Correlation Decay \textit{up to} Uniqueness \textit{in} Spin Systems

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Two-State Spin System

graph $G=(V,E)$  2 states \{0,1\}

configuration $\sigma : V \rightarrow \{0,1\}$

$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$

$b = (b_0, b_1) = (\lambda, 1)$

$w(\sigma) = \prod_{(u,v) \in E} A_{\sigma(u),\sigma(v)} \prod_{v \in V} b_{\sigma(v)}$

edge activity: $\begin{bmatrix} \beta & \gamma & 1 \end{bmatrix}$

external field: $\begin{bmatrix} \lambda & 1 \end{bmatrix}$
Two-State Spin System

graph \( G=(V,E) \) 2 states \( \{0,1\} \)

configuration \( \sigma : V \to \{0,1\} \)

\[
A = \begin{bmatrix}
A_{0,0} & A_{0,1} \\
A_{1,0} & A_{1,1}
\end{bmatrix} = \begin{bmatrix}
\beta & 1 \\
1 & \gamma
\end{bmatrix}
\]

\( b = (b_0, b_1) = (\lambda, 1) \)

\[
w(\sigma) = \prod_{(u,v) \in E} A_{\sigma(u),\sigma(v)} \prod_{v \in V} b_{\sigma(v)}
\]

Gibbs measure: \( \Pr(\sigma) = \frac{w(\sigma)}{Z(G)} \)

partition function: \( Z(G) = \sum_{\sigma \in \{0,1\}^V} w(\sigma) \)
$$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix} \quad b = (b_0, b_1) = (\lambda, 1)$$

$$w(\sigma) = \prod_{(u,v) \in E} A_{\sigma(u), \sigma(v)} \prod_{v \in V} b_{\sigma(v)}$$

**partition function:**  
$$Z(G) = \sum_{\sigma \in \{0,1\}^V} w(\sigma)$$

**marginal probability:**  
$$\Pr[\sigma(v) = 0 \mid \sigma(v_1), \ldots, \sigma(v_k)]$$

$$\Pr(\tau) = \prod_{k=1}^n \Pr[\sigma(v_k) = \tau(v_k) \mid \sigma(v_i) = \tau(v_i), 1 \leq i < k]$$

$$= \frac{w(\tau)}{Z} \quad 1/\text{poly}(n) \text{ additive error for marginal in poly-time}$$
ferromagnetic: \( \beta \gamma > 1 \)

FPRAS: \[ \text{[Jerrum-Sinclair'93]} \quad \text{[Goldberg-Jerrum-Paterson'03]} \]

anti-ferromagnetic: \( \beta \gamma < 1 \)

hardcore model: \( \beta = 0, \gamma = 1 \) \[ \text{[Weitz'06]} \]

Ising model: \( \beta = \gamma \) \[ \text{[Sinclair-Srivastava-Thurley'12]} \]

\((\beta, \gamma, \lambda)\) lies in the interior of
uniqueness region of \(\Delta\)-regular tree \(\exists\) FPTAS for graphs of max-degree \(\Delta\)

[Goldberg-Jerrum-Paterson’03]
FPRAS for arbitrary graphs

[Li-Lu-Y. ’12]: no external field
FPTAS for arbitrary graphs
anti-ferromagnetic: \( \beta \gamma < 1 \)

bounded \( \Delta \) or \( \Delta = \infty \)

\((\beta, \gamma, \lambda)\) lies in the interiors of \textit{uniqueness} regions of \(d\)-regular trees for all \(d \leq \Delta\).

\(\exists\) FPTAS for graphs of max-degree \(\Delta\)

[Sly-Sun’12] [Galanis-Stefankovic-Vigoda’12]:

\((\beta, \gamma, \lambda)\) lies in the interiors of \textit{non-uniqueness} regions of \(d\)-regular trees for some \(d \leq \Delta\).

