Improved FPTAS *for* Multi-spin Systems

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presented by:

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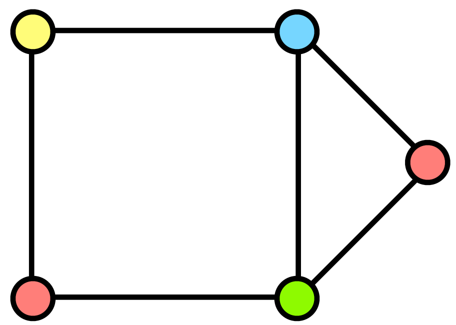
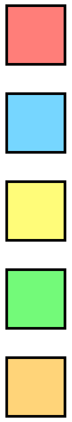
KTH

Colorings

instance: undirected G(V,E) with max-degree $\leq \Delta$



q colors: 

goal: counting the number of proper q-colorings for G

- exact counting is **#P-hard**
- when $q<\Delta$, decision of existence is **NP-hard**

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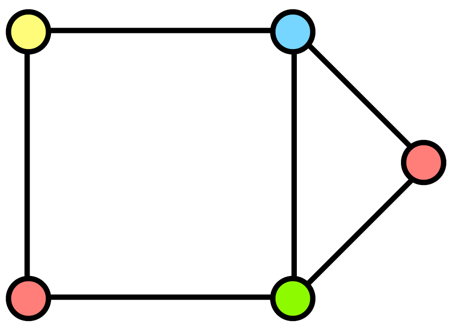

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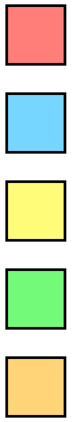

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- exact counting is **#P-hard**
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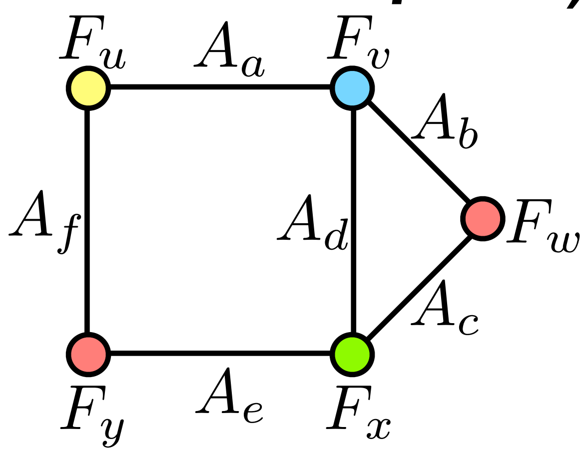
approximately counting the number of proper q-colorings for G when $q \geq \alpha\Delta+\beta$

equivalent to sampling an almost uniform random q-coloring

Spin System
(pairwise *Markov random field*)

instance:



- undirected G(V,E);
- q *states*: $[q]$;
- each edge $e \in E$ associated with an *activity*:

a symmetric nonnegative $q \times q$ matrix

$$A_e : [q] \times [q] \to \mathbb{R}_{\geq 0}$$

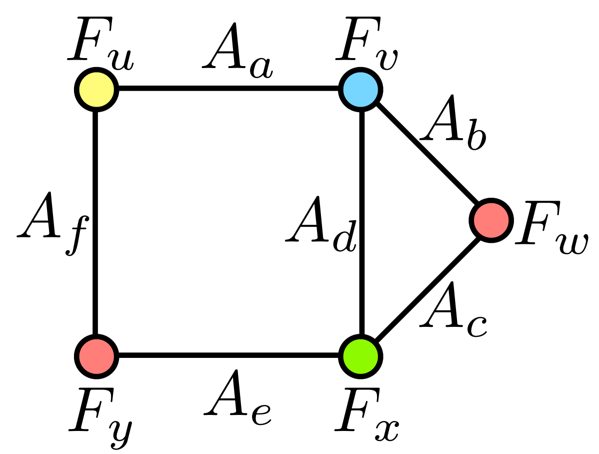
- each vertex $v \in V$ associated with an *external field*:

a nonnegative q-vector $F_v : [q] \to \mathbb{R}_{\geq 0}$

goal: computing the partition function:

$$Z = \sum_{\boldsymbol{x} \in [q]^V} \prod_{e=uv \in E} A_e(x_u, x_v) \prod_{v \in V} F_v(x_v)$$

Spin System



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partition function: count the # of solutions to an CSP

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enumerate all configurations

binary constraints

unary constraints

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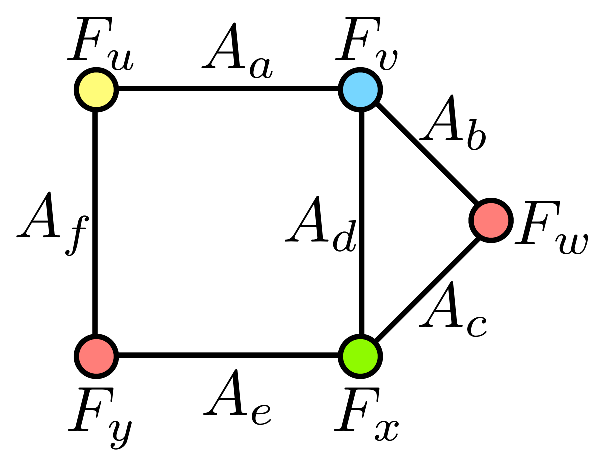

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coloring: $A = \begin{bmatrix} 0 & & 1 \\ & 0 & \\ & & \ddots & \\ 1 & & & 0 \end{bmatrix}$ $F = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

Examples of Spin systems

- 2-spin: q=2
  - *hardcore* model (independent set), *Ising* model, etc.
- multi-spin: general q
  - coloring: $A = \begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & 1 & \ddots & \\ & & & 0 \end{bmatrix}$ $F = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

