算法设计与分析(2024年春季学期)

Advanced Topics

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Outline

Why do we use randomized algorithms?

- simpler algorithms: quick-sort, minimum-cut, and Max 3-SAT.
- faster algorithms: polynomial identity testing, Freivald's matrix multiplication verification algorithm, sampling and fingerprinting.
- mathematical beauty: Nash equilibrium for 0-sum game
- proof of existence of objects: union bound, Lovasz local lemma.

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Price of using randomness

- The algorithm may be incorrect with some probability (Monto Carlo Algorithm)
- The algorithm may take a long time to terminate (Las Vegas Algorithm)

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  - Best known algorithm for matrix multiplication: $O(n^{2.3719})$.
- Freivald's algorithm: randomized algorithm with $O(n^2)$ time.

Freivald's Algorithm: one experiment

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- $AB = C$: algorithm outputs true with probability 1.
- $AB \neq C$: algorithm may incorrectly output true.

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Proof.

- $D := C - AB \neq 0$ $\qquad Cr = ABr \iff Dr = 0$
- $\exists i, j \in [n], D_{i,j} \neq 0$

$$D_i r = \sum_{j'=1}^{n} D_{i,j'} r_{j'} = X + Y, \quad X = \sum_{j' \in [n], j' \neq j} D_{i,j'} r_{j'}, Y = D_{i,j} r_j$$

$$\begin{aligned}
\Pr[D_i r \neq 0] &= \Pr[Y \neq -X] \\
&= \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \Pr[Y \neq -x | X = x] \\
&= \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \Pr[D_{i,j} r_j \neq -x | X = x] \\
&\geq \sum_{x \in \mathbb{Z}} \Pr[X = x] \cdot \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

□

- probabilities:

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3: **if** $ABr \neq Cr$ **then return false**
4: **return true**

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- to achieve δ probability of mistake, need $\log_2 \frac{1}{\delta} = O(\log \frac{1}{\delta})$ experiments.

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Def. A Monta Carlo algorithm is a randomized algorithm whose output may be incorrect with some probability.

- For a Monta Carlo algorithm that outputs true/false, we say the algorithm has one-sided error if it makes error only if the correct output is true (or false).

Outline

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85

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Quicksort

quicksort(A, n)

```
if n ≤ 1 then return A
x ← lower median of A
A_L ← elements in A that are less than x          \\ Divide
A_R ← elements in A that are greater than x       \\ Divide
B_L ← quicksort(A_L, A_L.size)                    \\ Conquer
B_R ← quicksort(A_R, A_R.size)                    \\ Conquer
t ← number of times x appear A
return the array obtained by concatenating B_L, the array
containing t copies of x, and B_R
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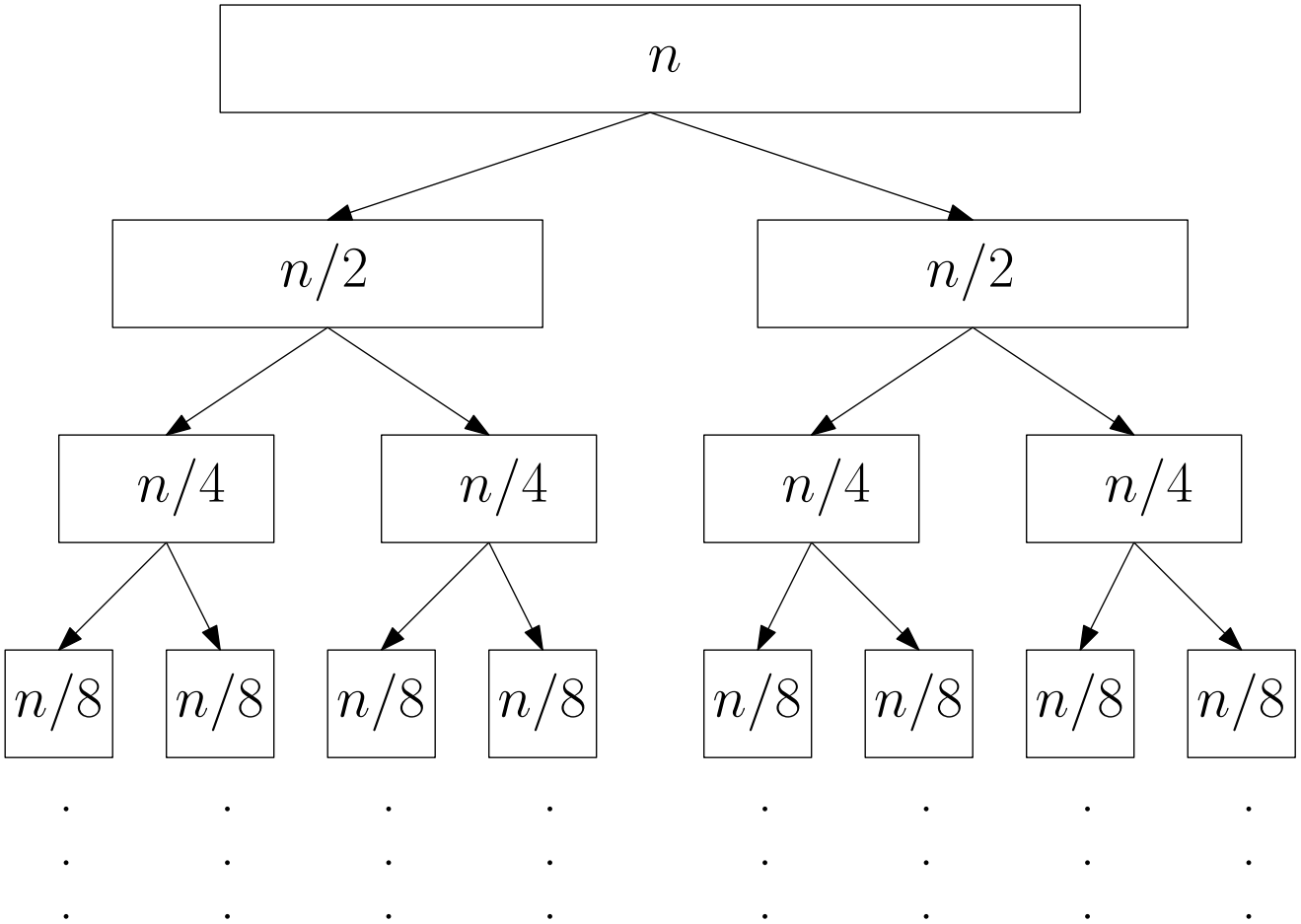
- Recurrence $T(n) \leq 2T(n/2) + O(n)$
- Running time $= O(n \log n)$



- Each level has total running time $O(n)$
- Number of levels $= O(\log n)$
- Total running time $= O(n \log n)$

Randomized Quicksort Algorithm

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Variant of Randomized Quicksort Algorithm

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Analysis of Variant

1: $x \leftarrow$ a random element of A
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Q: What is the probability that A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$?

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A: At least $1/2$

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Q: What is the expected number of iterations the above procedure takes?

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Q: What is the expected number of iterations the above procedure takes?

A: At most 2

- Suppose an experiment succeeds with probability $p \in (0, 1]$, independent of all previous experiments.

