Probability & Computing

Stochastic Process

$$\{X_t \mid t \in T\} \qquad X_t \in \Omega$$

time t state space Ω state $x \in \Omega$

- discrete time:

  T is countable $\qquad T = \{0, 1, 2, \ldots\}$

- discrete space:

  Ω is finite or countably infinite

$$X_0, X_1, X_2, \ldots$$

Markov Property

- dependency structure of $X_0, X_1, X_2, \ldots$
- Markov property: (memoryless)

X_{t+1} depends only on X_t

$$\forall t = 0, 1, 2, \ldots, \forall x_0, x_1, \ldots, x_{t-1}, x, y \in \Omega$$

$$\Pr[X_{t+1} = y \mid X_0 = x_0, \ldots, X_{t-1} = x_{t-1}, X_t = x]$$
$$= \Pr[X_{t+1} = y \mid X_t = x]$$

Markov chain: discrete time discrete space stochastic process with Markov property.

Transition Matrix

Markov chain: $X_0, X_1, X_2, \ldots$

$$\Pr[X_{t+1} = y \mid X_0 = x_0, \ldots, X_{t-1} = x_{t-1}, X_t = x]$$
$$= \Pr[X_{t+1} = y \mid X_t = x] = P^{(t)}_{xy} = P_{xy}$$

(time-homogenous)

$$\begin{array}{cc} P & y \in \Omega \\ x \in \Omega & \begin{bmatrix} & \vdots & \\ \cdots\cdots & P_{xy} & \\ & & \end{bmatrix} \end{array}$$

stochastic matrix $\quad P\mathbf{1} = \mathbf{1}$

chain: $X_0, \quad X_1, \quad X_2, \quad \ldots$

distribution: $\pi^{(0)} \quad \pi^{(1)} \quad \pi^{(2)} \quad \in [0,1]^{\Omega} \quad \sum_{x \in \Omega} \pi_x = 1$

$$\pi_x^{(t)} = \Pr[X_t = x]$$

$$\pi^{(t+1)} = \pi^{(t)} P$$

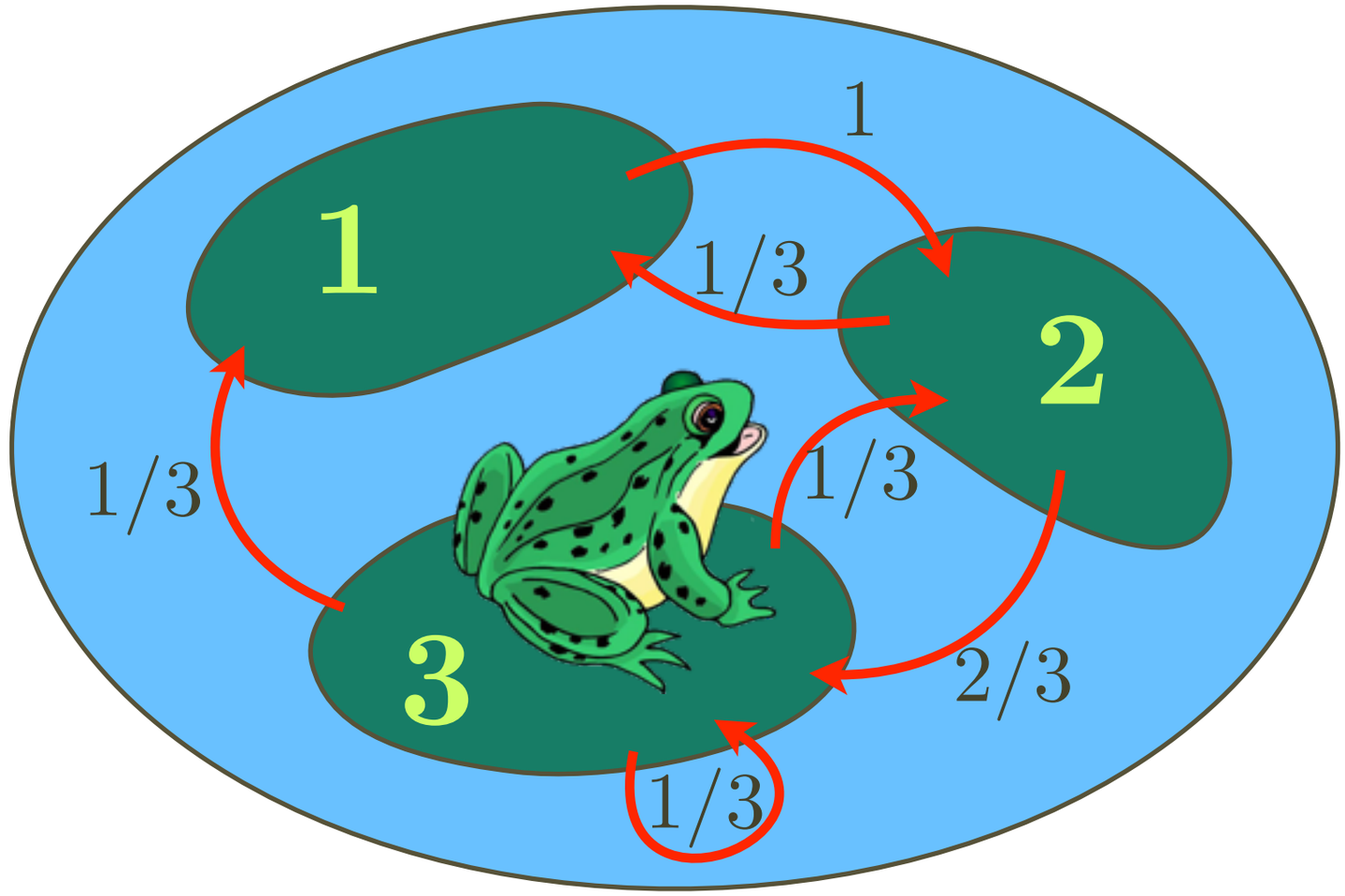
$$\begin{aligned}
\pi_y^{(t+1)} &= \Pr[X_{t+1} = y] \\
&= \sum_{x \in \Omega} \Pr[X_t = x] \Pr[X_{t+1} = y \mid X_t = x] \\
&= \sum_{x \in \Omega} \pi_x^{(t)} P_{xy} \\
&= (\pi^{(t)} P)_y
\end{aligned}$$

$$\pi^{(0)} \xrightarrow{P} \pi^{(1)} \xrightarrow{P} \ldots\ldots \pi^{(t)} \xrightarrow{P} \pi^{(t+1)} \xrightarrow{P} \ldots$$

- initial distribution: $\pi^{(0)}$
- transition matrix: P

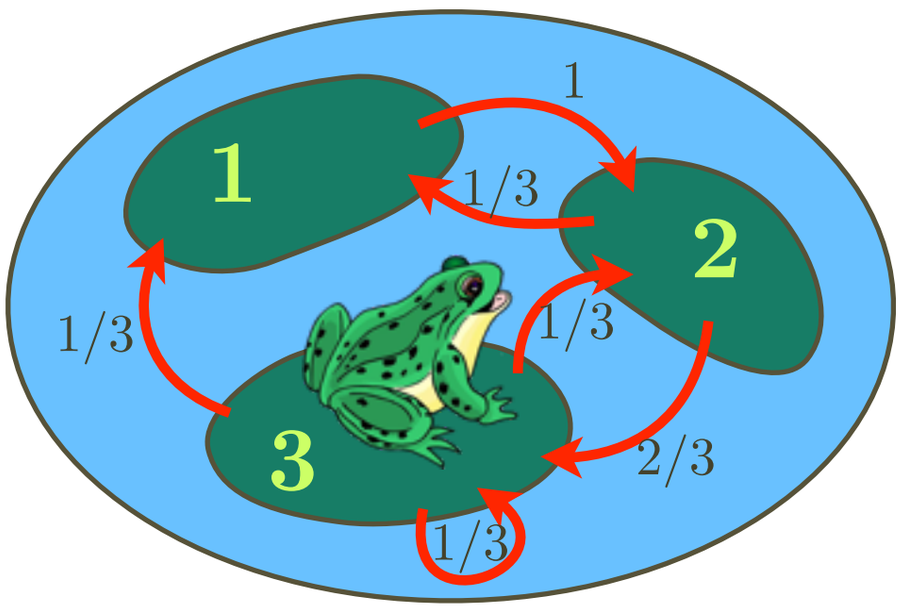
$$\pi^{(t)} = \pi^{(0)} P^t$$

Markov chain: $\mathfrak{M} = (\Omega, P)$



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Convergence



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^{20} \approx \begin{bmatrix} 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \end{bmatrix}$$

$$\forall \text{ distribution } \pi, \qquad \pi P^{20} \approx \left(\tfrac{1}{4}, \tfrac{3}{8}, \tfrac{3}{8}\right)$$

Stationary Distribution

Markov chain $\mathfrak{M} = (\Omega, P)$

- stationary distribution π:

$$\pi P = \pi \qquad \text{(fixed point)}$$

- Perron-Frobenius Theorem:
  - stochastic matrix P: $P\mathbf{1} = \mathbf{1}$
  - 1 is also a left eigenvalue of P (eigenvalue of P^T)
  - the left eigenvector $\pi P = \pi$ is nonnegative
- stationary distribution always exists

Perron-Frobenius

Perron-Frobenius Theorem:

- A : a *nonnegative* $n \times n$ matrix with spectral radius $\varrho(A)$
  - $\varrho(A) > 0$ is an eigenvalue of A;
  - there is a *nonnegative* (left and right) eigenvector associated with $\varrho(A)$;
- if further A is *irreducible*, then:
  - there is a *positive* (left and right) eigenvector associated with $\varrho(A)$ that is of multiplicity 1;
- for stochastic matrix A the spectral radius $\varrho(A)=1$.

Stationary Distribution

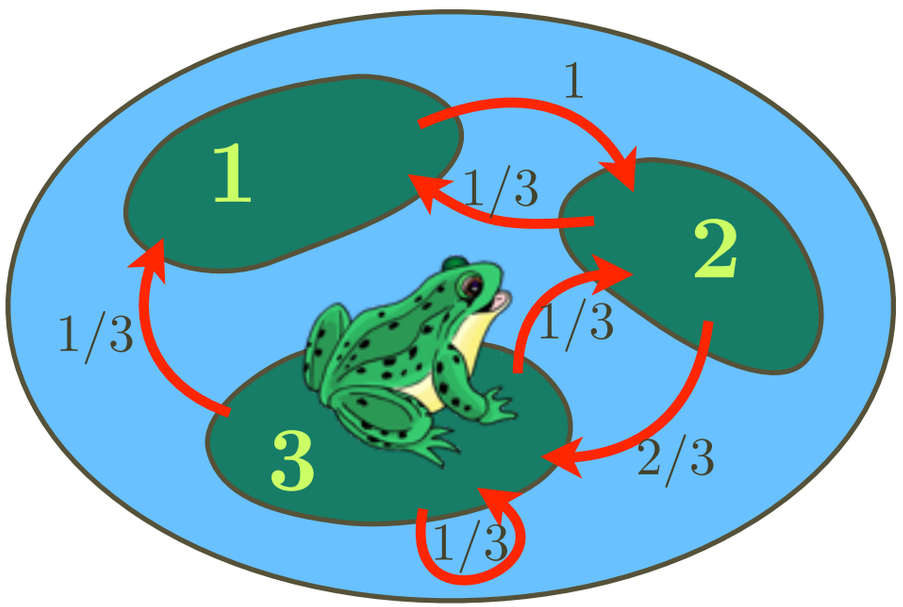
Markov chain $\mathfrak{M} = (\Omega, P)$

- stationary distribution π:

$$\pi P = \pi \qquad \text{(fixed point)}$$

- Perron-Frobenius Theorem:
  - stochastic matrix P: $P\mathbf{1} = \mathbf{1}$
  - 1 is also a left eigenvalue of P (eigenvalue of P^T)
  - the left eigenvector $\pi P = \pi$ is nonnegative
- stationary distribution always exists

