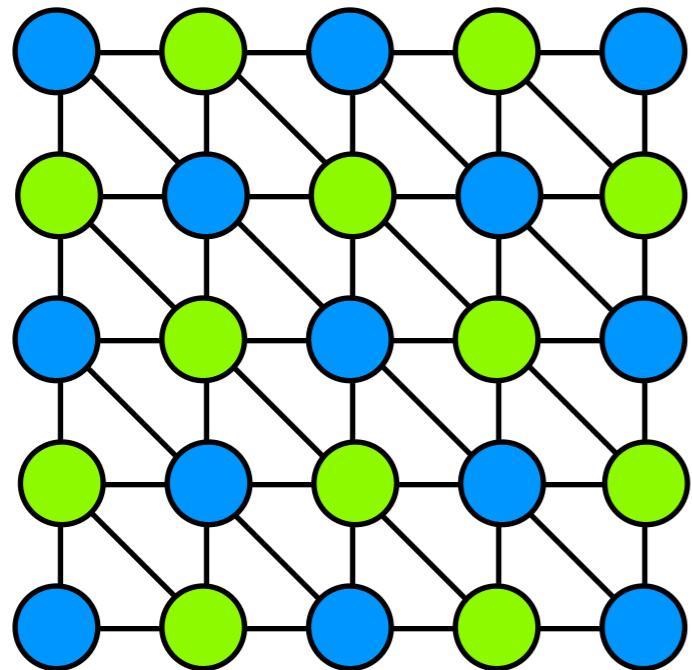


Correlation Decay *up to* Uniqueness *in* Spin Systems

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Joint work with
Liang Li (Peking University)
Pinyan Lu (Microsoft research Asia)

Two-State Spin System



graph $G=(V,E)$ 2 states {0,1}

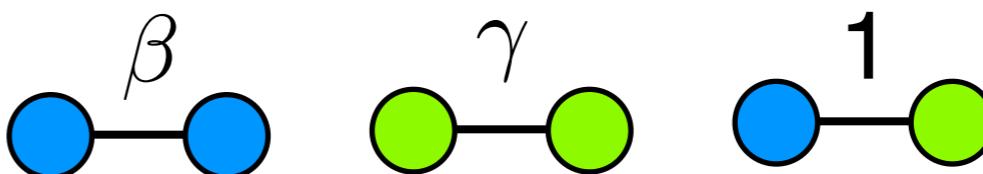
configuration $\sigma : V \rightarrow \{0, 1\}$

$$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$$

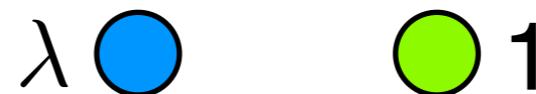
$$b = (b_0, b_1) = (\lambda, 1)$$

$$w(\sigma) = \prod_{(u,v) \in E} A_{\sigma(u), \sigma(v)} \prod_{v \in V} b_{\sigma(v)}$$

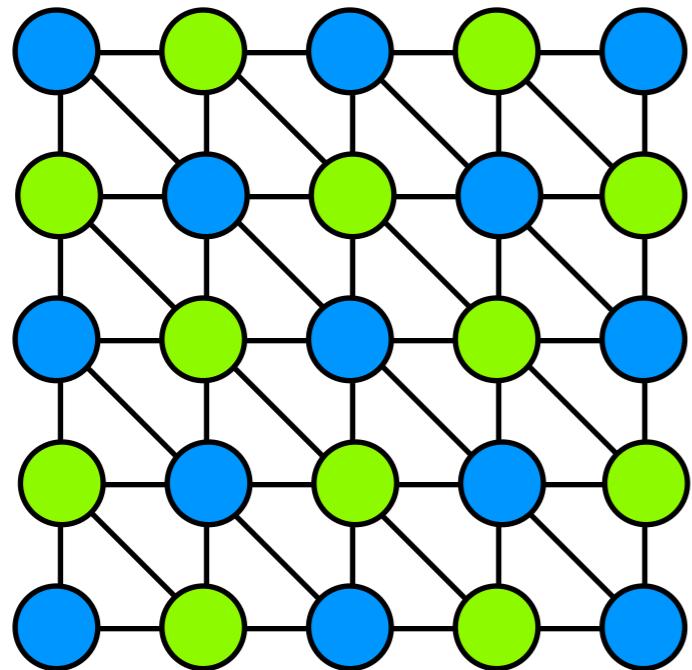
edge activity:



external field:



Two-State Spin System



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$$b = (b_0, b_1) = (\lambda, 1)$$

$$w(\sigma) = \prod_{(u,v) \in E} A_{\sigma(u), \sigma(v)} \prod_{v \in V} b_{\sigma(v)}$$

