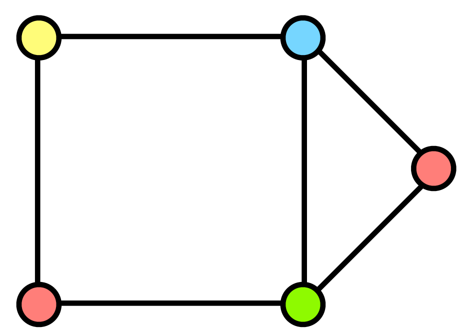
Spatial Mixing *of* Coloring Random Graphs

Yitong Yin
Nanjing University

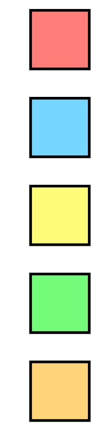
Colorings

undirected G(V,E)

max-degree: d



q colors:



temporal mixing of Glauber dynamics

α: 2 → 11/6

[Jerrum'95] [Salas-Sokal'97] [Bubley-Dyer'97]

[Vigoda'99]

approximately counting
or sampling almost uniform
proper q-colorings of G

when $q \geq \alpha d + \beta$

?

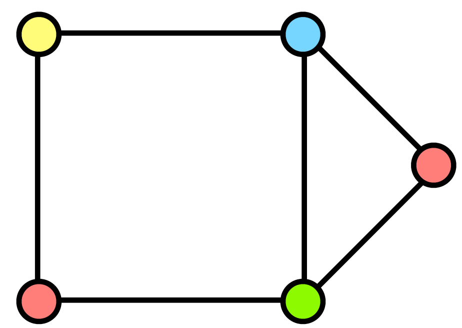
spatial mixing of Gibbs measure

conjecture: $\alpha=1$

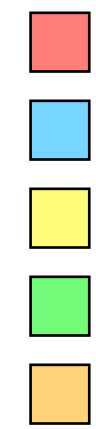
Spatial Mixing

undirected G(V,E)

max-degree:
d

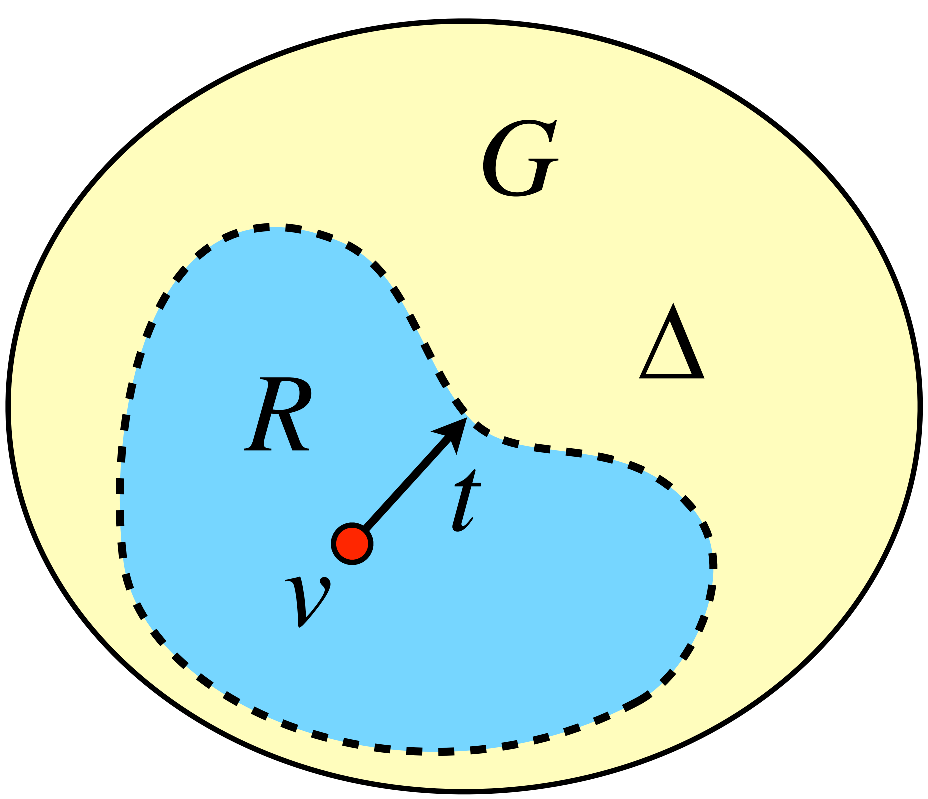


q colors:



Gibbs measure: uniform random proper *q*-coloring of G

$$c : V \to [q]$$



region $R \subset V$ $\quad \Delta \supseteq \partial R$

proper *q*-colorings $\sigma_\Delta, \tau_\Delta : \Delta \to [q]$

$$\Pr[c(v) = x \mid \sigma_\Delta] \approx \Pr[c(v) = x \mid \tau_\Delta]$$

error < exp (-t)

Spatial Mixing

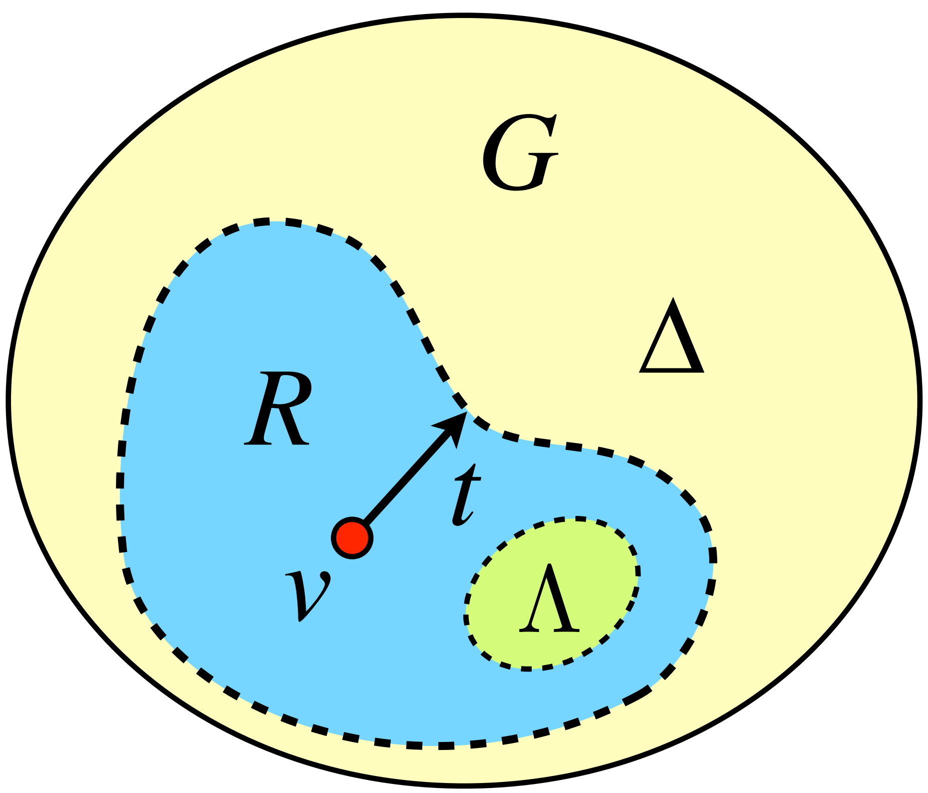
weak spatial mixing (WSM):

$$\Pr[c(v)=x \mid \sigma_\Delta] \approx \Pr[c(v)=x \mid \tau_\Delta]$$

strong spatial mixing (SSM):

$$\Pr[c(v)=x \mid \sigma_\Delta, \sigma_\Lambda] \approx \Pr[c(v)=x \mid \tau_\Delta, \sigma_\Lambda]$$

error < exp (-t)



