Probability & Computing
Stochastic Process

\[ \{X_t \mid t \in T\} \quad X_t \in \Omega \]

- **time** \( t \)  
- **state space** \( \Omega \)  
- **state** \( x \in \Omega \)

- **discrete time:**
  
  \( T \) is countable  
  \( T = \{0, 1, 2, \ldots\} \)

- **discrete space:**
  
  \( \Omega \) is finite or countably infinite

\[ X_0, X_1, X_2, \ldots \]
Markov Property

- dependency structure of $X_0, X_1, X_2, \ldots$
- **Markov property:** (memoryless)

\[ X_{t+1} \] depends only on \[ X_t \]

\[
\forall t = 0, 1, 2, \ldots, \forall x_0, x_1, \ldots, x_{t-1}, x, y \in \Omega
\]

\[
\Pr[X_{t+1} = y \mid X_0 = x_0, \ldots, X_{t-1} = x_{t-1}, X_t = x] = \Pr[X_{t+1} = y \mid X_t = x]
\]

**Markov chain:** discrete time discrete space stochastic process with Markov property.
Transition Matrix

Markov chain: \( X_0, X_1, X_2, \ldots \)

\[
\Pr[X_{t+1} = y \mid X_0 = x_0, \ldots, X_{t-1} = x_{t-1}, X_t = x] = \Pr[X_{t+1} = y \mid X_t = x] = P_{xy}^{(t)} = P_{xy}
\]

(time-homogenous)

\[
P = \begin{bmatrix}
    P_{x_1y_1} & \cdots & P_{x_1y_y} \\
    \vdots & \ddots & \vdots \\
    P_{x_ny_1} & \cdots & P_{x_ny_y}
\end{bmatrix}
\]

\(x \in \Omega\) \hspace{1cm} \(y \in \Omega\)

stochastic matrix \( P1 = 1 \)
chain: \( X_0, X_1, X_2, \ldots \)

distribution: \[
\begin{align*}
\pi^{(0)} & \quad \pi^{(1)} & \quad \pi^{(2)} & \quad \in [0, 1]^\Omega \\
\sum_{x \in \Omega} \pi_x &= 1
\end{align*}
\]

\[
\pi_x^{(t)} = \Pr[X_t = x]
\]

\[
\pi^{(t+1)} = \pi^{(t)} P
\]

\[
\begin{align*}
\pi_y^{(t+1)} &= \Pr[X_{t+1} = y] \\
&= \sum_{x \in \Omega} \Pr[X_t = x] \Pr[X_{t+1} = y \mid X_t = x] \\
&= \sum_{x \in \Omega} \pi_x^{(t)} P_{xy} \\
&= (\pi^{(t)} P)_y
\end{align*}
\]
\[ \pi^{(0)} \xrightarrow{P} \pi^{(1)} \xrightarrow{P} \ldots \pi^{(t)} \xrightarrow{P} \pi^{(t+1)} \xrightarrow{P} \ldots \]

- initial distribution: \( \pi^{(0)} \)

- transition matrix: \( P \)

\[ \pi^{(t)} = \pi^{(0)} P^t \]

Markov chain: \( \mathcal{M} = (\Omega, P) \)
$P = \begin{bmatrix}
0 & 1 & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}$
Convergence

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
1/3 & 0 & 2/3 \\
1/3 & 1/3 & 1/3
\end{bmatrix}
\]

\[
P^{30} \approx \begin{bmatrix}
0.2500 & 0.3050 & 0.4450 \\
0.2500 & 0.3550 & 0.3900 \\
0.2500 & 0.3560 & 0.3830
\end{bmatrix}
\]

\forall \text{ distribution } \pi, \quad \pi P^{20} \approx \left( \frac{1}{4}, \frac{3}{8}, \frac{3}{8} \right)
Stationary Distribution

Markov chain \( \mathcal{M} = (\Omega, P) \)

- stationary distribution \( \pi \):
  \[ \pi P = \pi \]  
  (fixed point)