```
repeat
    run an experiment
until the experiment succeeds
```

Lemma The expected number of experiments we run in the above procedure is $1/p$.

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Proof

$$
\begin{aligned}
\text{Expectation} &= p + (1-p)p \times 2 + (1-p)^2 p \times 3 + (1-p)^3 p \times 4 \\
&\quad + \cdots \\
&= p \sum_{i=1}^{\infty} (1-p)^{i-1} i \quad = \quad p \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} (1-p)^{i-1} \\
&= p \sum_{j=1}^{\infty} (1-p)^{j-1} \frac{1}{1-(1-p)} \quad = \quad \sum_{j=1}^{\infty} (1-p)^{j-1} \\
&= (1-p)^0 \frac{1}{1-(1-p)} = 1/p
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Variant Randomized Quicksort Algorithm

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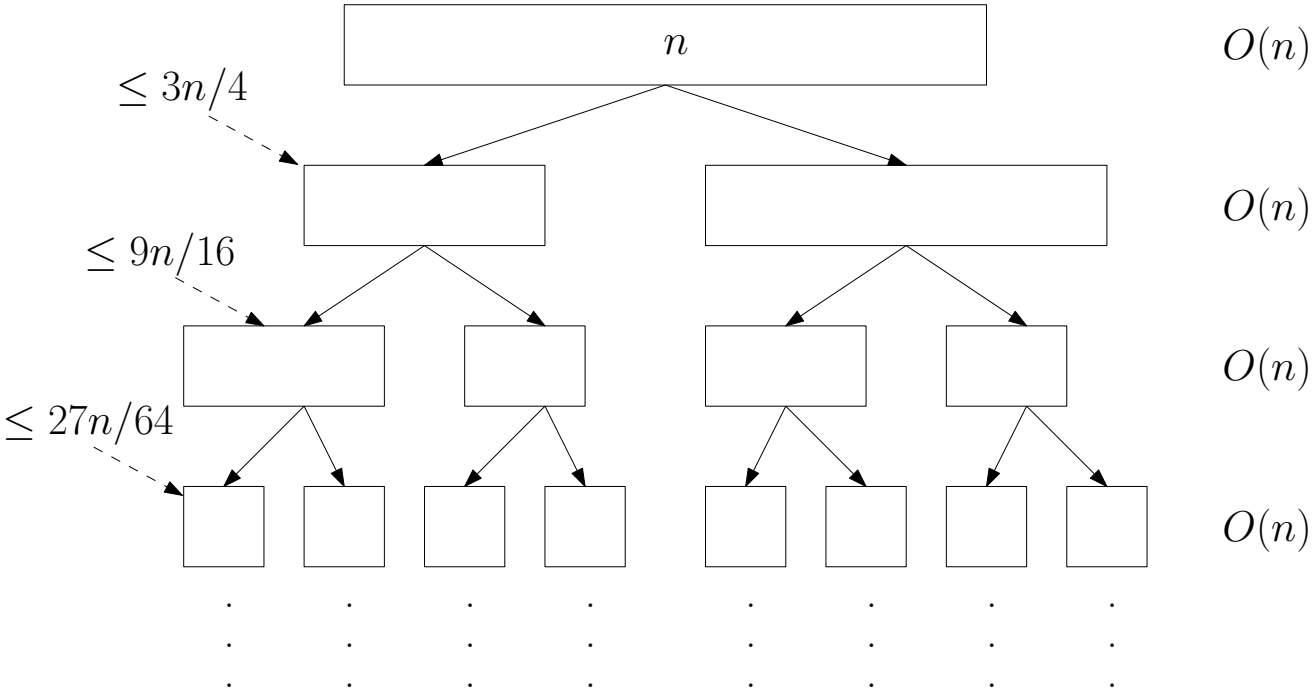
Analysis of Variant

- Divide and Combine: takes $O(n)$ time
- Conquer: break an array of size n into two arrays, each has size at most $3n/4$. Recursively sort the 2 sub-arrays.



- Number of levels $\leq \log_{4/3} n = O(\log n)$

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- Intuition: the quicksort algorithm should be better than the variant.

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T(n) &= \frac{1}{n}\sum_{i=1}^{n}\big(T(i-1)+T(n-i)\big)+O(n) \\
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- Can prove $T(n) \le c(n\log n)$ for some constant c by reduction

Analysis of Randomized Quicksort Algorithm

The induction step of the proof:

$$
\begin{aligned}
T(n) &\leq \frac{2}{n}\sum_{i=0}^{n-1} T(i) + c'n \leq \frac{2}{n}\sum_{i=0}^{n-1} ci\log i + c'n \\
&\leq \frac{2c}{n}\left(\sum_{i=0}^{\lfloor n/2\rfloor-1} i\log\frac{n}{2} + \sum_{i=\lfloor n/2\rfloor}^{n-1} i\log n\right) + c'n \\
&\leq \frac{2c}{n}\left(\frac{n^2}{8}\log\frac{n}{2} + \frac{3n^2}{8}\log n\right) + c'n \\
&= c\left(\frac{n}{4}\log n - \frac{n}{4} + \frac{3n}{4}\log n\right) + c'n \\
&= cn\log n - \frac{cn}{4} + c'n \leq cn\log n \qquad \text{if } c \geq 4c'
\end{aligned}
$$

Indirect Analysis Using Number of Comparisons

- Running time $= O(\text{number of comparisons})$
- $\forall 1 \leq i < j \leq n$, $D_{i,j}$ indicates if we compared the i-th smallest element with the j-th smallest element
- number of comparisons $= \sum_{1 \leq i < j \leq n} D_{i,j}$

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Lemma $\mathbb{E}[D_{i,j}] = \frac{2}{j-i+1}$.

Proof.

- A': sorted array for A. Focus on $A'[i..j]$.
- pivot outside $A'[i]$: $A'[i \cdots j]$ will be passed to left or right recursion; go to that recursion
- pivot inside $A'[i]$: $A'[i]$ and $A'[j]$ will be separated; call this critical recursion
- $A[i]$ and $A[j]$ are compared in the critical recursion with probability $\frac{2}{j-i+1}$. $\square$

$$\mathbb{E}\left[\text{number of comparisons}\right] = \mathbb{E}\left[\sum_{1\le i<j\le n} D_{i,j}\right]$$

$$= \sum_{1\le i<j\le n} \mathbb{E}\left[D_{i,j}\right] = 2\sum_{1\le i<j\le n} \frac{1}{j-i+1}$$

$$\le 2n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$$

$$= \Theta\left(n\log n\right).$$

- The algorithm is a Las-Vegas algorithm:

Def. A Las-Vegas algorithm is a randomized algorithm that always outputs a correct solution but has randomized running time.

Table: Comparisons between Monta Carlo and Las Vegas Algorithms.

	correctness	running time
Monta Carlo	may be wrong	usually has good worst-case running time
Las Vegas	always correct	may take a long time and usually only has good "expected running time"

Lemma Given a Las Vegas algorithm $\mathcal{A}$ with expected running time at most $T(n)$, we can design a Monta Carlo algorithm $\mathcal{A}'$ with worst-case running time $O(T(n))$ and error at most 0.99.

- 0.99 can be changed to any $c < 1$

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- 0.99 can be changed to any $c < 1$

Proof.

- run $\mathcal{A}$ for $100T(n)$ time
- if $\mathcal{A}$ terminated, output what $\mathcal{A}$ outputs
- otherwise, declare failure
- Markov Inequality:
  $\Pr[\mathcal{A}$ runs for more than $100T(n)$ time$] \leq 1/100$ $\square$

Randomized Selection Algorithm

selection(A, n, i)

