Homework 6

Course: Algorithm Design and Analysis **Semester**: Spring 2024

Instructor: Shi Li **Due Date**: **2024/6/23**

Student Name: __________ Student ID: __________

Problems	1	2	3	4	Total
Max. Score	15	30	25	30	100
Your Score					

Problem 1. Consider the following linear program.

$$
\begin{aligned}
\min \quad & 3x_1 + 2x_2 + 5x_3 \\
2x_1 + 2x_2 - 3x_3 &\geq 5 \\
x_1 - x_2 + 2x_3 &\geq 10 \\
3x_1 + 5x_2 + x_3 &\geq 8 \\
x_1, x_2, x_3 &\geq 0
\end{aligned} \tag{1}
$$

(1a) Write down the dual linear program for LP (1).

(1b) Assuming the variables in the dual LP are y_1, y_2 and y_3. Given the optimum primal solution x^* and the optimum dual solution y^* for the two LPs.

Problem 2. Consider the network flow problem on the directed graph $G = (V, E)$ with source $s \in V$, sink $t \in V$ and capacities $c : E \to \mathbb{R}_{\geq 0}$.

Another way to formulate the linear program for maximum flow is to variables correspondent to paths. Let $\mathcal{P}$ be the set of all simple paths from s to t. For every $P \in \mathcal{P}$, we define a variable f_P in the linear program, indicating the amount of flow we sent using the path f_P. For example, for the s-t network in Figure 1, there are 4 different s-t paths given in the left column of Table 1. The right column of Table 1 gives an maximum s-t flow.

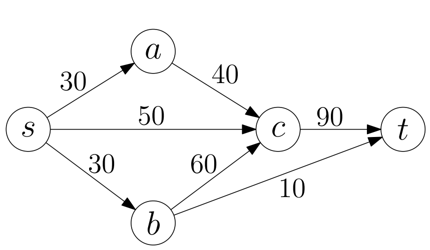

Figure 1: The s-t network.

path P	f_P
$s \to a \to c \to t$	30
$s \to c \to t$	50
$s \to b \to c \to t$	10
$s \to b \to t$	10

Table 1: The s-t paths for the network in Figure 1 and the optimum s-t flow f.

Answer the questions below.

(2a) Give the linear program for the s-t network flow problem, using variables $(f_P)_{P\in\mathcal{P}}$.

(2b) Give the dual linear program of the linear program in (2a).

(2c) Briefly explain the meaning of the variables and constraints in the dual linear program, and why it gives an upper bound on the value of the maximum flow.

Problem 3. In the class, we learned about the strong duality theorem. Now suppose we have a different form of the primal linear program:

$$\min \quad c^{\mathrm{T}}x, \qquad \text{s.t.} \quad Ax \geq b. \tag{2}$$

That is, we do not have $x \geq 0$. For simplicity, we assume both the feasible region of the LP is a non-empty polytope.

Using the strong duality theorem you learned in the class to prove that the value of LP (2) is equal to the value of LP(3):

$$\max \quad b^{\mathrm{T}}y, \qquad \text{s.t.} \quad A^{\mathrm{T}}y = c, y \geq 0. \tag{3}$$

Problem 4. Recall that in the weighted interval scheduling problem, we are given a set of n activities indexed as $1, 2, \cdots, n$. Each activity $i \in [n]$ has a starting time s_i, a finish time $f_i > s_i$, and a weight $v_i > 0$. The goal of the problem is to find a set of mutually compatible activities with the maximum total weight, where two distinct activities i and j are compatible iff $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint.

In this variant, we have m rooms to hold the activities and our goal is to schedule a maximum total weight of activities using the m rooms. Formally, we are additionally given an integer $m \geq 1$. Our goal is to find m disjoint subsets $J_1, J_2, \cdots, J_m \subseteq [n]$ so as to maximize $\sum_{k=1}^{m} \sum_{i\in J_k} v_i$, so that for every $k \in [m]$, activities in J_k are mutually compatible.

Using the linear programming technique to design an efficient algorithm to solve the problem. You can assume that you have an efficient LP solver that returns the optimum vertex solution.