\(\nexists\) FPRAS for graphs of max-degree \(\Delta\)

assuming \(\text{NP} \neq \text{RP}\)
Uniqueness Condition

\( (d+1) \)-regular tree

\[ f_d(x) = \lambda \left( \frac{\beta x + 1}{x + \gamma} \right)^d \]

\[ \hat{x}_d = f_d(\hat{x}_d) \]

\[ |f'_d(\hat{x}_d)| < 1 \]
anti-ferromagnetic:  \[ \beta \gamma < 1 \]

bounded \( \Delta \) or \( \Delta = \infty \)

\[ f_d(x) = \lambda \left( \frac{\beta x + 1}{x + \gamma} \right)^d \]

\[ \forall d < \Delta, \left| f'_d(\hat{x}_d) \right| < 1 \]

\[ \exists \text{ FPTAS for graphs of max-degree } \Delta \]

[Sly-Sun’12] [Galanis-Stefankovic-Vigoda’12]:

\[ \exists d < \Delta, \left| f'_d(\hat{x}_d) \right| > 1 \]

assuming \( \text{NP} \neq \text{RP} \)

\[ \nexists \text{ FPRAS for graphs of max-degree } \Delta \]
Correlation Decay

weak spatial mixing (WSM):
\[ \forall \sigma_{\partial B}, \tau_{\partial B} \in \{0, 1\}^{\partial B} \]
\[ \Pr_{\sigma}[\sigma(v) = 0 \mid \sigma_{\partial B}] \approx \Pr_{\sigma}[\sigma(v) = 0 \mid \tau_{\partial B}] \]

strong spatial mixing (SSM):
\[ \Pr_{\sigma}[\sigma(v) = 0 \mid \sigma_{\partial B}, \sigma_{\Lambda}] \approx \Pr_{\sigma}[\sigma(v) = 0 \mid \tau_{\partial B}, \sigma_{\Lambda}] \]

error < \exp(-t)

exponential correlation decay

uniqueness: WSM in reg. tree
Self-Avoiding Walk Tree
due to Weitz (2006)

$G = (V, E)$

$T = T_{SAW}(G, v)$

preserve the marginal dist. at $v$

on bounded degree graphs:

SSM $\rightarrow$ FPTAS
hardcore model, anti-ferro Ising model:
(for $\beta, \gamma < 1$)
hardcore model, anti-ferro Ising model:
(for $\beta, \gamma < 1$)

in reg. trees: WSM $\rightarrow$ SSM

SSM in $\Delta$-reg. tree

SSM in trees of degree $\leq \Delta$

fixing

SSM in graphs of degree $\leq \Delta$

SAW-tree
for general anti-ferro 2-state spin systems:
WSM in $d$-reg. trees for $d \leq \Delta$ \\
SSM in trees of degree $\leq \Delta$ \\
in reg. trees: \\
WSM $\rightarrow$ SSM
for general anti-ferro 2-state spin systems:

WSM in $d$-reg. trees for $d \leq \Delta$

SSM in trees of degree $\leq \Delta$

SSM in graphs of degree $\leq \Delta$

WSM in $d$-reg. trees for $d \leq \Delta$
$x \in [R, R + \delta]$ 

$\delta = \exp(-\Omega(n))$

$x = \frac{\Pr[\sigma(v) = 0 \mid \sigma_{\Lambda}]}{\Pr[\sigma(v) = 1 \mid \sigma_{\Lambda}]}$

$x = f(x_1, \ldots, x_d) = \lambda \prod_{i=1}^{d} \left( \frac{\beta x_i + 1}{x_i + \gamma} \right)$