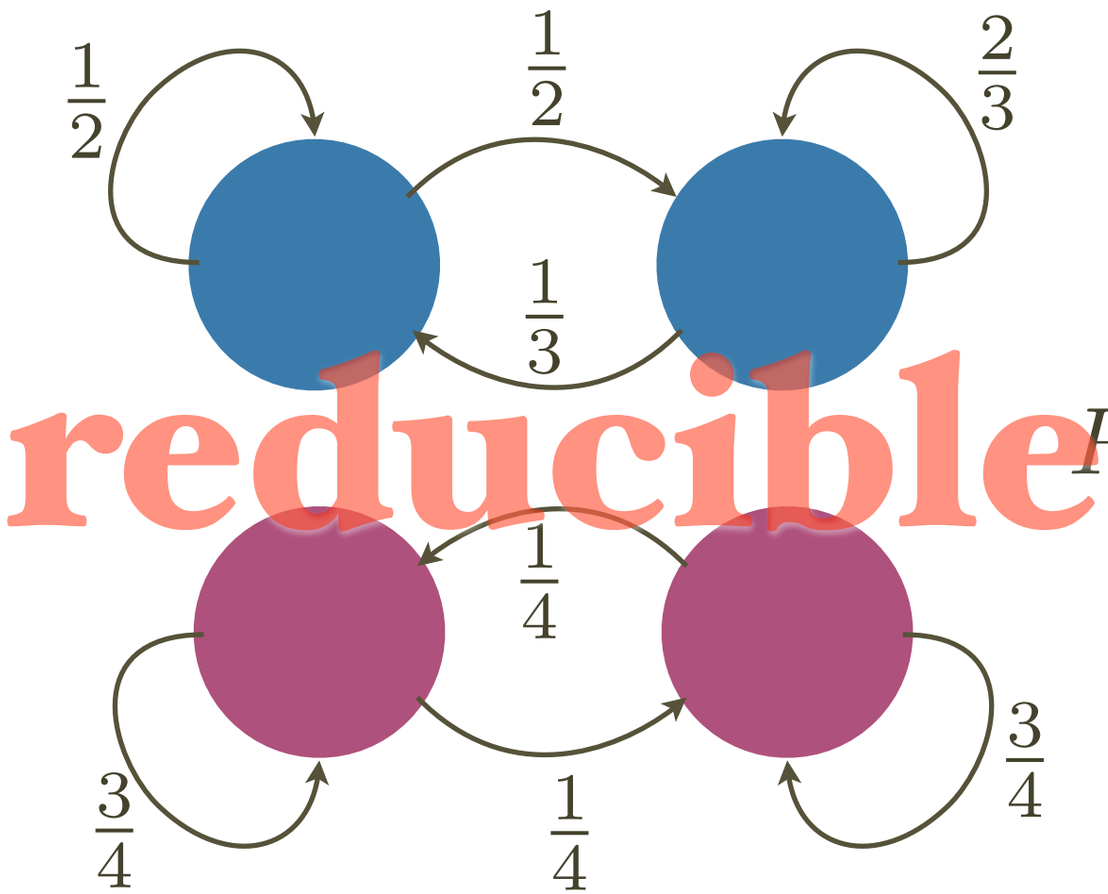
Convergence



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

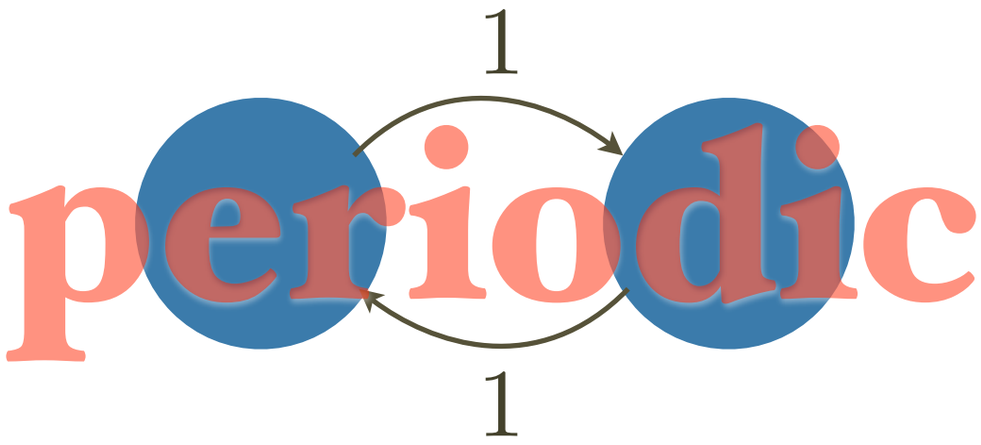
$$P^{20} \approx \begin{bmatrix} 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \end{bmatrix}$$

ergodic: convergent to stationary distribution



$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 1/4 & 3/4 \end{bmatrix}$$

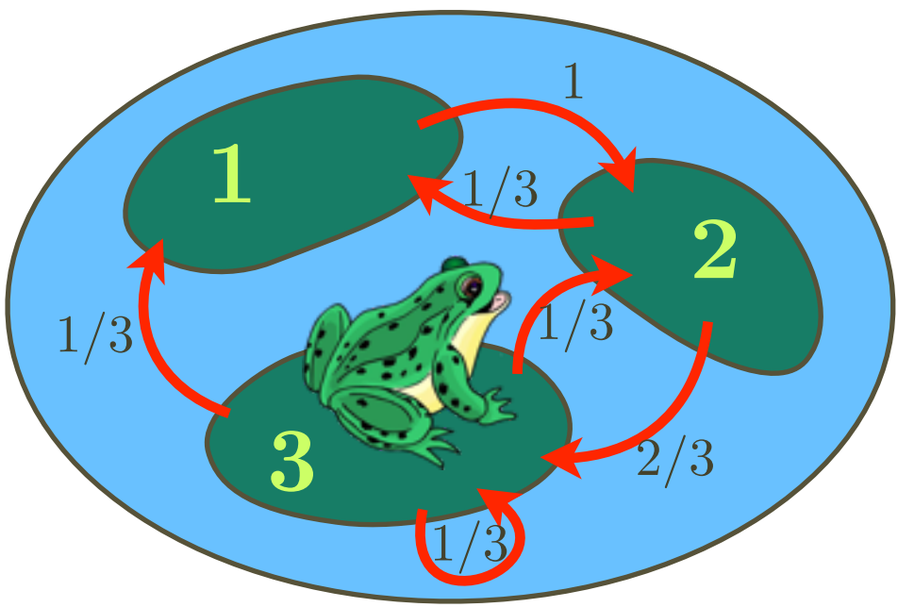
$$P^{20} \approx \begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

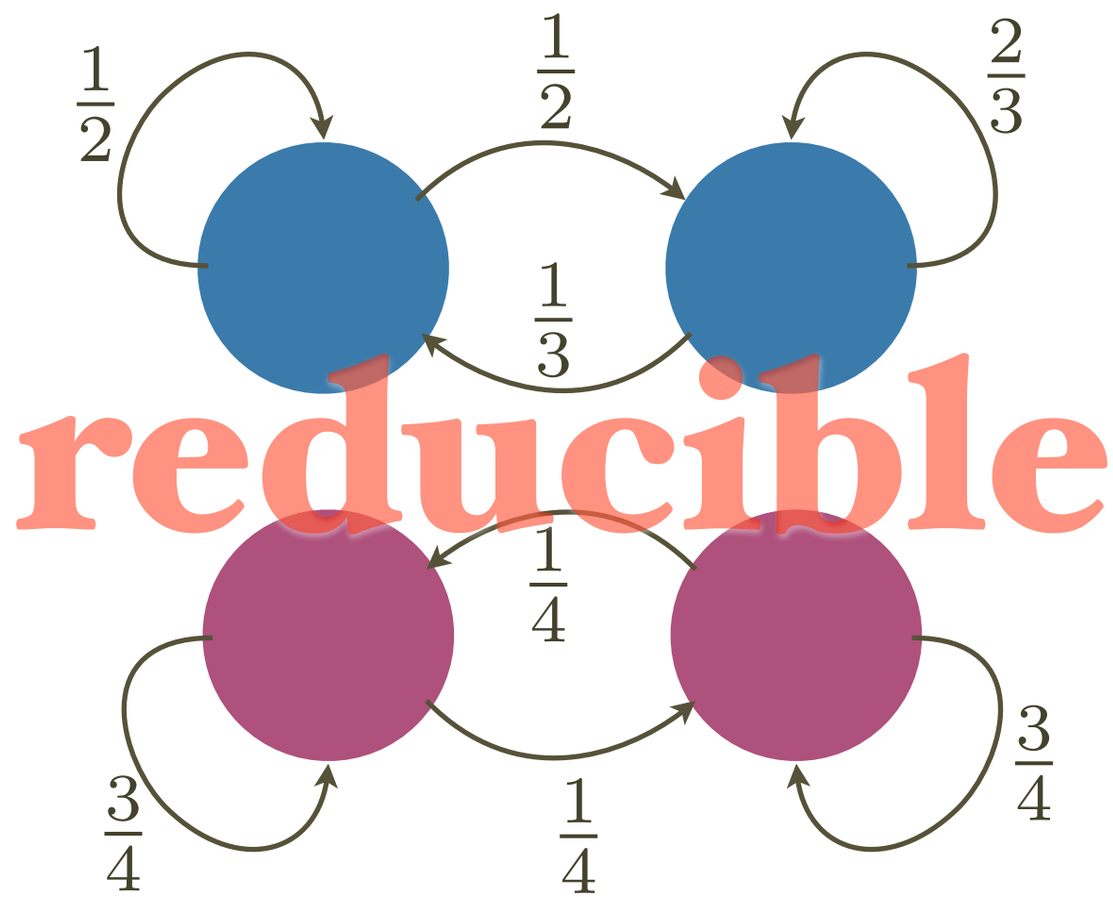


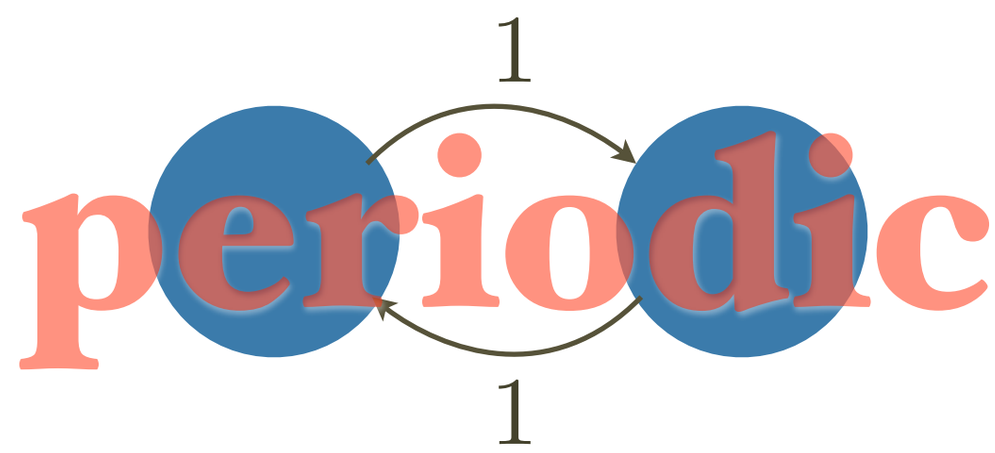
$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^{2k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P^{2k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$


1
2
3
1
1/3
1/3
1/3
2/3
1/3


reducible
1/2
1/2
2/3
1/3
1/4
3/4
1/4
3/4


periodic
1
1

Fundamental Theorem of Markov Chain:

If a *finite* Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *aperiodic*, then $\forall$ initial distribution $\pi^{(0)}$

(ergodic)

