# Advanced Algorithms

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# Recap

**Previous lecture**:
Cheeger's inequality on d-regular graphs

- Easy direction: a sparse cut implies $\lambda_2$ is small
- Hard direction: a small $\lambda_2$ means we can find a sparse cut from $x_2$
- Spectral partitioning
- Improvements of Cheeger's
- Generalizations of Cheeger's

**What next?**
Random walks on undirected graphs

- Fundamental theorem of Markov chains
- Spectral analysis
- Mixing time
- Random sampling

# Random Walks on Graphs

A random walk is a simple random process.

- Start from some vertex
- Move to a uniformly random neighbor of the current vertex
- Repeat many steps

We are interested in the long term behavior of this random process

For example, what is the probability of being in a vertex at a time $t$?

# Common Questions

Basic questions in applying random walks in designing algorithms:

1. Is there a limiting distribution of the random walk? What does it look like? (*Stationary distribution*)
2. How long does it take for the random walk to converge to the limiting distribution? (*Mixing time*)
3. Starting from a vertex $s$, what is the expected number of steps to first reach vertex $t$? (*Hitting time*)
4. How long does it take to reach every vertex of the graph at least once? (*Cover time*)

There are two main approaches to answer the first two questions.

- One approach is probabilistic, using the technique of "**coupling**" of two random processes
- Another approach is linear algebraic, using the **eigenvalues of the transition matrix**
- We first introduce the spectral approach to answer the first two questions
- Then the probabilistic approach when we introduce the **Markov chain Monte Carlo method**
- The last two questions are best answered by viewing the graph as an "**electrical network**"

# A random walk for finding bipartite matching

**Matching :** A subset of edges that do not overlap at any vertex

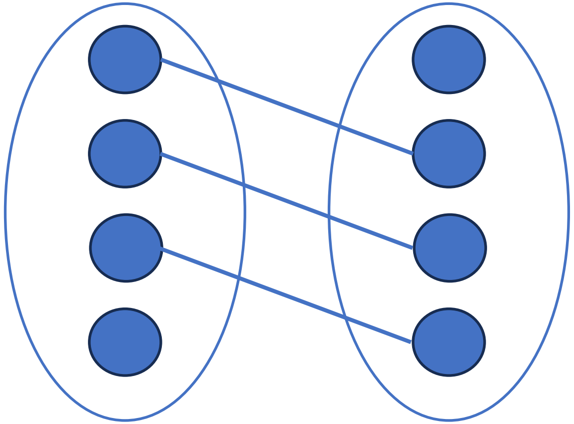
**Bipartite Matching problem:** Find the largest matching in a bipartite graph

We consider an O(n log n) algorithm for regular bipartite graphs (Goel, Kapralov, Khanna 2010), which is based on random walk

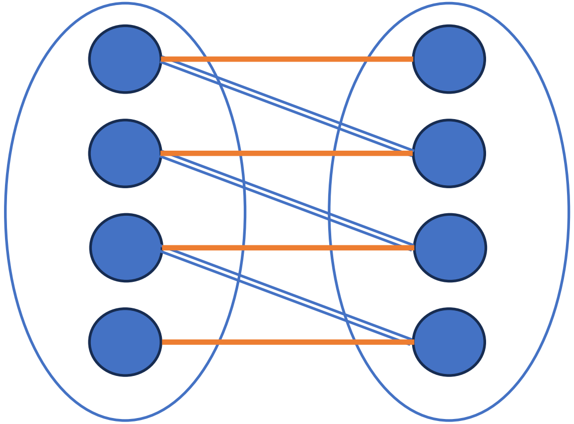
- Traditional: find augmenting paths by e.g. BFS/DFS

# Augmenting path

**Augmenting path for a matching $M$:** A path in the graph that alternates between edges in $M$ and outside of $M$, and starts and ends with edges outside of $M$

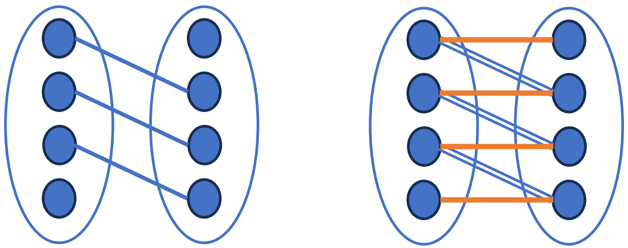


A matching $M$



An augmenting path for matching $M$

# Augmenting path



**Augmenting path for a matching $M$:** A path in the graph that alternates between edges in $M$ and outside of $M$, and starts and ends with edges outside of $M$

**Berge's Theorem** A matching $M$ is maximum iff there is no augmenting path for it

Proof (sketch): If there is an augmenting path, clearly it enlarges the matching. Conversely, if there is a larger matching $M'$, one can find an augmenting path for $M$ using the edges of $M \cup M'$.