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  - Potts model: inverse temperature β

$$A = \begin{bmatrix} e^{\beta} & & & \\ & e^{\beta} & & 1 \\ & 1 & \ddots & \\ & & & e^{\beta} \end{bmatrix}$$

arbitrary F

when β=-∞ and F=(1,1,... ,1), it is coloring

Results

sufficient conditions for FPTAS for classes of spin systems

- coloring: $q \geq \alpha\Delta+\beta$
  - randomized algorithms: by simulating a random walk (the Glauber dynamics) over colorings
    - $\alpha=11/6$ (Jerrum'95 ⇢ Bubley-Dyer'97 ⇢ Vigoda'99)
  - deterministic algorithms: by exploiting the correlation decay (spatial mixing) property
    - $\alpha\approx 2.8432$ (Gamarnik-Katz'07)
    - just correlation decay (no FPTAS): $\alpha\approx 1.763$ (Goldberg-Martin-Paterson'05, Gamarnik-Katz-Misra'12)

this paper: deterministic FPTAS for $\alpha\approx 2.58071$

Results

sufficient conditions for FPTAS for classes of spin systems

- general multi-spin system:

  in terms of $$c = \max_{\substack{e\in E\\ w,x,y,z\in[q]}} \frac{A_e(x,y)}{A_e(w,z)}$$

  - Gamarnik-Katz'07: $(c^{\Delta} - c^{-\Delta})\Delta q^{\Delta} < 1$

    this paper: $3\Delta(c^{\Delta} - 1) \leq 1$

    an exponential improvement!

  - on Potts model (with inverse temperature β):

    it implies: $3\Delta(\mathrm{e}^{|\beta|} - 1) \leq 1$

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    - confirming the conjecture of $|\beta| = O\left(\frac{1}{\Delta}\right)$ in [GK'07]
    - asymptotically matching the $\mathrm{e}^{\beta} < 1 - \frac{q}{\Delta}$ inaproximability bound for $\beta < 0$ in [Galanis-Stefankovic-Vigoda'13]

The standard first step:

reducing to the computing of marginal probability

$$Z = \sum_{\boldsymbol{x}\in[q]^V} \prod_{e=uv\in E} A_e(x_u, x_v) \prod_{v\in V} F_v(x_v)$$

for any configuration $\boldsymbol{x} \in [q]^V$

Gibbs measure:

$$\mathbb{P}[\boldsymbol{X} = \boldsymbol{x}] = \frac{\prod_{e=uv\in E} A_e(x_u, x_v) \prod_{v\in V} F_v(x_v)}{Z}$$

marginal probability: $\mathbb{P}[X_v = x_v]$

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marginal probability: $\mathbb{P}[X_v = x_v]$

Jerrum-Valiant-Vazirani'86

for self-reducible class of spin-systems:

efficient approximation of marginal probability (with additive error) ⇒ FPTAS for Z

The standard first step:

- self-reducible: general spin systems, Potts models
- not self-reducible: coloring
- self-reducible superclass of coloring: list-coloring

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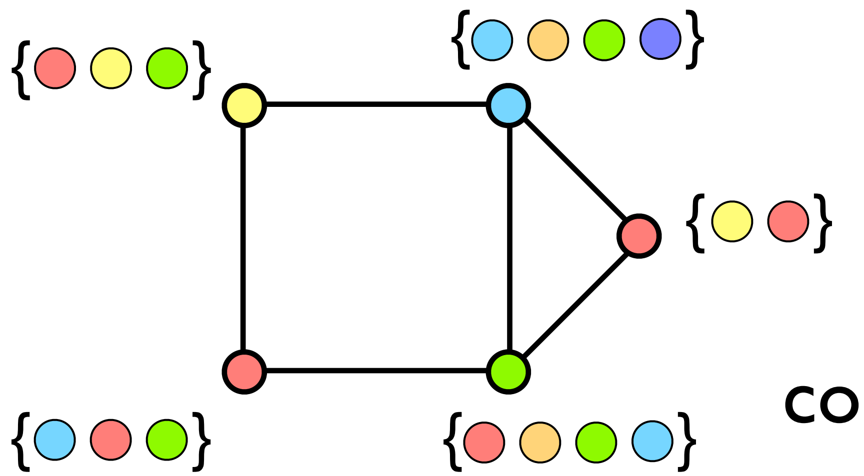

each vertex v associated with a list L_v of colors allowed to use on v

$$\text{coloring} \subset \underbrace{\text{list-coloring} \subset \text{Potts} \subset \text{multi-spin}}_{\textit{self-reducible}}$$

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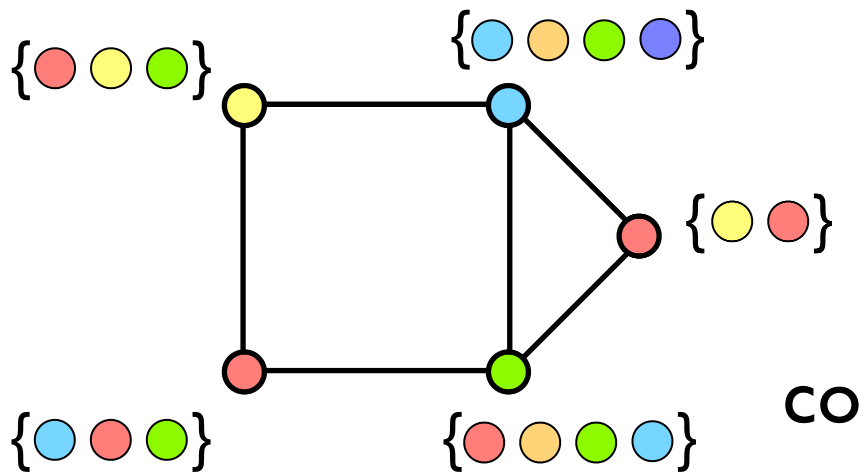

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for $\left\{ \begin{array}{c} \text{multi-spin system} \\ \text{Potts model} \\ \text{list-coloring} \end{array} \right\}$ approximate the marginal $\mathbb{P}[X_v = x]$ (with additive error)