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

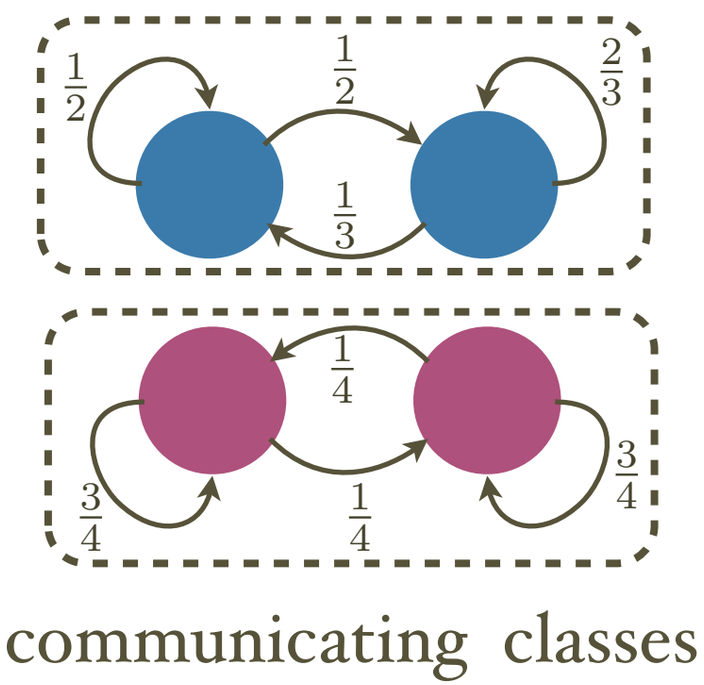
Irreducibility

- y is accessible from x:

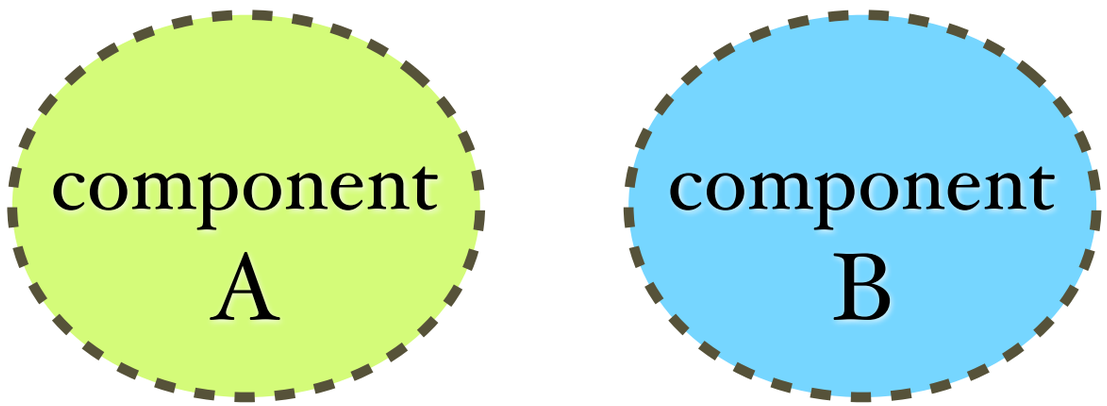
$$\exists t,\ \ P^t(x,y) > 0$$

- x communicates with y:
  - x is accessible from y
  - y is accessible from x

- MC is irreducible: all pairs of states communicate

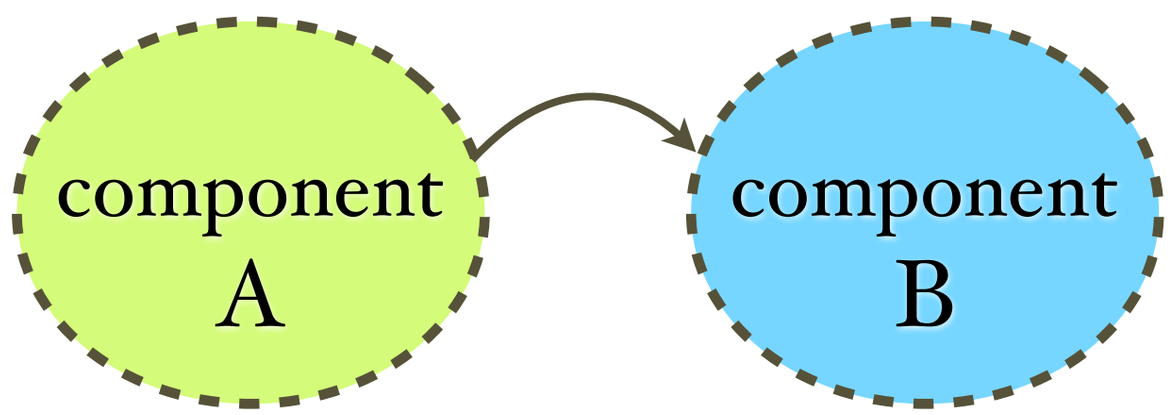


Reducible Chains



$$P = \begin{bmatrix} P_A & 0 \\ 0 & P_B \end{bmatrix}$$

stationary distributions: $\pi = \lambda \pi_A + (1 - \lambda)\pi_B$



stationary distribution: $\pi = (\mathbf{0}, \pi_B)$

Aperiodicity

- period of state *x*:

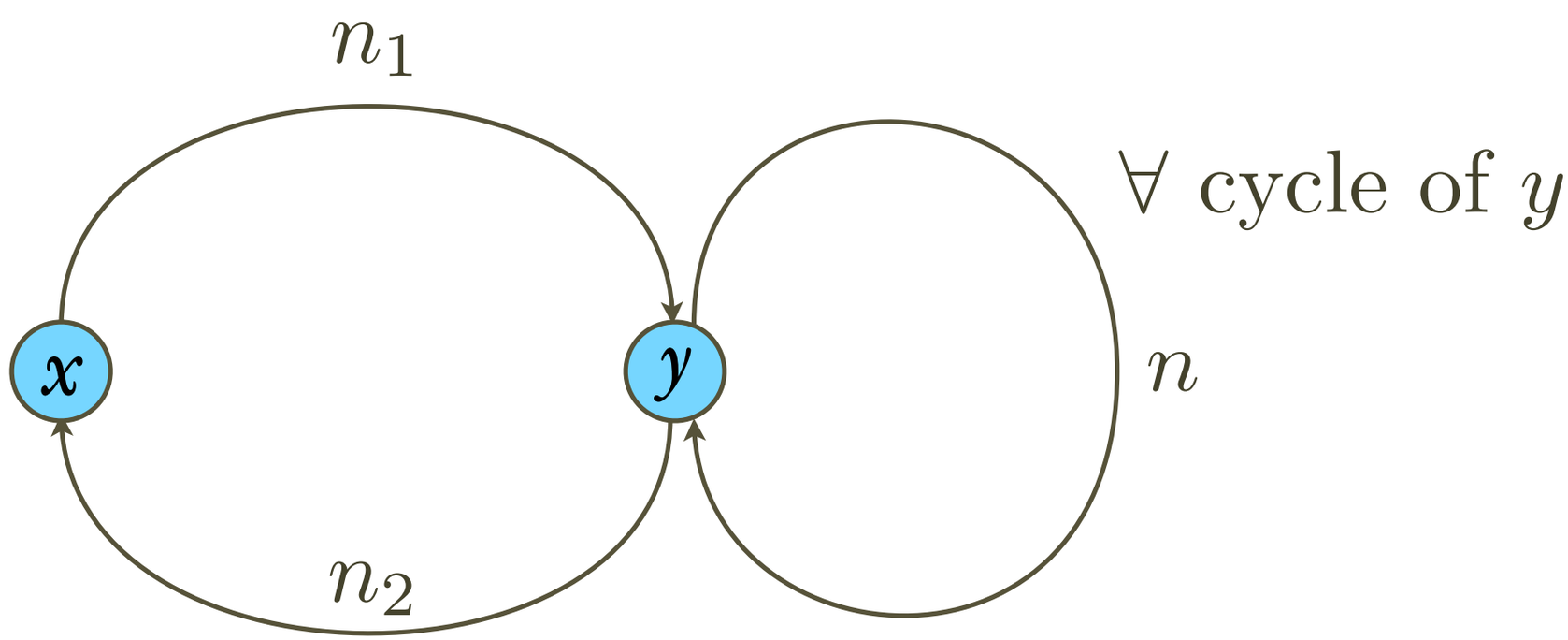
$$d_x = \gcd\{t \mid P^t(x,x) > 0\}$$

- aperiodic chain: all states have period 1
- period: the gcd of lengths of cycles

$$d_x = 3$$

Lemma (period is a class property)

x and y communicate $\Rightarrow$ $d_x = d_y$



$$\left.\begin{array}{l} d_x \mid (n_1 + n_2) \\ d_x \mid (n_1 + n_2 + n) \end{array}\right\} \Rightarrow d_x \mid n \quad \Rightarrow d_x \leq d_y$$

Fundamental Theorem of Markov Chain:

If a *finite* Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *aperiodic*, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finiteness $\Rightarrow$ existence

irreducibility $\Rightarrow$ uniqueness

} Perron-Frobenius

ergodicity $\Rightarrow$ convergence

Coupling of Markov Chains

Markov chain $\mathfrak{M} = (\Omega, P)$

initial $\pi^{(0)}$: $X_0, X_1, X_2, \ldots$

stationary π : $X'_0, X'_1, X'_2, \ldots$

faithful running of MC

$$\begin{tikzcd}
\text{initial } \pi^{(0)} : & X_0 \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, equal] & \cdots\cdots \arrow[r] \arrow[d, equal] & X_n \arrow[d, equal] \arrow[dr] & & \\
\text{stationary } \pi : & X'_0 \arrow[r] & X'_1 \arrow[r] & \cdots\cdots \arrow[r] & X'_n \arrow[r] & X'_{n+1} \arrow[r] & \cdots\cdots \quad \pi
\end{tikzcd}$$

X''_t

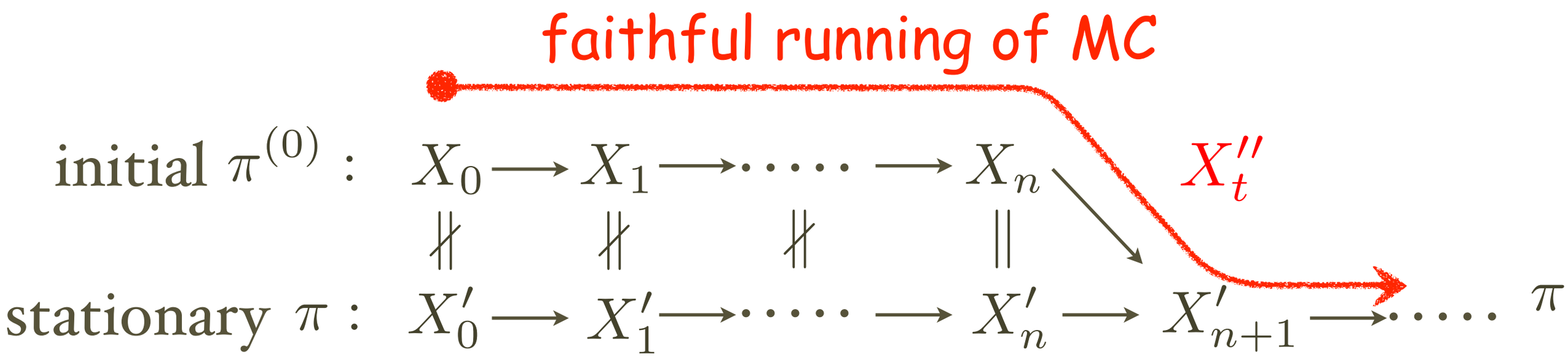
Goal: $\lim_{t \to \infty} \Pr[X''_t = X'_t] = 1$