Gibbs measure: $\Pr(\sigma) = \frac{w(\sigma)}{Z(G)}$

partition function: $Z(G) = \sum_{\sigma \in \{0,1\}^V} w(\sigma)$

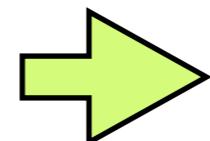
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partition function: $Z(G) = \sum_{\sigma \in \{0,1\}^V} w(\sigma)$

Gibbs measure: $\Pr(\sigma) = \frac{w(\sigma)}{Z(G)}$

marginal probability: $\Pr(\sigma(v) = 0 \mid \sigma_\Lambda)$

1/n additive error for
marginal in poly(n)-time



FPTAS for $Z(G)$

ferromagnetic: $\beta\gamma > 1$

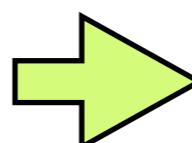
FPRAS: [Jerrum-Sinclair'93] [Goldberg-Jerrum-Paterson'03]

anti-ferromagnetic: $\beta\gamma < 1$

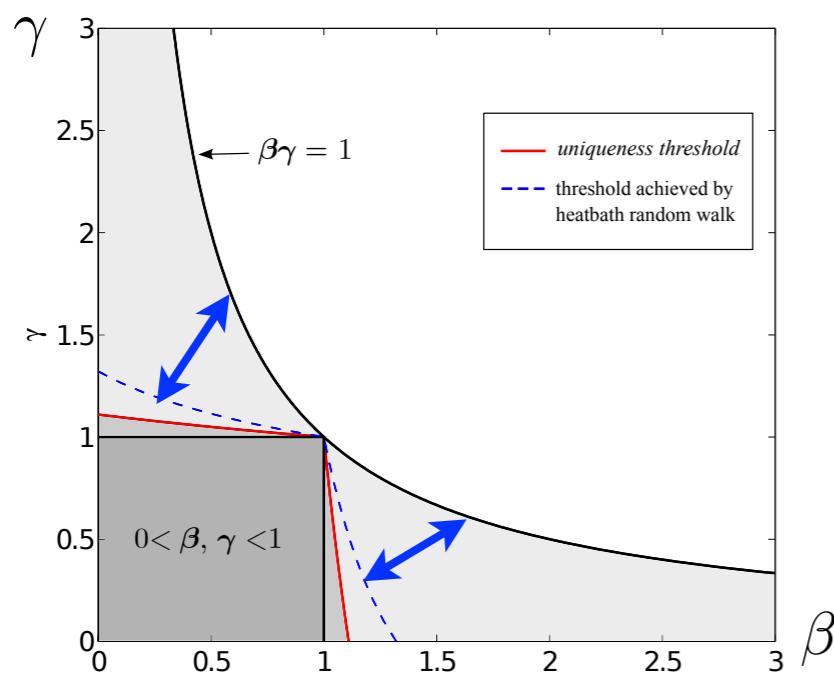
hardcore model: $\beta = 0, \gamma = 1$ [Weitz'06]

Ising model: $\beta = \gamma$ [Sinclair-Srivastava-Thurley'12]

(β, γ, λ) lies in the interior of
uniqueness region of Δ -regular tree



\exists FPTAS for graphs
of max-degree Δ



[Goldberg-Jerrum-Paterson'03]

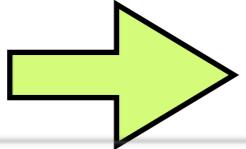
[Li-Lu-Y. '12]:

FPTAS for arbitrary graphs

anti-ferromagnetic: $\beta\gamma < 1$

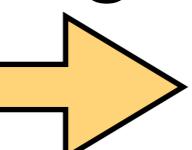
bounded Δ or $\Delta=\infty$

(β, γ, λ) lies in the interiors of **uniqueness** regions of d -regular trees for all $d \leq \Delta$.