```
if n = 1 then return A
x ← random element of A (called pivot)
A_L ← elements in A that are less than x                    ▷ Divide
A_R ← elements in A that are greater than x                 ▷ Divide
if i ≤ A_L.size then
    return selection(A_L, A_L.size, i)                      ▷ Conquer
else if i > n − A_R.size then
    return selection(A_R, A_R.size, i − (n − A_R.size))     ▷ Conquer
else
    return x
```

- expected running time $= O(n)$

Randomized Selection

- $X_j, j = 0, 1, 2, \cdots$: the size of A in the j-th recursion

$$\begin{aligned}
\mathbb{E}[X_{j+1}|X_j = n'] &\leq \frac{1}{n'} \sum_{k=1}^{n'} \max\{k-1, n'-k\} \\
&\leq \frac{1}{n'} \left(\int_{k=0}^{n'/2} (n'-k)\mathrm{d}k + \int_{k=n'/2}^{n'} k\mathrm{d}k \right) \\
&= \frac{1}{n'} \left(\left(n'k - \frac{k^2}{2} \right) \Big|_0^{n'/2} + \frac{k^2}{2} \Big|_{n'/2}^{n'} \right) \\
&= \frac{1}{n'} \left(\frac{n'^2}{2} - \frac{n'^2}{8} + \frac{n'^2}{2} - \frac{n'^2}{8} \right) = \frac{3n'}{4}.
\end{aligned}$$

- $\mathbb{E}[X_{j+1}] \leq \frac{3}{4}\,\mathbb{E}[X_j]$
- $X_0 = n \quad \Longrightarrow \quad \mathbb{E}[X_j] \leq \left(\frac{3}{4}\right)^j n$

$$\mathbb{E}[\text{running time of randomized selection}]$$