Coupling of Markov Chains

Markov chain $\mathfrak{M} = (\Omega, P)$

initial $\pi^{(0)}$: $X_0, X_1, X_2, \ldots$

stationary π : $X'_0, X'_1, X'_2, \ldots$



Goal: conditioning on $X_0 = x, X'_0 = y$

$$\exists n, \Pr[X_n = X'_n = y] > 0$$

Markov chain $\mathfrak{M} = (\Omega, P)$

$$
\begin{tikzcd}
\text{initial } \pi^{(0)}: & X_0 \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, equal] & \cdots\cdots \arrow[r] \arrow[d, equal] & X_n \arrow[d, equal] \arrow[dr] & & \\
\text{stationary } \pi: & X'_0 \arrow[r] & X'_1 \arrow[r] & \cdots\cdots \arrow[r] & X'_n \arrow[r] & X'_{n+1} \arrow[r] & \cdots\cdots \quad \pi
\end{tikzcd}
$$

conditioning on $X_0 = x, X'_0 = y$

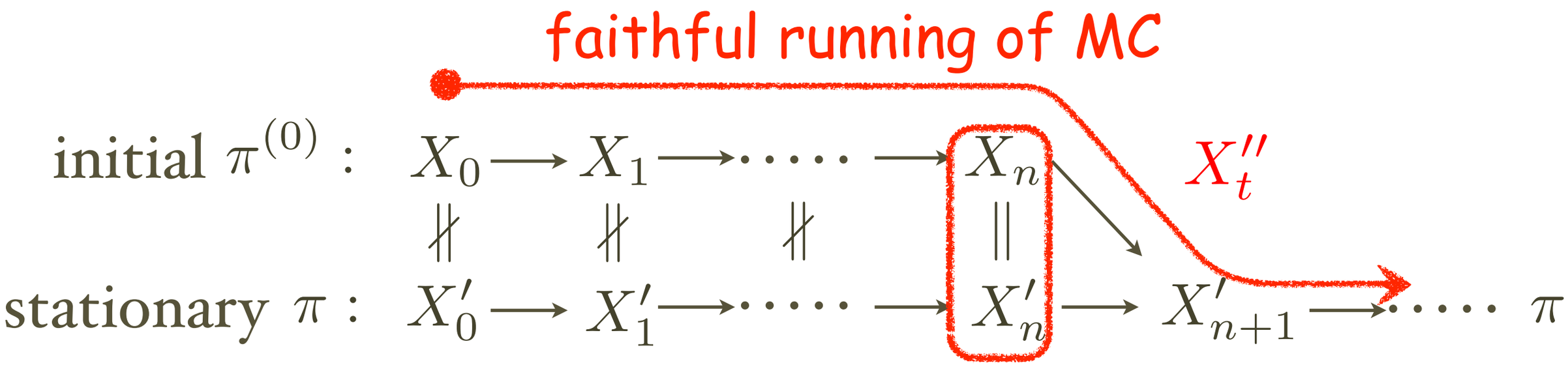
irreducible: $\exists n_1, \quad P^{n_1}(x, y) > 0$

aperiodic: $\exists n_2, \quad \begin{cases} P^{n_2}(y, y) > 0 \\ P^{n_1+n_2}(y, y) > 0 \end{cases}$

$n = n_1 + n_2$

$$
\begin{aligned}
\Pr[X_n = X'_n = y] &\geq \Pr[X_n = y] \Pr[X'_n = y] \\
&\geq P^{n_1}(x, y) P^{n_2}(y, y) P^{n}(y, y) \quad > 0
\end{aligned}
$$

Markov chain $\mathfrak{M} = (\Omega, P)$



conditioning on $X_0 = x, X_0' = y$

$$\exists n, \quad \Pr[X_n = X_n'] \geq \epsilon > 0$$

$$\lim_{t \to \infty} \Pr[X_t'' = X_t'] = 1$$

Fundamental Theorem of Markov Chain:

If a *finite* Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *aperiodic*, then $\forall$ initial distribution $\pi^{(0)}$

(ergodic)

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finiteness ⇒ existence

irreducibility ⇒ uniqueness

} Perron-Frobenius

ergodicity ⇒ convergence

Fundamental Theorem of Markov Chain:

If a Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *ergodic*, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finite: ergodic = aperiodic

infinite: ergodic $\Leftarrow$ aperiodic + *non-null persistent*

Infinite Markov Chains

- f_{xy} : prob. of ever reaching y starting from x
- h_{xy} : expected time of first reaching y starting from x

$$f_{xy}(t) \triangleq \Pr[y \notin \{X_1, \ldots, X_{t-1}\} \wedge X_t = y \mid X_0 = x]$$

$$f_{xy} \triangleq \sum_{t=1}^{\infty} f_{xy}(t) \qquad h_{xy} \triangleq \sum_{t=1}^{\infty} t f_{xy}(t)$$

- x is *transient* if $f_{xx} < 1$
- x is *recurrent* or *persistent* if $f_{xx} = 1$
  - x is *null-persistent* if x is persistent and $h_{xx} = \infty$
  - x is *non-null-persistent* if x is persistent and $h_{xx} < \infty$
- a chain is *non-null-persistent* if all states are

Fundamental Theorem of Markov Chain:

If a Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *ergodic*, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finite: ergodic = aperiodic

infinite: ergodic $\Leftarrow$ aperiodic + *non-null persistent*

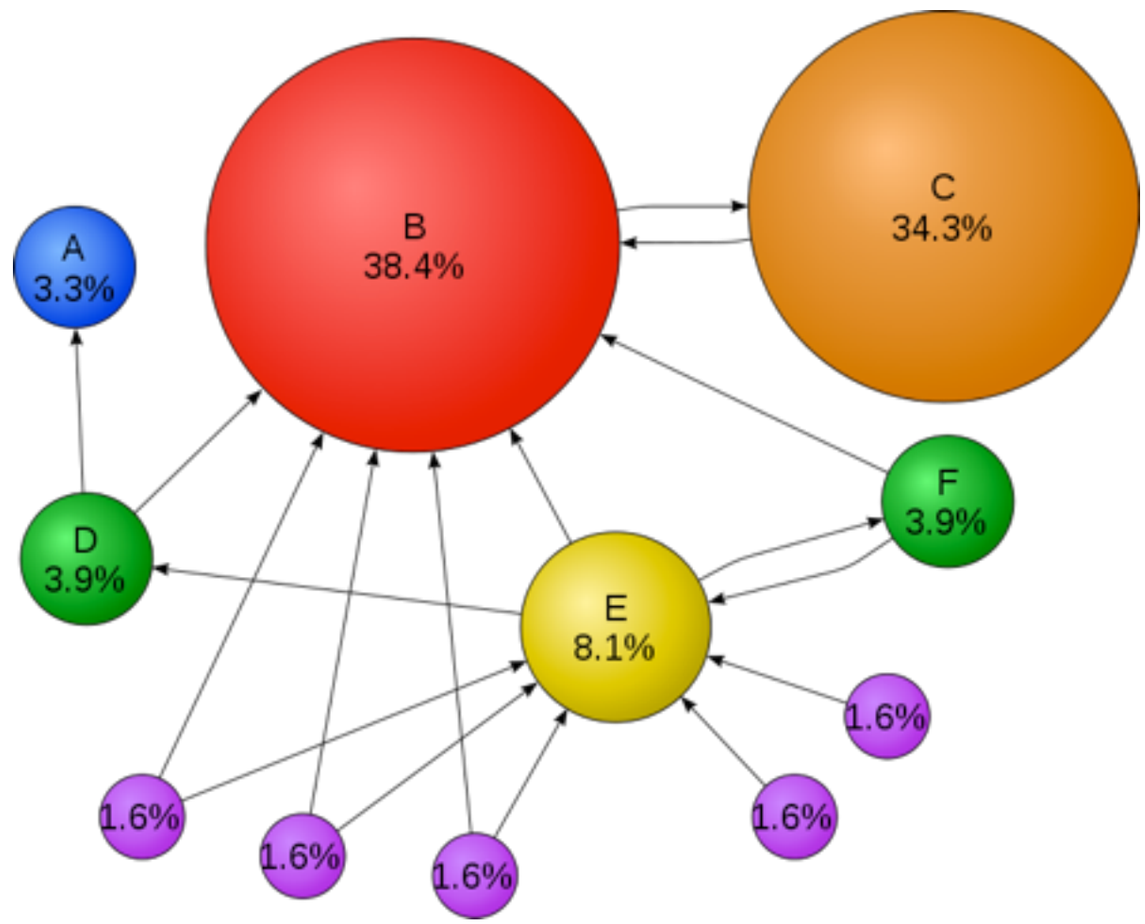
irreducible + *non-null persistent* $\Rightarrow$ $\exists$ unique stationary distribution

aperiodic + *non-null persistent* $\Rightarrow$ ergodic

PageRank

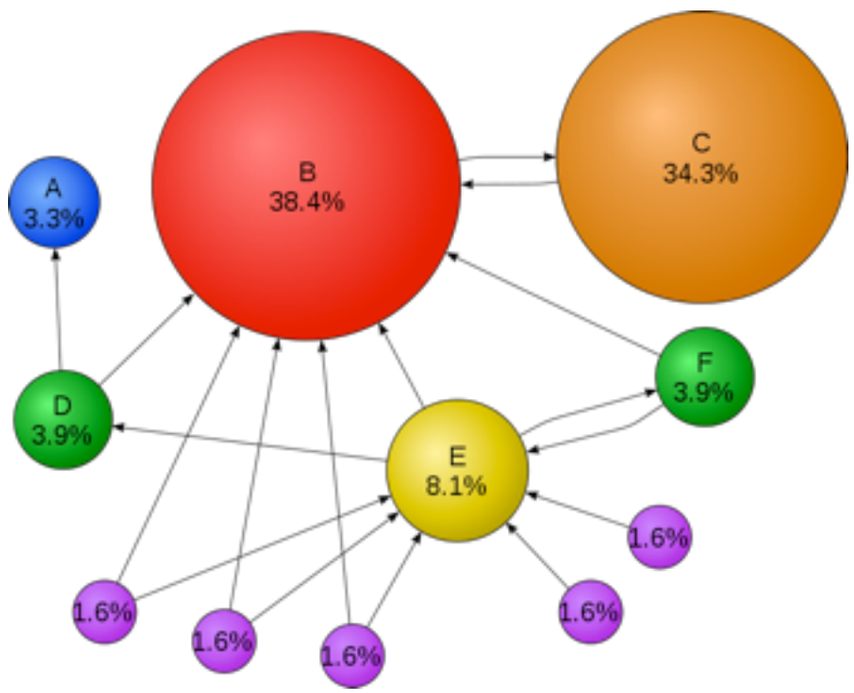
Rank: importance of a page

- A page has higher rank if pointed by more high-rank pages.
- High-rank pages have greater influence.
- Pages pointing to few others have greater influence.



PageRank (simplified)

the web graph $G(V, E)$



rank of a page: $r(v)$

$$r(v) = \sum_{u:(u,v)\in E} \frac{r(u)}{d_+(u)}$$

$d_+(u)$: out-degree of u

random walk: $$P(u,v) = \begin{cases} \frac{1}{d_+(u)} & \text{if } (u,v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

stationary distribution: $rP = r$

a tireless random surfer

Random Walk on Graph

- undirected graph $G(V, E)$
- walk: $v_1, v_2, \ldots \in V$ that $v_{i+1} \sim v_i$
- random walk: v_{i+1} is uniformly chosen from $N(v_i)$

$$P(u, v) = \begin{cases} \frac{1}{d(u)} & u \sim v \\ 0 & u \not\sim v \end{cases}$$

$$P = D^{-1} A$$

adjacency matrix A

$$D(u, v) = \begin{cases} d(u) & u = v \\ 0 & u \neq v \end{cases}$$

Random Walk on Graph

- stationary:
  - convergence (*ergodicity*);
  - stationary distribution;
- hitting time: time to reach a vertex;
- cover time: time to reach all vertices;
- mixing time: time to converge.