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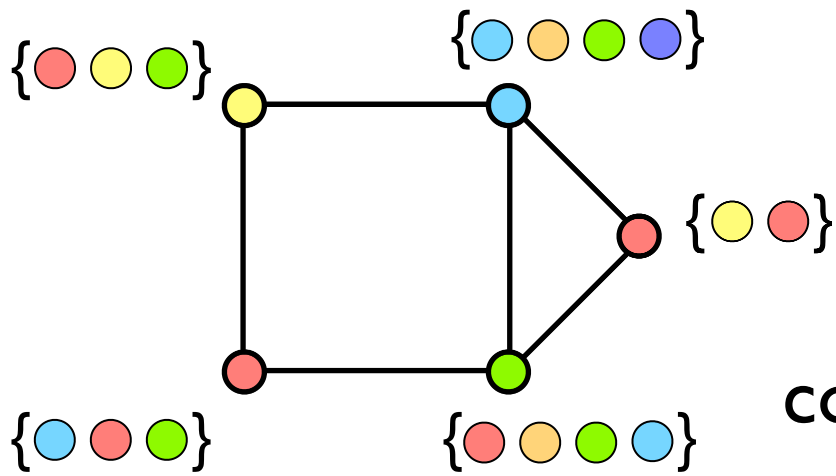

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classic way: random walk new way: correlation decay

Recursion for List-Coloring

list-coloring instance Ω

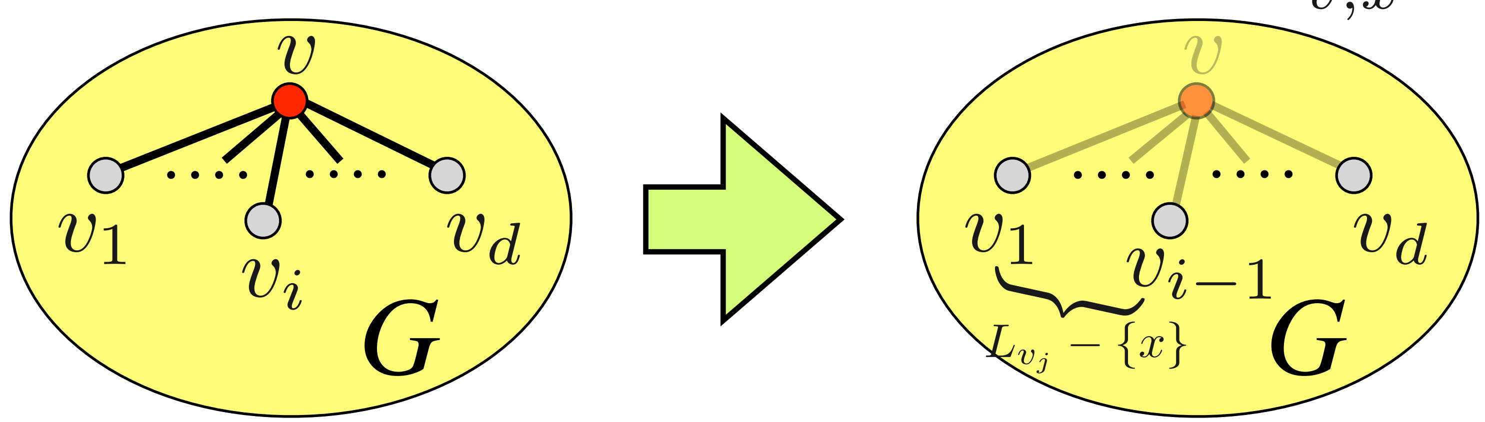
v's neighbors: $v_1, v_2, \ldots, v_d$

color: x

$\Omega^i_{v,x}$: delete v

$\forall j<i$, delete x

from list L_{v_j}



Gamarnik-Katz'07:

$$\mathbb{P}_\Omega(X_v = x) = \frac{\prod_{i=1}^d \mathbb{P}_{\Omega^i_{v,x}}(X_{v_i} \neq x)}{\sum_{y\in L_v} \prod_{i=1}^d \mathbb{P}_{\Omega^i_{v,x}}(X_{v_i} \neq y)}$$

$$= \frac{\prod_{i=1}^d \left(1 - \mathbb{P}_{\Omega^i_{v,x}}(X_{v_i} = x)\right)}{\sum_{y\in L_v} \prod_{i=1}^d \left(1 - \mathbb{P}_{\Omega^i_{v,x}}(X_{v_i} = y)\right)}$$

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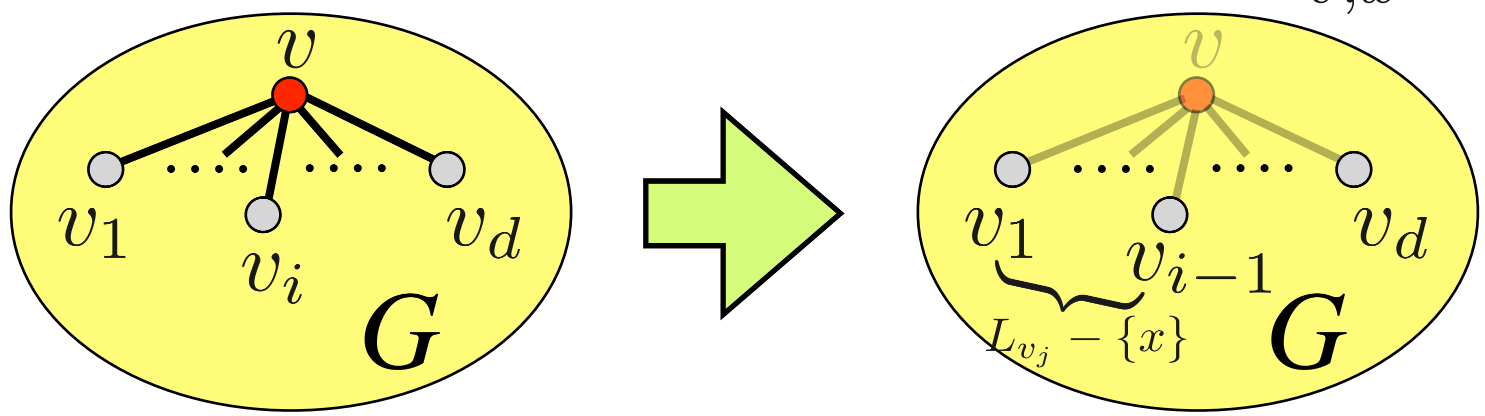