$$\leq \mathbb{E}\left[O(1)\sum_{j=0}^{\infty} X_j\right] \leq O(1)\sum_{j=0}^{\infty}\mathbb{E}[X_j]$$

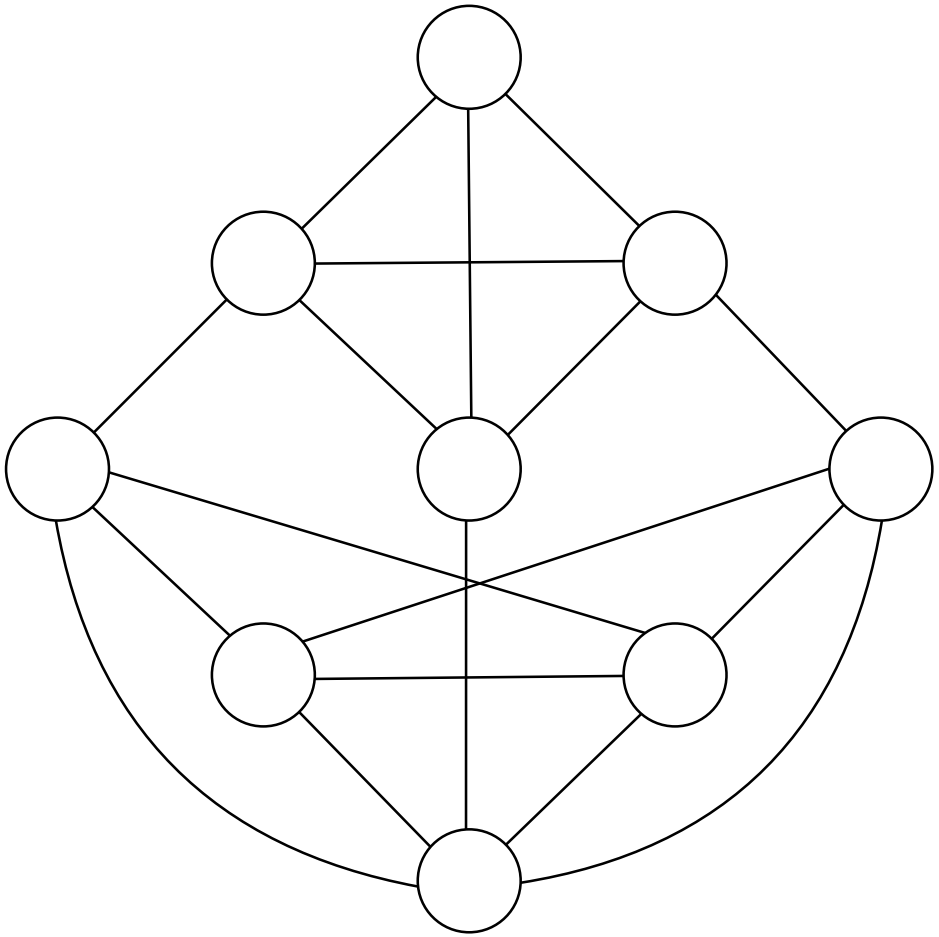
$$\leq O(1)\sum_{j=0}^{\infty}\left(\frac{3}{4}\right)^j n = O(1)\cdot 4n = O(n).$$

Outline

Global Min-Cut Problem

Input: a connected graph $G = (V, E)$

Output: the minimum number of edges whose removal will disconnect G



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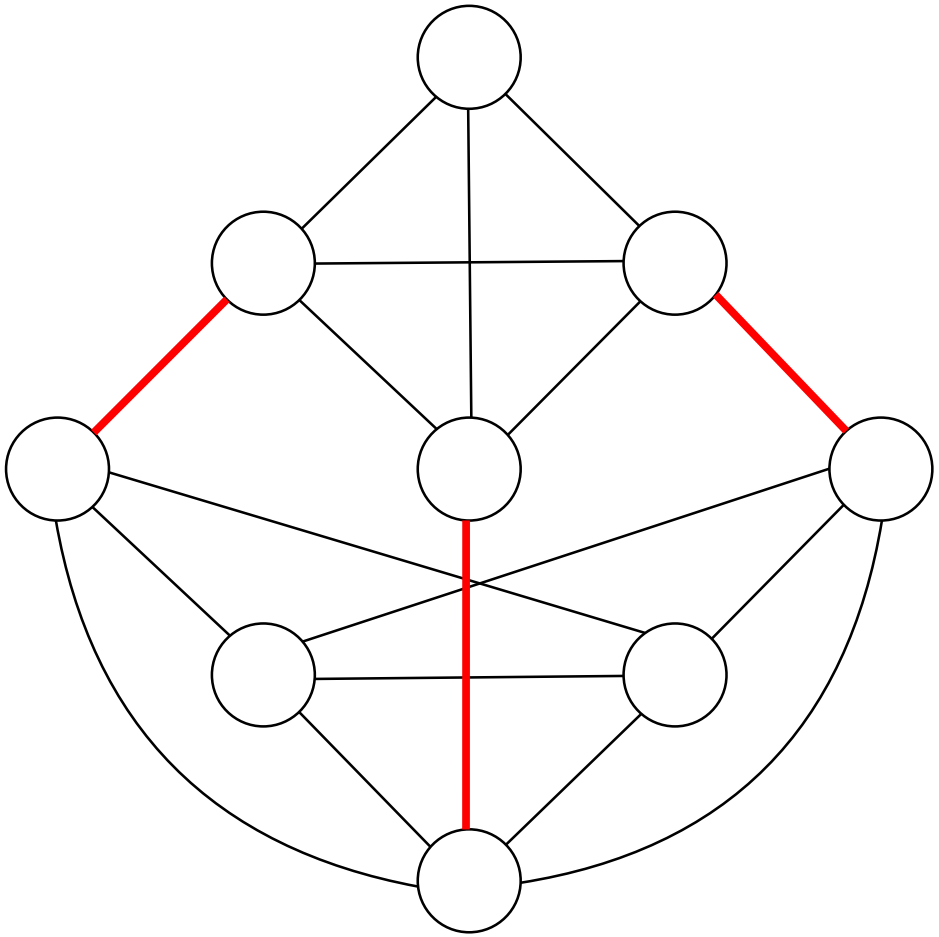

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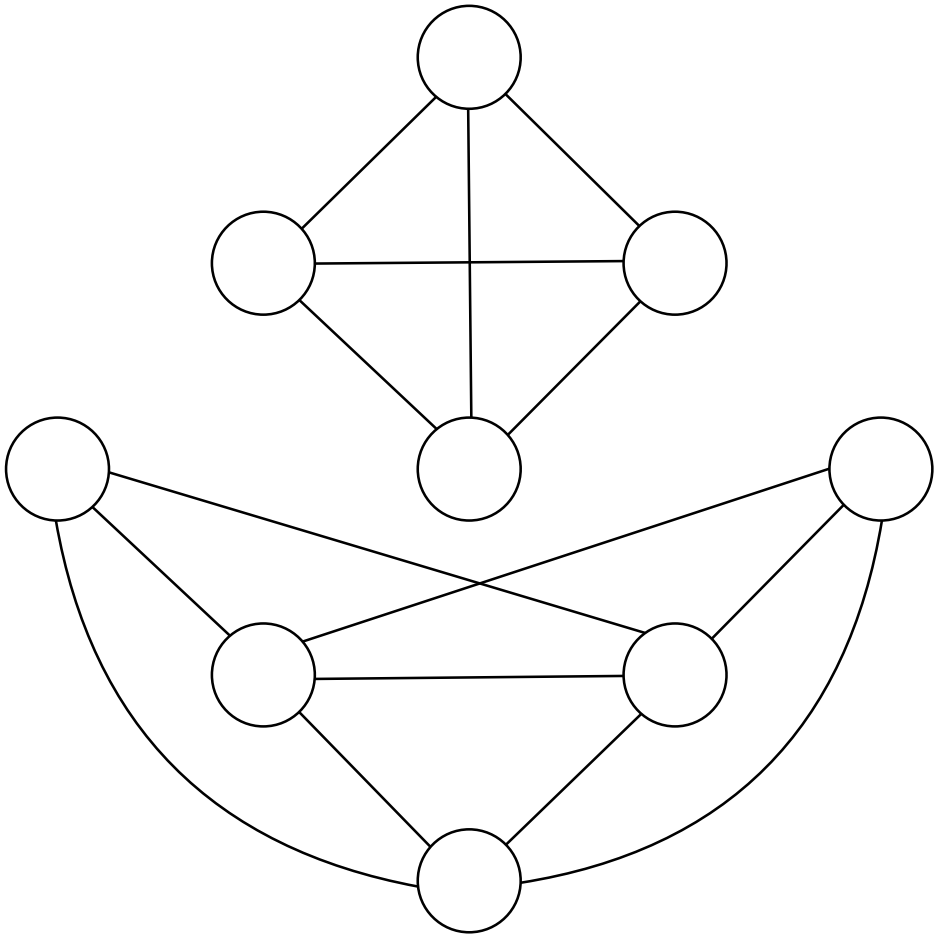
Input: a connected graph $G = (V, E)$