SSM: the value of

$$\boxed{\Pr[c(v)=x \mid \sigma_\Lambda]}$$

is approximable
by local information

critical to counting and sampling

Spatial Mixing of Coloring

q-coloring of G $\quad q \geq \alpha d + O(1) \quad$ max-degree: d

average degree?

SSM: $\alpha > 1.763...$ (solution to $x^x = \mathrm{e}$)

- [Goldberg, Martin, Paterson 05] triangle-free amenable graphs
- [Ge, Stefankovic 11] regular tree
- [Gamarnik, Katz, Misra 12] triangle-free graphs

Spatial-mixing-based FPTAS:

- [Gamarnik, Katz 07] $\alpha > 2.8432...$, triangle-free graphs
- [Lu, Y. 14] $\alpha > 2.58071...$

SSM $\Rightarrow$ algorithm

- [Goldberg, Martin, Paterson 05] amenable graph, SSM $\Rightarrow$ FPRAS
- [Y., Zhang 13] planar graph (apex-minor-free), SSM $\Rightarrow$ FPTAS

Random Graph $G(n,d/n)$

average degree: d max-degree: $\Theta\left(\frac{\ln n}{\ln \ln n}\right)$ *whp*

q-colorable whp for a $q=O(d/\ln d)$

rapid mixing of (block) Glauber dynamics:

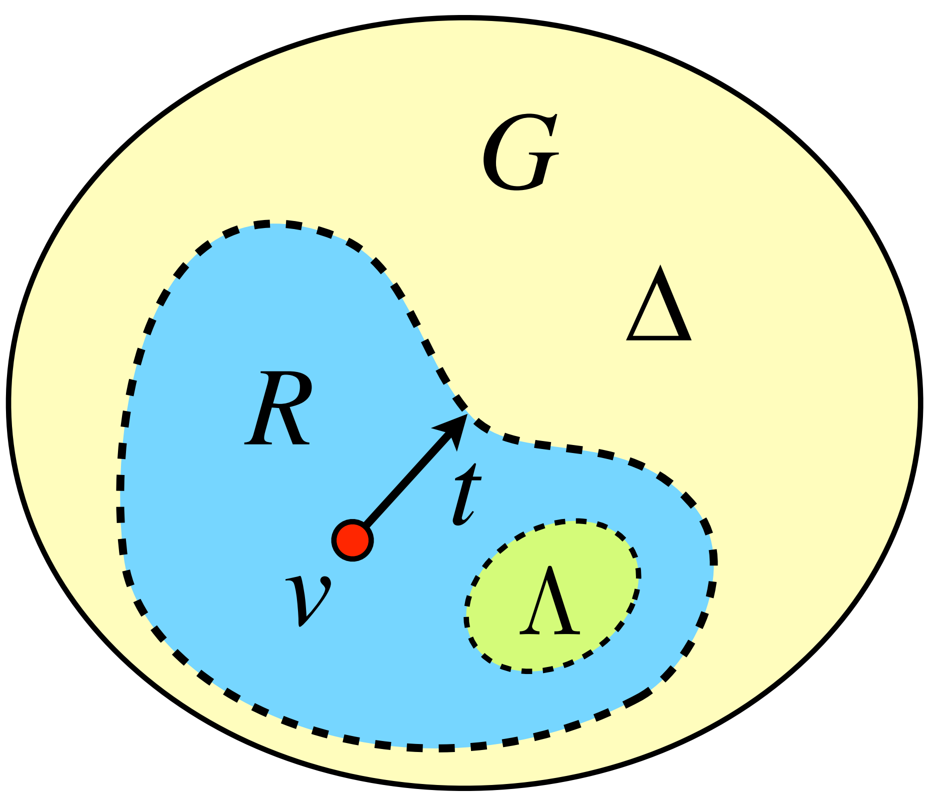
- [Dyer, Flaxman, Frieze, Vigoda 06] $q=O(\ln\ln n/\ln\ln\ln n)$
- [Efthymiou, Spirakis 07] [Mossel, Sly 08] $q=\mathrm{poly(d)}$
- [Efthymiou 14] $q > 5.5d+1$

spatial mixing?

Negative Result for SSM

strong spatial mixing (SSM): for any vertex v

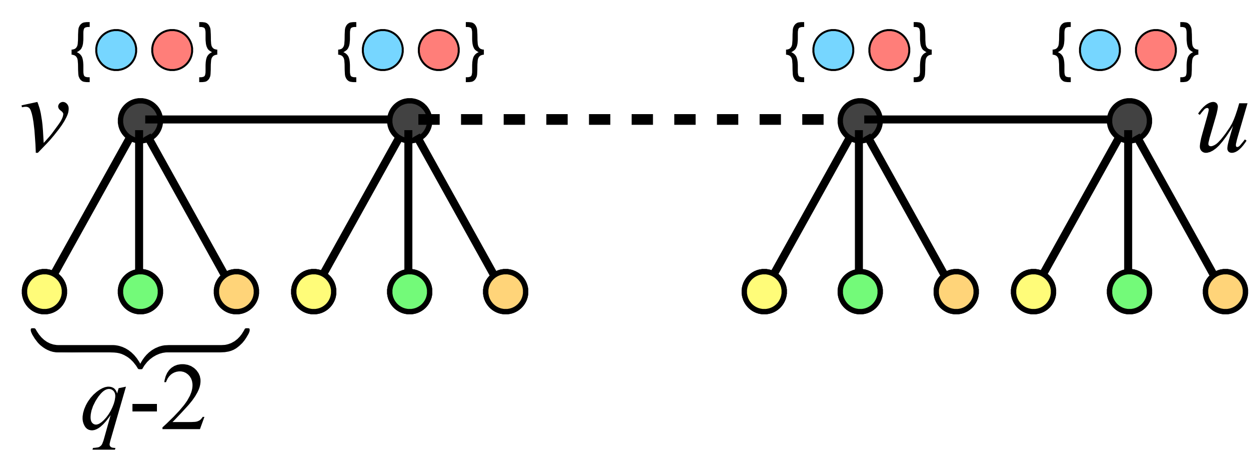
$$\Pr[c(v) = x \mid \sigma_\Delta, \sigma_\Lambda] \approx \Pr[c(v) = x \mid \tau_\Delta, \sigma_\Lambda]$$



in $G(n, d/n)$ for any q=O(1)

q colors:

whp, ∃: $\Omega(\ln n)$ long



This counter-example only affect the *strong* spatial mixing.