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telescopic products

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Recursion for general multi-spin system

a natural generalization of list-coloring:

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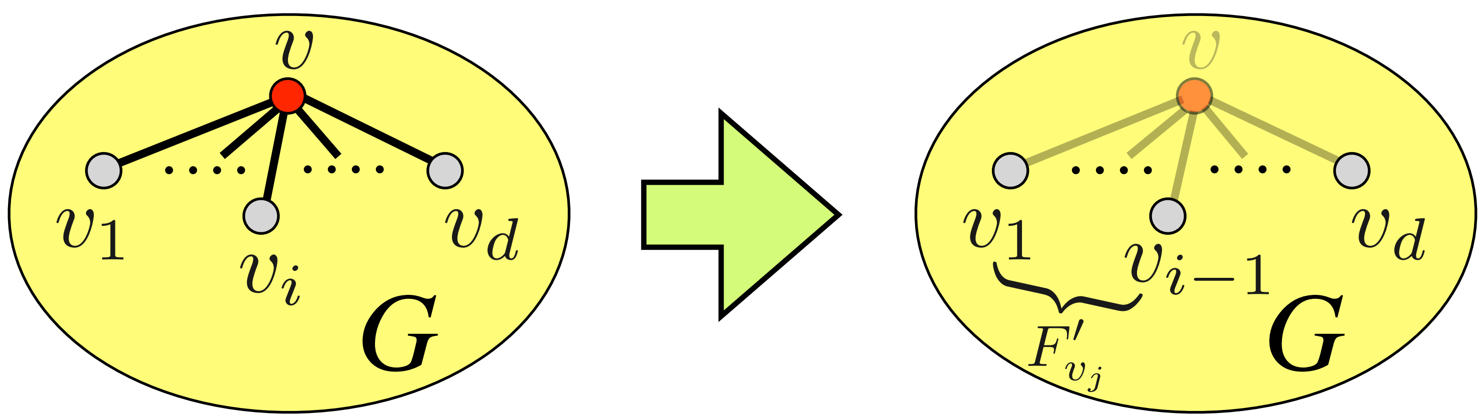
$F'_{v_j}(y) = A_{vv_j}(x,y)F_{v_j}(y)$



augmented by edge activity

$$\mathbb{P}_\Omega(X_v = x) = \frac{F_v(x)\prod_{i=1}^d\left(A_{vv_i}(x,x) - \sum_{z\neq x}\left(A_{vv_i}(x,x)-A_{vv_i}(x,z)\right)\mathbb{P}_{\Omega^i_{v,x}}(X_{v_i}=z)\right)}{\sum_{y\in[q]}F_v(y)\prod_{i=1}^d\left(A_{vv_i}(y,y) - \sum_{z\neq y}\left(A_{vv_i}(y,y)-A_{vv_i}(y,z)\right)\mathbb{P}_{\Omega^i_{v,y}}(X_{v_i}=z)\right)}$$

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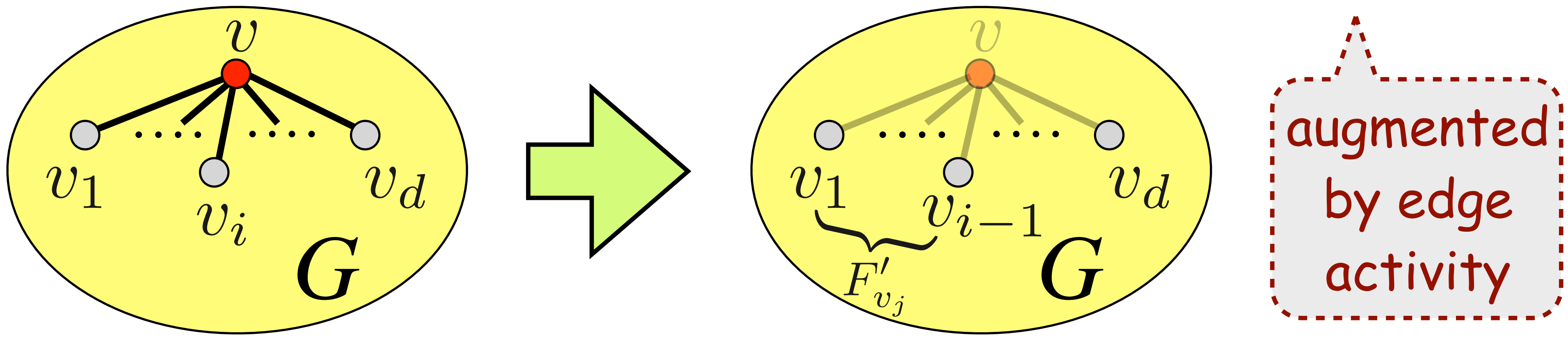

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for list-coloring: special case

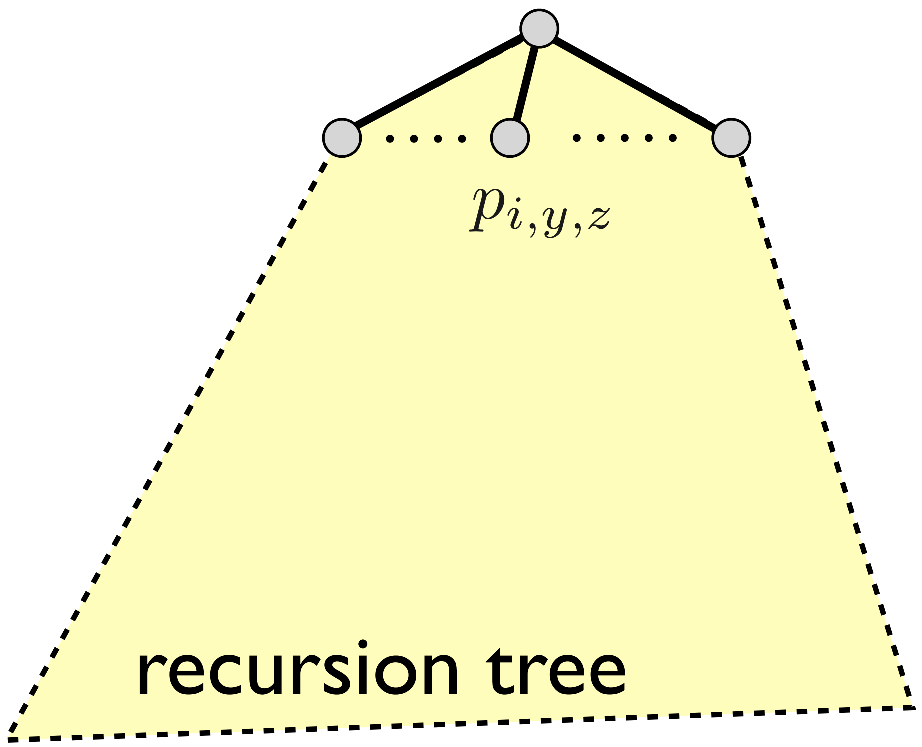
$$\mathbb{P}_\Omega(X_v = x) = \frac{\prod_{i=1}^d\left(1-\mathbb{P}_{\Omega^i_{v,x}}(X_{v_i}=x)\right)}{\sum_{y\in L_v}\prod_{i=1}^d\left(1-\mathbb{P}_{\Omega^i_{v,x}}(X_{v_i}=y)\right)}$$