Output: the minimum number of edges whose removal will disconnect G



Solving Global Min-Cut Using s-t Min-Cut

1: let G' be the directed graph obtained from G by replacing every edge with two anti-parallel edges
2: **for** a fixed $s \in V$ and every pair $t \in V \setminus \{s\}$ **do**
3: obtain the minimum cut separating s and t in G, by solving the maximum flow instance with graph G',source s and sink t
4: output the smallest minimum cut we found

- Time $= O(n) \times$ (Time for Maximum Flow)

Karger's Randomized Algorithm for Min-Cut

$G' = (V', E') \leftarrow G$
while $|V'| > 2$ **do**
    pick $uv \in E'$ uniformly at random
    contract uv in G', keeping parallel edges, but not self-loops
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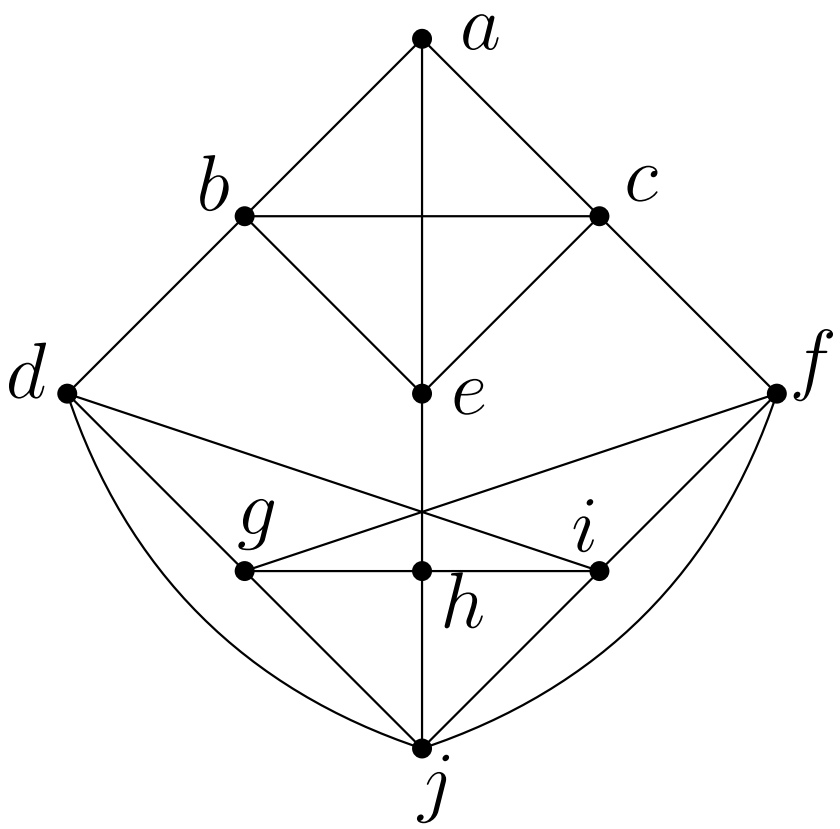

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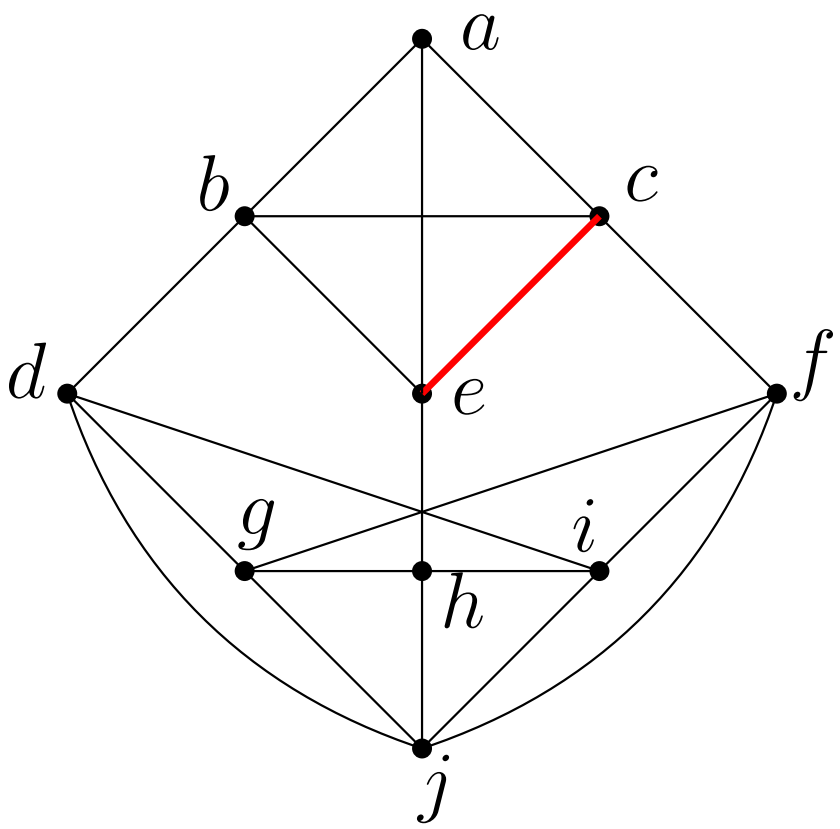

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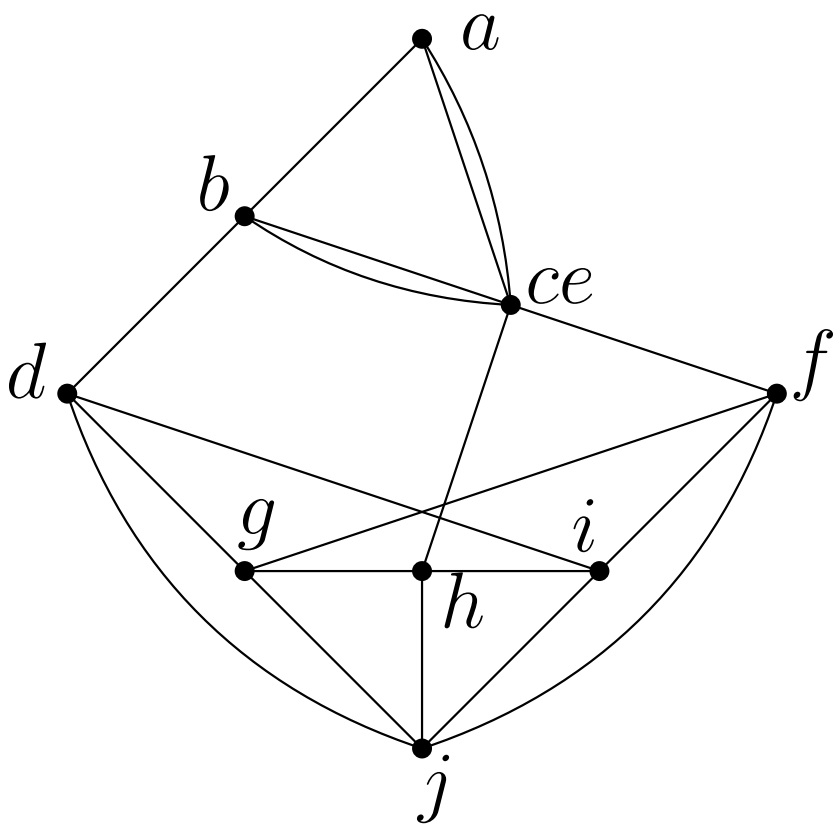

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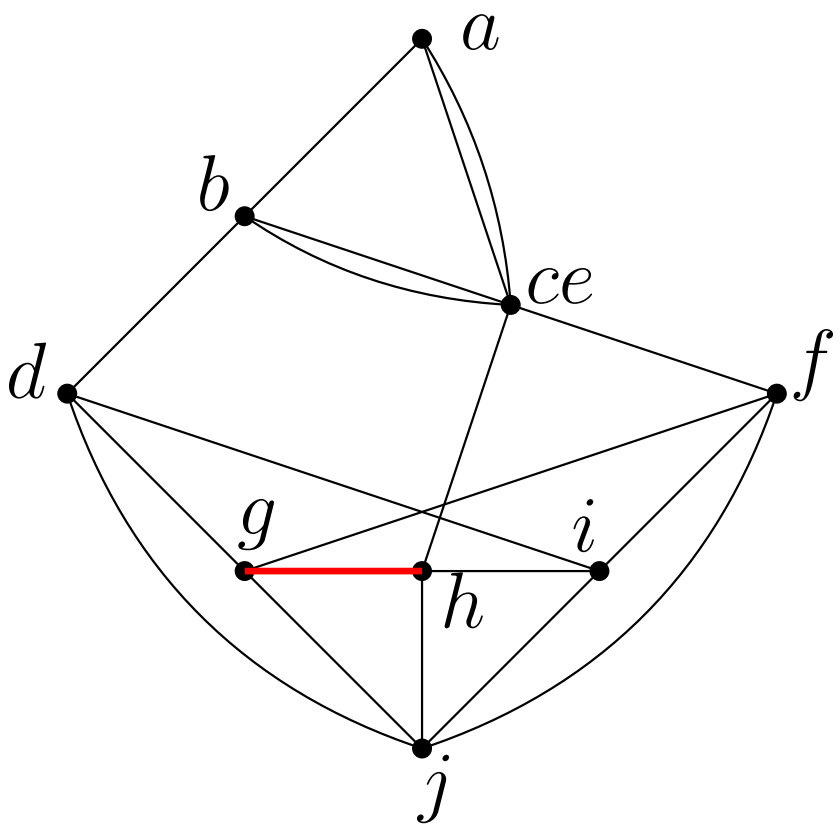

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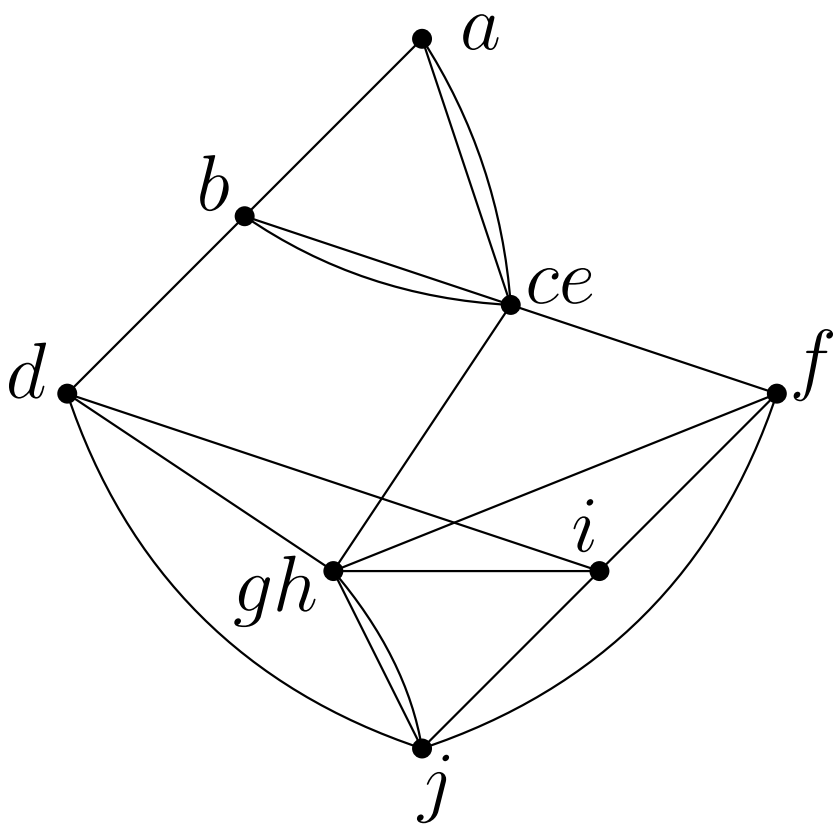