- Perron-Frobenius Theorem:
  - stochastic matrix \( P \): \( P1 = 1 \)
  - 1 is also a left eigenvalue of \( P \) (eigenvalue of \( P^T \))
  - the left eigenvector \( \pi P = \pi \) is nonnegative

- stationary distribution always exists
Perron-Frobenius Theorem:

- \( A \) : a nonnegative \( n \times n \) matrix with spectral radius \( \rho(A) \)
- \( \rho(A) > 0 \) is an eigenvalue of \( A \);
- there is a nonnegative (left and right) eigenvector associated with \( \rho(A) \);
- if further \( A \) is irreducible, then:
  - there is a positive (left and right) eigenvector associated with \( \rho(A) \) that is of multiplicity 1;
- for stochastic matrix \( A \) the spectral radius \( \rho(A) = 1 \).
Stationary Distribution

Markov chain \( \mathcal{M} = (\Omega, P) \)

- **stationary distribution** \( \pi \):
  \[ \pi P = \pi \] (fixed point)

- Perron-Frobenius Theorem:
  - stochastic matrix \( P \): \( P 1 = 1 \)
  - 1 is also a **left eigenvalue** of \( P \) (eigenvalue of \( P^T \))
  - the **left eigenvector** \( \pi P = \pi \) is nonnegative
  - stationary distribution always exists
Convergence

\[ P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \]

\[ P^{20} \approx \begin{bmatrix} 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \end{bmatrix} \]

ergodic: convergent to stationary distribution
$P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{3} & \frac{2}{3} & 0 & 0 \\
0 & 0 & \frac{3}{4} & \frac{1}{4} \\
0 & 0 & \frac{1}{4} & \frac{3}{4}
\end{bmatrix}$

$P^{20} \approx \begin{bmatrix}
0.4 & 0.6 & 0 & 0 \\
0.4 & 0.6 & 0 & 0 \\
0 & 0 & 0.5 & 0.5 \\
0 & 0 & 0.5 & 0.5
\end{bmatrix}$

*reducible*
$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

\[ P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ P^{2k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad P^{2k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]
reducible

periodic
Fundamental Theorem of Markov Chain:

If a \textit{finite} Markov chain $\mathcal{M} = (\Omega, P)$ is \textit{irreducible} and \textit{aperiodic}, then $\forall$ initial distribution $\pi^{(0)}$ (ergodic)

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where $\pi$ is a \textit{unique} stationary distribution satisfying

$$\pi P = \pi$$
Irreducibility

- $y$ is accessible from $x$:
  $$\exists t, \ P^t(x, y) > 0$$

- $x$ communicates with $y$:
  - $x$ is accessible from $y$
  - $y$ is accessible from $x$

- MC is irreducible: all pairs of states communicate
Reducible Chains

\[ P = \begin{bmatrix} P_A & 0 \\ 0 & P_B \end{bmatrix} \]

stationary distributions: \( \pi = \lambda \pi_A + (1 - \lambda) \pi_B \)

stationary distribution: \( \pi = (0, \pi_B) \)
Aperiodicity

- **period** of state $x$:
  
  $$d_x = \gcd\{t \mid P^t(x, x) > 0\}$$

- **aperiodic** chain: all states have period 1

- **period**: the gcd of lengths of cycles

  $$d_x = 3$$

  $x$  □  □  △  □  □  △  □  □  □  △  \ldots\ldots
Lemma (period is a class property)

$x$ and $y$ communicate $\implies d_x = d_y$

\[ d_x \mid (n_1 + n_2) \]
\[ d_x \mid (n_1 + n_2 + n) \]

\{ $\implies d_x \mid n \implies d_x \leq d_y$ \}

\forall$ cycle of $y$
If a finite Markov chain $\mathcal{M} = (\Omega, P)$ is irreducible and aperiodic, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where $\pi$ is a unique stationary distribution satisfying $\pi P = \pi$