Random Walk on Graph

$G(V,E)$

$$P(u,v) = \begin{cases} \frac{1}{d(u)} & u \sim v \\ 0 & u \not\sim v \end{cases}$$

- for finite chain:

  irreducible and aperiodic ⇒ converge

- irreducible ⇔ G is connected

$$P^t(u,v) > 0 \iff A^t(u,v) > 0$$

- aperiodic ⇔ G is non-bipartite

  bipartite ⇒ no odd cycle ⇒ period =2

  $\left.\begin{array}{r}\text{non-bipartite} \Rightarrow \exists(2k+1)\text{-cycle} \\ \text{undirected} \Rightarrow \exists\ 2\text{-cycle}\end{array}\right\} \Rightarrow$ aperiodic

Lazy Random Walk

- undirected graph $G(V, E)$
- lazy random walk: flip a coin to decide whether to stay

$$P(u,v) = \begin{cases} \frac{1}{2} & u = v \\ \frac{1}{2d(u)} & u \sim v \\ 0 & \text{otherwise} \end{cases}$$

$$P = \frac{1}{2}(I + D^{-1}A)$$

adjacency matrix A

$$D(u,v) = \begin{cases} d(u) & u = v \\ 0 & u \neq v \end{cases}$$

always aperiodic!

Random Walk on Graph

$G(V,E)$

$$P(u,v) = \begin{cases} \frac{1}{d(u)} & u \sim v \\ 0 & u \not\sim v \end{cases}$$

Stationary distribution π:

$$\forall v \in V, \quad \pi_v = \frac{d(v)}{2m}$$

$$\sum_{v \in V} \pi_v = \sum_{v \in V} \frac{d(v)}{2m} = \frac{1}{2m} \sum_{v \in V} d(v) = 1$$

$$(\pi P)_v = \sum_{u \in V} \pi_u P(u,v) = \sum_{u \in N(v)} \frac{d(u)}{2m} \frac{1}{d(u)} = \frac{d(v)}{2m} = \pi_v$$

regular graph ⇒ uniform distribution

Random Walk on Graph

$G(V,E)$

$$P(u,v) = \begin{cases} \frac{1}{2} & u = v \\ \frac{1}{2d(u)} & u \sim v \\ 0 & \text{otherwise} \end{cases}$$

Stationary distribution π:

$$\forall v \in V, \quad \pi_v = \frac{d(v)}{2m}$$

$$(\pi P)_v = \sum_{u \in V} \pi_u P(u,v) = \frac{1}{2}\frac{d(v)}{2m} + \frac{1}{2}\sum_{u \in N(v)} \frac{d(u)}{2m}\frac{1}{d(u)}$$

$$= \frac{d(v)}{2m} = \pi_v$$

Reversibility

ergodic flow

detailed balance equation:

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

time-reversible Markov chain:

$$\exists \pi, \forall, x, y \in \Omega, \quad \pi(x)P(x,y) = \pi(y)P(y,x)$$

stationary distribution:

$$(\pi P)y = \sum_x \pi(x)P(x,y) = \sum_x \pi(y)P(y,x) = \pi(y)$$

time-reversible: when start from π

$$\Pr[X_0 = x_0 \wedge X_1 = x_1 \wedge \ldots \wedge X_n = x_n]$$

$$= \Pr[X_0 = x_n \wedge X_1 = x_{n-1} \wedge \ldots \wedge X_n = x_0]$$

Reversibility

ergodic flow

detailed balance equation:

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

time-reversible Markov chain:

$$\exists \pi, \forall, x, y \in \Omega, \quad \pi(x)P(x,y) = \pi(y)P(y,x)$$

stationary distribution:

$$(\pi P)y = \sum_x \pi(x)P(x,y) = \sum_x \pi(y)P(y,x) = \pi(y)$$

time-reversible: when start from π

$$(X_0, X_1, \ldots, X_n) \sim (X_n, X_{n-1}, \ldots, X_0)$$

Reversibility

ergodic flow

detailed balance equation:

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

time-reversible Markov chain:

$$\exists \pi, \forall, x, y \in \Omega, \quad \pi(x)P(x,y) = \pi(y)P(y,x)$$

stationary distribution:

$$(\pi P)y = \sum_x \pi(x)P(x,y) = \sum_x \pi(y)P(y,x) = \pi(y)$$

the Markov chain can be uniquely represented as an edge-weighted undirected graph $G(V,E)$ with ergodic flows as edge weights

Random Walk on Graph

$G(V,E)$

$$P(u,v) = \begin{cases} \frac{1}{d(u)} & u \sim v \\ 0 & u \not\sim v \end{cases}$$

$$P(u,v) = \begin{cases} \frac{1}{2} & u = v \\ \frac{1}{2d(u)} & u \sim v \\ 0 & \text{otherwise} \end{cases}$$

detailed balance equation:

$$\pi(u)P(u,v) = \pi(v)P(v,u)$$

$u = v:$

$u \not\sim v:$

hold for free

$u \sim v:$

$$\pi(u) \propto \frac{1}{P(u,v)}$$

Random Walk on Graph

- undirected graph $G(V, E)$ (not necessarily regular)
- lazy random walk

Goal: a uniform stationary distribution π

Goal: an arbitrary stationary distribution π

Random Walk

- stationary:
  - convergence (*ergodicity*);
  - stationary distribution;
- hitting time: time to reach a vertex;
- cover time: time to reach all vertices;
- mixing time: time to converge.

Hitting and Covering

consider a random walk on $G(V,E)$

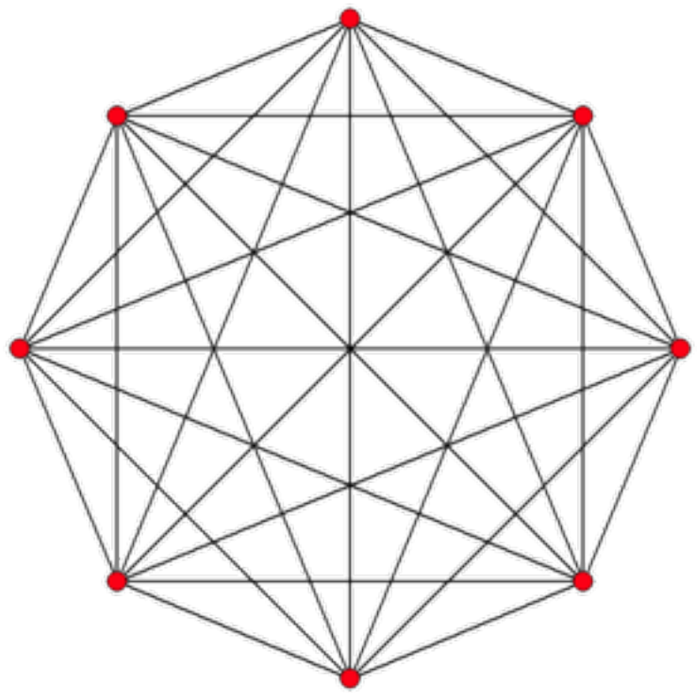
- hitting time: expected time to reach v from u

$$\tau_{u,v} = \mathbf{E}\Big[\, \min\{n > 0 \mid X_n = v\} \,\Big|\, X_0 = u \,\Big]$$

- cover time: expected time to visit all vertices

$$C_u = \mathbf{E}\Big[\, \min\{n \mid \{X_0, \ldots, X_n\} = V\} \,\Big|\, X_0 = u \,\Big]$$

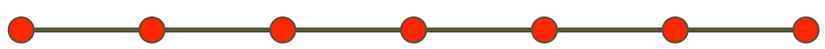
$$C(G) = \max_{u \in V} C_u$$



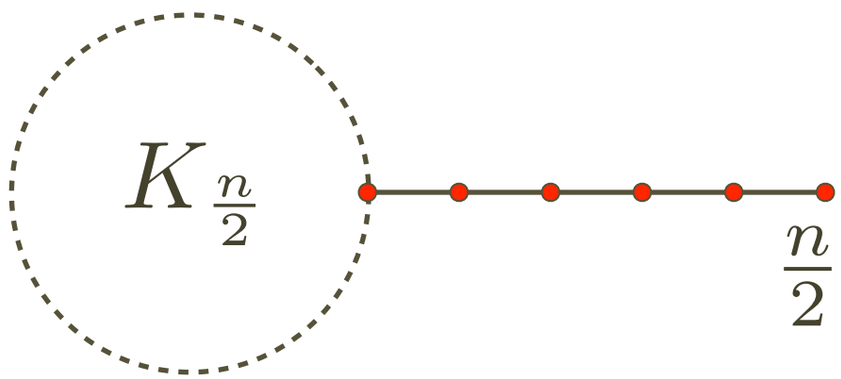
$$G = K_n$$

$$C(G) = \Theta(n \log n)$$

G : path or cycle



$$C(G) = \Theta(n^2)$$



G : "lollipop"

$$C(G) = \Theta(n^3)$$

Hitting Time

$$\tau_{u,v} = \mathbf{E}\Big[\ \min\{n > 0 \mid X_n = v\} \ \Big|\ X_0 = u\Big]$$

stationary distribution π $\qquad \pi_v = \dfrac{d(v)}{2m}$

Renewal Theorem:

irreducible $\Rightarrow$ $\tau_{v,v} = \dfrac{1}{\pi_v} = \dfrac{2m}{d(v)}$

Renewal Theorem:

irreducible $\Rightarrow$ $\tau_{v,v} = \dfrac{1}{\pi_v} = \dfrac{2m}{d(v)}$

$\pi \sim X_0, X_1, X_2, \ldots$

T_x : first t>0 that $X_t = x$

integer $X \geq 0$

$$\mathbb{E}[X] = \sum_{n \geq 0} \Pr[X > n]$$

$$\pi(x) = \Pr[X_0 = x]$$

$$\tau_{x,x} = \mathbb{E}[T_x \mid X_0 = x] = \sum_{n \geq 1} \Pr[T_x \geq n \mid X_0 = x]$$

$$\tau_{x,x}\pi_x = \sum_{n \geq 1} \Pr[T_x \geq n \mid X_0 = x] \Pr[X_0 = x]$$

$$= \sum_{n \geq 1} \Pr[T_x \geq n \wedge X_0 = x]$$

$$= \Pr[X_0 = x] + \sum_{n \geq 2} (\Pr[\forall 1 \leq t \leq n-1, X_t \neq x] - \Pr[\forall 0 \leq t \leq n-1, X_t \neq x])$$

$$= \Pr[X_0 = x] + \sum_{n \geq 2} (\Pr[\forall 0 \leq t \leq n-2, X_t \neq x] - \underbrace{\Pr[\forall 0 \leq t \leq n-1, X_t \neq x]}_{a_{n-1}})$$

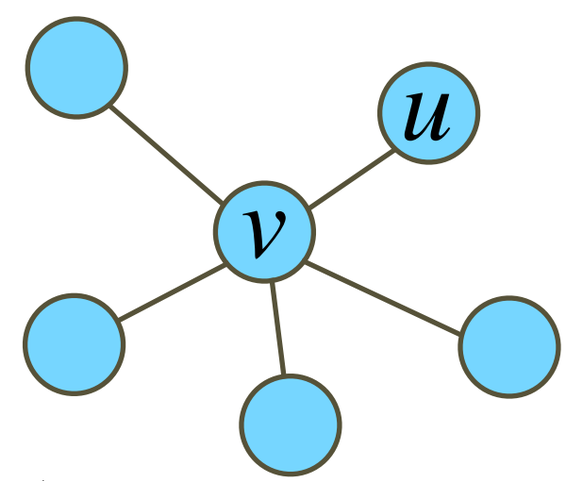
$$= \Pr[X_0 = x] + \Pr[X_0 \neq x] - \lim_{n \to \infty} a_n = 1$$ (irreducible)

random walk on $G(V,E)$

$$\tau_{v,v} = \frac{1}{\pi_v} = \frac{2m}{d(v)}$$

Lemma: $uv \in E \Rightarrow \tau_{u,v} < 2m$

$$\tau_{v,v} = \sum_{wv \in E} \frac{1}{d(v)}(1 + \tau_{w,v})$$



$$\Rightarrow \quad 2m = \sum_{wv \in E} (1 + \tau_{w,v}) \quad \Rightarrow \quad \tau_{u,v} < 2m$$