Correlation Decay

vector $\boldsymbol{p} = (p_{i,y,z})_{1\le i\le d;\, y,z\in[q];\, y\neq z}$ where $p_{i,y,z} = \mathbb{P}_{\Omega^i_{v,y}}(X_{v_i} = z)$

$$f(\boldsymbol{p}) = \frac{F_v(x)\prod_{i=1}^d\Big(A_{vv_i}(x,x) - \sum_{z\neq x}\big(A_{vv_i}(x,x) - A_{vv_i}(x,z)\big)p_{i,x,z}\Big)}{\sum_{y\in[q]}F_v(y)\prod_{i=1}^d\Big(A_{vv_i}(y,y) - \sum_{z\neq y}\big(A_{vv_i}(y,y) - A_{vv_i}(y,z)\big)p_{i,y,z}\Big)}$$

$\mathbb{P}_\Omega[X_v = x] = f(\boldsymbol{p})$



- an exponential-time exact algorithm

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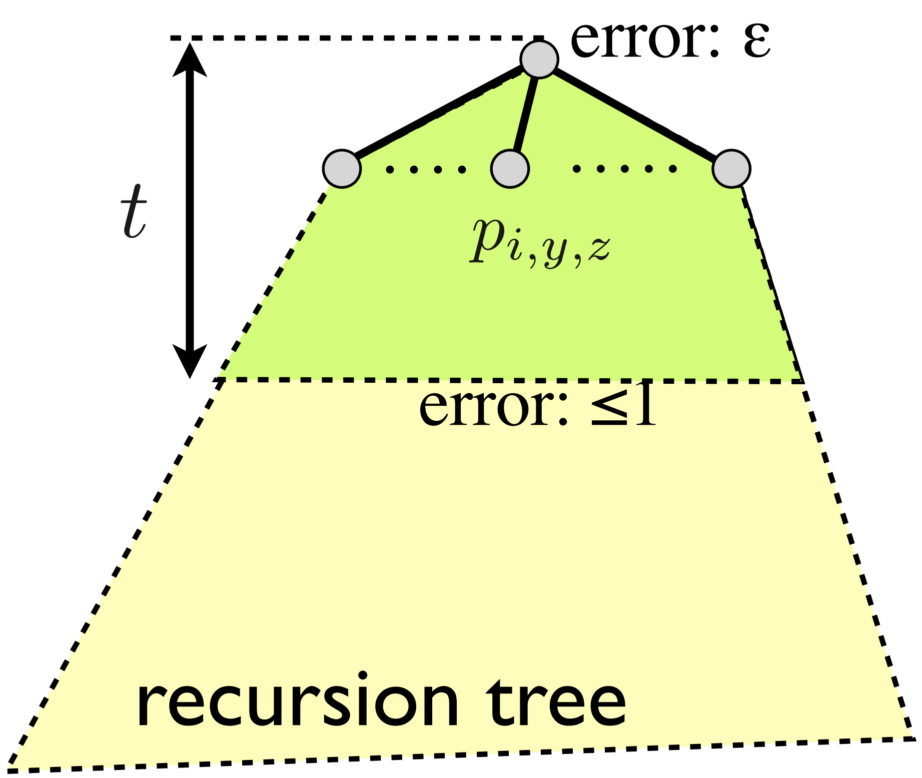
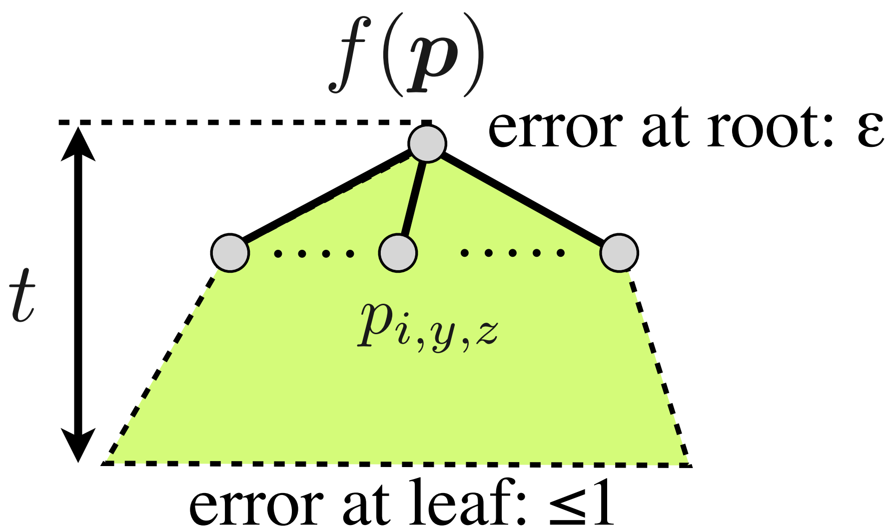


- an exponential-time exact algorithm
- truncation:
  - compute up-to level *t*
  - use arbitrary estimation at level *t*

correlation decay:

error at root ε = exp(-Ω(t))