Finiteness $\rightarrow$ Existence
Irreducibility $\rightarrow$ Uniqueness
Ergodicity $\rightarrow$ Convergence

\{ Perron-Frobenius }
Coupling of Markov Chains

Markov chain \( \mathcal{M} = (\Omega, P) \)

initial \( \pi^{(0)} \): \( X_0, X_1, X_2, \ldots \)

stationary \( \pi \): \( X'_0, X'_1, X'_2, \ldots \)

faithful running of MC

initial \( \pi^{(0)} \): \( X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \)

stationary \( \pi \): \( X'_0 \rightarrow X'_1 \rightarrow \cdots \rightarrow X'_n \rightarrow X'_{n+1} \rightarrow \cdots \)

\( X'' \)

Goal: \( \lim_{t \rightarrow \infty} \Pr[X''_t = X'_t] = 1 \)
Coupling of Markov Chains

Markov chain \( \mathcal{M} = (\Omega, P) \)

initial \( \pi^{(0)} \): \( X_0, X_1, X_2, \ldots \)

stationary \( \pi \): \( X'_0, X'_1, X'_2, \ldots \)

faithful running of MC

Goal: conditioning on \( X_0 = x, X'_0 = y \)
\[ \exists n, \Pr[X_n = X'_n = y] > 0 \]
Markov chain \( \mathcal{M} = (\Omega, P) \)

Initial \( \pi^{(0)}: X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \)

Stationary \( \pi: X'_0 \rightarrow X'_1 \rightarrow \cdots \rightarrow X'_n \rightarrow X'_{n+1} \rightarrow \cdots \)

Conditioning on \( X_0 = x, X'_0 = y \)

Irreducible: \( \exists n_1, \quad P^{n_1}(x, y) > 0 \)

Aperiodic: \( \exists n_2, \quad \left\{ \begin{array}{l} P^{n_2}(y, y) > 0 \\ P^{n_1+n_2}(y, y) > 0 \end{array} \right. \)

\( n = n_1 + n_2 \)

\( \Pr[X_n = X'_n = y] \geq \Pr[X_n = y] \Pr[X'_n = y] \geq P^{n_1}(x, y) P^{n_2}(y, y) P^n(y, y) > 0 \)
Markov chain \( \mathcal{M} = (\Omega, P) \)

faithful running of MC

initial \( \pi^{(0)} : X_0 \to X_1 \to \ldots \to X_n \to X''_t \)

stationary \( \pi : X'_0 \to X'_1 \to \ldots \to X'_n \to X'_{n+1} \to \ldots \pi \)

conditioning on \( X_0 = x, X'_0 = y \)

\( \exists n, \quad \Pr[X_n = X'_n] \geq \epsilon > 0 \)

\( \lim_{t \to \infty} \Pr[X''_t = X'_t] = 1 \)
If a finite Markov chain $\mathcal{M} = (\Omega, P)$ is irreducible and aperiodic, then $\forall$ initial distribution $\pi(0)$ is a unique stationary distribution satisfying

$$\lim_{t \to \infty} \pi(0) P^t = \pi$$

where $\pi$ is a unique stationary distribution satisfying $\pi P = \pi$.

Finiteness $\rightarrow$ Existence $\rightarrow$ Uniqueness $\rightarrow$ Convergence \{ Perron-Frobenius \}
Fundamental Theorem of Markov Chain:

If a Markov chain $\mathcal{M} = (\Omega, P)$ is \textit{irreducible} and \textit{ergodic}, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where $\pi$ is a \textit{unique} stationary distribution satisfying $\pi P = \pi$

finite: \quad ergodic = aperiodic

infinite: \quad ergodic $\iff$ aperiodic + \textit{non-null persistent}
Infinite Markov Chains

• \( f_{xy} \): prob. of ever reaching \( y \) starting from \( x \)