**Remark**: Given this theorem, we can find a maximum **bipartite** matching by repeatedly finding augmenting paths using e.g. DFS/BFS.

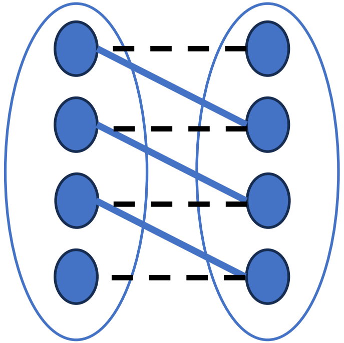
This takes $O(n)$ iterations of DFS/BFS.

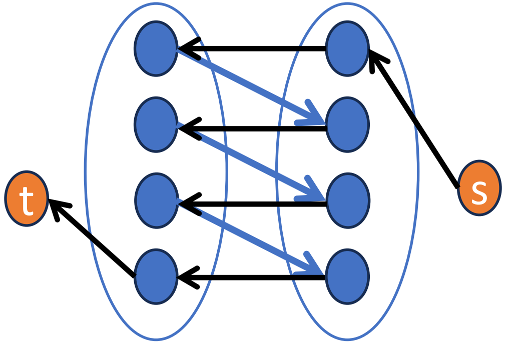
One can speed this up by finding many augmenting paths at once (Hopcroft-Karp)

# A random walk for finding bipartite matching

Idea: replace BFS/DFS by a random walk (on a different **Eulerian directed graph**, i.e. indegree=outdegree for every vertex)

- Each edge in the matching points to the right
- Each edge not in the matching points to the left
- There is a source node $s$ that connects to every unmatched vertex on the right
- A sink node $t$ that is connected from every unmatched vertex on the left



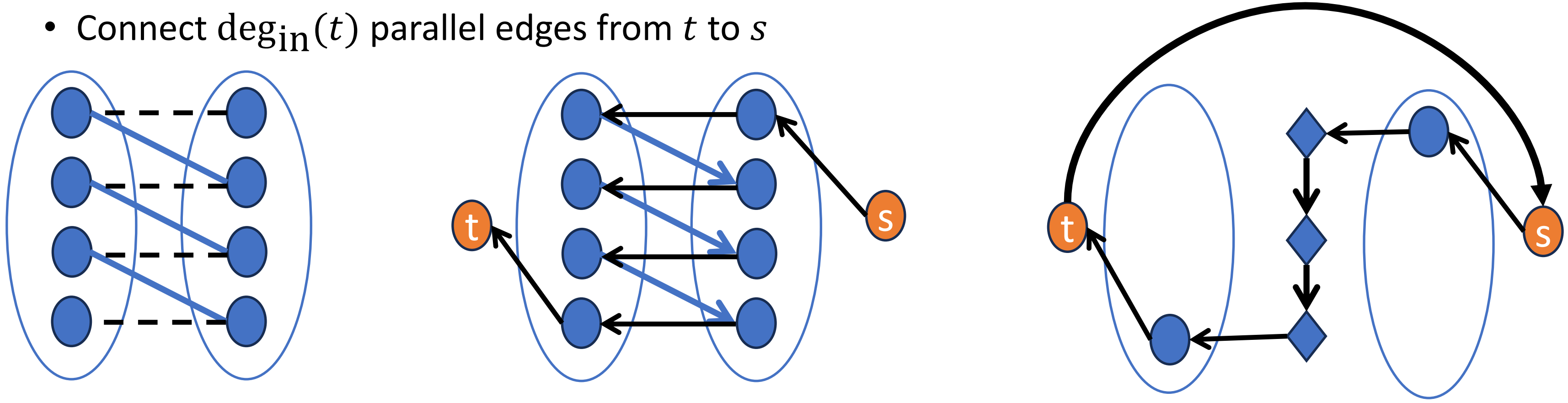



Then $G_1$ has an augmenting path iff $G_2$ has an $s$-$t$ path

# A random walk for finding bipartite matching

Idea: replace BFS/DFS by a random walk (on a different **Eulerian directed graph**, i.e. indegree=outdegree for every vertex)