Correlation Decay



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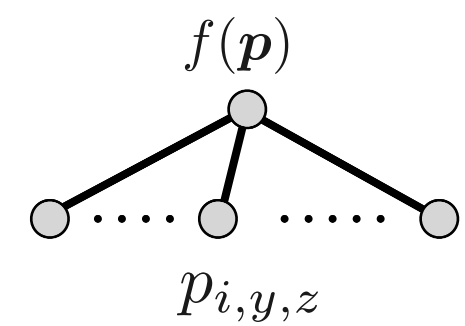
if running the recursion up-to level *t*

a sufficient condition: at any step



$$\kappa \triangleq \sum_{i,y,z} \left| \frac{\partial f(\boldsymbol{p})}{\partial p_{i,y,z}} \right| < 1$$

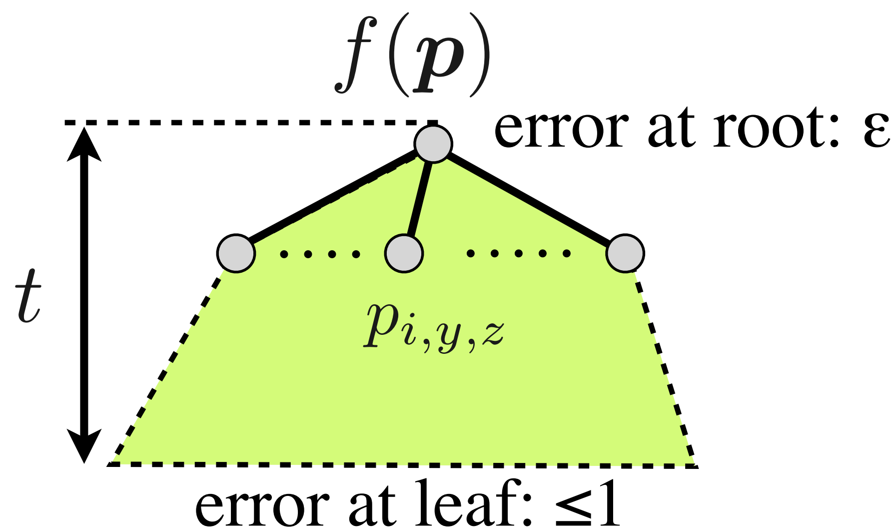
(stepwise decay)

then due to the Mean Value Theorem

$$\epsilon \leq \sum_{i,y,z} \left| \frac{\partial f(\boldsymbol{p})}{\partial p_{i,y,z}} \right| \epsilon_{i,y,z} \leq \kappa \cdot \max_{i,y,z} \epsilon_{i,y,z}$$

induction!

Correlation Decay



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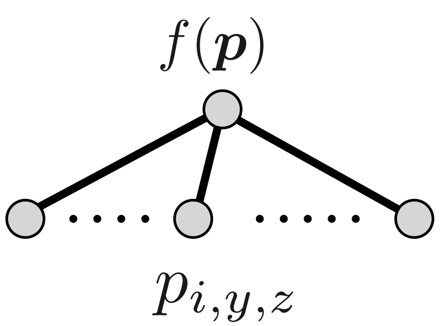

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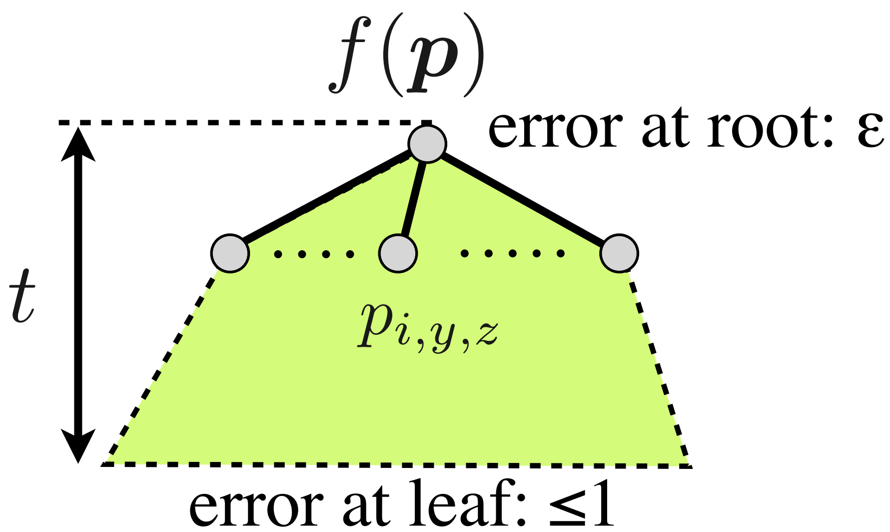

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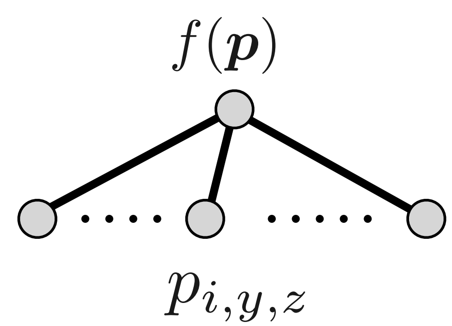