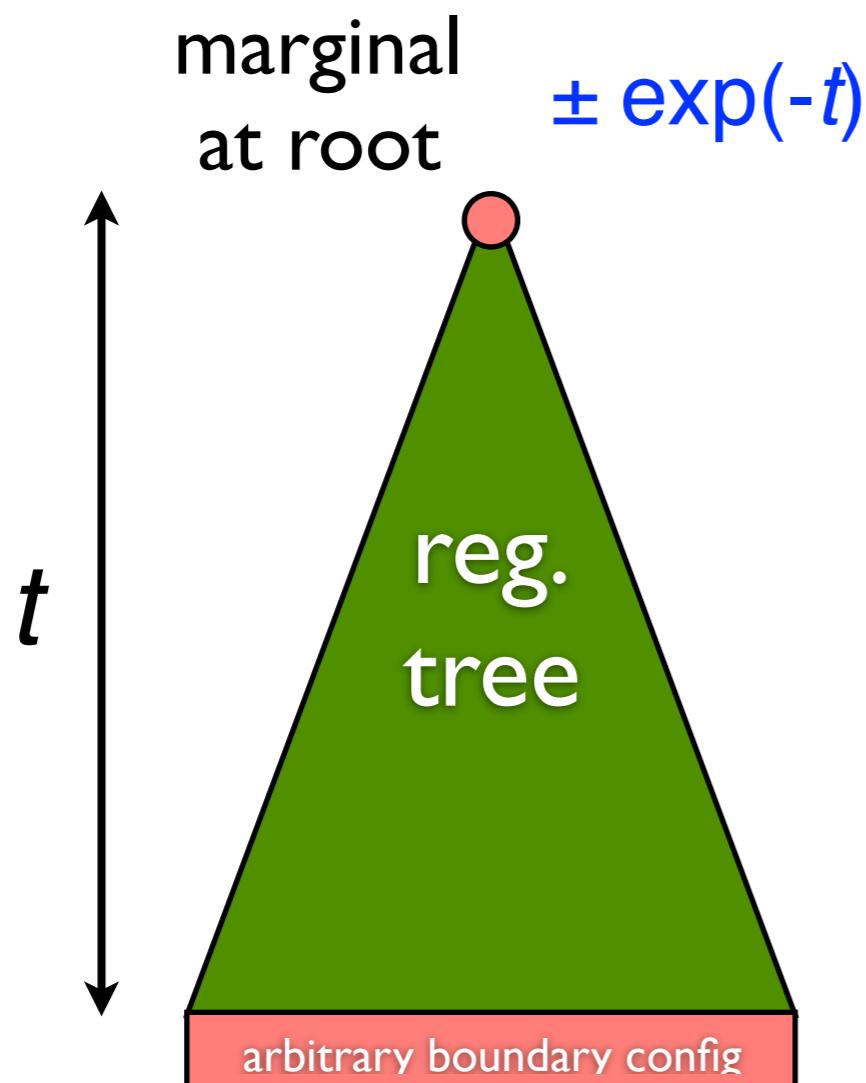
 \exists FPTAS for graphs of max-degree Δ

[Sly-Sun'12] [Galanis-Stefankovic-Vigoda'12]:

(β, γ, λ) lies in the interiors of **non-uniqueness** regions of d -regular trees for some $d \leq \Delta$.

assuming
 $\text{NP} \neq \text{RP}$  \nexists FPRAS for graphs of max-degree Δ

Uniqueness Condition



$(d+1)$ -regular tree

$$f_d(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d$$

$$\hat{x}_d = f_d(\hat{x}_d)$$

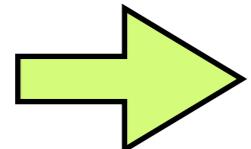
$$|f'_d(\hat{x}_d)| < 1$$

anti-ferromagnetic: $\beta\gamma < 1$

bounded Δ or $\Delta=\infty$

$$f_d(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d$$

$$\forall d < \Delta, |f'_d(\hat{x}_d)| < 1$$



\exists FPTAS for graphs of max-degree Δ

[Sly-Sun'12] [Galanis-Stefankovic-Vigoda'12]:

$$\exists d < \Delta, |f'_d(\hat{x}_d)| > 1$$

assuming
 $\text{NP} \neq \text{RP}$

\nexists FPRAS for graphs of max-degree Δ

Correlation Decay

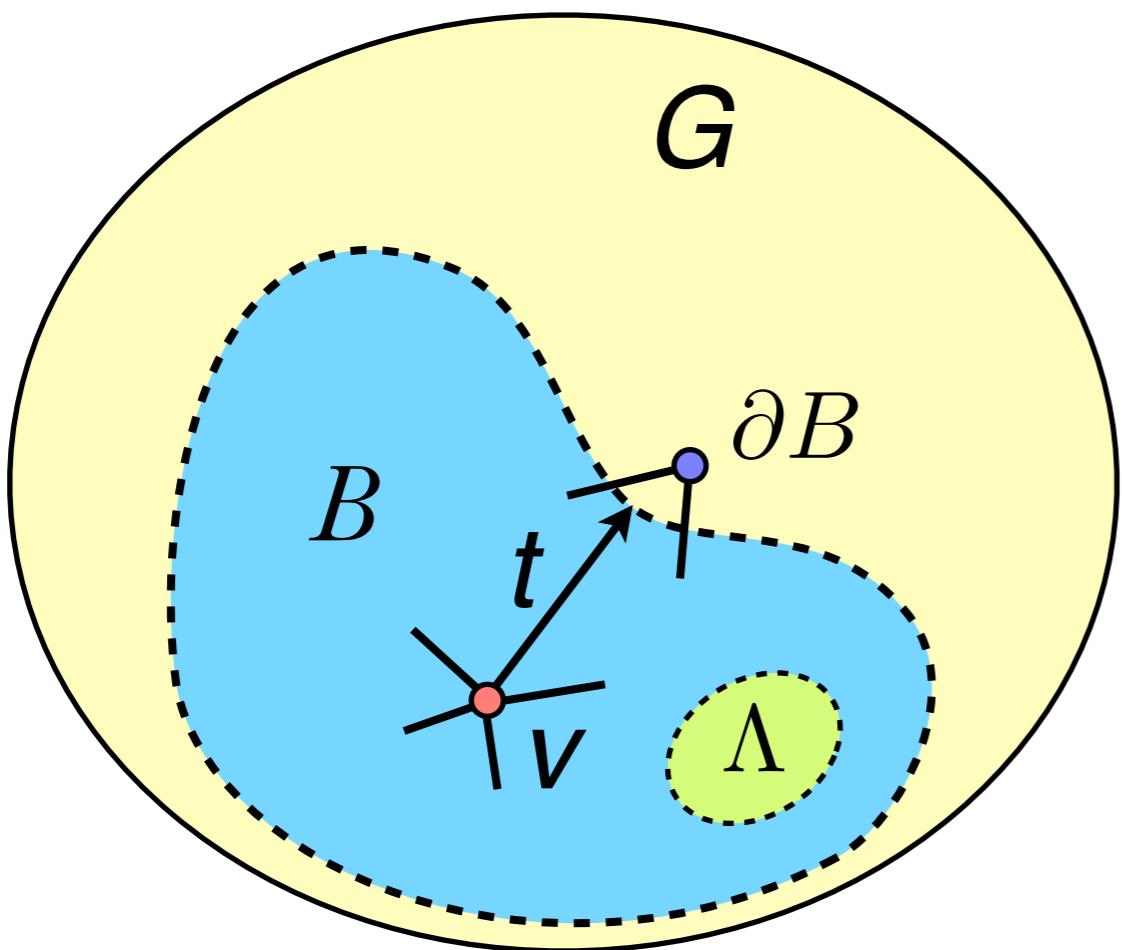
weak spatial mixing (WSM):

$$\forall \sigma_{\partial B}, \tau_{\partial B} \in \{0, 1\}^{\partial B}$$

$$|\Pr(\sigma(v) = 0 \mid \sigma_{\partial B}) - \Pr(\sigma(v) = 0 \mid \tau_{\partial B})| \leq \exp(-\Omega(t))$$

strong spatial mixing (SSM):

$$|\Pr(\sigma(v) = 0 \mid \sigma_{\partial B}, \sigma_{\Lambda}) - \Pr(\sigma(v) = 0 \mid \tau_{\partial B}, \sigma_{\Lambda})| \leq \exp(-\Omega(t))$$

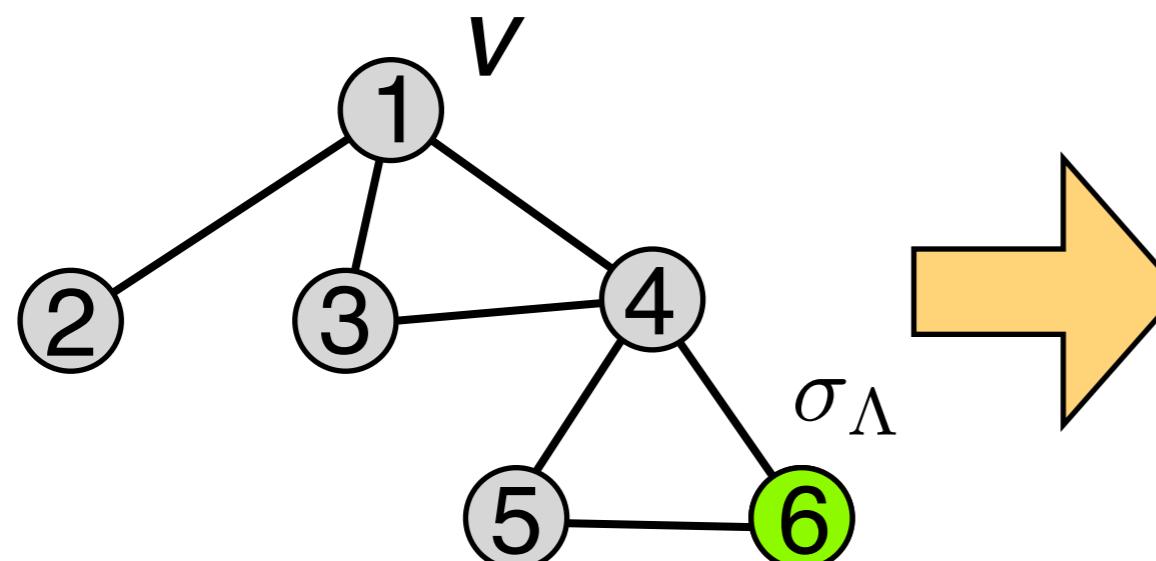


Uniqueness:
WSM in reg. tree

Self-Avoiding Walk Tree

due to Weitz (2006)