$x$'s in level $n \in [0, \infty)$
Potential Analysis

\[ f(x) \]

\[ f \uparrow \]

\[ x \]

\[ F_n(x) = f \circ f \circ \cdots \circ f(x) \]

\[ F_n(x + \delta) - F_n(x) = F_n'(x_0) \cdot \delta \]

\[ = \delta \cdot \prod_{t=0}^{n-1} f'(x_t) \quad x_t = f(x_{t-1}) \]

\[ = \delta \cdot \frac{\Phi(x_0)}{\Phi(x_n)} \cdot \prod_{t=0}^{n-1} \frac{\Phi(f(x_t))}{\Phi(x_t)} f'(x_t) \]
Potential Analysis

\[ f(x) \xrightarrow{\phi} g(y) \]

\[ f \uparrow \quad \phi^{-1} \quad \uparrow g \]

\[ x \quad \leftarrow \quad y \]

\[ G_n(x) = g \circ g \circ \cdots \circ g(x) \]

\[ G_n(x + \delta) - G_n(x) = G'_n(x_0) \cdot \delta \]

\[ G'_n(x_0) = \prod_{t=0}^{n-1} g'(x_t) \]

\[ = \prod_{t=0}^{n-1} \left[ \phi(f(\phi^{-1}(y_t))) \right]' \]

\[ = \prod_{t=0}^{n-1} \frac{\Phi(f(x_t))}{\Phi(x_t)} f'(x_t) \quad \phi'(x) = \Phi(x) \]
\[ \phi'(x) = \Phi(x) = \frac{1}{\sqrt{x(\beta x + 1)(x + \gamma)}} \]

\[
\begin{align*}
  f(x_1, \ldots, x_d) &= \lambda \prod_{i=1}^{d} \left( \frac{\beta x_i + 1}{x_i + \gamma} \right) \\
g(y_1, \ldots, y_d) &= \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d)))
\end{align*}
\]

\[
\begin{align*}
g(y_1, \ldots, y_d) - g(y_1 + \delta_1, \ldots, y_d + \delta_d) &= -\nabla \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d))) \cdot (\delta_1, \ldots, \delta_d) \\
&\leq \alpha(d; x_1, \ldots, x_d) \cdot \max_{1 \leq i \leq d} \{\delta_i\}
\end{align*}
\]

amortized decay rate
\( \alpha(d; x_1, ..., x_d) \) \hspace{1cm} \text{amortized decay rate}

\[
\begin{align*}
\alpha(d; x_1, ..., x_d) &= (1 - \beta \gamma) \left( \lambda \prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma} \right)^{\frac{1}{2}} \\
&= \frac{(1 - \beta \gamma) \left( \lambda \prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma} \right)^{\frac{1}{2}}}{\left( \beta \lambda \prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma} + 1 \right)^{\frac{1}{2}}} \cdot \sum_{i=1}^{d} \frac{x_i^{\frac{1}{2}}}{(\beta x_i + 1)^{\frac{1}{2}} (x_i + \gamma)^{\frac{1}{2}}}
\end{align*}
\]

\text{Cauchy-Schwarz arithmetic and geometric means}

\[
\leq \alpha_d(x) \defeq \alpha(d; x, \ldots, x)_{d}
\]

\[
\begin{align*}
\alpha_d(x) &= \sqrt{\frac{d(1 - \beta \gamma)x}{(\beta x + 1)(x + \gamma)}} \sqrt{\frac{d(1 - \beta \gamma)\lambda \left( \frac{\beta x + 1}{x + \gamma} \right)^d}{\left( \beta \lambda \left( \frac{\beta x + 1}{x + \gamma} \right)^d + 1 \right) \left( \lambda \left( \frac{\beta x + 1}{x + \gamma} \right)^d + \gamma \right)}} \\
&= \frac{\Phi(f(x))}{\Phi(x)} \left| f'(x) \right|
\end{align*}
\]
\[
\alpha_d(x) = \sqrt{\frac{d(1 - \beta \gamma)x}{(\beta x + 1)(x + \gamma)}} \sqrt{\frac{d(1 - \beta \gamma)\lambda \left(\frac{\beta x + 1}{x + \gamma}\right)^d}{\left(\beta \lambda \left(\frac{\beta x + 1}{x + \gamma}\right)^d + 1\right) \left(\lambda \left(\frac{\beta x + 1}{x + \gamma}\right)^d + \gamma\right)}}
\]