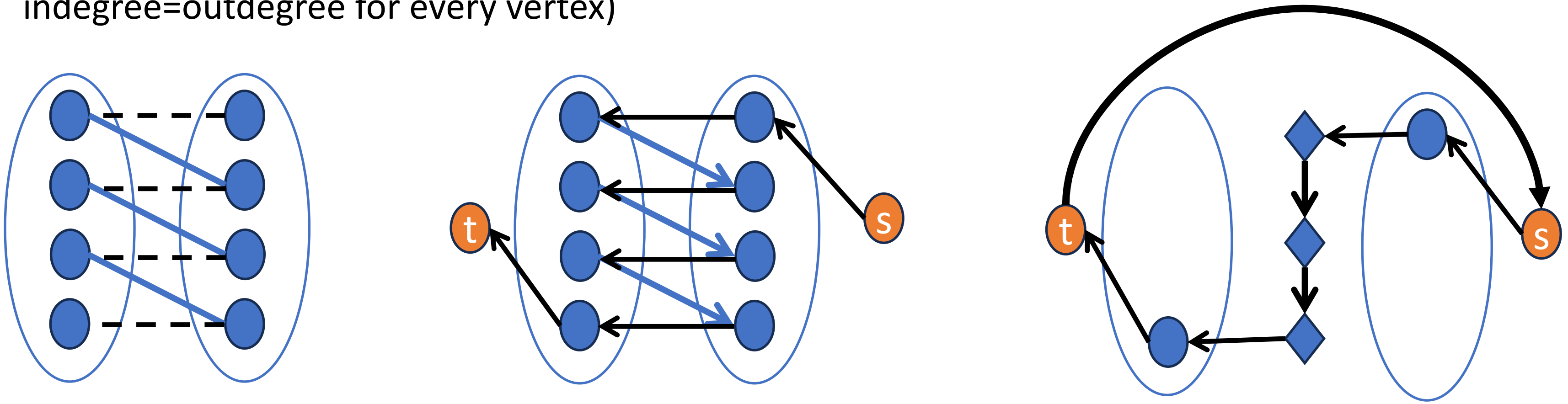
- From $G_2$ we contract edges in the matching
- Increase parallel edges from $s$ to every unmatched vertex $v$ on the right to $\deg_{\text{out}}(v)$
- Increase parallel edges from every unmatched vertex $u$ on the left to the sink $t$ to $\deg_{\text{in}}(v)$
- Connect $\deg_{\text{in}}(t)$ parallel edges from $t$ to $s$




Then $G_1$ has an augmenting path iff $G_2$ has an $s$-$t$ path iff $G_3$ has a cycle from $s$ to $s$

# A random walk for finding bipartite matching

Idea: replace BFS/DFS by a random walk (on a different **Eulerian directed graph**, i.e. indegree=outdegree for every vertex)




Then $G_1$ has an augmenting path iff $G_2$ has an $s$-$t$ path iff $G_3$ has a cycle from $s$ to $s$

Further, $G_3$ is Eulerian if $G_1$ is a regular graph

Question: Expected time to find a cycle from $s$ to $s$, using random walk? (Return time)

# Matrix Formulation of random walk

In each step, we move to a uniform random neighbor, and repeat.

Let $p_t \in \mathbb{R}^n$ be the probability distribution at time $t$.

Then, for all $v \in V$,

$$p_{t+1}(v) = \sum_{u:uv \in E} p_t(u) \cdot \frac{1}{\deg(u)}.$$

Let $A$ be the adjacency matrix and $D$ be the diagonal degree matrix.