• \( h_{xy} \): expected time of first reaching \( y \) starting from \( x \)

\[
f_{xy}(t) \triangleq \Pr[y \notin \{X_1, \ldots, X_{t-1}\} \land X_t = y \mid X_0 = x]
\]

\[
f_{xy} \triangleq \sum_{t=1}^{\infty} f_{xy}(t) \quad h_{xy} \triangleq \sum_{t=1}^{\infty} tf_{xy}(t)
\]

• \( x \) is transient if \( f_{xx} < 1 \)

• \( x \) is recurrent or persistent if \( f_{xx} = 1 \)

• \( x \) is null-persistent if \( x \) is persistent and \( h_{xx} = \infty \)

• \( x \) is non-null-persistent if \( x \) is persistent and \( h_{xx} < \infty \)

• a chain is non-null-persistent if all states are
Fundamental Theorem of Markov Chain:

If a Markov chain \( \mathcal{M} = (\Omega, P) \) is *irreducible* and *ergodic*, then \( \forall \) initial distribution \( \pi^{(0)} \)

\[
\lim_{t \to \infty} \pi^{(0)} P^t = \pi
\]

where \( \pi \) is a *unique* stationary distribution satisfying

\[
\pi P = \pi
\]

finite: \( \text{ergodic} = \text{aperiodic} \)

infinite: \( \text{ergodic} \iff \text{aperiodic} + \text{non-null persistent} \)

irreducible + *non-null persistent* \( \Rightarrow \exists \) unique stationary distribution

aperiodic + *non-null persistent* \( \Rightarrow \) ergodic
PageRank

**Rank**: importance of a page

- A page has higher rank if pointed by more high-rank pages.
- High-rank pages have greater influence.
- Pages pointing to few others have greater influence.
PageRank (simplified)

the web graph $G(V, E)$

rank of a page: $r(v)$

$$r(v) = \sum_{u: (u,v) \in E} \frac{r(u)}{d_+(u)}$$

$d_+(u)$: out-degree of $u$

random walk: $P(u, v) = \begin{cases} \frac{1}{d_+(u)} & \text{if } (u, v) \in E, \\ 0 & \text{otherwise.} \end{cases}$

stationary distribution: $rP = r$

a tireless random surfer
Random Walk on Graph

- undirected graph $G(V, E)$
- walk: $v_1, v_2, \ldots \in V$ that $v_{i+1} \sim v_i$
- random walk: $v_{i+1}$ is uniformly chosen from $N(v_i)$

$$P(u, v) = \begin{cases} 
\frac{1}{d(u)} & u \sim v \\
0 & u \not\sim v 
\end{cases}$$

$$P = D^{-1}A$$

adjacency matrix $A$

$$D(u, v) = \begin{cases} 
d(u) & u = v \\
0 & u \neq v 
\end{cases}$$
Random Walk on Graph

- stationary:
  - convergence (ergodicity);
  - stationary distribution;

- **hitting time**: time to reach a vertex;

- **cover time**: time to reach all vertices;

- **mixing time**: time to converge.
Random Walk on Graph

\[ G(V,E) \]

\[ P(u, v) = \begin{cases} 
\frac{1}{d(u)} & u \sim v \\
0 & u \not\sim v 
\end{cases} \]

• for finite chain:
  irreducible and aperiodic \(\Rightarrow\) converge

• irreducible \(\iff\) \(G\) is connected

\[ P^t(u, v) > 0 \iff A^t(u, v) > 0 \]

• aperiodic \(\iff\) \(G\) is non-bipartite

  bipartite \(\Rightarrow\) no odd cycle \(\Rightarrow\) period = 2

  non-bipartite \(\Rightarrow\) \(\exists\) \(2k+1\)-cycle

  undirected \(\Rightarrow\) \(\exists\) 2-cycle \(\Rightarrow\) aperiodic
Lazy Random Walk