$$G=(V,E)$$

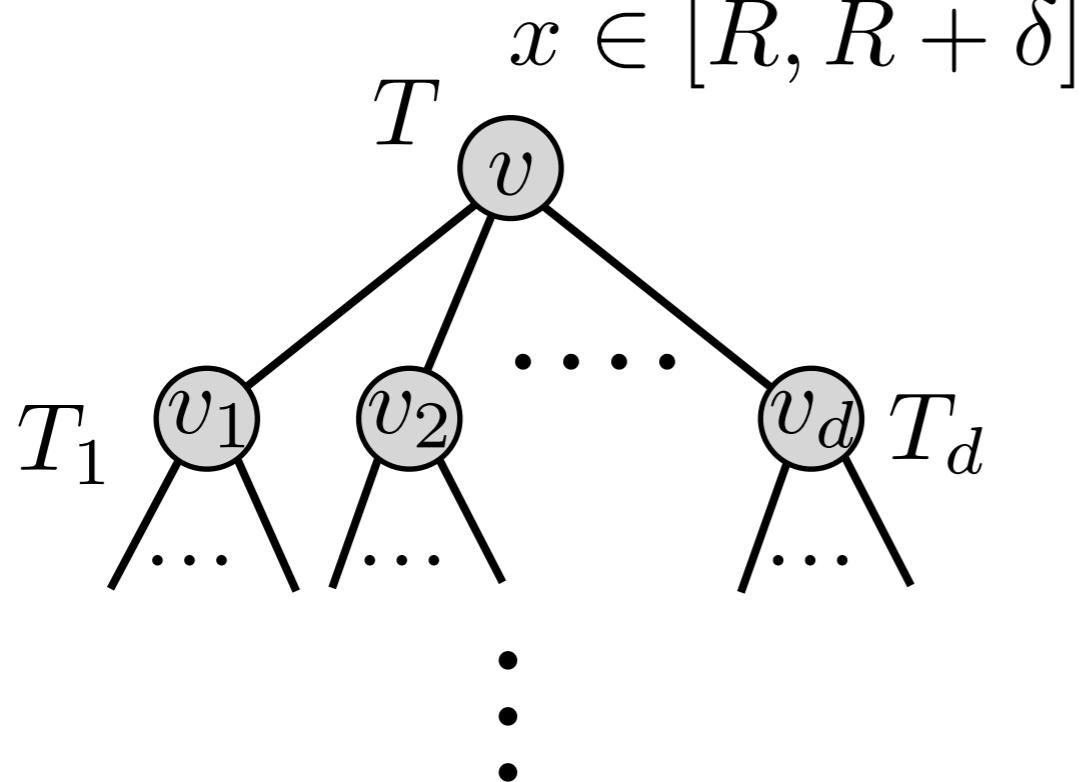


preserve the marginal dist. at v

on bounded degree graphs:

SSM → FPTAS

$T = T_{\text{SAW}}(G, v)$



$$\delta = \exp(-\Omega(n))$$

$$x = \frac{\Pr[\sigma(v) = \textcolor{blue}{0} \mid \sigma_\Lambda]}{\Pr[\sigma(v) = \textcolor{green}{1} \mid \sigma_\Lambda]}$$

x 's in level $n \in [0, \infty)$

$$x = f(x_1, \dots, x_d) = \lambda \prod_{i=1}^d \left(\frac{\beta x_i + 1}{x_i + \gamma} \right)$$

