y_{n+1} \mid X_0, X_1, \ldots, X_n \,\right] = Y_n$$

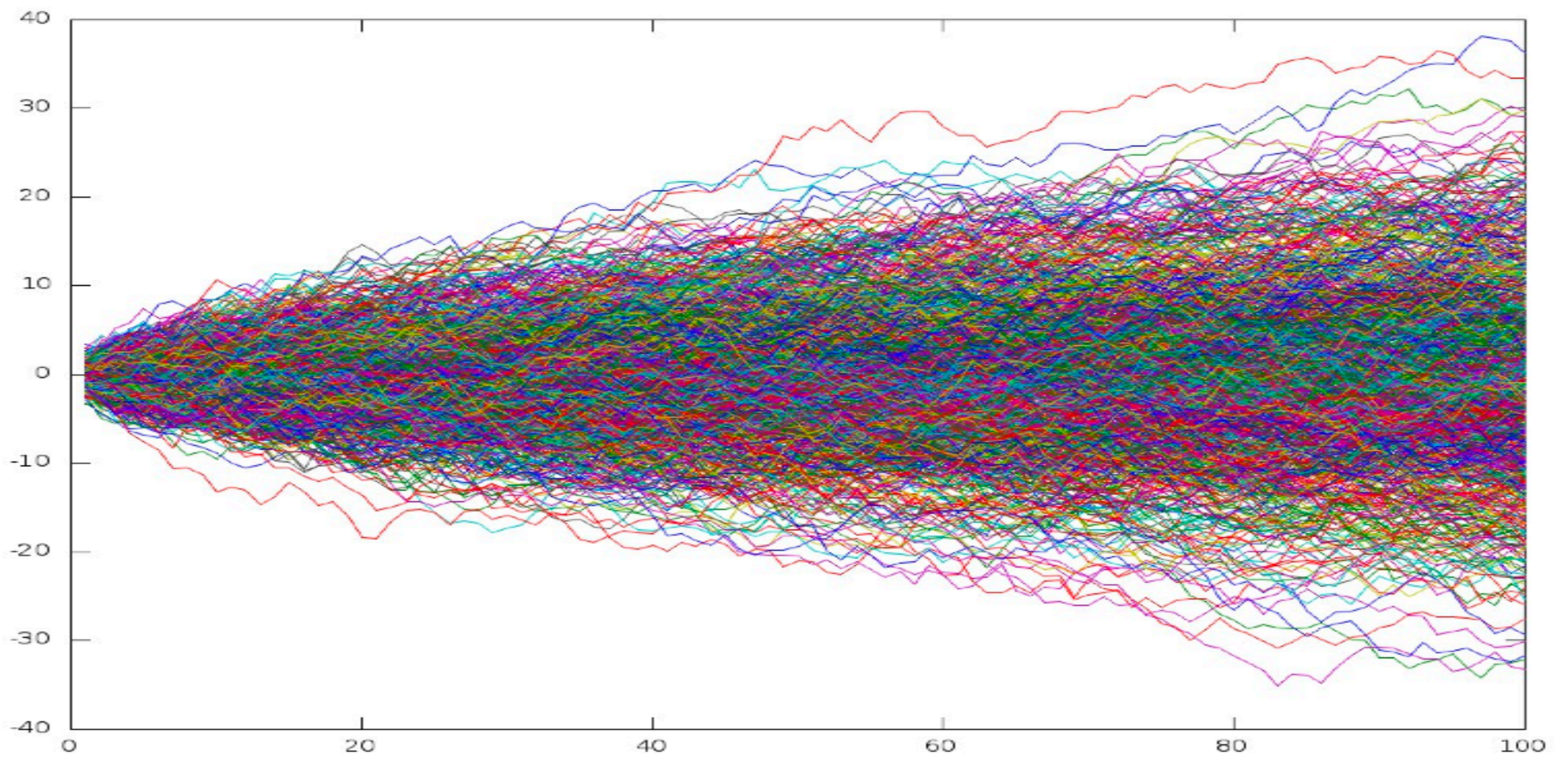
- **Concentration of measure** (tail inequality): under what condition

$$\Pr\left(\, | Y_n - Y_0 | \geq t \right) \leq ?$$

- **Optional stopping theorem** (**OST**): under what condition for a **stopping time** τ

$$\mathbb{E}[Y_\tau] = \mathbb{E}[Y_0]$$

Fair Gambling Game



- If $\{Y_n : n \geq 0\}$ is a **martingale** with respect to $\{X_n : n \geq 0\}$, then $\forall n \geq 0$,

$$\mathbb{E}\left[Y_n\right] = \mathbb{E}\left[Y_0\right]$$

Proof: By total expectation $\mathbb{E}\left[Y_n\right] = \mathbb{E}\left[\mathbb{E}\left[\,Y_n \mid X_0, X_1, \ldots, X_{n-1}\,\right]\right]$

As a martingale, $\mathbb{E}\left[\,Y_n \mid X_0, X_1, \ldots, X_{n-1}\,\right] = Y_{n-1}$

$$\Longrightarrow \mathbb{E}\left[Y_n\right] = \mathbb{E}\left[\mathbb{E}\left[\,Y_n \mid X_0, X_1, \ldots, X_{n-1}\,\right]\right] = \mathbb{E}\left[Y_{n-1}\right]$$