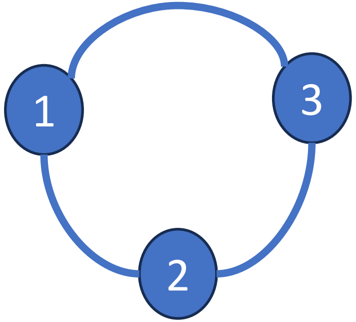
Then $p_{t+1} = p_t(D^{-1}A)$ and thus $p_t = p_0(D^{-1}A)^t$, if $p_t$ is a row vector.

For the spectral analysis, it will be more convenient to have $p_t$ as a column vector.

So we write $p_{t+1} = (AD^{-1})p_t$ and $p_t = (AD^{-1})^t p_0$.

This is called a **Markov chain**, because it forgets about the past (given the current state)

# Matrix Formulation of random walk




Example: Consider an undirected 3-cycle

If we start at vertex 1

with probability ½ we go to vertex 2

with probability ½ we go to vertex 3

In matrix/vector terms

our starting distribution is $p_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

after one-step of the random walk, the distribution becomes $p_1 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$

# Matrix Formulation of random walk

$p_{t+1} = (AD^{-1})p_t$ and $p_t = (AD^{-1})^t p_0$

Transition matrix $W \coloneqq AD^{-1}$

To understand $W^t$, it suffices to study the powers of $\mathcal{A} = D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$, because:

$$\mathcal{A} = D^{-\frac{1}{2}} W\, D^{\frac{1}{2}} \Rightarrow W^t = D^{\frac{1}{2}} \mathcal{A}^t D^{-\frac{1}{2}}$$

Further, as a real symmetric matrix $\mathcal{A} = V\Sigma V^T$ we have:

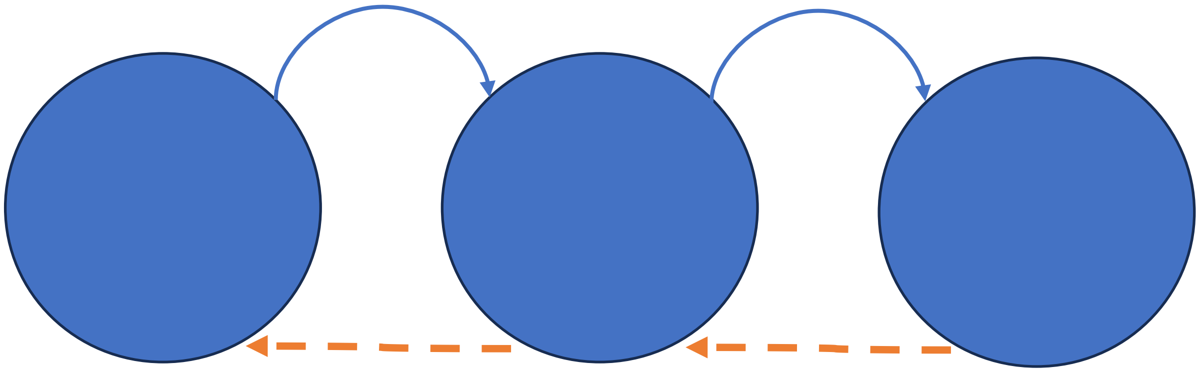
$$\mathcal{A}^k = V\Sigma^k V^T = \sum_{1 \le i \le n} \lambda_i^k v_i v_i^T$$

We study the convergence of the random walk, by utilizing the spectral knowledge of $\mathcal{A}$

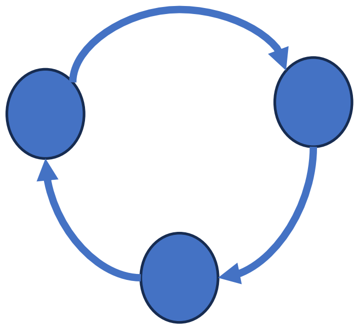
# Irreducible Markov Chains

We assume all Markov chains are finite in this course.

A Markov chain is called **irreducible** if the underlying directed graph is strongly connected.



# Aperiodic Markov Chains



The period of a state $i$ is defined as $period(i) \coloneqq \gcd\{\, t \mid P^t_{i,i} > 0\}$.

A state is a **aperiodic** if $period(i) = 1$.

A Markov chain is **aperiodic** if all states are aperiodic; otherwise it is called periodic.