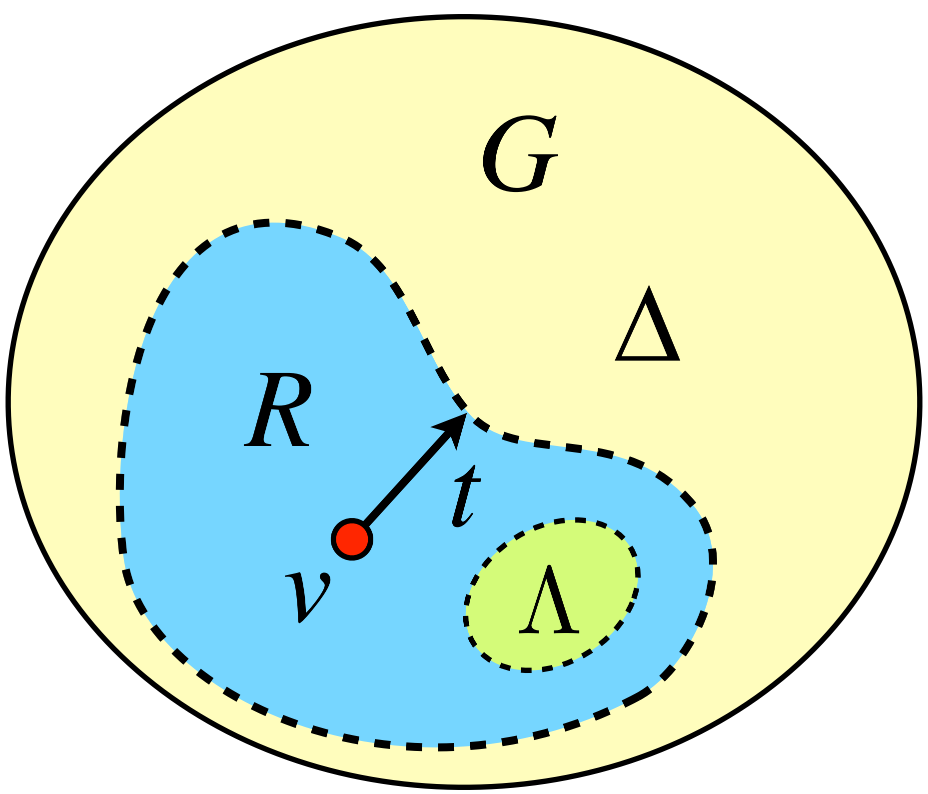
Main Result

$q \geq \alpha d + \beta$ for $\alpha > 2$ and some $\beta = O(1)$ (23 is enough)

fix any $v \in [n]$, and then sample $G(n,d/n)$

whp: $G(n,d/n)$ is q-colorable, and for any σ, τ

$$|\Pr[c(v) = x \mid \sigma] - \Pr[c(v) = x \mid \tau]| = \exp(-\Omega(t))$$



$t = \mathrm{dist}(v, \Delta) = \omega(1)$

is the shortest distance from v to where σ,τ differ

Strong Spatial Mixing w.r.t any fixed vertex!

Error Function

error function [Gamarnik, Katz, Misra 12]:

two distributions $\mu_1, \mu_2 : \Omega \to [0,1]$

$$\mathcal{E}(\mu_1, \mu_2) = \max_{x,y\in\Omega} \left(\log \frac{\mu_1(x)}{\mu_2(x)} - \log \frac{\mu_1(y)}{\mu_2(y)} \right)$$

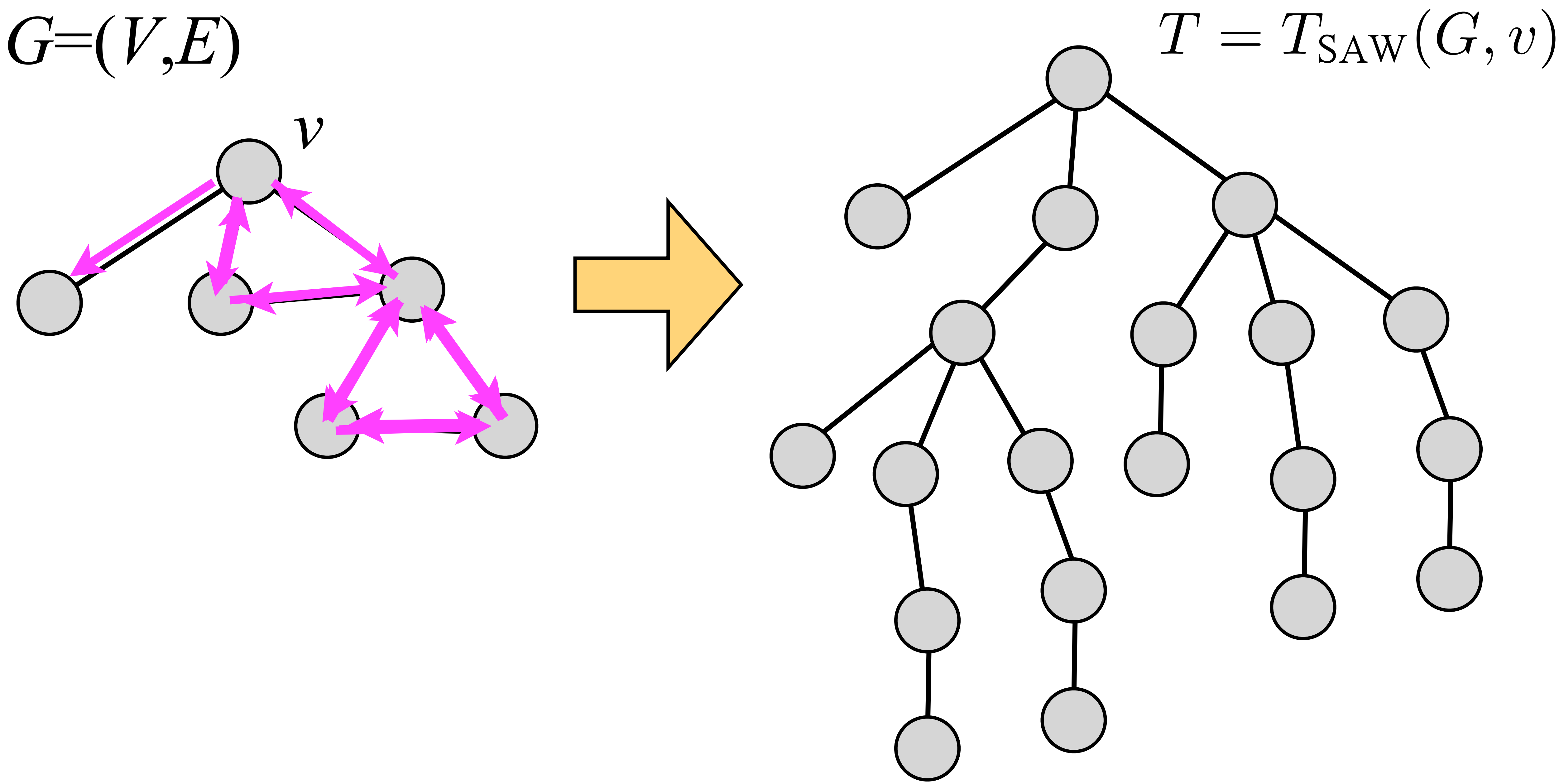
marginal distributions $\mu_v^\sigma(x) = \Pr[c(v) = x \mid \sigma]$ and μ_v^τ