Cover Time

$$C_u = \mathbf{E}\Big[\ \min\{n \mid \{X_0, \ldots, X_n\} = V\} \ \Big|\ X_0 = u\Big] \qquad C(G) = \max_{u \in V} C_u$$

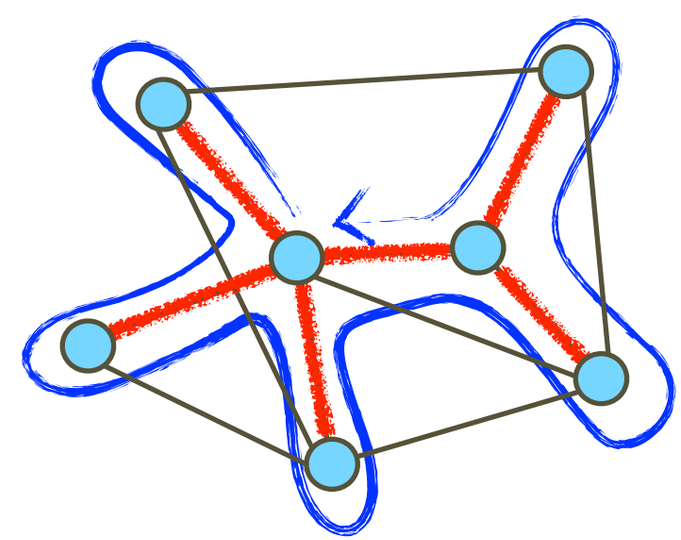
Theorem: $C(G) \leq 4nm$

pick a spanning tree T of G

Eulerian tour: $v_1 \to v_2 \to \cdots \to v_{2(n-1)} \to v_{2n-1} = v_1$

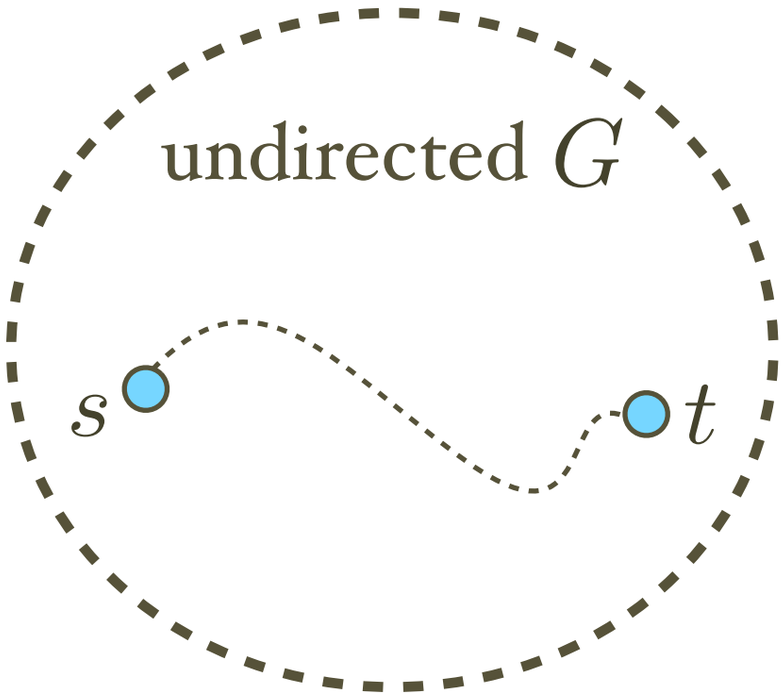
$$C(G) \leq \sum_{i=1}^{2(n-1)} \tau_{v_i, v_{i+1}}$$

$$< 4nm$$



USTCON
(undirected *s*-*t* connectivity)

- Instance:
  - undirected $G(V, E)$;
  - vertices: s, t
- s-t connected ?
- deterministic:
  - traverse: linear space
- log-space ?



Theorem (Aleliunas-Karp-Lipton-Lovász-Rackoff 1979)

USTCON can be solved by a poly-time Monte Carlo randomized algorithm with bounded one-sided error, which uses O(log n) extra space.

- start a random walk at s;
- if reach t in $4n^3$ steps
  return "yes"
- else return "no"

space: O(log n)

unconnected ⇒ "no"

connected ⇒

Cover time:

$$C(G) \leq 4nm < 2n^3$$

Markov's inequality ⇒

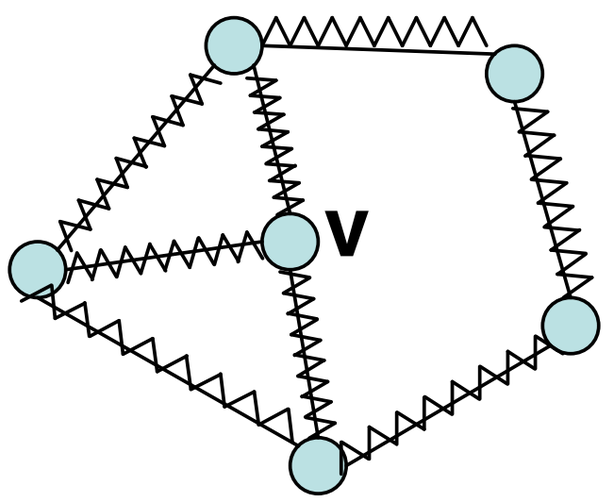
Pr["no"] < 1/2

Electric Network

edge uv : resistance R_{uv}

vertex v : potential ϕ_v

edge orientation $u \to v$: current flow $C_{u \to v}$



potential difference

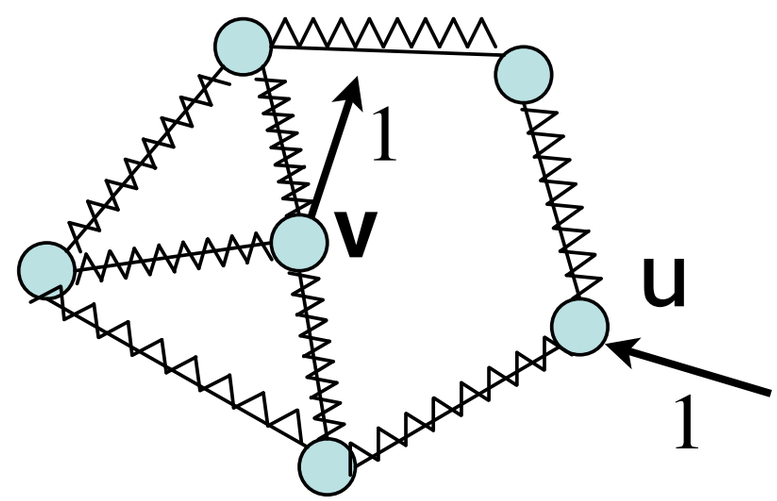
$$\phi_{u,v} = \phi_u - \phi_v$$

Kirchhoff's Law: $\forall$ vertex *v*, flow-in = flow-out

Ohm's Law: $\forall$ edge *uv*, $C_{u \to v} = \dfrac{\phi_{u,v}}{R_{uv}}$

Effective Resistance

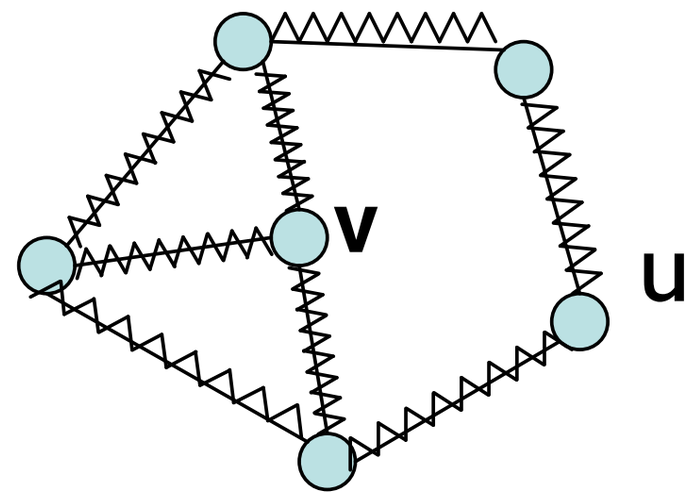
electrical network:



effective resistance $R(u,v)$:

potential difference between u and v
required to send 1 unit of flow current
from u to v

each edge e
resistance
$R_e = 1$

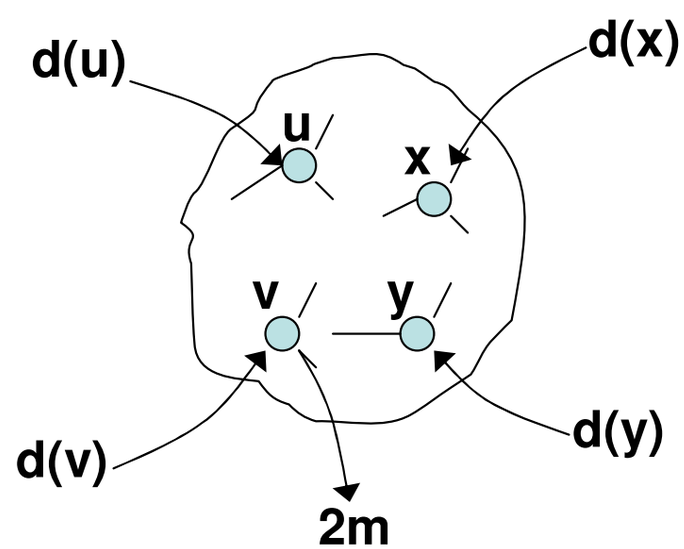


Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

$$\forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u,v)$$

graph $G(V,E)$

construct an electrical network:



for every edge e :

$$R_e = 1$$

each vertex *u* :

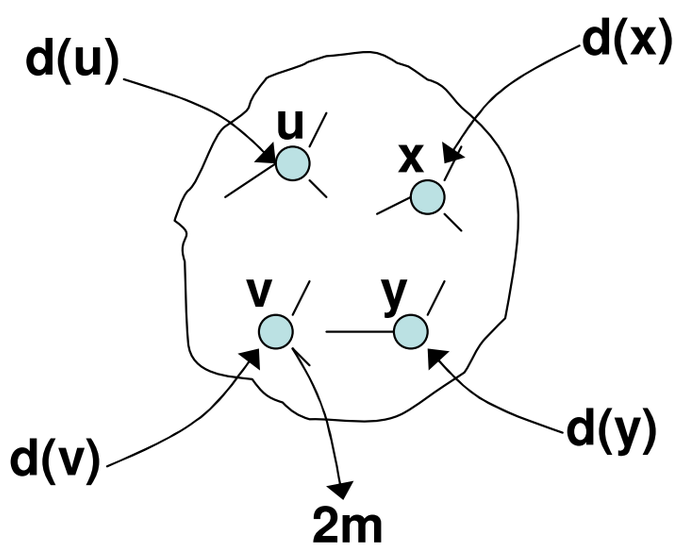
inject *d*(*u*) units current flow into *u*

a special vertex *v* :