Example: What is the period of a directed 3-cyle? What about an undirected 3-cycle?
Example: What if we add a self-loop to the directed 3-cyle?

By irreducibility and aperiodicity, we have the following property.

**Lemma**. For any finite, irreducible, aperiodic Markov chain, there exists a $T < \infty$ such that

$$(P^t)_{i,j} > 0 \text{ for all } i, j \text{ and for all } t \geq T.$$

# Stationary Distribution

A probability distribution $\vec{\pi}$ is a **stationary distribution** if $\vec{\pi} = \vec{\pi} \cdot P$

A stationary distribution is a "steady/equilibrium/fixed point" distribution, and $\vec{\pi} = \vec{\pi} \cdot P^t$ for any $t$.

A limiting distribution is a stationary distribution.

# Distance and Convergence

Given two probability distributions $\vec{p}$ and $\vec{q}$, the **total variation distance** is defined as

$$d_{TV}(\vec{p},\vec{q}) = \frac{1}{2}\|p-q\|_1 = \frac{1}{2}\sum_{i=1}^{n}|p_i - q_i|.$$

We say $\overrightarrow{p_t}$ **converges** to $\vec{q}$ if

$$\lim_{t\to\infty} d_{TV}(\overrightarrow{p_t},\vec{q}) = 0.$$

Side note: we can also measure progress in the 2-norm given by

$$\left\|\frac{p}{\pi}-1\right\|_{2,\pi}^2 \coloneqq \sum_{i=1}^{n}\pi_i\left(\frac{p_i}{\pi_i}-1\right)^2$$

Similar to a "variance"

By Cauchy-Schwarz,

$$\sum_{i=1}^{n}|p_i-\pi_i| = \sum_{i=1}^{n}\pi_i\left|\frac{p_i}{\pi_i}-1\right| \le \sqrt{\sum_{i=1}^{n}\pi_i\left|\frac{p_i}{\pi_i}-1\right|^2} = \left\|\frac{p}{\pi}-1\right\|_{2,\pi}$$

This means that if 2-norm is small, so is the 1-norm (but not vice versa).

# Return Time

The **return time** from $i$ to $i$ is defined as

$$H_i \coloneqq \min\{t \geq 1 \mid X_t = i, X_0 = i\}.$$

The **expected return time** is defined as $h_i \coloneqq \mathbb{E}[H_i]$.

Note that the expected return time can be seen as a special case of (expected) hitting time, where the start/end vertices are the same

# Fundamental Theorem of Markov Chains

**Theorem**. For any finite, irreducible, aperiodic Markov chain, it holds that

1. There exists a stationary distribution $\vec{\pi}$.
2. The distribution $\overrightarrow{p_t}$ will converge to $\vec{\pi}$ as $t \to \infty$, regardless of the distribution $\overrightarrow{p_0}$.
3. There is a unique stationary distribution.
4. $\pi(i) = \frac{1}{h_i}$.

# Intuition

Imagine you are playing a game of guessing where a random walk might have started, and the only information that you know is the "current location of a random walk"

Then, two random walks become indistinguishable after they meet at the same vertex, because the "current location of a random walk" can no longer be used to distinguish them

By the lemma from irreducibility and aperiodicity, after some time $T$, there is a non-zero probability

for two random walks to meet at the same vertex, no matter what their current vertices are.

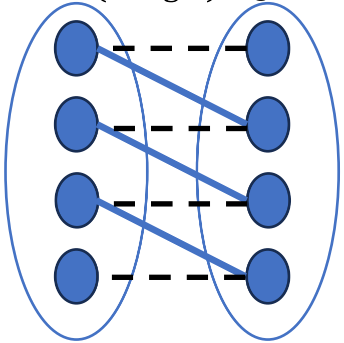
So, eventually, two random walks will meet at the same vertex with probability one,

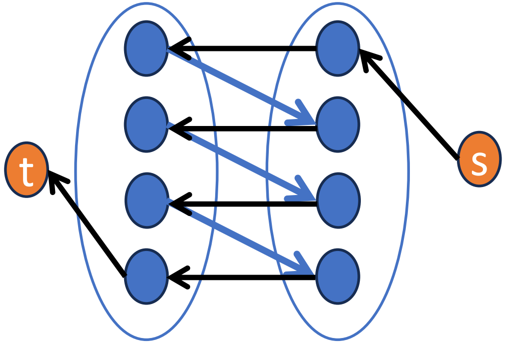
and they would converge to the same stationary distribution.