Stopping Time

- A nonnegative integer-valued random variable T is a **stopping time** with respect to the sequence $\{X_t : t = 0,1,2,\ldots\}$ if for any $n \geq 0$ the occurrence of the event $T = n$ is determined by the evaluation of $X_0, X_1, \ldots, X_n$
  - Formally, $\{X_t : t = 0,1,2,\ldots\}$ defines a filtration of σ-fields $\Sigma_0 \subseteq \Sigma_1 \subseteq \cdots$ such that $(X_0, X_1, \ldots, X_n)$ is Σ_n-measurable (and Σ_n is the smallest such σ-field). Then T is a stopping time if $\{T = n\} \in \Sigma_n$ for any $n \geq 0$.
  - Intuitively, T is a stopping time, if whether stopping at time n is determined by the outcomes of $X_0, X_1, \ldots, X_n$

Optional Stopping Theorem (OST)
(Martingale Stopping Theorem)

- Optional Stopping Theorem (OST): Let $\{Y_t : t \geq 0\}$ be a **martingale** and T be a **stopping time**, both with respect to $\{X_t : t \geq 0\}$. Then
$$\mathbb{E}\left[Y_T\right] = \mathbb{E}\left[Y_0\right]$$
if any one of the following conditions holds:
  - (bounded time) there is a finite n such that $T < n$ a.s.
  - (bounded range) $T < \infty$ a.s. and there is a finite c s.t. $|Y_t| < c$ for all $t \leq T$
  - (bounded differences) $\mathbb{E}[T] < \infty$ and there is a finite c such that
$$\mathbb{E}[\,|Y_{t+1} - Y_t|\mid X_0, X_1, \ldots, X_t] < c \text{ for all } t \geq 0$$

Optional Stopping Theorem (OST)

(Martingale Stopping Theorem)

- <u>Optional Stopping Theorem</u> **(OST)**: Let $\{Y_t : t \geq 0\}$ be a **martingale** and T be a **stopping time**, both with respect to $\{X_t : t \geq 0\}$. Then

$$\mathbb{E}\left[Y_T\right] = \mathbb{E}\left[Y_0\right]$$

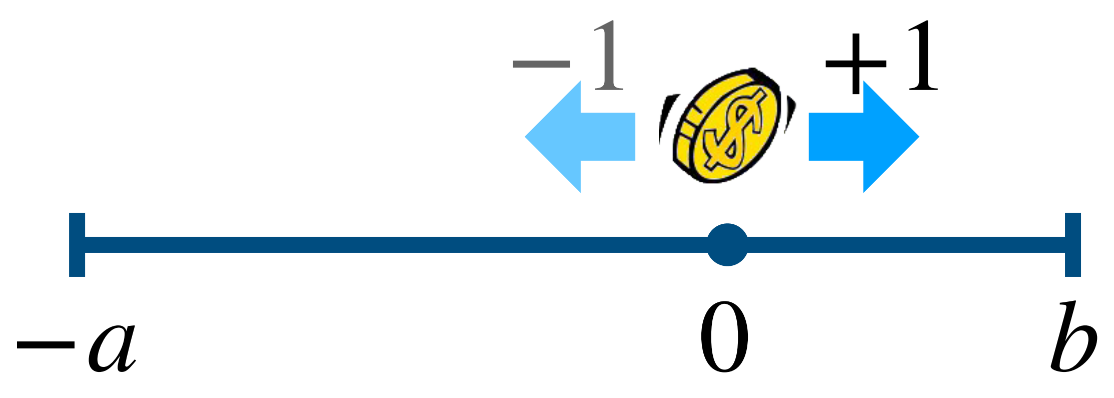
  (general condition) if all the following conditions hold:

  - $\Pr(T < \infty) = 1$
  - $\mathbb{E}[\,|Y_T|\,] < \infty$
  - $\lim_{n\to\infty} \mathbb{E}\left[Y_n \cdot I[T > n]\right] = 0$

- The proof of this general OST utilizes ***Doob's optional sampling*** argument

Gambler's Ruin

(Symmetric Random Walk in One-Dimension)



- Let $Y_t = \sum_{i=1}^{t} X_i$ where $X_i \in \{-1, +1\}$ are i.i.d. uniform (**Rademacher**) RVs
- Let T be the first time t that $Y_t = -a$ or $Y_t = b$
- $\{Y_t : t \geq 0\}$ is a **martingale** and T is a **stopping time** (both w.r.t. $\{X_i : i \geq 1\}$) satisfying that $|Y_t| \leq \max\{a, b\}$ for all $0 \leq t \leq T$ and $T < \infty$ a.s.