\[
= \sqrt{\frac{d(1 - \beta \gamma)x}{(\beta x + 1)(x + \gamma)}} \sqrt{\frac{d(1 - \beta \gamma)\lambda f_d(x)}{\left(\beta f_d(x) + 1\right) \left(f_d(x) + \gamma\right)}}
\]

\[
\leq \sqrt{\frac{d(1 - \beta \gamma)\hat{x}}{(\beta \hat{x} + 1)(\hat{x} + \gamma)}}
\]

\[
= \sqrt{|f'_d(\hat{x}_d)|}
\]

\[
f_d(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma}\right)^d
\]

\[
\hat{x}_d = f_d(\hat{x}_d)
\]

\[
|f'_d(\hat{x}_d)| < 1
\]

\[
\delta \leq \alpha \cdot \max_{1 \leq i \leq d} \{\delta_i\}
\]

\[
\alpha < 1
\]
anti-ferromagnetic: \( \beta \gamma < 1 \)

\[
f_d(x) = \lambda \left( \frac{\beta x + 1}{x + \gamma} \right)^d
\]

\( \forall d < \Delta, |f'_d(\hat{x}_d)| < 1 \)

- SSM in trees of max-degree \( \Delta \)
- SSM in graphs of max-degree \( \Delta \)
- \( \exists \) FPTAS for graphs of max-degree \( \Delta \)

bounded \( \Delta \)

\[
|f'_\Delta(\hat{x}_\Delta)| < 1
\]

[Weitz’06] + [Sinclair-Srivastava-Thurley’12] + translation
requirement of potential function:

\[ f(x) = \lambda \left( \frac{\beta x + 1}{x + \gamma} \right)^d \]

\[ \hat{x} = f(\hat{x}) \]

uniqueness: \[ |f'(\hat{x})| < 1 \]

amortized decay: \[ |f'(x)| \cdot \frac{\Phi(f(x))}{\Phi(x)} < 1 \]
requirement of potential function:

\[ f(x) = \lambda \left( \frac{\beta x + 1}{x + \gamma} \right)^d \quad \hat{x} = f(\hat{x}) \]

phase-trans: \[ |f'(\hat{x})| = 1 \]

amortized decay: \[ |f'(x)| \cdot \frac{\Phi(f(x))}{\Phi(x)} \]

\[ |f'(\hat{x})| \cdot \frac{\Phi(f(\hat{x}))}{\Phi(\hat{x})} = 1 \]

\[ \left[ f'(x) \cdot \frac{\Phi(f(x))}{\Phi(x)} \right]' \bigg|_{x=\hat{x}} = 0 \]

\[ (\ln(\Phi(\hat{x})))' = -\frac{f''(\hat{x})}{2} = \frac{1}{2} \left( \frac{1}{\hat{x}} + \frac{1}{\hat{x} + \gamma} + \frac{\beta}{\beta \hat{x} + 1} \right) \]
requirement of potential function:

\[
(ln(\Phi(\hat{x}))))' = \frac{1}{2} \left( \frac{1}{\hat{x}} + \frac{1}{\hat{x} + \gamma} + \frac{\beta}{\beta \hat{x} + 1} \right)
\]

strengthen the requirement:

\[
(ln(\Phi(x))))' = \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x + \gamma} + \frac{\beta}{\beta x + 1} \right)
\]

\[
\Phi(x) = \frac{C}{\sqrt{x(\beta x + 1)(x + \gamma)}}
\]
Computationally Efficient Correlation Decay

\[\delta \leq \alpha_d(x) \cdot \max_{1 \leq i \leq d} \{\delta_i\}\]