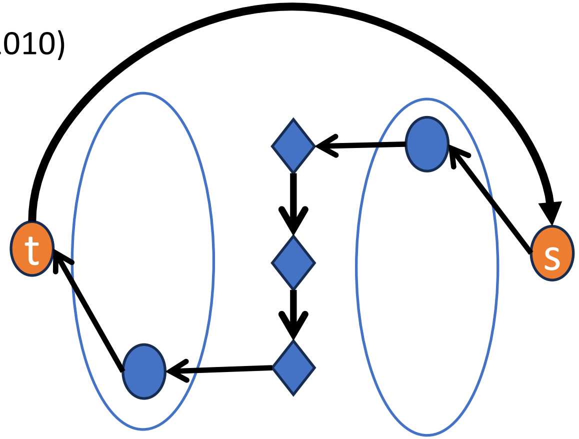
This argument can be made precise by the "coupling" technique we introduce later.

# Return time for finding augmenting path

An $O(n \log n)$ algorithm for regular bipartite graphs (Goel, Kapralov, Khanna 2010)






Then $G_1$ has an augmenting path iff $G_2$ has an $s$-$t$ path iff $G_3$ has a cycle from $s$ to $s$

Further, $G_3$ is Eulerian if $G_1$ is a regular graph

Expected time to find an augmenting path => expected return time => stationary value in Eulerian directed graph

**Claim**: $\pi(v) = \frac{\deg_{\text{out}}(v)}{E}$ is the unique stationary distribution for an Eulerian directed graph

Open problem: Extension to non-regular bipartite graphs?

Aperiodicity: Add self-loop at $t$

# Return time for finding augmenting path

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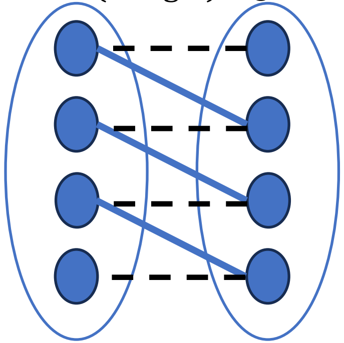
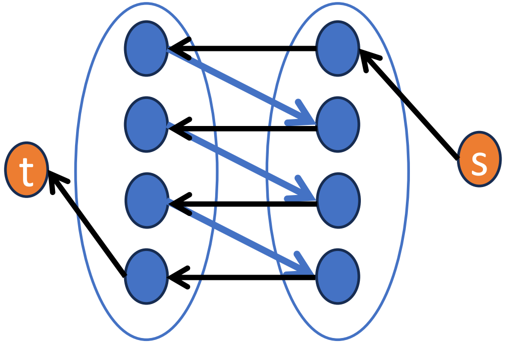
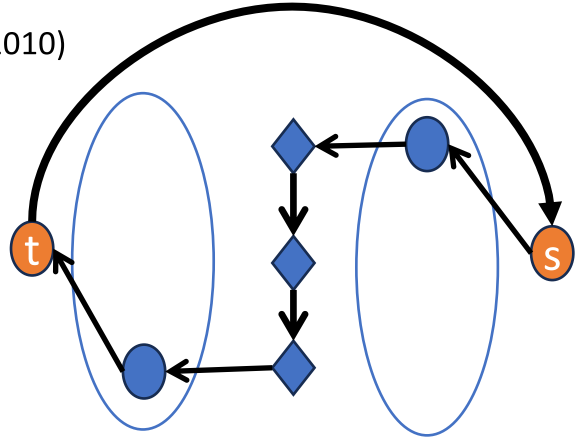





Then $G_1$ has an augmenting path iff $G_2$ has an $s$-$t$ path iff $G_3$ has a cycle from $s$ to $s$

**Claim**: $\pi(v) = \frac{\mathrm{deg}_{\mathrm{out}}(v)}{E}$ is the unique stationary distribution for an Eulerian directed graph