(OST) $\Longrightarrow \mathbb{E}[Y_T] = \mathbb{E}[Y_0] = 0$

$$\mathbb{E}[Y_T] = b \cdot \Pr(Y_T = b) - a \cdot \Pr(Y_T \neq b) \quad \Longrightarrow \quad \Pr(Y_T = b) = \frac{a}{a+b}$$

Wald's Equation

(Linearity of expectation with randomly many random variables)

- <u>Wald's equation</u>: Let $X_1, X_2, \ldots$ be i.i.d. non-negative with $\mu = \mathbb{E}[X_i] < \infty$. Let T if a stopping time with respect to $X_1, X_2, \ldots$. If $\mathbb{E}[T] < \infty$, then

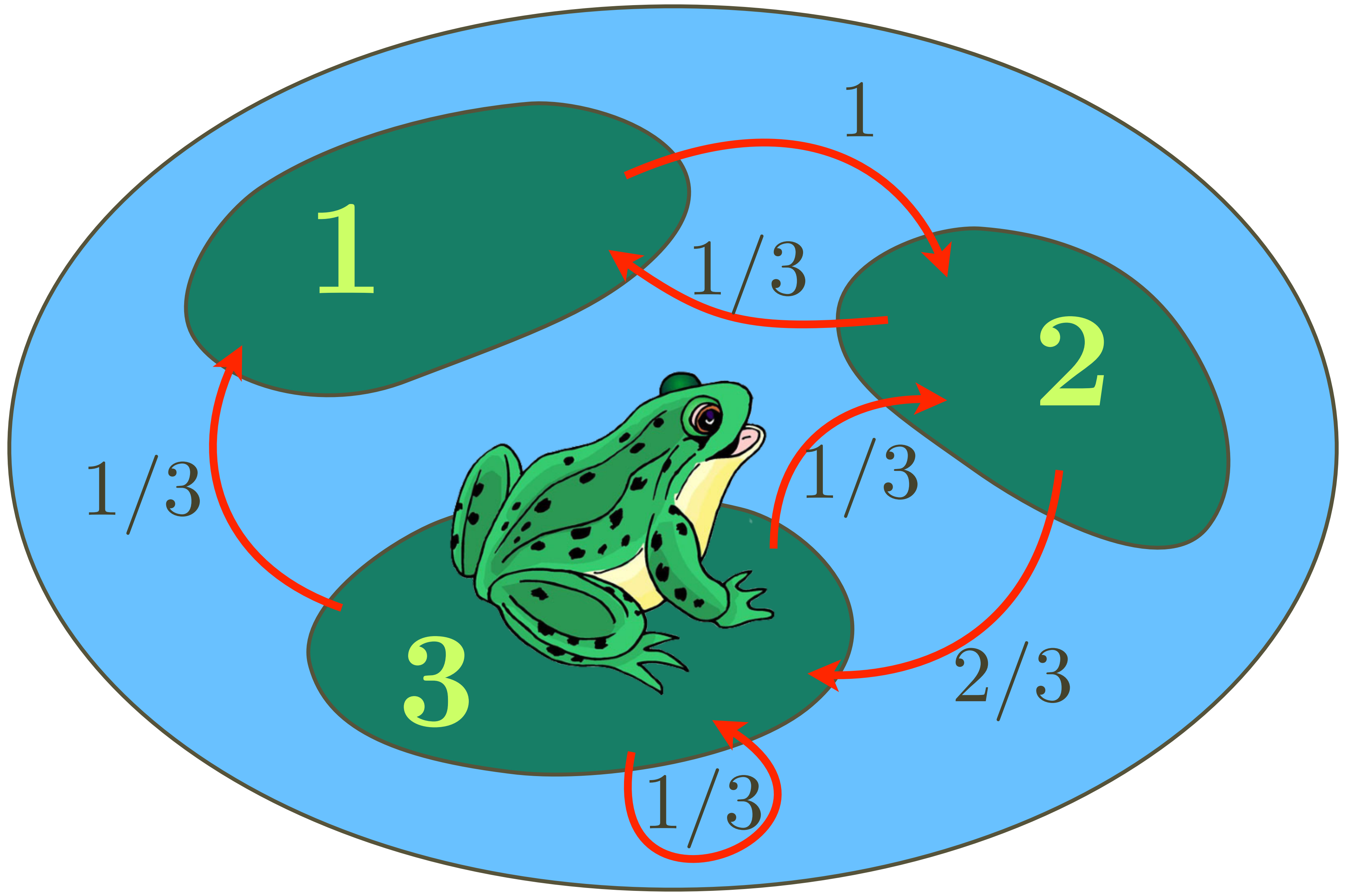
$$\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbb{E}[T] \cdot \mu$$

- **Proof**: For $t \geq 1$, let $Y_t = \Sigma_{i=1}^{t}(X_i - \mu)$, which is a martingale. Observe that:

$$\mathbb{E}[T] < \infty \text{ and } \mathbb{E}[\,|\,Y_{t+1} - Y_t\,|\mid X_1, \ldots, X_t] = \mathbb{E}[\,|\,X_{t+1} - \mu\,|\,] \leq 2\mu$$

By **OST**: $\mathbb{E}[Y_T] = \mathbb{E}[Y_1] = 0$. Note that $\mathbb{E}[Y_T] = \mathbb{E}\left[\Sigma_{i=1}^{T} X_i\right] - \mathbb{E}[T] \cdot \mu$

Markov Chain



Markov Chain (马尔可夫链)

- A discrete-time random process $X_0, X_1, X_2, \dots$ is a <u>**Markov chain**</u> if

$$\Pr(X_{t+1} = x_{t+1} \mid X_t = x_t, \dots, X_0 = x_0) = \Pr(X_{t+1} = x_{t+1} \mid X_t = x_t)$$

- The **Markov property (memoryless property)**:
  - The next state X_{t+1} depends on the current state X_t but is independent of the history $X_0, X_1, \dots, X_{t-1}$ of how the process arrived at state X_t
  - X_{t+1} is **conditionally independent** of $X_0, X_1, \dots, X_{t-1}$ given X_t

$$X_0 \to X_1 \to \cdots \to X_{t-1} \to X_t \to X_{t+1}$$

Transition Matrix (转移矩阵)