remove all $2m$ units current flow from *v*

Lemma: $\forall u \in V, \quad \phi_{u,v} = \tau_{u,v}$

Lemma: $\forall u \in V, \quad \phi_{u,v} = \tau_{u,v}$



$$d(u) = \sum_{uw \in E} C_{u \to w} \quad \text{(Kirchhoff)}$$

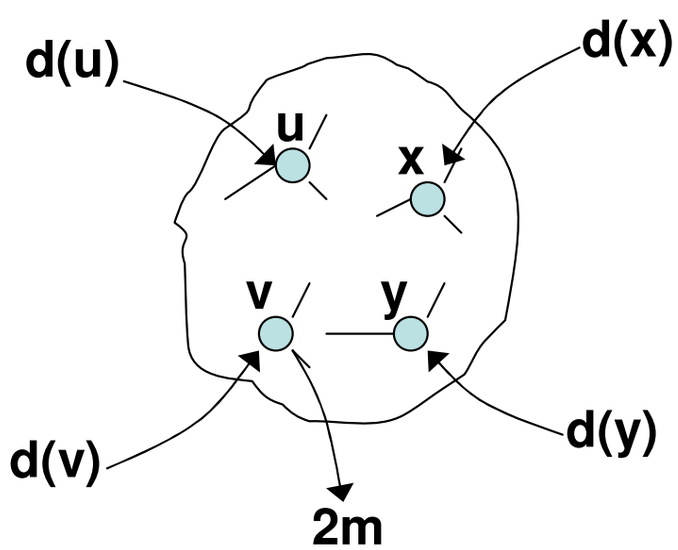
$$= \sum_{uw \in E} \phi_{u,w} \quad \text{(Ohm)}$$

$$= \sum_{uw \in E} (\phi_{u,v} - \phi_{w,v})$$

$$= d(u)\phi_{u,v} - \sum_{uw \in E} \phi_{w,v}$$

$$\Rightarrow \quad \phi_{u,v} = 1 + \frac{1}{d(u)} \sum_{uw \in E} \phi_{w,v}$$

Lemma: $\forall u \in V, \quad \phi_{u,v} = \tau_{u,v}$



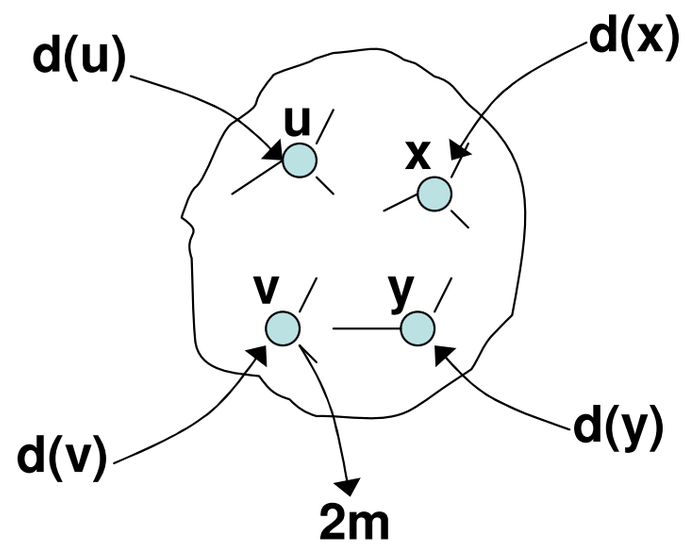
$$\phi_{u,v} = 1 + \frac{1}{d(u)} \sum_{uw \in E} \phi_{w,v}$$

$$\tau_{u,v} = \mathbf{E}\Big[\min\{n > 0 \mid X_n = v\} \,\Big|\, X_0 = u \Big]$$

$$\tau_{u,v} = \sum_{wu \in E} \frac{1}{d(u)} (1 + \tau_{w,v})$$

$$= 1 + \frac{1}{d(u)} \sum_{wu \in E} \tau_{w,v}$$

Lemma: $\forall u \in V, \quad \phi_{u,v} = \tau_{u,v}$

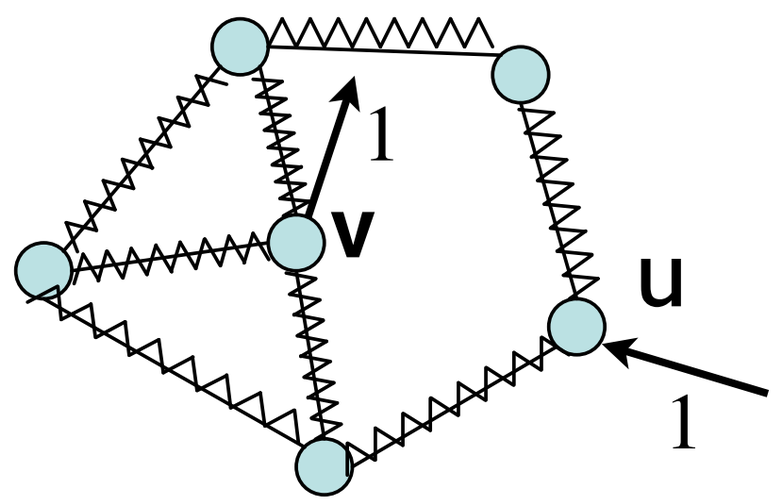


$$\left.\begin{aligned} \phi_{u,v} &= 1 + \frac{1}{d(u)} \sum_{uw \in E} \phi_{w,v} \\ \tau_{u,v} &= 1 + \frac{1}{d(u)} \sum_{wu \in E} \tau_{w,v} \end{aligned}\right\} \text{ has the same unique solution}$$

Effective Resistance

electrical network:

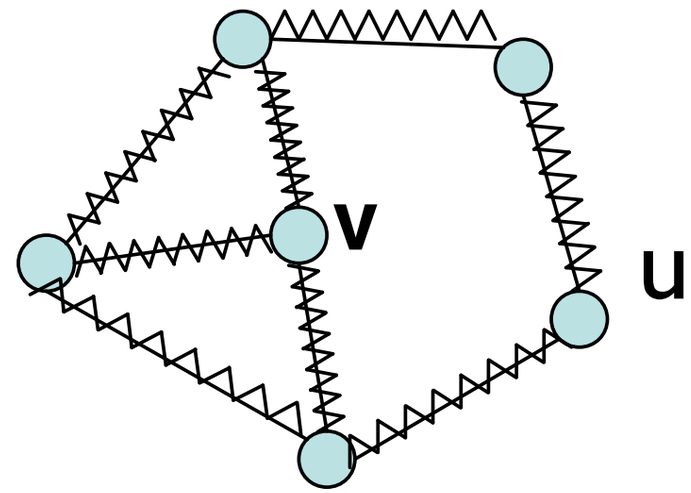
each edge e
resistance
$R_e = 1$



effective resistance $R(u,v)$:

potential difference between u and v
required to send 1 unit of flow current
from u to v

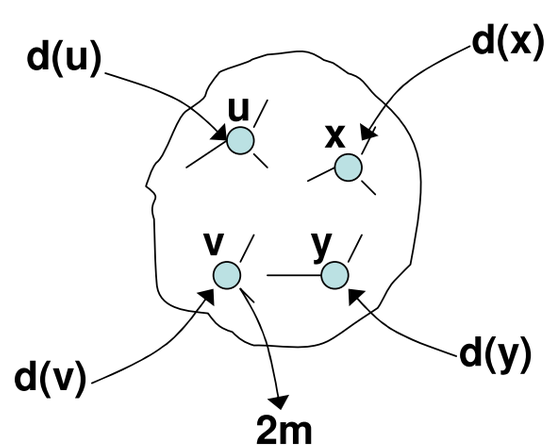
each edge e
resistance
$R_e = 1$



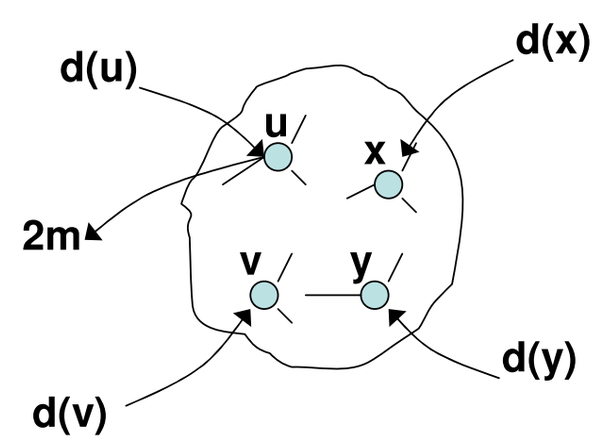
Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

$$\forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u,v)$$

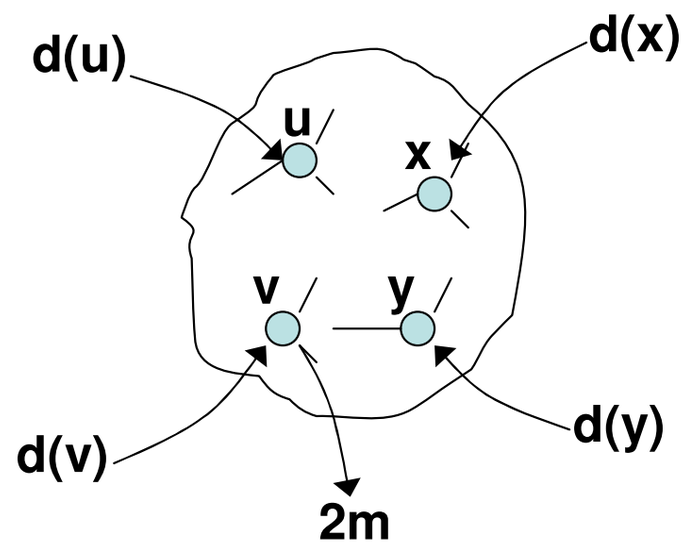
A:



B:



Lemma: $\forall u \in V, \quad \phi_{u,v} = \tau_{u,v}$

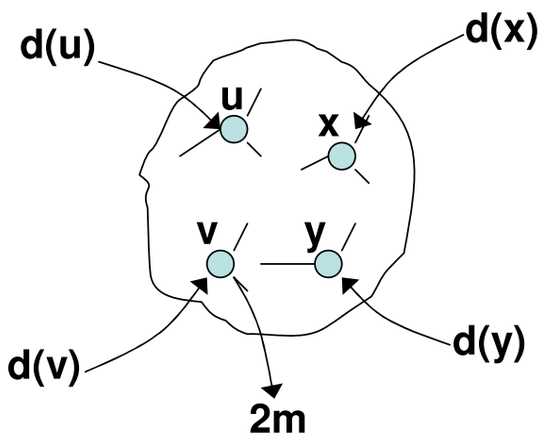


$$\left.\begin{aligned} \phi_{u,v} &= 1 + \frac{1}{d(u)} \sum_{uw \in E} \phi_{w,v} \\ \tau_{u,v} &= 1 + \frac{1}{d(u)} \sum_{wu \in E} \tau_{w,v} \end{aligned}\right\} \text{ has the same unique solution}$$

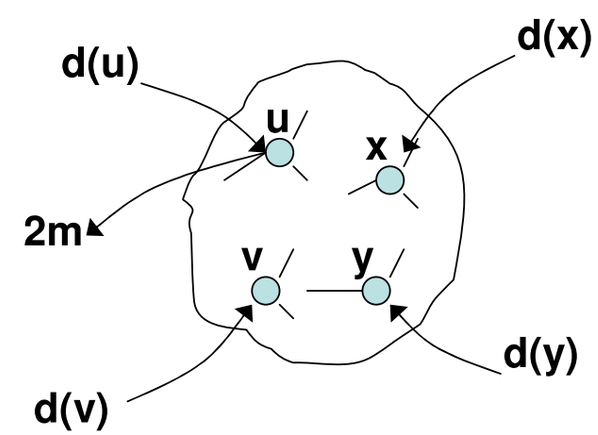
Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

$$\forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u,v)$$

A:



B:



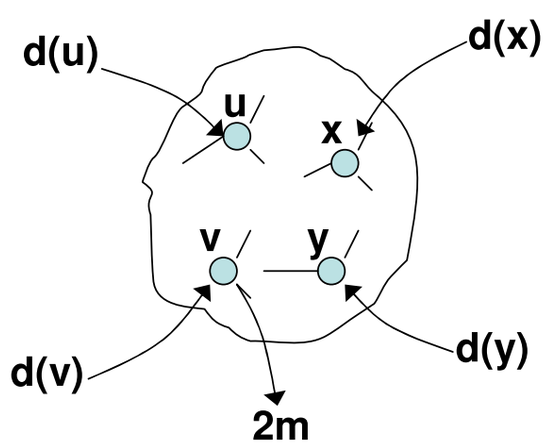
$$\phi^A_{u,v} = \tau_{u,v}$$

$$\phi^B_{v,u} = \tau_{v,u}$$

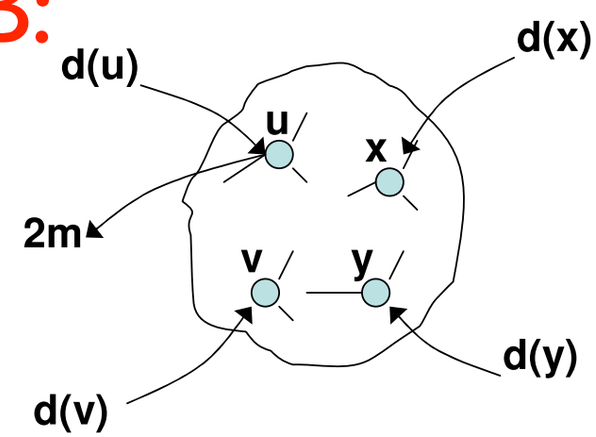
Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

$$\forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u,v)$$

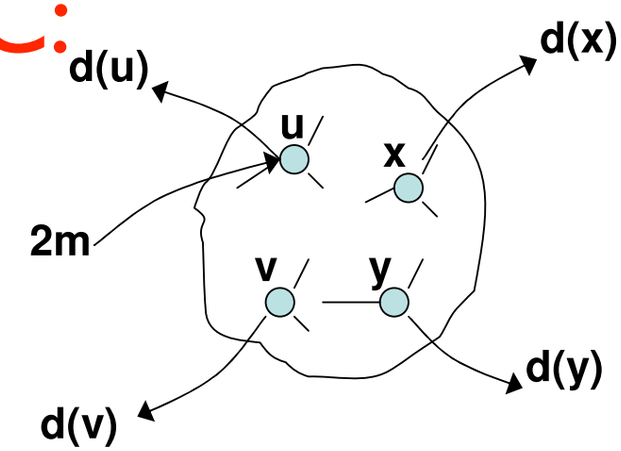
A:



B:

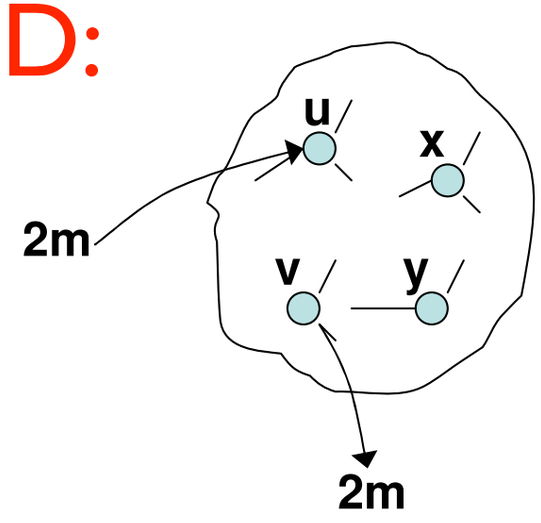


C:



$$\phi^A_{u,v} = \tau_{u,v} \qquad \phi^B_{v,u} = \tau_{v,u} \qquad \phi^C_{u,v} = \phi^B_{v,u} = \tau_{v,u}$$

D:



D = A + C

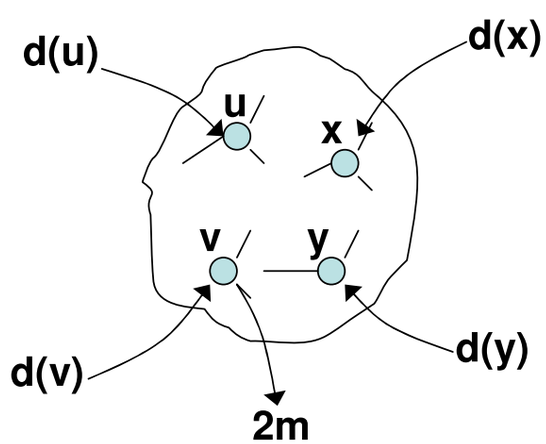
$$\phi^D_{u,v} = \phi^A_{u,v} + \phi^C_{u,v} \quad = \tau_{u,v} + \tau_{v,u}$$

$\phi^D_{u,v}$: potential difference between *u* and *v* to send $2m$ units current flow from *u* to *v*

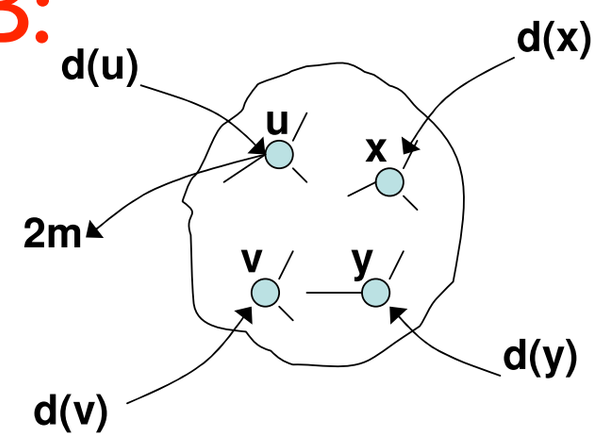
Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

$$\forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u,v)$$

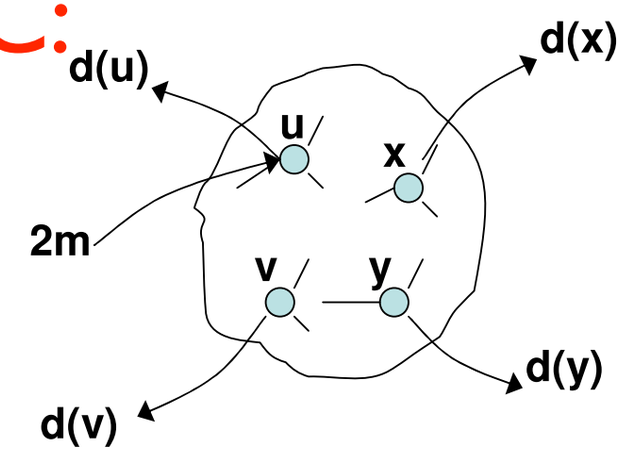
A:



B:

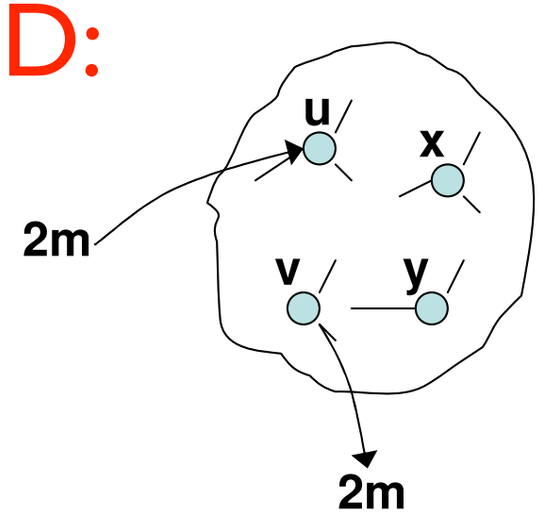


C:



$$\phi^A_{u,v} = \tau_{u,v} \qquad \phi^B_{v,u} = \tau_{v,u} \qquad \phi^C_{u,v} = \phi^B_{v,u} = \tau_{v,u}$$

D:



D = A + C

$$\phi^D_{u,v} = \phi^A_{u,v} + \phi^C_{u,v} = \tau_{u,v} + \tau_{v,u}$$

$$R(u,v) = \frac{\phi^D_{u,v}}{2m}$$

Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

$$\forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u,v)$$

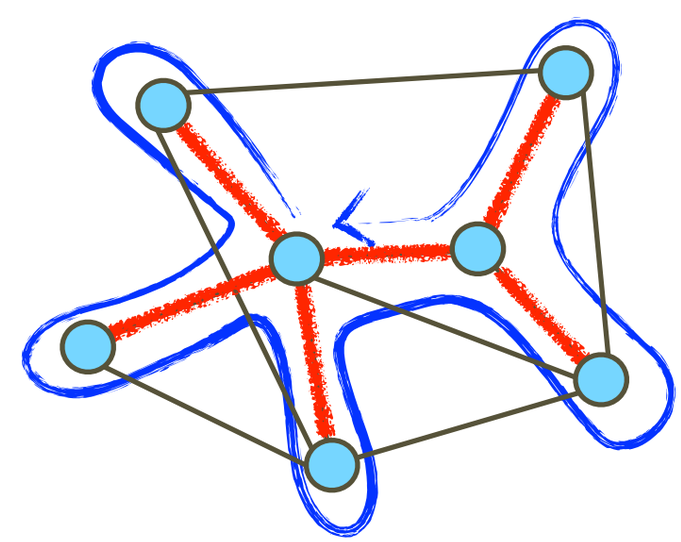
Theorem: $C(G) \leq 2nm$

pick a spanning tree T of G

Eulerian tour: $v_1 \to v_2 \to \cdots \to v_{2(n-1)} \to v_{2n-1} = v_1$

$$C(G) \leq \sum_{i=1}^{2(n-1)} \tau_{v_i, v_{i+1}} = \sum_{uv \in T} (\tau_{u,v} + \tau_{v,u})$$

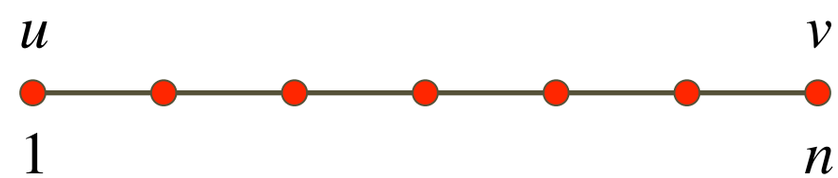
$$= 2m \sum_{uv \in T} R(u,v) \leq 2mn$$



Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

$$\forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u,v)$$

G: path



$$C(G) \leq 2nm = 2n^2 \ = O(n^2)$$

$$C(G) = \Theta(n^2)$$

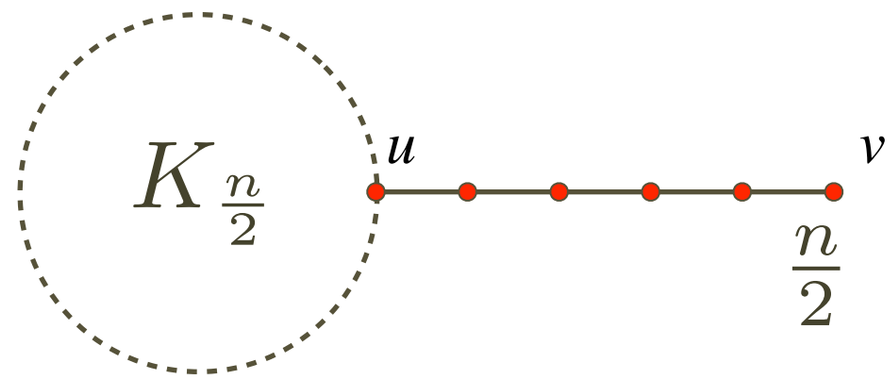
$$C(G) \geq \tau_{u,v} \ = \tau_{v,u} \ = \Omega(n^2)$$

$$\tau_{u,v} + \tau_{v,u} = 2mR(u,v) \ = 2n(n-1)$$

Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

$$\forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u,v)$$

G: lollipop



$$C(G) \leq 2nm \ = O(n^3)$$

$$C(G) = \Theta(n^3)$$

$$C(G) \geq \tau_{u,v} \ = \Omega(n^3)$$

$$\tau_{u,v} + \tau_{v,u} = 2mR_{u,v} \ = \Omega(n^3) \qquad \tau_{v,u} = O(n^2)$$