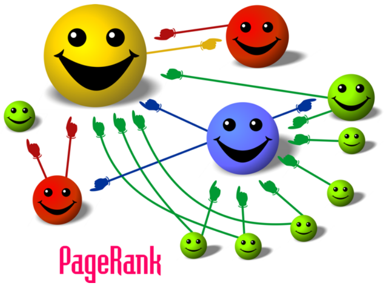
Then, by the fundamental theorem, if we are given a matching of size $i$

$$h_s = \frac{1}{\pi(s)} \leq O\left(\frac{nd}{d(n-i)}\right) = O\left(\frac{n}{n-i}\right)$$

We have to find at most $n$ augmenting paths, so the total running time sums up to $\sum_{i=1}^{n} O\left(\frac{n}{n-i}\right) \leq O(n \log n)$

Open problem: Extension to non-regular bipartite graphs?

# Another application: Pagerank



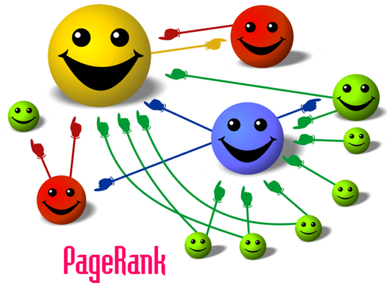

Webpages: vertices

Hyperlinks: edges

This gives a directed graph. A search engine wants to rank pages based on their "importance" or "reputation"

- A page that is being cited by many other people, it is probably more important
  - Metric: in-degree of a vertex
- A page that is being cited by other important pages, it is probably more important
  - Metric: ???

# Pagerank



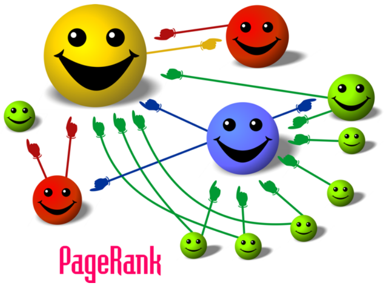

Consider the following iterative algorithm:

- Each page is initialized with a pagerank of 1/n
- Then repeat the following until convergence:
  - Each page will distribute its pagerank equally to its outgoing neighbors
  - Each page updates its pagerank by the total sum of pagerank

$$\text{Pagerank}_{t+1}(j) = \sum_{i:ij\in E} \text{Pagerank}_t(i) / \deg_{out}(i)$$

Let $P_{i,j} = \begin{cases} \frac{1}{\deg_{out}(i)}, & \text{if } ij \in E \\ 0, & \text{otherwise} \end{cases}$, Then, $\overrightarrow{\text{Pagerank}_{t+1}} = \overrightarrow{\text{Pagerank}_t} \cdot P$

# Pagerank




- Can be interpreted as a random walk on directed graphs
- When the graph is finite, irreducible, aperiodic, the pagerank values are unique by the fundamental thm.
- This shows that the pagerank values is a function of the graph structure, not based on initial values.
- Also, we have some intuition about pagerank values, which are the reciprocal of the expected return time.
- Practical modification makes the graph irreducible and aperiodic, without changing the relative importance of the webpages

# Application in 2-SAT

Say you are given an assignment, some clauses are violated

A random walk algorithm is to repeat until all clauses are satisfied:

- Pick an arbitrary violated clause, choose a literal uniformly at random, then flip its assignment

To analyze this algorithm, take any satisfying assignment $\tau$, consider the evolution of $\|\tau - \sigma_t\|_1$

We would like to know the first time $\|\tau - \sigma_t\|_1 = 0$, in expectation

This is dominated by the same hitting time of a simple symmetric random walk on $[0, n] \cap \mathbb{Z}$

See Chapter 7.1.1 of Probability and Computing for a coupling + recursion analysis

See **Gambler's ruin** and **optional stopping theorem** for an analysis of such hitting time

Open: Can the random walk idea give constant factor approximation to MAX-2SAT

# Back to Markov chain: What is the stationary distribution?

As in Eulerian directed graph, the stationary distribution of undirected graphs is easy to describe

Let $\vec{d} \in \mathbb{R}^{\mathrm{n}}$ be the degree vector and $m = |E|$

**Claim**. The distribution $\vec{\pi} = \frac{\vec{d}}{2m}$ is a stationary distribution of the random walk on undirected graphs

In the stationary distribution, the probability of going across an edge is the same for every edge

# Fundamental Theorem of Markov Chains in undirected graphs

Does $p_t \to \vec{\pi} = \frac{\vec{d}}{2m}$ as $t \to \infty$ regardless of $p_0$?