Potential Analysis

$$f(x)$$

$$\begin{array}{c} \uparrow \\ f \\ \downarrow \end{array}$$

$$F_n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n}(x)$$

$$x$$

$$F_n(x + \delta) - F_n(x) = F'_n(x_0) \cdot \delta$$

$$= \delta \cdot \prod_{t=0}^{n-1} f'(x_t) \quad x_t = f(x_{t-1})$$

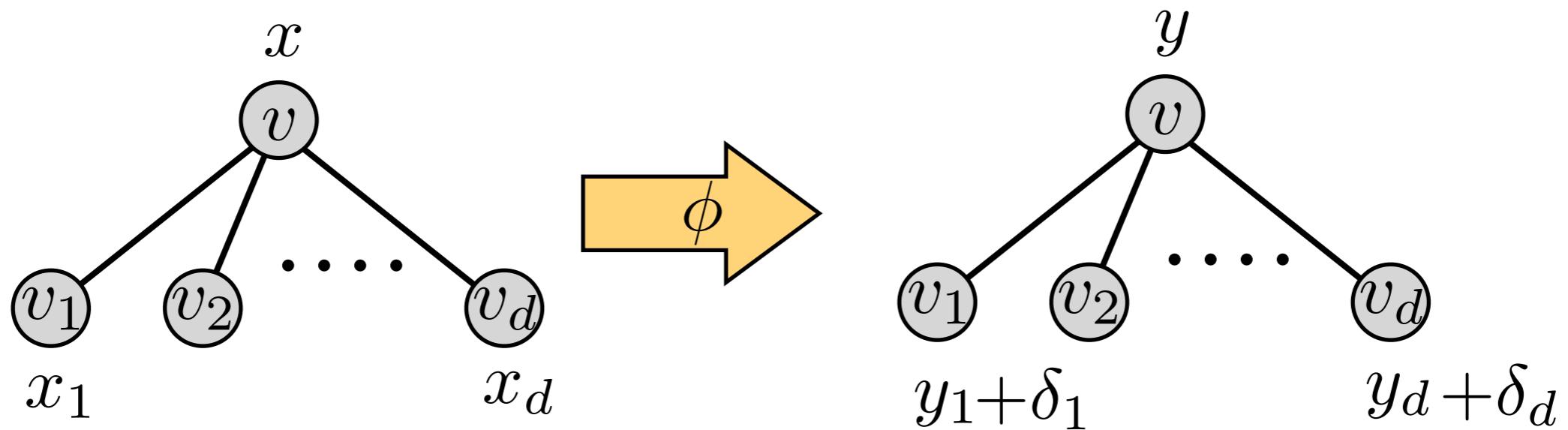
$$= \delta \cdot \frac{\Phi(x_0)}{\Phi(x_n)} \cdot \prod_{t=0}^{n-1} \frac{\Phi(f(x_t))}{\Phi(x_t)} f'(x_t)$$

Potential Analysis

$$\begin{array}{ccc} f(x) & \xrightarrow{\phi} & g(y) \\ f \uparrow & & \uparrow g \\ x & \xleftarrow{\phi^{-1}} & y \end{array} \quad G_n(x) = \underbrace{g \circ g \circ \cdots \circ g}_{n}(x)$$
$$G_n(x + \delta) - G_n(x) = G'_n(x_0) \cdot \delta$$

$$\begin{aligned} G'_n(x_0) &= \prod_{t=0}^{n-1} g'(x_t) \\ &= \prod_{t=0}^{n-1} [\phi(f(\phi^{-1}(y_t)))]' \\ &= \prod_{t=0}^{n-1} \frac{\Phi(f(x_t))}{\Phi(x_t)} f'(x_t) \quad \phi'(x) = \Phi(x) \end{aligned}$$

$$\phi'(x) = \Phi(x) = \frac{1}{\sqrt{x(\beta x + 1)(x + \gamma)}}$$



$$f(x_1, \dots, x_d) = \lambda \prod_{i=1}^d \left(\frac{\beta x_i + 1}{x_i + \gamma} \right) \quad g(y_1, \dots, y_d) = \phi(f(\phi^{-1}(y_1), \dots, \phi^{-1}(y_d)))$$

$$\begin{aligned} & g(y_1, \dots, y_d) - g(y_1 + \delta_1, \dots, y_d + \delta_d) \\ &= -\nabla \phi(f(\phi^{-1}(y_1), \dots, \phi^{-1}(y_d))) \cdot (\delta_1, \dots, \delta_d) \end{aligned}$$

$$\leq \alpha(d; x_1, \dots, x_d) \cdot \max_{1 \leq i \leq d} \{\delta_i\}$$

amortized decay rate

$$\alpha(d; x_1, \dots, x_d)$$

amortized decay rate

$$= \frac{(1 - \beta\gamma) \left(\lambda \prod_{i=1}^d \frac{\beta x_i + 1}{x_i + \gamma} \right)^{\frac{1}{2}}}{\left(\beta \lambda \prod_{i=1}^d \frac{\beta x_i + 1}{x_i + \gamma} + 1 \right)^{\frac{1}{2}} \left(\lambda \prod_{i=1}^d \frac{\beta x_i + 1}{x_i + \gamma} + \gamma \right)^{\frac{1}{2}}} \cdot \sum_{i=1}^d \frac{x_i^{\frac{1}{2}}}{(\beta x_i + 1)^{\frac{1}{2}} (x_i + \gamma)^{\frac{1}{2}}}$$

Convexity analysis

$$\leq \alpha_d(x) \triangleq \alpha(d; \underbrace{x, \dots, x}_d)$$

$$= \sqrt{\frac{d(1 - \beta\gamma)x}{(\beta x + 1)(x + \gamma)}} \sqrt{\frac{d(1 - \beta\gamma)\lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d}{\left(\beta \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d + 1 \right) \left(\lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d + \gamma \right)}}$$