- A discrete-time random process $X_0, X_1, X_2, \ldots$ is a <u>**Markov chain**</u> if

$$\Pr(X_{t+1} = x_{t+1} \mid X_t = x_t, \ldots, X_0 = x_0) = \Pr(X_{t+1} = x_{t+1} \mid X_t = x_t)$$

(time-homogeneous) $\quad = P(x_t, x_{t+1}) \;\; = P^{(t)}(x_t, x_{t+1})$

- P is called the <u>**transition matrix**</u>: (assuming discrete-space)

$$P(x, y) = \Pr(X_{t+1} = y \mid X_t = x) \text{ for any } x, y \in \mathcal{S}$$

where $\mathcal{S}$ is the discrete **state space** on which $X_0, X_1, X_2, \ldots$ take values

- P is a (**row-**)**stochastic** matrix: $\;P \geq 0\;$ and $\;P\mathbf{1} = \mathbf{1}$

Transition Matrix (转移矩阵)

- For a **Markov chain** $X_0, X_1, X_2, \ldots$ with discrete **state space** $\mathcal{S}$

$$\Pr(X_{t+1} = y \mid X_t = x) = P(x, y)$$

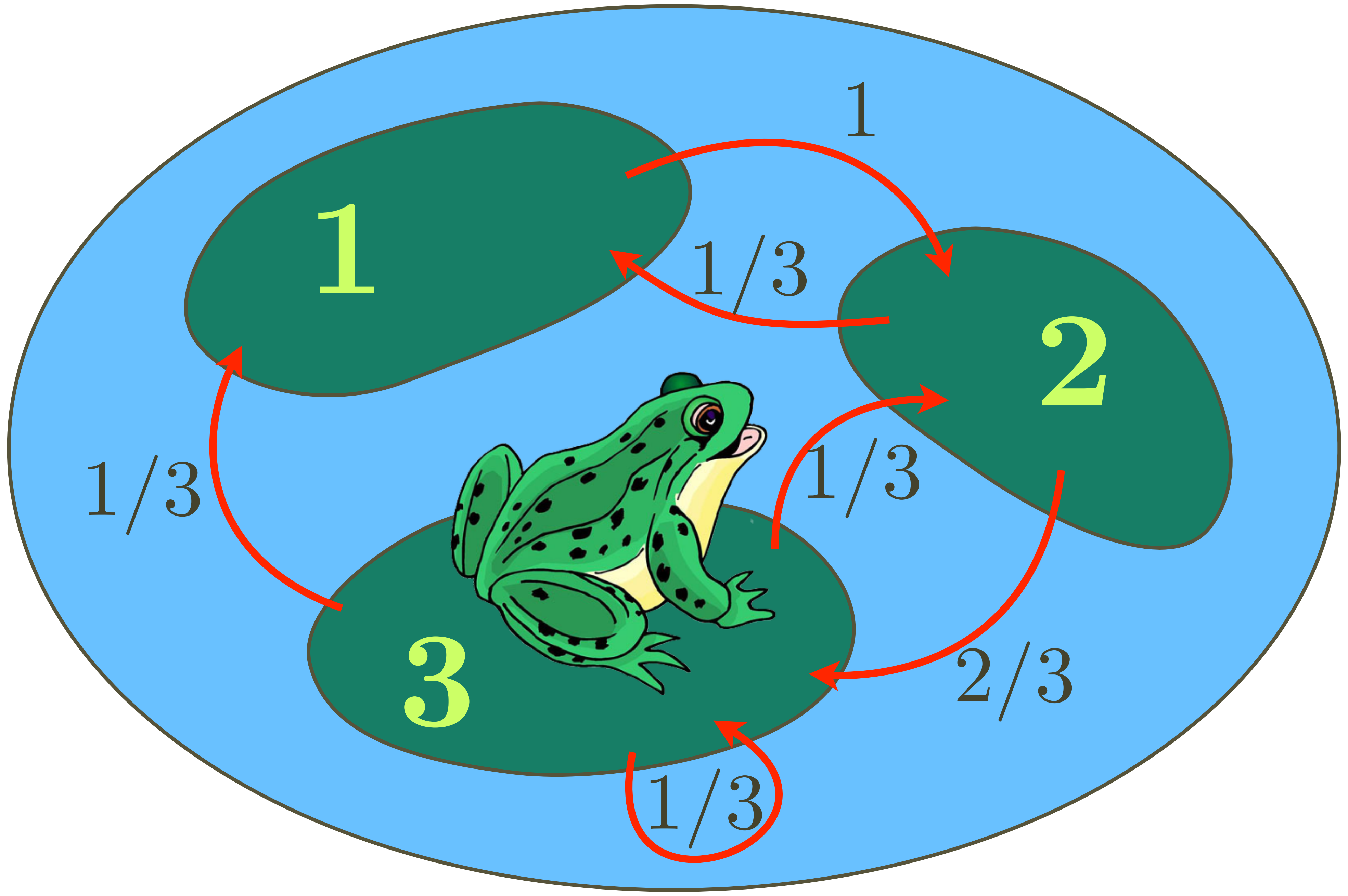
  where $P \in \mathbb{R}_{\geq 0}^{\mathcal{S} \times \mathcal{S}}$ is the **transition matrix**, which is a (**row-**)**stochastic** matrix