- undirected graph $G(V, E)$

- lazy random walk: flip a coin to decide whether to stay

$$P(u, v) = \begin{cases} 
\frac{1}{2} & u = v \\
\frac{1}{2d(u)} & u \sim v \\
0 & \text{otherwise}
\end{cases}$$

$$P = \frac{1}{2}(I + D^{-1}A)$$

adjacency matrix $A$

$$D(u, v) = \begin{cases} 
d(u) & u = v \\
0 & u \neq v
\end{cases}$$

always aperiodic!
Random Walk on Graph

\[ G(V,E) \]

\[ P(u, v) = \begin{cases} 
\frac{1}{d(u)} & u \sim v \\
0 & u \not\sim v 
\end{cases} \]

Stationary distribution \( \pi \):

\[ \forall v \in V, \quad \pi_v = \frac{d(v)}{2m} \]

\[ \sum_{v \in V} \pi_v = \sum_{v \in V} \frac{d(v)}{2m} = \frac{1}{2m} \sum_{v \in V} d(v) = 1 \]

\[ (\pi P)_v = \sum_{u \in V} \pi_u P(u, v) = \sum_{u \in N(v)} \frac{d(u)}{2m} \frac{1}{d(u)} = \frac{d(v)}{2m} = \pi_v \]

regular graph \quad \xrightarrow{\quad \text{uniform distribution} \quad}
Random Walk on Graph

\[ G(V,E) \quad P(u, v) = \begin{cases} \frac{1}{2} & u = v \\ \frac{1}{2d(u)} & u \sim v \\ 0 & \text{otherwise} \end{cases} \]

Stationary distribution \( \pi \):

\[
\forall v \in V, \quad \pi_v = \frac{d(v)}{2m}
\]

\[
(\pi P)_v = \sum_{u \in V} \pi_u P(u, v) = \frac{1}{2} \frac{d(v)}{2m} + \frac{1}{2} \sum_{u \in N(v)} \frac{d(u)}{2m} \frac{1}{d(u)}
\]

\[
= \frac{d(v)}{2m} = \pi_v
\]
Reversibility

\textbf{ergodic flow}

\begin{align*}
\text{detailed balance equation:} \\
\pi(x)P(x, y) &= \pi(y)P(y, x)
\end{align*}

\textbf{time-reversible Markov chain:}

\[ \exists \pi, \forall, x, y \in \Omega, \quad \pi(x)P(x, y) = \pi(y)P(y, x) \]

\textbf{stationary distribution:}

\[ (\pi P) y = \sum_{x} \pi(x)P(x, y) = \sum_{x} \pi(y)P(y, x) = \pi(y) \]

\textbf{time-reversible: when start from } \pi

\[ \Pr[X_0 = x_0 \land X_1 = x_1 \land \ldots \land X_n = x_n] \]

\[ = \Pr[X_0 = x_n \land X_1 = x_{n-1} \land \ldots \land X_n = x_0] \]
Reversibility

detailed balance equation:
\[ \pi(x)P(x, y) = \pi(y)P(y, x) \]

**time-reversible** Markov chain:
\[ \exists \pi, \forall x, y \in \Omega, \quad \pi(x)P(x, y) = \pi(y)P(y, x) \]

stationary distribution:
\[ (\pi P)y = \sum_{x} \pi(x)P(x, y) = \sum_{x} \pi(y)P(y, x) = \pi(y) \]

time-reversible: when start from $\pi$
\[ (X_0, X_1, \ldots, X_n) \sim (X_n, X_{n-1}, \ldots, X_0) \]
Reversibility

detailed balance equation:
\[ \pi(x)P(x, y) = \pi(y)P(y, x) \]

time-reversible Markov chain:
\[ \exists \pi, \forall x, y \in \Omega, \quad \pi(x)P(x, y) = \pi(y)P(y, x) \]

stationary distribution:
\[ (\pi P)y = \sum_x \pi(x)P(x, y) = \sum_x \pi(y)P(y, x) = \pi(y) \]

the Markov chain can be uniquely represented as an edge-weighted undirected graph \( G(V, E) \) with ergodic flows as edge weights
Random Walk on Graph

\[ G(V,E) \]