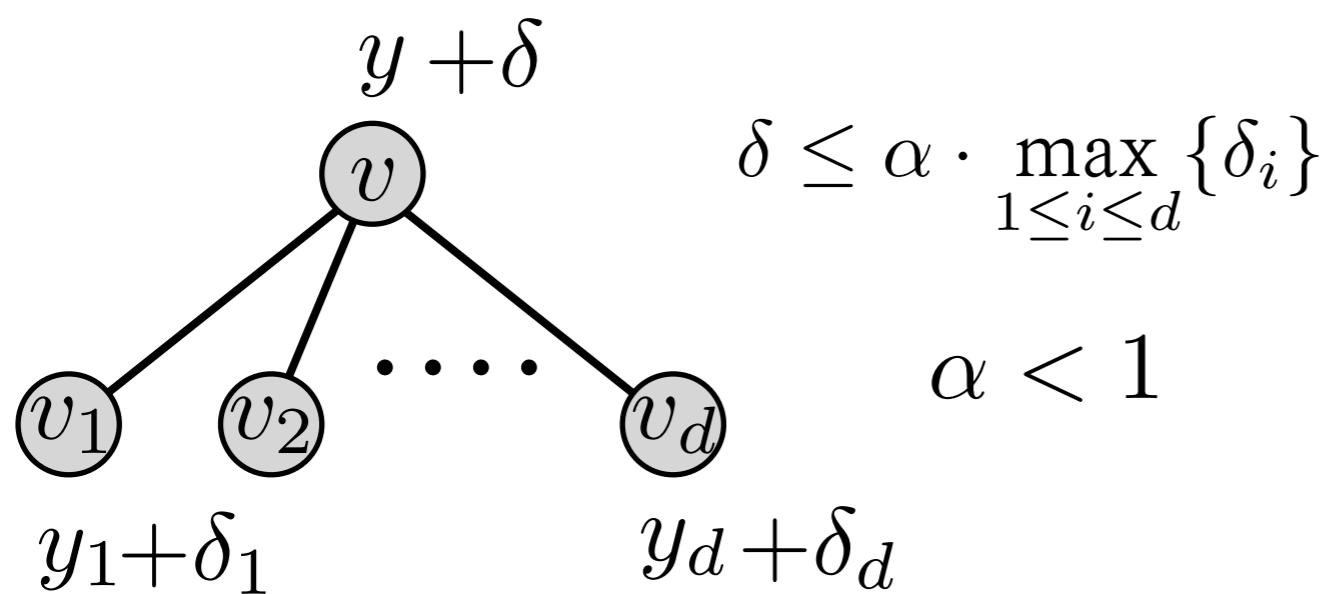
$$= \frac{\Phi(f(x))}{\Phi(x)} |f'(x)|$$

$$\alpha_d(x) = \sqrt{\frac{d(1 - \beta\gamma)x}{(\beta x + 1)(x + \gamma)}} \sqrt{\frac{d(1 - \beta\gamma)\lambda \left(\frac{\beta x + 1}{x + \gamma}\right)^d}{\left(\beta\lambda \left(\frac{\beta x + 1}{x + \gamma}\right)^d + 1\right) \left(\lambda \left(\frac{\beta x + 1}{x + \gamma}\right)^d + \gamma\right)}}$$

$$= \sqrt{\frac{d(1 - \beta\gamma)x}{(\beta x + 1)(x + \gamma)}} \sqrt{\frac{d(1 - \beta\gamma)f_d(x)}{(\beta f_d(x) + 1)(f_d(x) + \gamma)}}$$

$$\leq \sqrt{\frac{d(1 - \beta\gamma)\hat{x}}{(\beta\hat{x} + 1)(\hat{x} + \gamma)}}$$

$$= \sqrt{|f'_d(\hat{x}_d)|}$$



$$f_d(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d$$

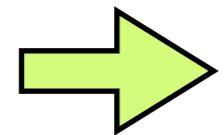
$$\hat{x}_d = f_d(\hat{x}_d)$$

$$|f'_d(\hat{x}_d)| < 1$$

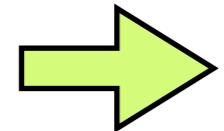
anti-ferromagnetic: $\beta\gamma < 1$

$$f_d(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma} \right)^d$$

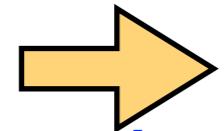
$$\forall d < \Delta, |f'_d(\hat{x}_d)| < 1$$