- Let $\pi^{(t)}(x) = \Pr(X_t = x)$ be the mass function (***pmf***) of X_t. By total probability:

$$\pi^{(t+1)}(y) = \Pr(X_{t+1} = y) = \sum_{x \in S} \Pr(X_{t+1} = y \mid X_t = x) \Pr(X_t = x) \ = \pi^{(t)} P$$

$$\pi^{(0)} \xrightarrow{P} \pi^{(1)} \xrightarrow{P} \cdots \xrightarrow{P} \pi^{(t)} \xrightarrow{P} \pi^{(t+1)} \xrightarrow{P} \cdots$$

Random Walk (随机游走)



- WLOG: a Markov chain is a <u>random walk</u> on state space $\mathcal{S}$
- Each state $x \in \mathcal{S}$ corresponds to a vertex
- Given the current state $x \in \mathcal{S}$, the probability of next state being $y \in \mathcal{S}$ is:

$$P(x, y) = \Pr(X_{t+1} = y \mid X_t = x)$$

- Initially, $\pi^{(0)}(x) = \Pr(X_0 = x)$, for $t \geq 0$:

$$\pi^{(t+1)} = \pi^{(t)} P$$

Stationary Distribution (稳态分布)

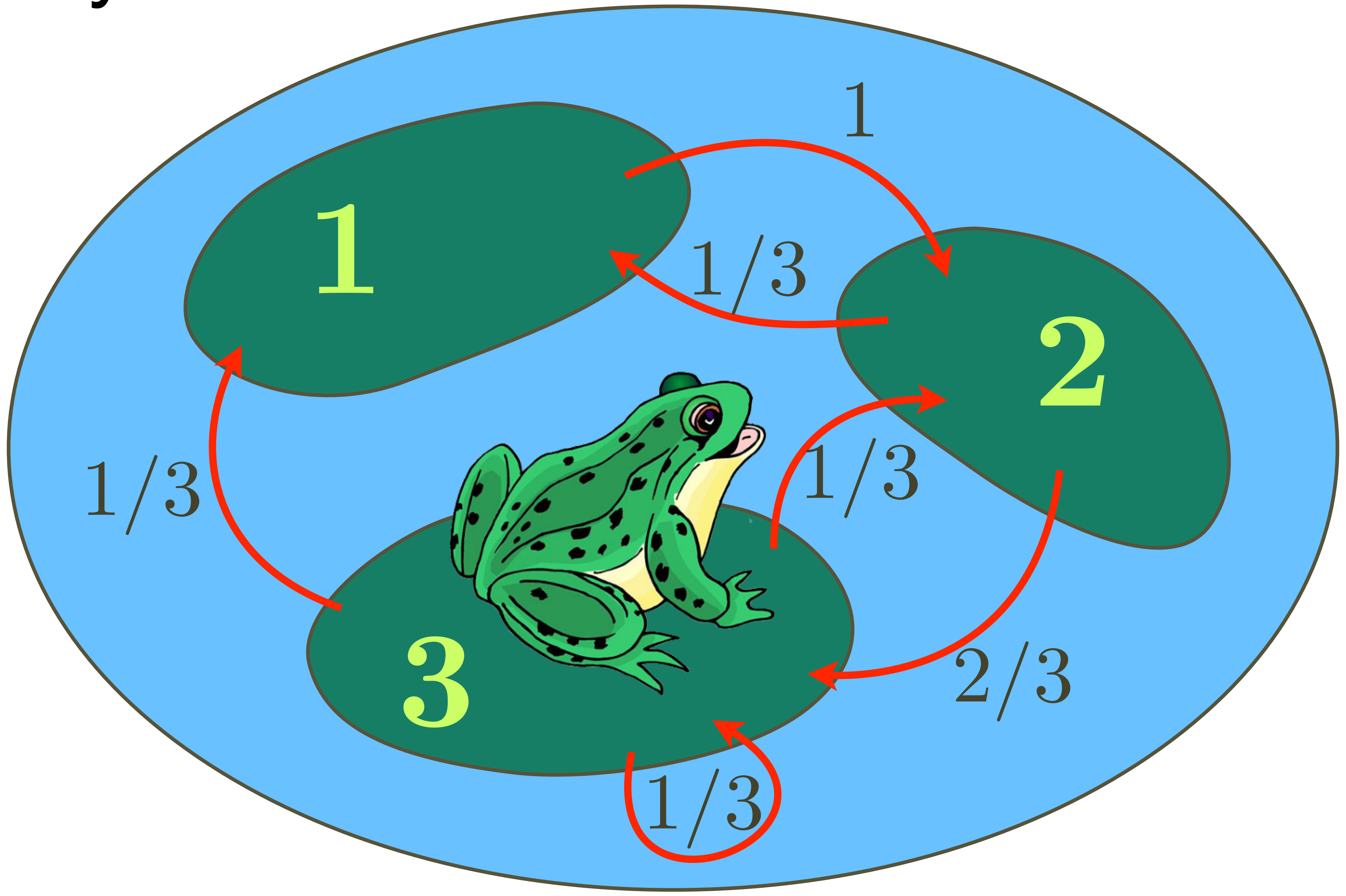
- A distribution (***pmf***) π on state space $\mathcal{S}$ is called a **<u>stationary distribution</u>** of the Markov chain P if

$$\pi P = \pi$$

- π is a **fixpoint** (**equilibrium**) of the linear dynamic system

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \quad \pi = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8} \right)$$

$$P^{20} \approx \begin{bmatrix} 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \end{bmatrix}$$