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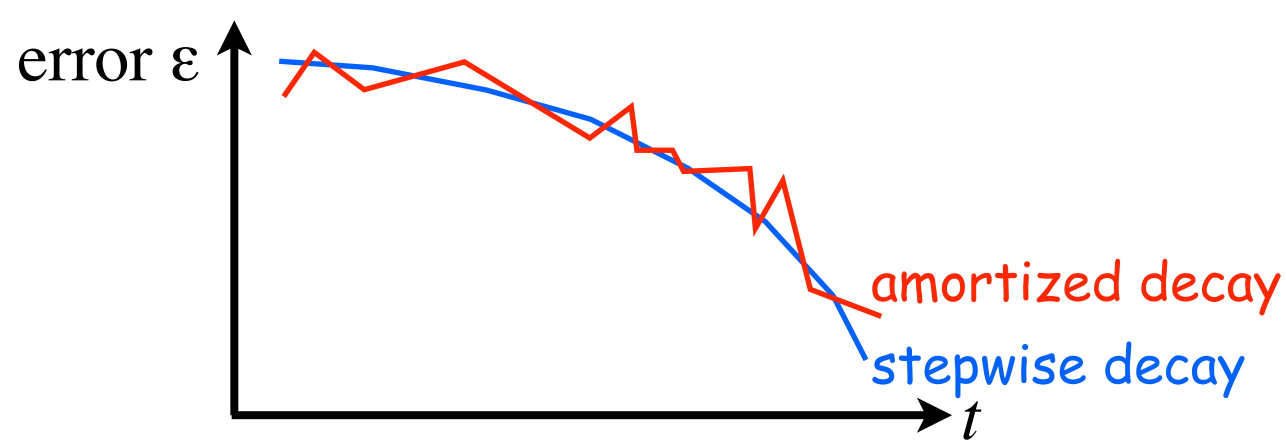
induction!

amortized behavior
correlation decay?

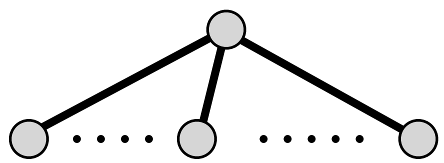


The Potential Method

original:

$$p = f(\vec{p})$$

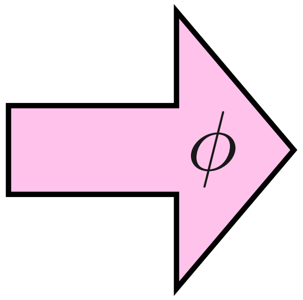
error ϵ



$p_{i,y,z}$

error $\epsilon_{i,y,z}$

$$\xi = \phi(p)$$

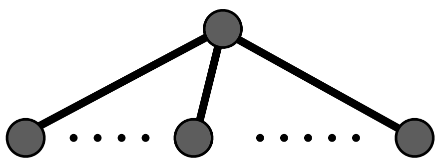


$$\xi_{i,y,z} = \phi(p_{i,y,z})$$

potential:

$$\xi = g(\vec{\xi})$$

error δ

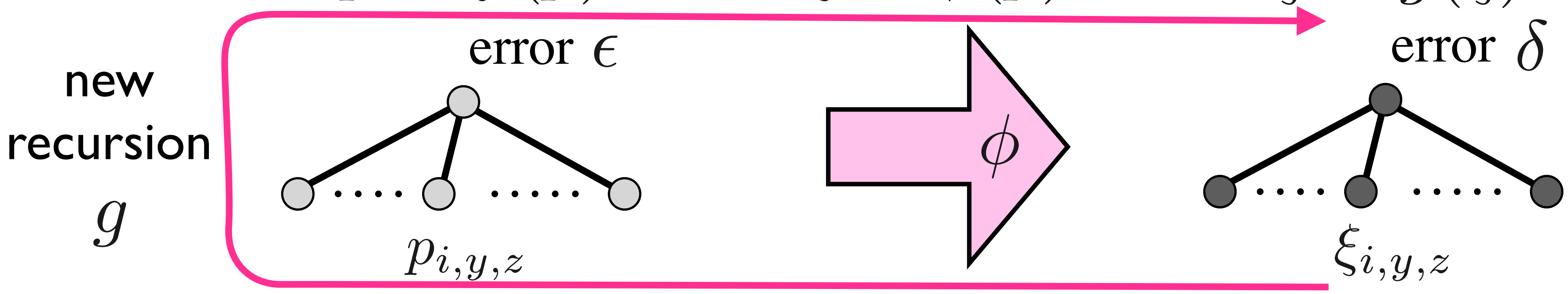


$\xi_{i,y,z}$

error $\delta_{i,y,z}$

The Potential Method

original: $p = f(\vec{p})$ $\xi = \phi(p)$ **potential:** $\xi = g(\vec{\xi})$



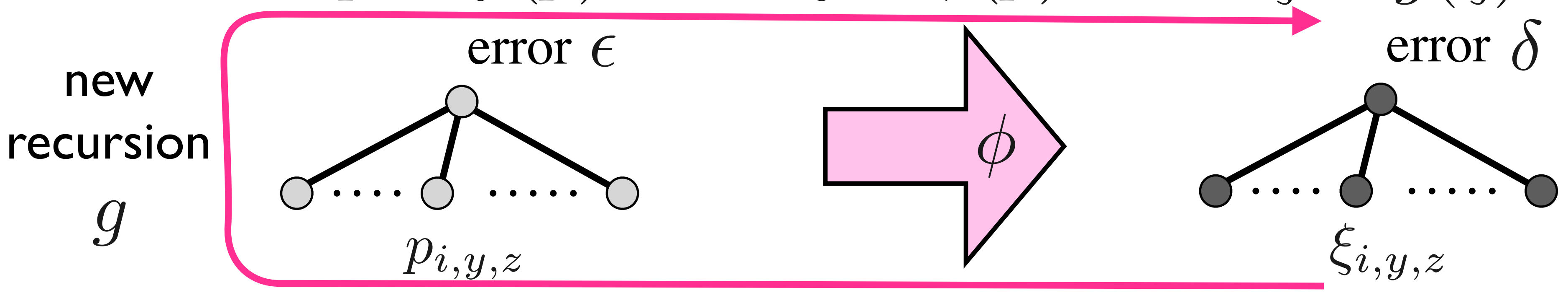
error $\epsilon_{i,y,z}$ $\xi_{i,y,z} = \phi(p_{i,y,z})$ error $\delta_{i,y,z}$

$$\xi = g(\vec{\xi}) = \phi(f(\phi^{-1}(\xi_{i,y,z}), \forall i, y, z)))$$

let $\Phi(x) = \dfrac{\mathrm{d}\,\phi(x)}{\mathrm{d}\,x}$ by Mean Value Thm: $\delta \leq \sum_{i,y,z} \left| \dfrac{\partial f(\boldsymbol{p})}{\partial p_{i,y,z}} \right| \dfrac{\Phi(f(\boldsymbol{p}))}{\Phi(p_{i,y,z})} \delta_{i,y,z}$