\[ P(u, v) = \begin{cases} 
\frac{1}{d(u)} & u \sim v \\
0 & u \not\sim v 
\end{cases} \]

\[ P(u, v) = \begin{cases} 
\frac{1}{2} & u = v \\
\frac{1}{2d(u)} & u \sim v \\
0 & \text{otherwise} 
\end{cases} \]

detailed balance equation:

\[ \pi(u)P(u, v) = \pi(v)P(v, u) \]

\[ u = v : \quad \text{hold for free} \]

\[ u \not\sim v : \]

\[ u \sim v : \quad \pi(u) \propto \frac{1}{P(u, v)} \]
Random Walk on Graph

- undirected graph $G(V, E)$ (not necessarily regular)
- lazy random walk

Goal: a uniform stationary distribution $\pi$

Goal: an arbitrary stationary distribution $\pi$
Random Walk

- stationary:
  - convergence (ergodicity);
  - stationary distribution;
- hitting time: time to reach a vertex;
- cover time: time to reach all vertices;
- mixing time: time to converge.
Hitting and Covering

consider a random walk on $G(V,E)$

- **hitting time**: expected time to reach $v$ from $u$
  \[
  \tau_{u,v} = \mathbb{E} \left[ \min \{ n > 0 \mid X_n = v \} \mid X_0 = u \right] 
  \]

- **cover time**: expected time to visit all vertices
  \[
  C_{u} = \mathbb{E} \left[ \min \{ n \mid \{ X_0, \ldots, X_n \} = V \} \mid X_0 = u \right] 
  \]

\[
C(G) = \max_{u \in V} C_u
\]
\[ G = K_n \]

\[ C(G) = \Theta(n \log n) \]

\[ G : \text{path or cycle} \]

\[ C(G) = \Theta(n^2) \]

\[ G : \text{“lollipop”} \]

\[ C(G) = \Theta(n^3) \]
Hitting Time

\[ \tau_{u,v} = \mathbb{E} \left[ \min \{ n > 0 \mid X_n = v \} \mid X_0 = u \right] \]

stationary distribution \( \pi \)

\[ \pi_v = \frac{d(v)}{2m} \]

Renewal Theorem:

irreducible

\[ \tau_{v,v} = \frac{1}{\pi_v} = \frac{2m}{d(v)} \]
Renewal Theorem:

\[
\pi(v, v) = \frac{1}{\pi_v} = \frac{2m}{d(v)}
\]

\[
\pi(x) = \Pr[X_0 = x]
\]

\[
\tau_{x, x} = \mathbb{E}[T_x \mid X_0 = x] = \sum_{n \geq 1} \Pr[T_x \geq n \mid X_0 = x]
\]

\[
\tau_{x, x} \pi_x = \sum_{n \geq 1} \Pr[T_x \geq n \mid X_0 = x] \Pr[X_0 = x]
\]

\[
= \sum_{n \geq 1} \Pr[T_x \geq n \land X_0 = x]
\]

\[
= \Pr[X_0 = x] + \sum_{n \geq 2} (\Pr[\forall 1 \leq t \leq n - 1, X_t \neq x] - \Pr[\forall 0 \leq t \leq n - 1, X_t \neq x])
\]

\[
= \Pr[X_0 = x] + \sum_{n \geq 2} (\Pr[\forall 0 \leq t \leq n - 2, X_t \neq x] - \Pr[\forall 0 \leq t \leq n - 1, X_t \neq x])
\]

\[
= \Pr[X_0 = x] + \Pr[X_0 \neq x] - \lim_{n \to \infty} a_n = 1 \quad \text{(irreducible)}
\]
random walk on $G(V,E)$

\[
\tau_{v,v} = \frac{1}{\pi_v} = \frac{2m}{d(v)}
\]