Not necessarily: not when the Markov chain is reducible and periodic.

In undirected graphs, being irreducible just means that the graph is connected.

In undirected connected graphs, aperiodic just means that the graph is non-bipartite

(To see why, recall what happens to an undirected 3-cycle)

So the fundamental theorem of Markov chain just becomes the following in undirected graphs.

**Theorem**. For any finite, connected, non-bipartite graph, $p_t \to \vec{\pi} = \frac{\vec{d}}{2m}$ as $t \to \infty$ regardless of $p_0$

# Lazy Random Walks

We can remove the non-bipartiteness assumption by doing a **lazy random walk**.

In each step, we stay at the same vertex with probability 1/2,

and we move to a uniform random neighbor with probability 1/2.

In matrix form, $p_t = \left(\frac{1}{2}I + \frac{1}{2}AD^{-1}\right)^t p_0$.

Theorem. For any finite and connected graph, $p_t = \left(\frac{1}{2}I + \frac{1}{2}AD^{-1}\right)^t p_0 \to \frac{\vec{d}}{2m}$ as $t \to \infty$ regardless of $p_0$.

It will be clear in the proof what the lazy random walk does to remove the non-bipartiteness assumption.

Intuitively, we can see it as making the random walk "very" aperiodic.

# Spectra Analysis

Let $W = AD^{-1}$ be the random walk matrix and $Z = \frac{1}{2}I + \frac{1}{2}AD^{-1}$ be the lazy random walk matrix.
To understand $p_t = W^t p_0$, it is very useful to understand the spectrum of $W$.
One problem is that $W$ is not symmetric.
But $W$ is **similar** to a symmetric matrix: $D^{-\frac{1}{2}}WD^{\frac{1}{2}} = D^{-\frac{1}{2}}(AD^{-1})D^{\frac{1}{2}} = D^{-\frac{1}{2}}AD^{-\frac{1}{2}} = \mathcal{A}$.
**Claim**. $W$ and $\mathcal{A}$ have the same spectrum

Note that $W$ may not have an orthonormal basis of eigenvectors.

# Spectrum of $W$

Let $W = \frac{1}{d} A = I - \frac{1}{d} L$.
What do we know about the spectrum $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$?

- We know that $1 \geq \alpha_1$ and $\alpha_n \geq -1$
- We know that $1 = \alpha_1$
- We know that $\alpha_1 > \alpha_2$ if and only if the graph is connected
- We know that $\alpha_1 = -\alpha_n$ if and only if the graph is bipartite

# Proof of Fundamental Theorem

**Theorem**. For any finite, connected, non-bipartite graph, $p_t \to \vec{\pi} = \frac{\vec{d}}{2m}$ as $t \to \infty$ regardless of $p_0$.
Proof sketch. We just need to show $W^t p_0 \to c_1 v_1$ as $t \to \infty$. Then find out what is $c_1 v_1$.
Note that $W^t p_0 = c_1 \alpha_1^t v_1 + c_2 \alpha_2^t v_2 + \cdots + c_n \alpha_n^t v_n$.

- $\alpha_1 = 1$.
- $\alpha_2 < 1$ if and only if the graph is connected.
- $\alpha_n > -1$ if and only if the graph is non-bipartite.

So the spectral conditions correspond exactly to the combinatorial assumptions of the theorem!

This implies that

- $W^t p_0 \to c_1 v_1$ as $t \to \infty$.

- The convergence is faster if $\alpha_2 < 1 - \epsilon$ and $\alpha_n > -1 + \epsilon$ for a larger $\epsilon > 0$.

# Proof for Lazy Random Walks

**Theorem**. For any finite and connected graph,

$$p_t = \left(\frac{1}{2}I + \frac{1}{2}AD^{-1}\right)^t p_0 \rightarrow \frac{\vec{d}}{2m} \text{ as } t \rightarrow \infty \text{ regardless of } p_0.$$

Proof is similar, but with a different spectrum.

Exercise: What is the new transition matrix, and what is its spectrum?