The Potential Method

original: $p = f(\vec{p})$ $\qquad \xi = \phi(p) \qquad$ **potential:** $\xi = g(\vec{\xi})$



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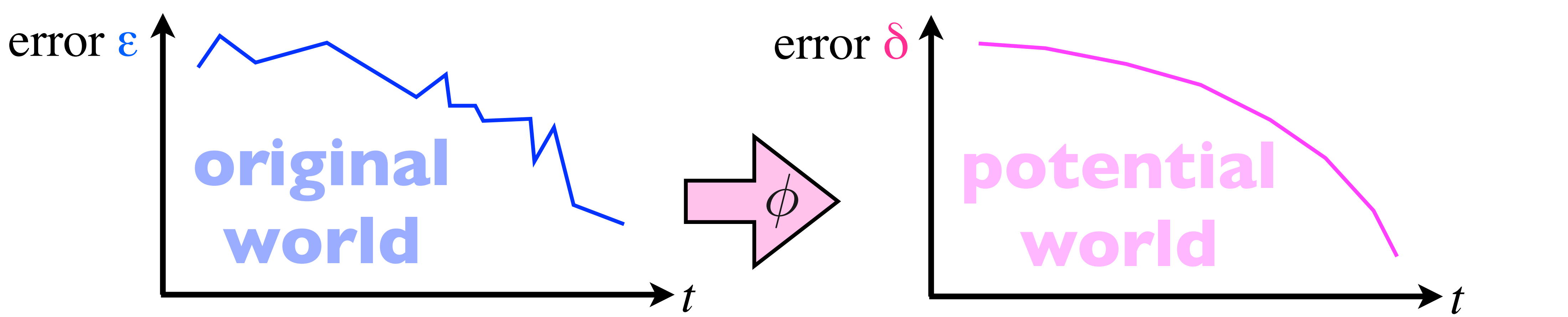
with good choice of potential function ϕ :



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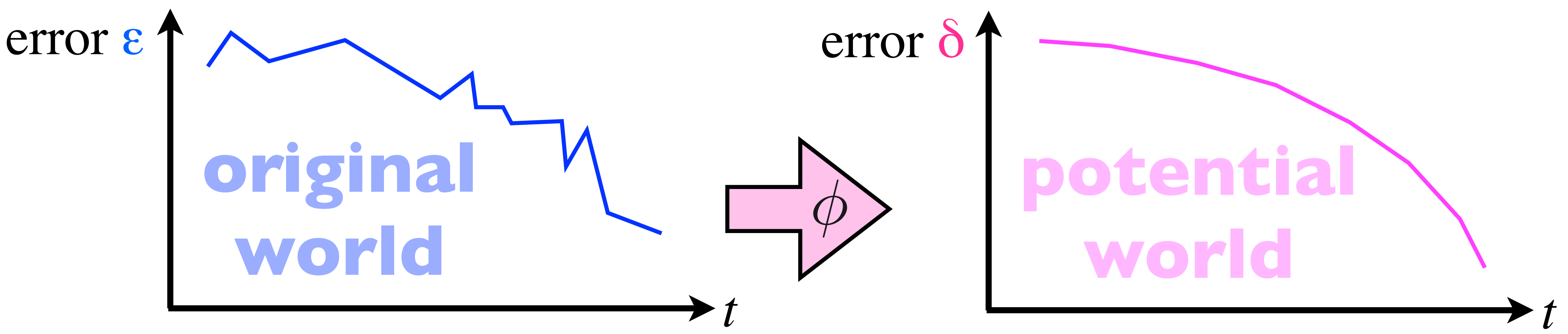

- The potential method has been used for analyzing the correlation decay in 2-spin systems (Restrepo-Shin-Tetali-Vigoda-Yang'11, Sinclair-Srivastava-Thurley'12, Li-Lu-Yin'12, Li-Lu-Yin'13, Sinclair-Srivastava-Yin'13).
- This is the first time it is used for multi-spin systems.

Amortized Correlation Decay

amortized decay condition:

$\exists$ a positive-valued function $\Phi(p)$, s.t.

- at any step, we have $\sum_{i,y,z} \left| \frac{\partial f(\boldsymbol{p})}{\partial p_{i,y,z}} \right| \frac{\Phi(f(\boldsymbol{p}))}{\Phi(p_{i,y,z})} < 1$
- the values of $\Phi(p)$ and $\frac{1}{\Phi(p)}$ are bounded over domain

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amortized decay condition → exponential correlation decay

→ efficient approximation of marginal probability

→ FPTAS

Establishing the Decay

for general multi-spin systems: with max-degree Δ

choose $\Phi(p) = \dfrac{1}{p+\eta}$ with small enough $\eta > 0$

denoted $c = \max\limits_{\substack{e\in E\\ w,x,y,z\in[q]}} \dfrac{A_e(x,y)}{A_e(w,z)}$

$3\Delta(c^{\Delta} - 1) \le 1$ $\Rightarrow$ amortized decay condition

(by easy calculation)

for Potts model (with inverse temperature β):

directly translated to $3\Delta(\mathrm{e}^{|\beta|} - 1) \le 1$

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* Other potential functions may further improve the constant factor (but may be harder to analyze).

Establishing the Decay

for list-coloring: with max-degree Δ,
each vertex v with color list L_v

choose $\Phi(p) = \dfrac{1}{(1-p)\sqrt{p}}$

observing that for list-coloring satisfying the condition, marginals are always bounded away from both 0 and 1

$\forall v, \quad |L_v| \geq \alpha\Delta + 1$ ⇒ amortized decay condition

$\alpha \approx 2.58071$

(by more involved calculation)

for coloring: replacing $|L_v|$ with q

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for coloring: replacing $|L_v|$ with q

* The potential functions are chosen in an ad hoc way.

Open Problem

- Find a more systematic way for designing good potential functions.
- Further improve the bounds for correlation decay and FPTAS for multi-spin systems.
- For coloring: $\alpha=2$ is a barrier for the approach due to the overheads caused by total differentiation. Overcome this barrier.