Lemma: \( uv \in E \rightarrow \tau_{u,v} < 2m \)

\[
\tau_{v,v} = \sum_{wv \in E} \frac{1}{d(v)} (1 + \tau_{w,v})
\]

\[
2m = \sum_{wv \in E} (1 + \tau_{w,v}) \rightarrow \tau_{u,v} < 2m
\]
**Cover Time**

\[ C_u = \mathbb{E}\left[ \min \{ n \mid \{X_0, \ldots, X_n\} = V \} \mid X_0 = u \right] \]

\[ C(G) = \max_{u \in V} C_u \]

**Theorem:** \[ C(G) \leq 4nm \]

pick a spanning tree \( T \) of \( G \)

**Eulerian tour:** \[ v_1 \to v_2 \to \cdots \to v_{2(n-1)} \to v_{2n-1} = v_1 \]

\[ C(G) \leq \sum_{i=1}^{2(n-1)} \tau_{v_i, v_{i+1}} \]

\[ < 4nm \]
USTCON
(undirected $s$-$t$ connectivity)

- Instance:
  - undirected $G(V, E)$;
  - vertices: $s, t$

- $s$-$t$ connected?

- deterministic:
  - traverse: linear space

- log-space?
**Theorem** (Aleliunas-Karp-Lipton-Lovász-Rackoff 1979)

USTCON can be solved by a poly-time Monte Carlo randomized algorithm with bounded one-sided error, which uses $O(\log n)$ extra space.

- start a random walk at $s$;
- if reach $t$ in $4n^3$ steps return “yes”
- else return “no”

Cover time: $C(G) \leq 4nm < 2n^3$

Markov’s inequality $\Rightarrow$ $\Pr[\text{“no”}] < 1/2$

unconnected $\Rightarrow$ “no”

connected $\Rightarrow$
Electric Network

eedge $uv$ : resistance $R_{uv}$

vertex $v$ : potential $\phi_v$

dedge orientation $u \rightarrow v$ : current flow $C_{u \rightarrow v}$

potential difference $\phi_{u,v} = \phi_u - \phi_v$

Kirchhoff’s Law: $\forall$ vertex $v$, flow-in = flow-out

Ohm’s Law: $\forall$ edge $uv$, $C_{u \rightarrow v} = \frac{\phi_{u,v}}{R_{uv}}$
Effective Resistance

electrical network:

effective resistance $R(u,v)$:

potential difference between $u$ and $v$
required to send 1 unit of flow current from $u$ to $v$
\textbf{Theorem} (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

\[ \forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u, v) \]
graph $G(V, E)$

construct an electrical network:

for every edge $e$:

$R_e = 1$

each vertex $u$:

inject $d(u)$ units current flow into $u$

a special vertex $v$:

remove all $2m$ units current flow from $v$

Lemma: $\forall u \in V$, $\phi_{u,v} = \tau_{u,v}$
Lemma: \( \forall u \in V, \quad \phi_{u,v} = \tau_{u,v} \)

\[
d(u) = \sum_{uw \in E} C_{u \rightarrow w} \quad \text{(Kirchhoff)}
\]

\[
d(u) = \sum_{uw \in E} \phi_{u,w} \quad \text{(Ohm)}
\]

\[
d(u) \phi_{u,v} - \sum_{uw \in E} \phi_{w,v}
\]

\[
\phi_{u,v} = 1 + \frac{1}{d(u)} \sum_{uw \in E} \phi_{w,v}
\]
Lemma: \( \forall u \in V, \quad \phi_{u,v} = \tau_{u,v} \)

\[
\phi_{u,v} = 1 + \frac{1}{d(u)} \sum_{uw \in E} \phi_{w,v}
\]

\[
\tau_{u,v} = E \left[ \min \{ n > 0 \mid X_n = v \} \mid X_0 = u \right]
\]

\[
\tau_{u,v} = \sum_{wu \in E} \frac{1}{d(u)} (1 + \tau_{w,v})
\]

\[
= 1 + \frac{1}{d(u)} \sum_{wu \in E} \tau_{w,v}
\]
Lemma: \( \forall u \in V, \phi_{u,v} = \tau_{u,v} \)