$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \le \exp(-\Omega(t))$$

$$\Downarrow$$

$$\boxed{\left|\Pr[c(v) = x \mid \sigma] - \Pr[c(v) = x \mid \tau]\right| = \exp\left(-\Omega(t)\right)}$$

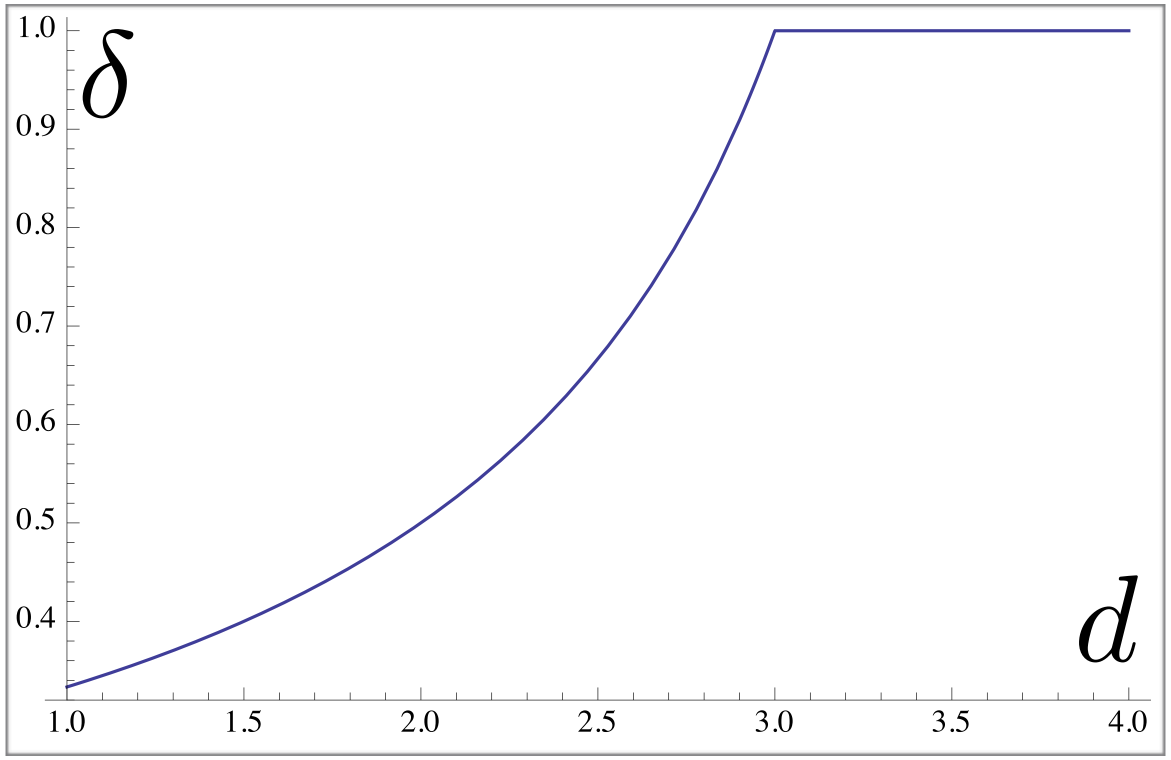
Self-Avoiding Walk Tree

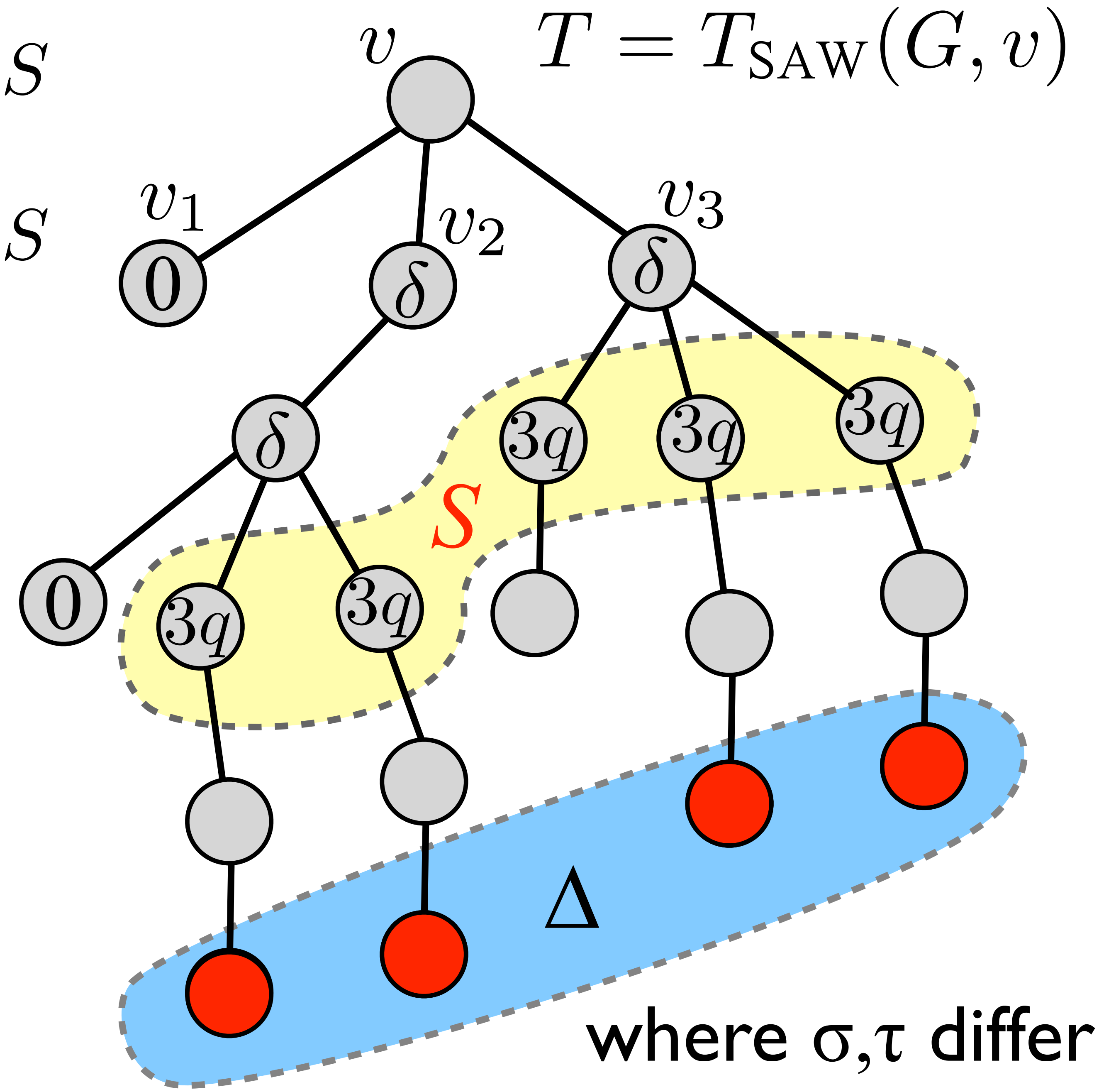


Error Propagation along Self-voiding Walks

$$\mathcal{E}_{T,S} = \begin{cases} \sum_i \delta(v_i) \cdot \mathcal{E}_{T_i,S} & v \notin S \\ 3q & v \in S \end{cases}$$

$$\delta(u) = \begin{cases} \frac{1}{q-d(u)-1} & q > d(u)+1 \\ 1 & \text{o.w.} \end{cases}$$





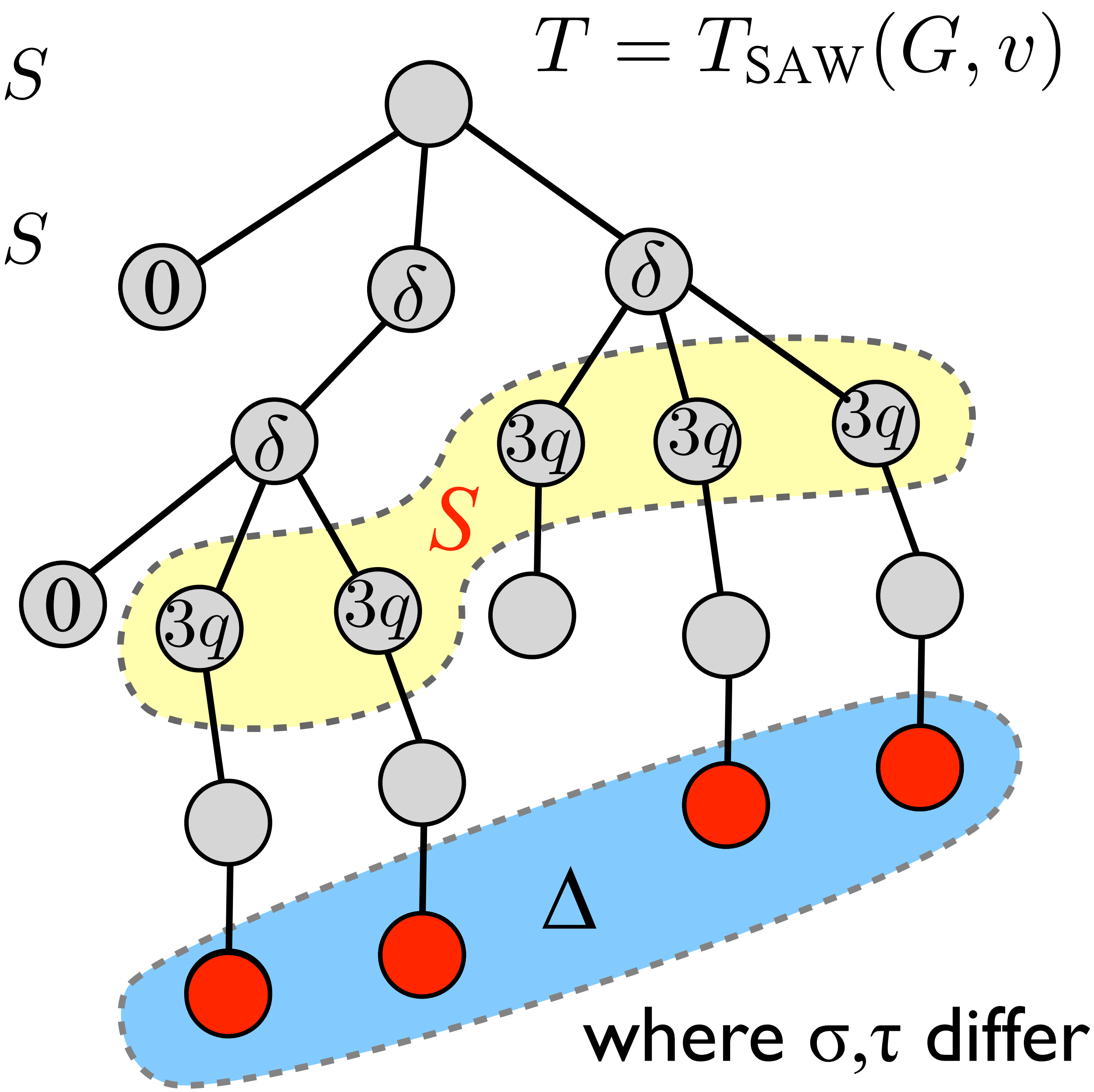
Error Propagation along Self-voiding Walks

$$\mathcal{E}_{T,S} = \begin{cases} \sum_i \delta(v_i) \cdot \mathcal{E}_{T_i,S} & v \notin S \\ 3q & v \in S \end{cases}$$

$$\delta(u) = \begin{cases} \frac{1}{q-d(u)-1} & q > d(u)+1 \\ 1 & \text{o.w.} \end{cases}$$

S: *permissive* cut-set

$$\boxed{\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \mathcal{E}_{T,S}}$$



μ_v^σ, μ_v^τ : marginal distributions at v in G conditioning on σ,τ

Proof of Main Result

μ_v^σ, μ_v^τ : marginal distributions at v in G conditioning on σ,τ

error function: $\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) = \max_{x,y\in[q]} \left(\log \frac{\mu_v^\sigma(x)}{\mu_v^\tau(x)} - \log \frac{\mu_v^\sigma(y)}{\mu_v^\tau(y)} \right)$