Convergence Theorem

- <u>Markov chain convergence theorem</u>:

  If a Markov chain $X_0, X_1, X_2 \ldots$ on state space $\mathcal{S}$ is ***irreducible*** and ***ergodic***, then there is a unique stationary distribution π on $\mathcal{S}$ such that

$$\pi(x) = \lim_{t \to \infty} \Pr(X_t = x \mid X_0 = x_0) \text{ for any } x_0 \in \mathcal{S}$$

- **Irreducibility:** the chain is <u>irreducible</u> if P is an irreducible matrix (不可约矩阵)

  $\iff$ the state space $\mathcal{S}$ is **strongly connected** under P

- **Ergodicity:** the chain is <u>ergodic</u> if all states are *aperiodic* (无周期)

  and *positive recurrent* (正常返)

Ergodicity

- Let $X_0, X_1, X_2 \ldots$ be a Markov chain on state space $\mathcal{S}$ with transition matrix P.
- The **period** $d(x)$ of a state $x \in \mathcal{S}$ is $d(x) = \text{gdc}\{t \geq 1 \mid P^t(x,x) > 0\}$
  - A state $x \in \mathcal{S}$ is called **aperiodic** if $d(x) = 1$
  - $P(x,x) > 0 \implies x$ is aperiodic
- A state $x \in \mathcal{S}$ is called **recurrent** if $\Pr(\exists t \geq 1, X_t = x \mid X_0 = x) = 1$
  and further called **positive recurrent** if $\mathbb{E}\left[\min\{t \geq 1 : X_t = x\} \mid X_0 = x\right] < \infty$
- *Shizuo Kakutani* (角谷静夫): random walk is recurrent on $\mathbb{Z}^2$ but non-recurrent on $\mathbb{Z}^3$
  "*A drunk man will find his way home, but a drunk bird may get lost forever.*"
- On finite state space $\mathcal{S}$: irreducible $\implies$ all states are positive recurrent

Convergence Theorem

- <u>**Markov chain convergence theorem**</u>:

  If a Markov chain $X_0, X_1, X_2 \ldots$ on state space $\mathcal{S}$ is ***irreducible*** and ***ergodic***, then there is a unique stationary distribution π on $\mathcal{S}$ such that

$$\pi(x) = \lim_{t \to \infty} \Pr(X_t = x \mid X_0 = x_0) \text{ for any } x_0 \in \mathcal{S}$$

- **Finite Markov chain** (with finite state space $\mathcal{S}$):

  lazy (i.e. $P(x, x) > 0$) and **strongly connected** P

  $\Longrightarrow$ always converge to the unique stationary distribution $\pi = \pi P$

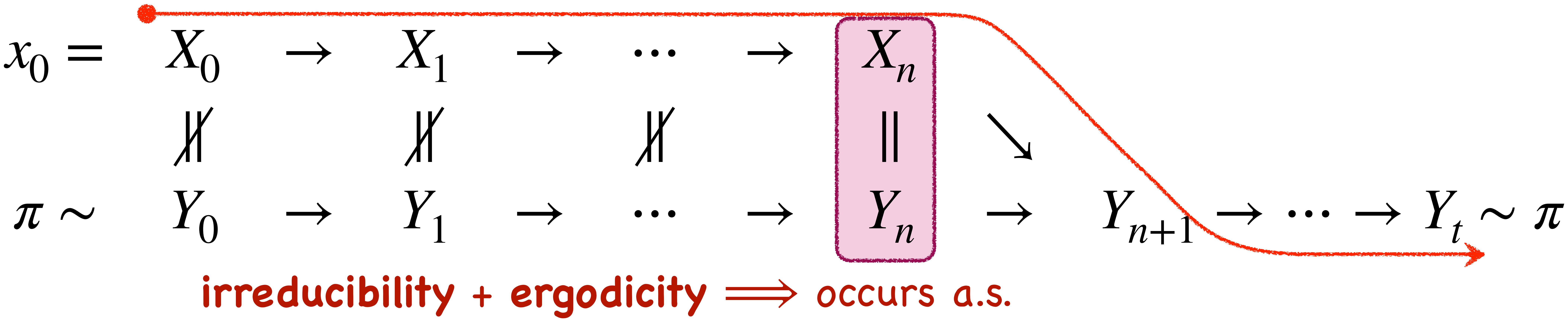
Convergence Theorem

- <u>Markov chain convergence theorem</u>:

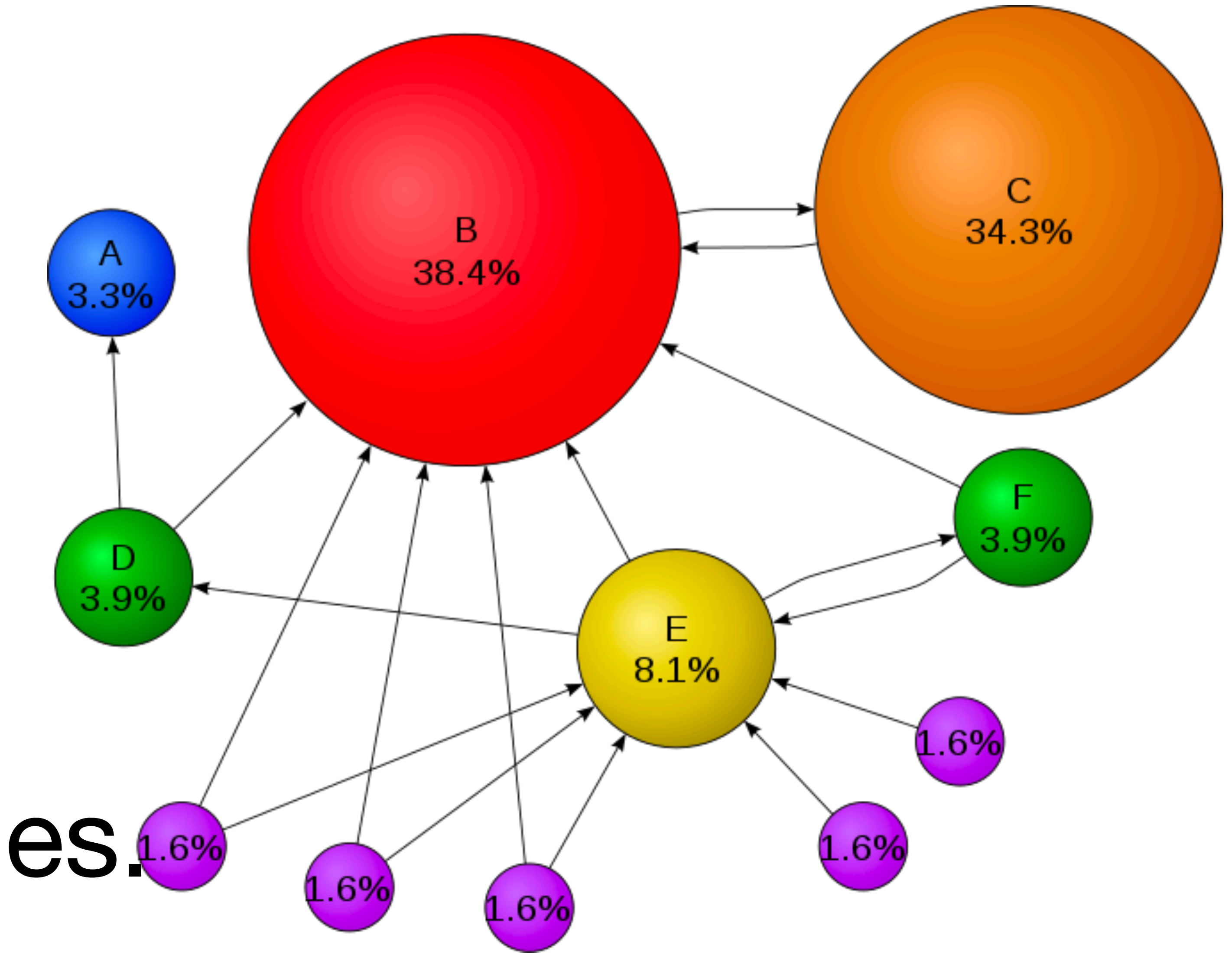
  If a Markov chain $X_0, X_1, X_2 \ldots$ on state space $\mathcal{S}$ is ***irreducible*** and ***ergodic***, then there is a unique stationary distribution π on $\mathcal{S}$ such that

$$\pi(x) = \lim_{t \to \infty} \Pr(X_t = x \mid X_0 = x_0) \quad \text{for any } x_0 \in \mathcal{S}$$

- **Proof**: (By coupling)



PageRank



- Each webpage $x \in \mathcal{S}$ is assigned a rank $r(x)$:
  - A page has high rank if pointed by many high-rank pages.
  - High-rank pages have greater influence.
  - Pages pointing to few others have greater influence.
- Linear system: $r(x) = \sum_{y \to x} \frac{r(y)}{d^+(y)}$ where $d^+(y)$ is the **out-degree** of page y
- **Stationary distribution** $rP = r$ for the random walk (tireless internet surfer)

$$P(x,y) = \begin{cases} \frac{1}{d^+(x)} & \text{if } x \to y \\ 0 & \text{o.w.} \end{cases}$$

Mixing of Markov Chain

- <u>Markov chain convergence theorem</u>:

  If a Markov chain $X_0, X_1, X_2 \ldots$ on state space $\mathcal{S}$ is ***irreducible*** and ***ergodic***, then there is a unique stationary distribution π on $\mathcal{S}$ such that