\[
\phi_{u,v} = 1 + \frac{1}{d(u)} \sum_{uw \in E} \phi_{w,v} \quad \text{has the same unique solution}
\]

\[
\tau_{u,v} = 1 + \frac{1}{d(u)} \sum_{wu \in E} \tau_{w,v}
\]
Effective Resistance

electrical network:

each edge $e$

resistance $R_e = 1$

effective resistance $R(u,v)$:

potential difference between $u$ and $v$
required to send 1 unit of flow from $u$ to $v$
Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

\[ \forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u, v) \]
Lemma: \( \forall u \in V, \quad \phi_{u,v} = \tau_{u,v} \)

\[
\phi_{u,v} = 1 + \frac{1}{d(u)} \sum_{uw \in E} \phi_{w,v}
\]

\[
\tau_{u,v} = 1 + \frac{1}{d(u)} \sum_{wu \in E} \tau_{w,v}
\]

\[\Rightarrow\] has the same unique solution
**Theorem** (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

\[ \forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u,v) \]

**A:**

\[ \phi^A_{u,v} = \tau_{u,v} \]

**B:**

\[ \phi^B_{v,u} = \tau_{v,u} \]
Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

∀u, v, ∈ V, \(\tau_{u,v} + \tau_{v,u} = 2mR(u, v)\)

A: \[\phi^A_{u,v} = \tau_{u,v}\]

B: \[\phi^B_{v,u} = \tau_{v,u}\]

C: \[\phi^C_{u,v} = \phi^B_{v,u} = \tau_{v,u}\]

D: \[\phi^D_{u,v} = \phi^A_{u,v} + \phi^C_{u,v} = \tau_{u,v} + \tau_{v,u}\]

\(\phi^D_{u,v}\): potential difference between \(u\) and \(v\) to send \(2m\) units current flow from \(u\) to \(v\)
**Theorem** (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

\[ \forall u, v \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u, v) \]

**A:**
\[ \phi^A_{u,v} = \tau_{u,v} \]

**B:**
\[ \phi^B_{v,u} = \tau_{v,u} \]

**C:**
\[ \phi^C_{u,v} = \phi^B_{v,u} = \tau_{v,u} \]

**D:**
\[ \phi^D_{u,v} = \phi^A_{u,v} + \phi^C_{u,v} = \tau_{u,v} + \tau_{v,u} \]

**D = A + C**

\[ R(u, v) = \frac{\phi^D_{u,v}}{2m} \]
**Theorem** (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

\[ \forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u, v) \]

**Theorem:** \[ C(G) \leq 2nm \]

pick a spanning tree \( T \) of \( G \)

**Eulerian tour:** \[ v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{2(n-1)} \rightarrow v_{2n-1} = v_1 \]

\[
C(G) \leq \sum_{i=1}^{2(n-1)} \tau_{v_i, v_{i+1}} = \sum_{uv \in T} (\tau_{u,v} + \tau_{v,u})
\]

\[ = 2m \sum_{uv \in T} R(u, v) \leq 2mn \]
Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

\[ \forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u, v) \]

\[ G: \text{ path} \]

\[ C(G) \leq 2nm = 2n^2 = O(n^2) \]

\[ C(G) \geq \tau_{u,v} = \tau_{v,u} = \Omega(n^2) \]

\[ \tau_{u,v} + \tau_{v,u} = 2mR(u, v) = 2n(n - 1) \]
Theorem (Chandra-Raghavan-Ruzzo-Smolensky-Tiwari 1989)

\[ \forall u, v, \in V, \quad \tau_{u,v} + \tau_{v,u} = 2mR(u, v) \]

\[ G: \text{lollipop} \]

\[ C(G) \leq 2nm = O(n^3) \]

\[ C(G) \geq \tau_{u,v} = \Omega(n^3) \]

\[ \tau_{u,v} + \tau_{v,u} = 2mR_{u,v} = \Omega(n^3) \]

\[ \tau_{v,u} = O(n^2) \]