$T = T_{\mathrm{SAW}}(G, v)$ S: *permissive* cut-set

correlation decay:

$$\boxed{\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \mathcal{E}_{T,S}}$$

for $T = T_{\mathrm{SAW}}(G, v)$ where G=$G(n,d/n)$

whp: always exists a permissive cut-set S

probabilistic method:

$$\boxed{\mathcal{E}_{T,S} = \exp(-\Omega(t))}$$

$$\boxed{\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \le \mathcal{E}_{T,S}}$$

for $v \in S$

then $\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \le 3q$

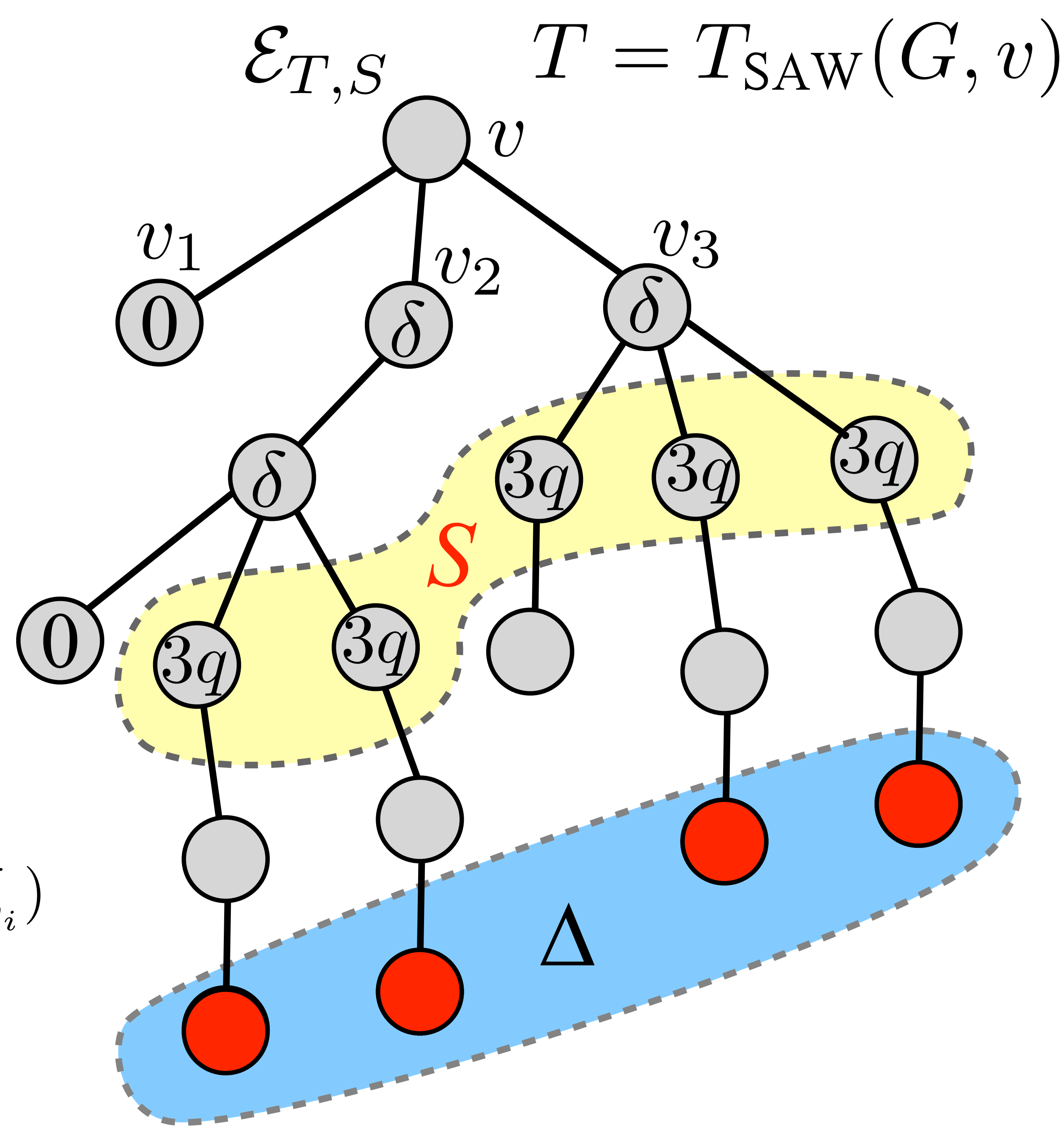
if $q > d(u) + 1$ for all u

$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \le \sum_i \frac{1}{q - d(v_i) - 1} \cdot \mathcal{E}(\mu_{v_i}^\sigma, \mu_{v_i}^\tau)$$

where $\mu_{v_i}^\sigma, \mu_{v_i}^\tau$ defined in $G \setminus \{v\}$

(with altered color lists)

$$\delta(u) = \begin{cases} \frac{1}{q-d(u)-1} & q > d(u)+1 \\ 1 & \text{o.w.} \end{cases}$$



[Gamarnik, Katz, Misra 12]:

if $q > d(u) + 1$ for all u $\quad \mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \le \sum_i \frac{1}{q - d(v_i) - 1} \cdot \mathcal{E}(\mu_{v_i}^\sigma, \mu_{v_i}^\tau)$

$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) = \max_{x,y\in\Omega} \left(\log \frac{\mu_v^\sigma(x)}{\mu_v^\tau(x)} - \log \frac{\mu_v^\sigma(y)}{\mu_v^\tau(y)} \right) = \max_{x,y\in\Omega} \left(\log \frac{\mu_v^\sigma(x)}{\mu_v^\sigma(y)} - \log \frac{\mu_v^\tau(x)}{\mu_v^\tau(y)} \right)$$

where $\frac{\mu_v^\sigma(x)}{\mu_v^\sigma(y)} = \frac{\Pr(c(v) = x \mid \sigma)}{\Pr(c(v) = y \mid \sigma)} = \frac{\Pr_{G\setminus\{v\}}(\forall i, c(v_i) \neq x \mid \sigma)}{\Pr_{G\setminus\{v\}}(\forall i, c(v_i) \neq y \mid \sigma)}$

$$= \prod_i \frac{1 - \Pr_{G\setminus\{v\}}(c(v_i) = x \mid \sigma)}{1 - \Pr_{G\setminus\{v\}}(c(v_i) = y \mid \sigma)}$$

(telescopic product)

$$= \sum_i \left[\log\left(1 - \mu_{v_i}^\sigma(x)\right) - \log\left(1 - \mu_{v_i}^\tau(x)\right)\right] - \sum_i \left[\log\left(1 - \mu_{v_i}^\sigma(y)\right) - \log\left(1 - \mu_{v_i}^\tau(y)\right)\right]$$

$$= \sum_i \frac{\mu_i}{1 - \mu_i} \log \frac{\mu_{v_i}^\tau(x)}{\mu_{v_i}^\sigma(x)} - \sum_i \frac{\mu_i'}{1 - \mu_i'} \log \frac{\mu_{v_i}^\tau(y)}{\mu_{v_i}^\sigma(y)}$$

(mean value theorem)