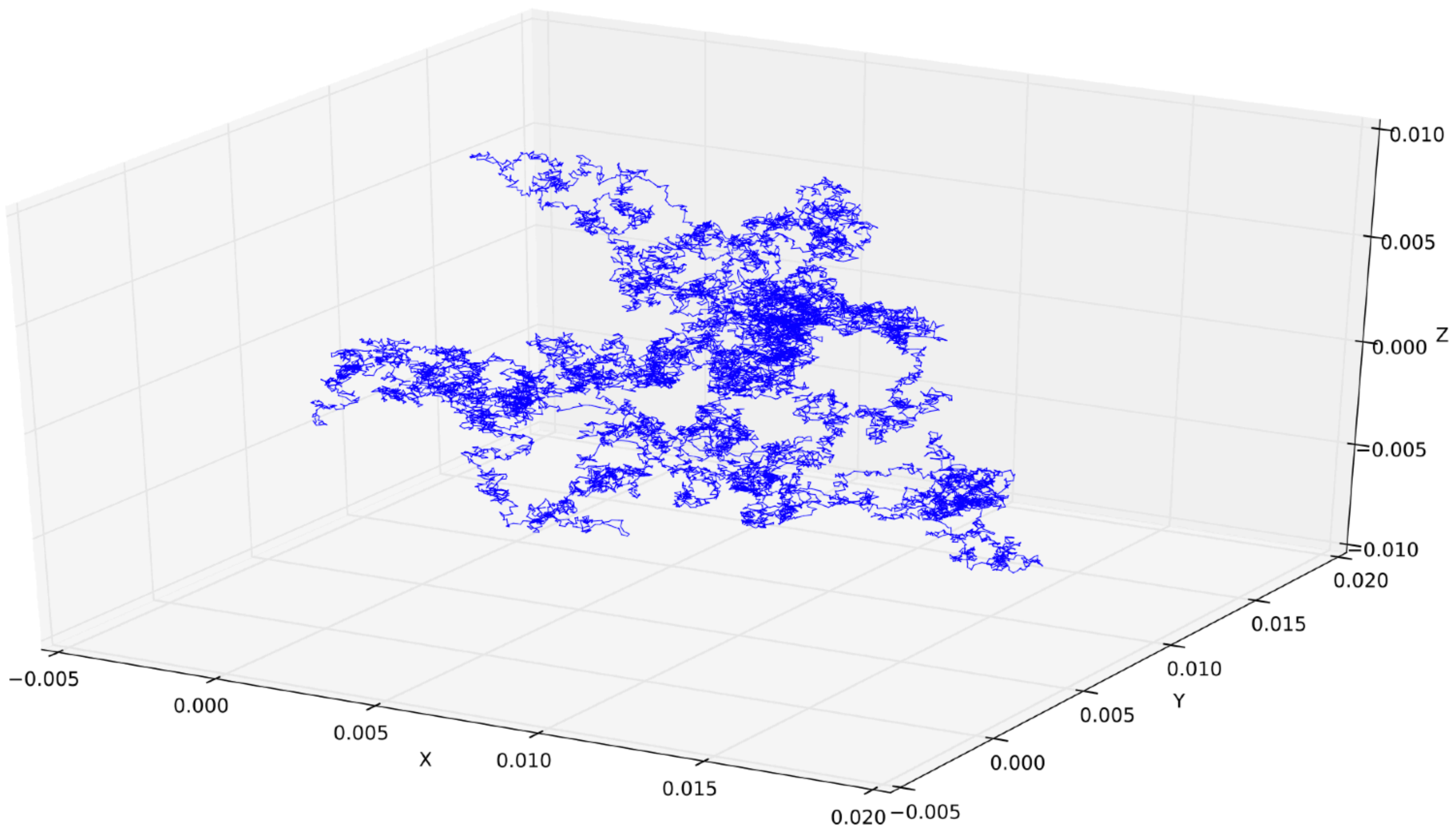
$$\pi(x) = \lim_{t\to\infty} \Pr(X_t = x \mid X_0 = x_0) \text{ for any } x_0 \in \mathcal{S}$$

- How fast is the convergence rate?

- <u>Mixing time</u>: let $\pi_x^{(t)}(y) = (\mathbf{1}_x P^t)_y = \Pr(X_t = y \mid X_0 = x)$

$$\tau(\epsilon) = \max_{x\in S} \min\left\{ t \geq 1 \,\middle|\, \left\| \pi_x^{(t)} - \pi \right\|_1 \leq 2\epsilon \right\}$$

Random Processes



Random Processes

- **Stationary processes:** $(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \sim (X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h})$
  - i.i.d. variables, stationary Markov chains, stationary Gaussian process, …
- **Renewal** (or **counting**) **processes:** $N(t) = \max\{n \mid X_1 + \cdots + X_n \leq t\}$ where $\{X_i : i \geq 1\}$ are i.i.d. nonnegative-valued random variables
  - Poisson processes (the only renewal processes that are Markov chains)
- **Wiener process (Brownian motion)**: continuous-time continuous-space $\{W(t) \in \mathbb{R} : t \geq 0\}$ with **time-homogeneity** and **independent increments**

  $W(s_i) - W(t_i)$ are independent whenever the intervals $(s_i, t_i]$ are disjoint

Diffusion Processes

(Stochastic processes with continuous sample paths)

- Let $(\Omega, \Sigma, \Pr)$ be a probability space. A random process $X : \mathcal{T} \times \Omega \to \mathcal{S}$ with time range $\mathcal{T}$ and state space $\mathcal{S}$ is called a <u>**diffusion process**</u> if there is an $A \in \Sigma$ with $\Pr(A) = 1$ such that for all $\omega \in A$,

$$X(\omega) : \mathcal{T} \to \mathcal{S}$$

  is a continuous function (between topological spaces).

- The **Wiener processes** are one-dimensional diffusions.

- **Itô (伊藤) calculus** may apply!