Problem 1. In a connected graph $G = (V, E)$, then number of global minimum cuts is at most $\binom{n}{2} \leq n^2$. Let $\alpha \geq 1$ be an integer. We say a cut in $G$ is an $\alpha$-approximate cut, if its cut value is at most $\alpha$ times the value of the global minimum cut. Prove that the number of $\alpha$-approximate minimum cuts in a connected graph $G = (V, E)$ is at most $n^2\alpha$.

Problem 2. We are given an array $A$ of $n$ integers, and we are promised that some integer appears in $A$ at least $\frac{n}{3}$ times. Design a simple Las Vegas algorithm that finds such an integer in $O(n)$ time in expectation.

Problem 3. Let $G = (V, E)$ be a graph with tree-width $tw$.

(3a) Prove that there is a $(tw + 1)$-coloring for the vertices of $G$.

(3b) Suppose we are additionally given a tree-decomposition $(T, (V_t)_{t \in U})$ of $G$ with tree-width $tw$. It is possible that $G$ can be colored using $k$ colors, for some given integer $k \leq tw$. Design an $f(tw)\text{poly}(n)$-time algorithm to check if some a $k$-coloring exists, where $f(tw)$ can be any function on $tw$.

Problem 4. Suppose we are given a directed acyclic graph with specified source node $s$ and sink node $t$, and each arc $e$ has an associated cost $c_e$ and length $l_e$. We are also given a length bound $L$. Give an FPTAS for the problem of finding a minimum-cost path from $s$ to $t$ of total length at most $L$. 