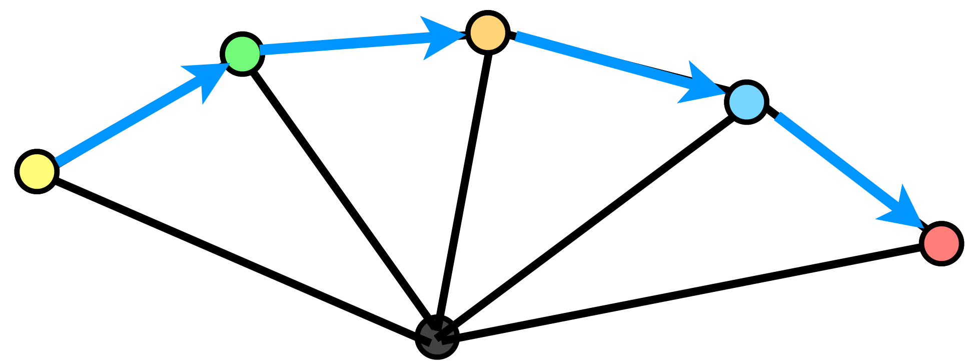
where $\mu_i, \mu_i' \le \max\{\mu_{v_i}^\tau(x), \mu_{v_i}^\sigma(x), \mu_{v_i}^\tau(y), \mu_{v_i}^\sigma(y)\} \le \frac{1}{q - d(v_i)}$

$$\le \sum_i \frac{1}{q - d(v_i) - 1} \max_{x,y} \left(\log \frac{\mu_{v_i}^\sigma(x)}{\mu_{v_i}^\tau(x)} - \log \frac{\mu_{v_i}^\sigma(y)}{\mu_{v_i}^\tau(y)} \right) \le \sum_i \frac{1}{q - d(v_i) - 1} \mathcal{E}\left(\mu_{v_i}^\sigma, \mu_{v_i}^\tau\right)$$

For unbounded degree:

q colors:

when calculating correlation decay along path:



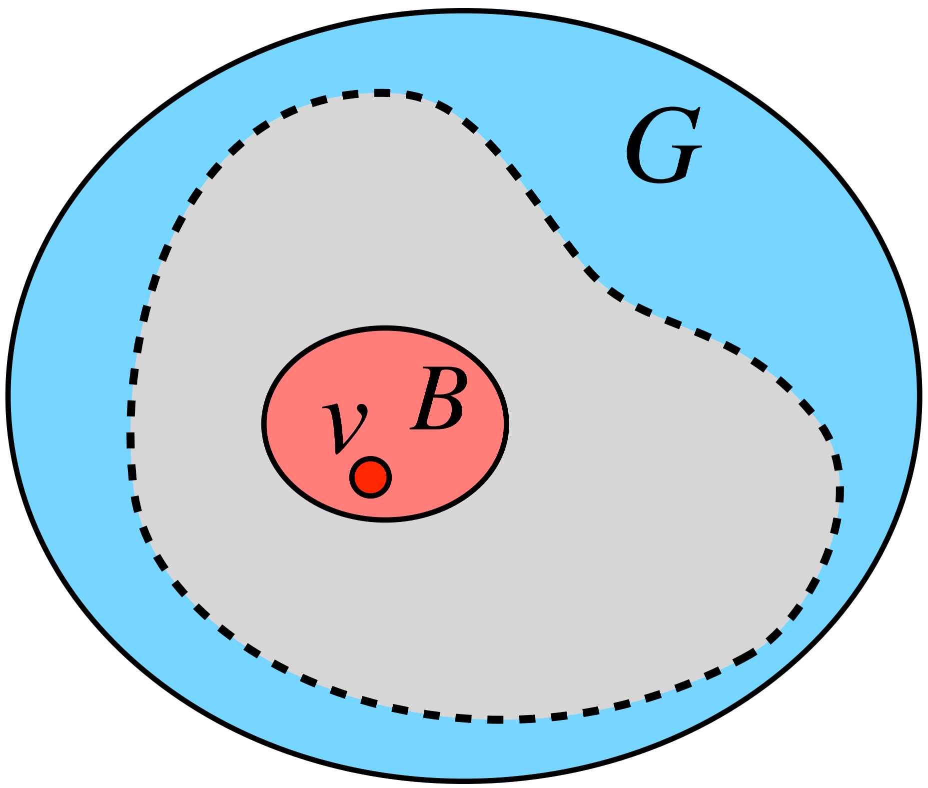
available colors = { }

end up with an infeasible coloring

effectively $\times\infty$ in calculating correlation decay:

- error function [Gamarnik-Katz-Misra'12]
- recursive coloring [Goldberg-Martin-Paterson'05]
- computation tree [Gamarnik-Katz'07]
- computation tree with potential function [Lu-Y.'14]

Block-wise Correlation Decay



vertex v grows to a *permissive* block $B \ni v$

$$\forall u \in \partial B, \quad q > d(u) + 1$$

minimal permissive block B around v

$$\forall u \in B \setminus \{v\}, \quad q \leq d(u) + 1$$

consider marginal distributions μ_B^σ, μ_B^τ of colorings of B

$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \leq \mathcal{E}(\mu_B^\sigma, \mu_B^\tau)$$

(averaging principle)

$$\mathcal{E}(\mu_B^\sigma, \mu_B^\tau) \leq \sum_i \frac{1}{q - d(v_i) - 1} \cdot \mathcal{E}(\mu_{v_i}^\sigma, \mu_{v_i}^\tau)$$

(telescopic product + mean value theorem)