\[
\alpha_d(x) = \sqrt{\frac{d(1 - \beta \gamma)x}{(\beta x + 1)(x + \gamma)}} \cdot \sqrt{\frac{d(1 - \beta \gamma)f_d(x)}{(\beta f_d(x) + 1)(f_d(x) + \gamma)}}
\]

\[
\leq \alpha^{\lceil \log_M (d+1) \rceil} \quad \text{for some} \quad \alpha < 1
\]

\[M > 1\]
Computationally Efficient Correlation Decay

\[ \delta \leq \alpha_d(x) \cdot \max_{1 \leq i \leq d} \{\delta_i\} \]

\[ \alpha_d(x) \leq \alpha^{\lfloor \log_M (d+1) \rfloor} \]

for some \( \alpha < 1 \) \( M > 1 \)

for small \( d < M \) one-step recursion decays \( \alpha \)

for large \( d \geq M \) one-step recursion decays \( \alpha^{\lfloor \log_M (d+1) \rfloor} \)

behaves like \( \lfloor \log_M (d + 1) \rfloor \) steps!
Computationally Efficient Correlation Decay

Correlation decay in new metric size grows exponentially:

\( \text{distance} = O(\log n) \quad \text{1/poly-precision in poly-time} \)
anti-ferromagnetic: \( \beta \gamma < 1 \)

WSM in \( d \)-reg. trees for \( d \leq \Delta \)

\( \exists \) FPTAS for graphs of max-degree \( \Delta \)

bounded \( \Delta \) or \( \Delta = \infty \)

[Weitz’06] hardcore model

[Sinclair-Srivastava-Thurley’12] Ising model

uniqueness condition:

WSM in \( \Delta \)-reg. tree
\[ f_d(x) = \lambda \left( \frac{\beta x + 1}{x + \gamma} \right)^d \]
\[ \hat{x}_d = f_d(\hat{x}_d) \]
\[ |f'_d(\hat{x}_d)| < 1 \]

\[ |f'_d(\hat{x}_d)| \]
due to [Guo'12]

\[ \beta, \gamma \leq 1 \]

\[ \gamma > 1 \]
\[ \beta \gamma < 1 \]

\[ \beta, \gamma \leq 1 \]
WSM in $\Delta$-reg. tree $\Rightarrow$
WSM in $d$-reg. tree for $d \leq \Delta$
\[ \gamma > 1 \]
\[ \beta \gamma < 1 \]
WSM in $D$-reg. tree $\Rightarrow$
WSM in all $d$-reg. trees
Recall that \( c \) is a real number throughout this section.

Below is Figure 1: The Monotonicity of the Contraction Ratio.

For both two curves, we fix \( d \). In particular, we want to know if \( \beta \gamma < 1 \), \( \beta, \gamma \leq 1 \), \( \gamma > 1 \). Notice the shape of the curve changed dramatically. Moreover, \( |f_d'(\hat{x}_d)| \) satisfies that \( \Delta \)-reg. tree but not in \((\Delta-1)\)-reg. tree.

For simplicity, we consider \( \Delta = 1 \), while \( \beta, \gamma, \lambda \leq 1 \). It turns out that if \( 0 < \lambda < 1 \), there exists a unique maximum point.

\((\beta, \gamma, \lambda)\) that WSM in \( \Delta \)-reg. tree but not in \((\Delta-1)\)-reg. tree.

in reg. trees:

WSM \[\rightarrow\] SSM

[Weitz’06] + [Sinclair-Srivastava-Thurley’12] + translation

SSM in \( \Delta \)-reg. tree

SSM in graphs of degree \( \leq \Delta \)
Open Problems

• Characterization of SSM in ferromagnetic 2-state spin systems.

• SSM in multi-state spin systems:
  • difficulty: no SAW-tree;
  • implications: WSM vs. SSM in reg. trees, monotonicity of WSM/SSM w.r.t degree.

• Apply potential analysis and computationally efficient correlation decay to other problems.
Thank you!