SSM in trees of max-degree Δ

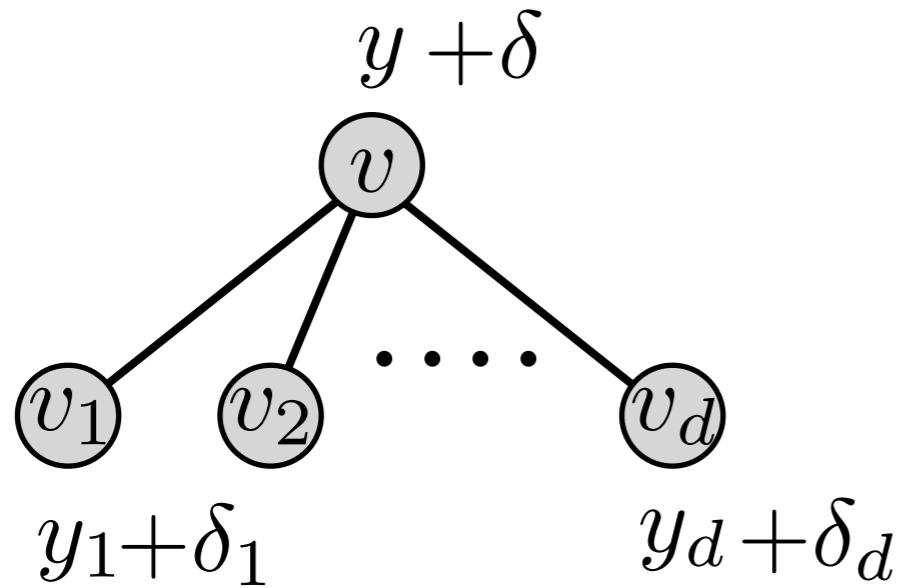


SSM in graphs of max-degree Δ



\exists FPTAS for graphs of max-degree Δ
bounded Δ

Computationally Efficient Correlation Decay

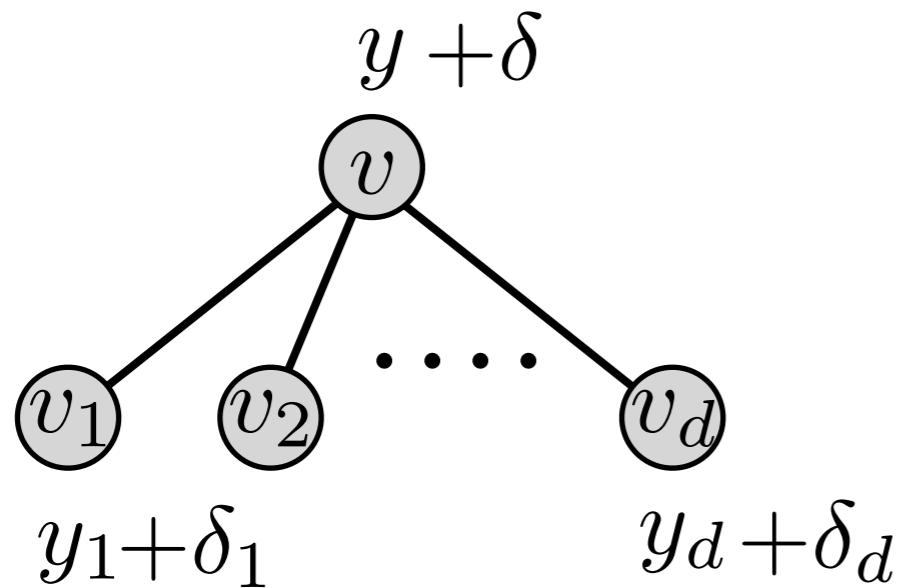


$$\delta \leq \alpha_d(x) \cdot \max_{1 \leq i \leq d} \{\delta_i\}$$

$$\alpha_d(x) = \sqrt{\frac{d(1 - \beta\gamma)x}{(\beta x + 1)(x + \gamma)}} \sqrt{\frac{d(1 - \beta\gamma)f_d(x)}{(\beta f_d(x) + 1)(f_d(x) + \gamma)}}$$

$$\leq \alpha^{\lceil \log_M(d+1) \rceil} \quad \text{for some} \quad \begin{aligned} \alpha &< 1 \\ M &> 1 \end{aligned}$$

Computationally Efficient Correlation Decay



$$\delta \leq \alpha_d(x) \cdot \max_{1 \leq i \leq d} \{\delta_i\}$$

$$\alpha_d(x) \leq \alpha^{\lceil \log_M(d+1) \rceil} \quad \text{for some } \alpha < 1 \quad M > 1$$

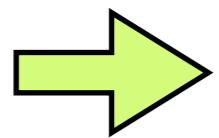
for **small** $d < M$ one-step recursion decays α

for **large** $d \geq M$ one-step recursion decays $\alpha^{\lceil \log_M(d+1) \rceil}$

behaves like $\lceil \log_M(d+1) \rceil$ steps!

anti-ferromagnetic: $\beta\gamma < 1$

WSM in d -reg. trees for $d \leq \Delta$



\exists FPTAS for graphs of max-degree Δ

bounded Δ or $\Delta=\infty$

[Weitz'06] hardcore model

[Sinclair-Srivastava-Thurley'12] Ising model

[Li-Lu-Y.'12] unbounded-degree graphs, no external field

Open Problems

- Characterization of SSM in **ferromagnetic** 2-state spin systems.
- SSM in multi-state spin systems:
 - difficulty: no SAW-tree;
 - implications: WSM vs. SSM in reg. trees, monotonicity of WSM/SSM w.r.t degree.
- Apply potential analysis and computationally efficient correlation decay to other problems.

Thank you! (again)