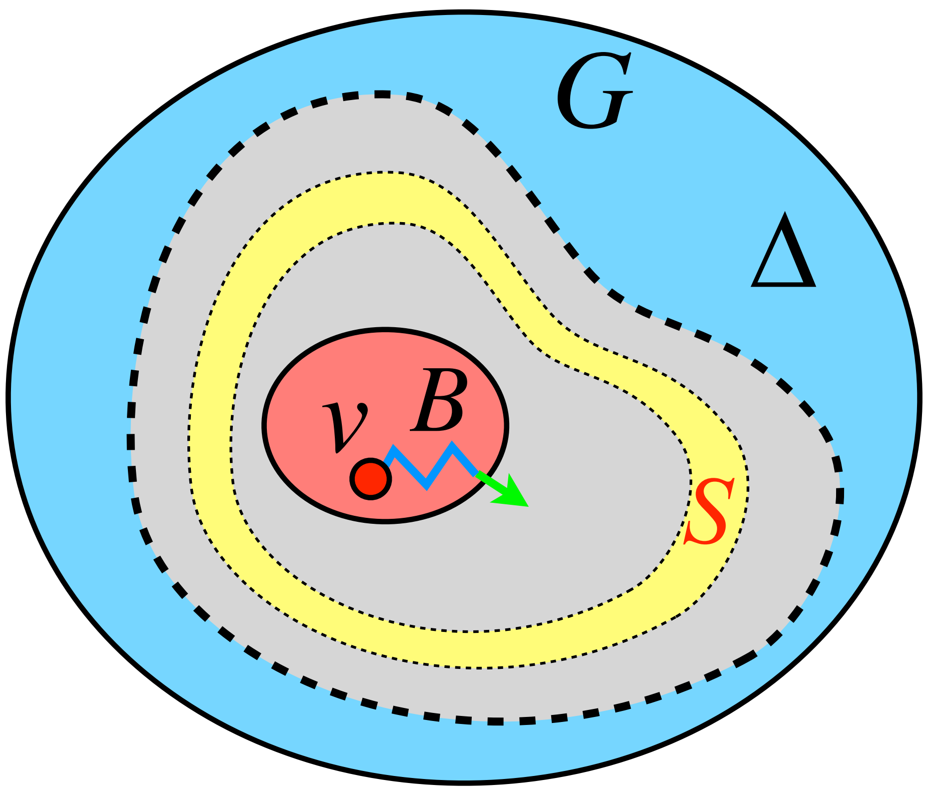
boundary vertices of *B*

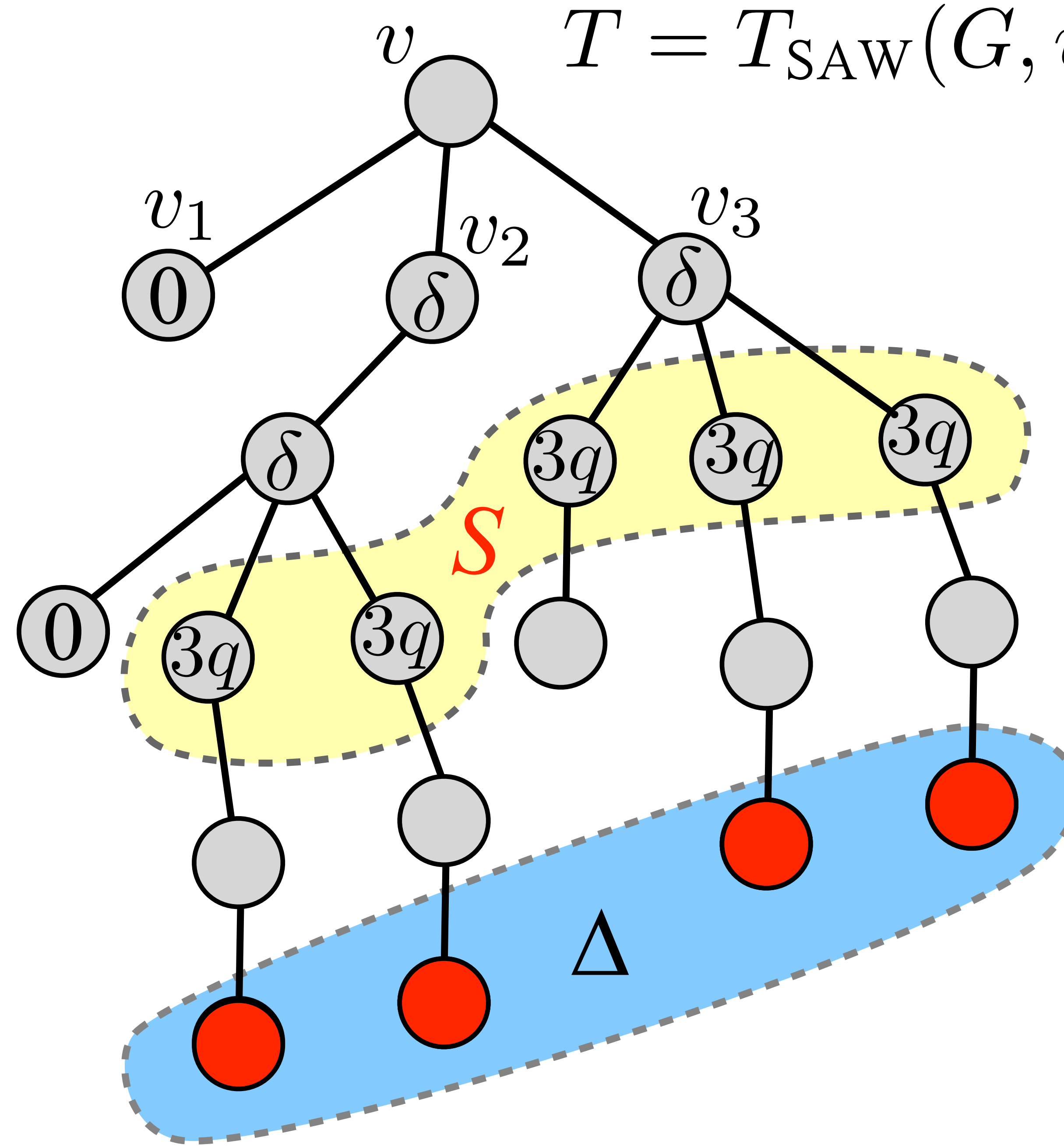
$$\boxed{\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \le \mathcal{E}_{T,S}}$$

$$\delta(u) = \begin{cases} \frac{1}{q-d(u)-1} & q > d(u)+1 \\ 1 & \text{o.w.} \end{cases}$$

for $v \in S$ $\quad \mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \le 3q$

$T = T_{\text{SAW}}(G, v)$





$$\mathcal{E}(\mu_v^\sigma, \mu_v^\tau) \le \mathcal{E}(\mu_B^\sigma, \mu_B^\tau)$$

$$\le \sum_i \frac{1}{q - d(v_i) - 1} \cdot \mathcal{E}(\mu_{v_i}^\sigma, \mu_{v_i}^\tau)$$

where v_i are boundary vertices of B

and $\mu_{v_i}^\sigma, \mu_{v_i}^\tau$ defined in $G \setminus B$

Random Self-Avoiding Walks

for $T = T_{\mathrm{SAW}}(G, v)$ where G=$G(n,d/n)$

whp: always exists a permissive cut-set S

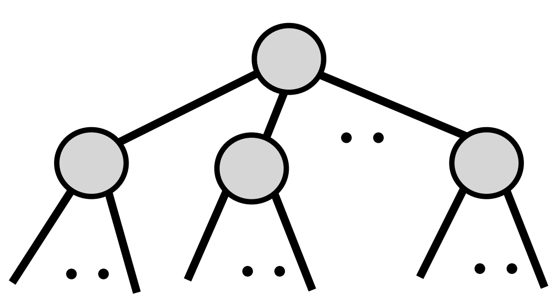
$$\mathcal{E}_{T,S} = \exp(-\Omega(t))$$

whp

$$\mathbb{E}[\mathcal{E}_{T,S}] = \exp(-\Omega(t))$$

$T = T_{\mathrm{SAW}}(G, v)$ is like a Galton-Watson random tree

with binomial degree distribution $B(n\text{-}1,d/n)$



each $d(u)$~ $B(n\text{-}1,d/n)$

when $q > \alpha d + O(1)$ for $\alpha > 2$

a permissive cut-set S of depth $> t/2$ exists

$$\delta(u) = \begin{cases} \frac{1}{q-d(u)-1} & q > d(u)+1 \\ 1 & \text{o.w.} \end{cases} \qquad \mathbb{E}[\delta(u)] < \frac{1}{q-d}$$

Summary

$q \geq \alpha d + O(1)$ for $\alpha > 2$

- SSM for q-colorings of $G(n,d/n)$ w.r.t. fixed vertex:
  - a block-wise decay of correlation for colorings of graphs with unbounded degree
- Algorithmic implication is still open:
  - With SSM, local information is sufficient to estimate marginals. What local structure of G(n,d/n) can be exploited to efficiently compute marginals?
  - Path-coupling of block Glauber Dynamics replies on correlation decay.

Thank you!

Any questions?