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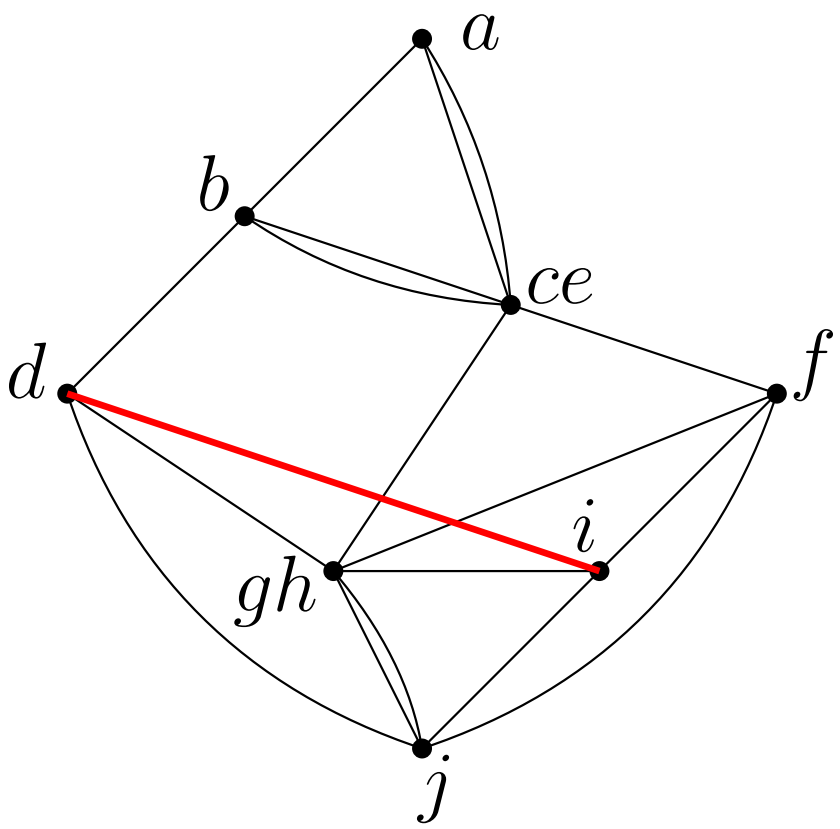

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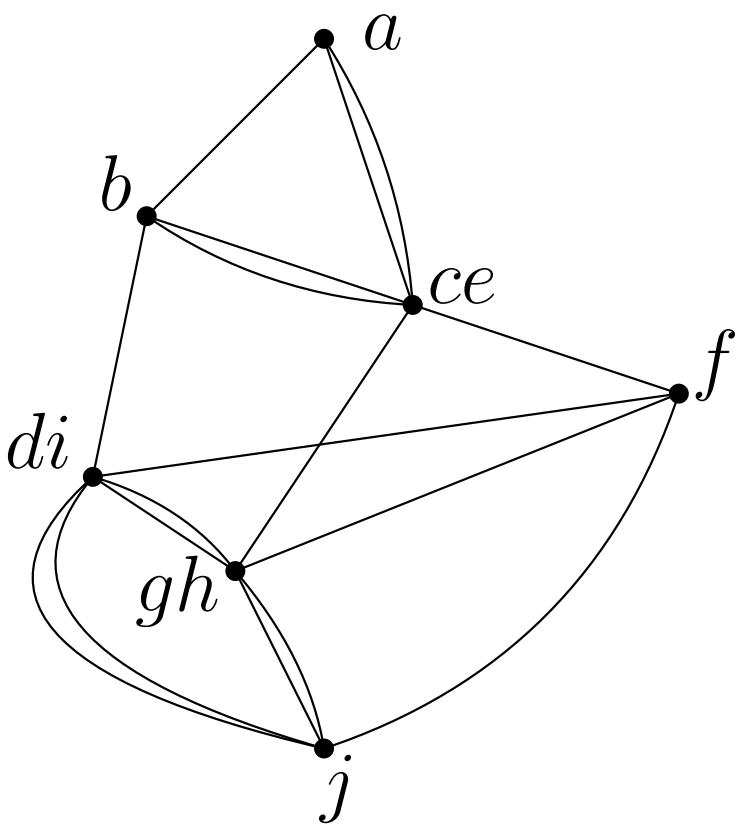

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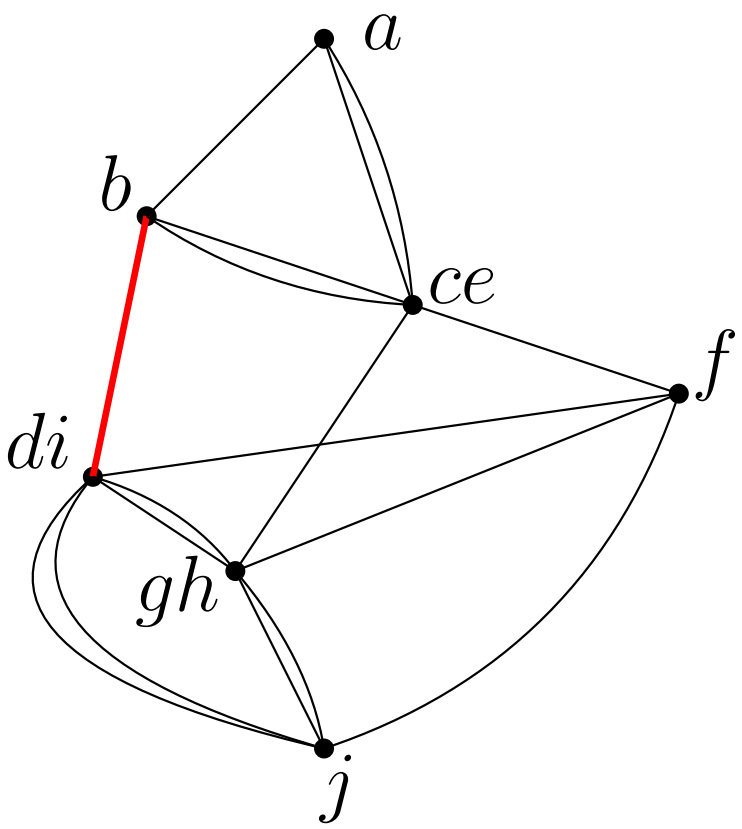

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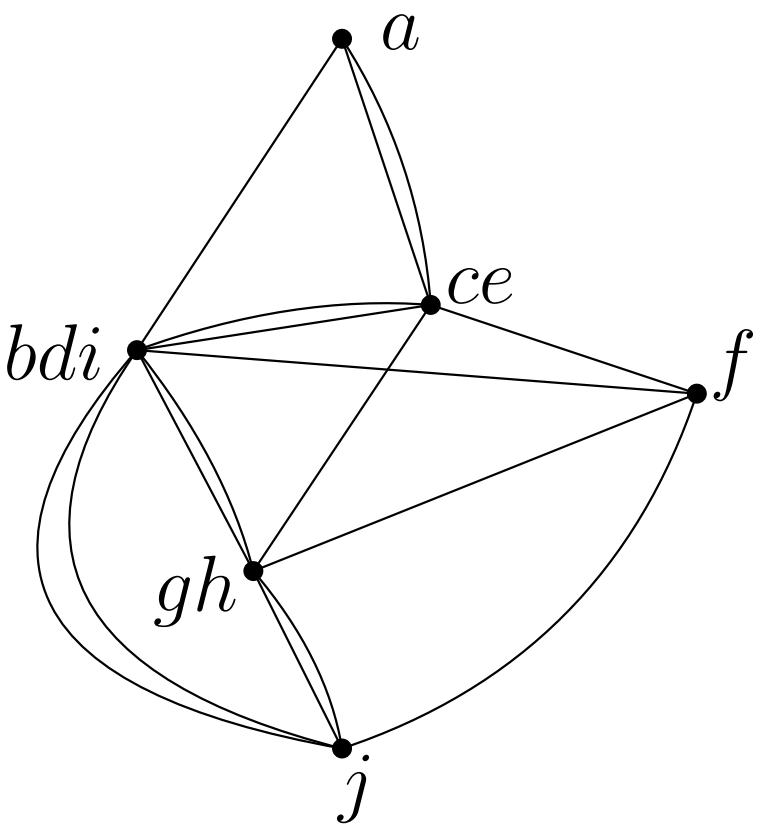

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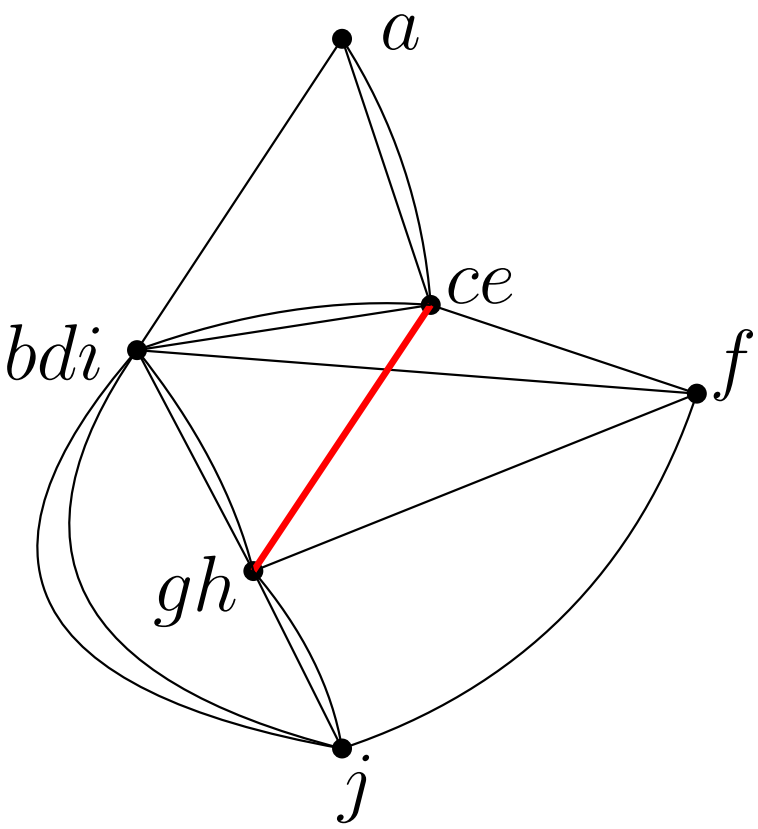

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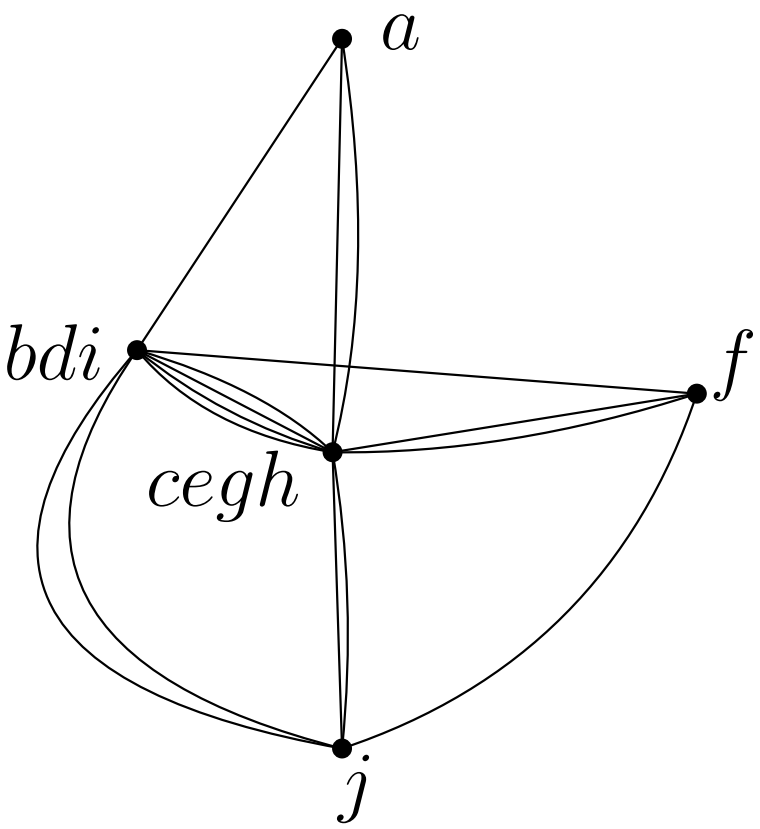

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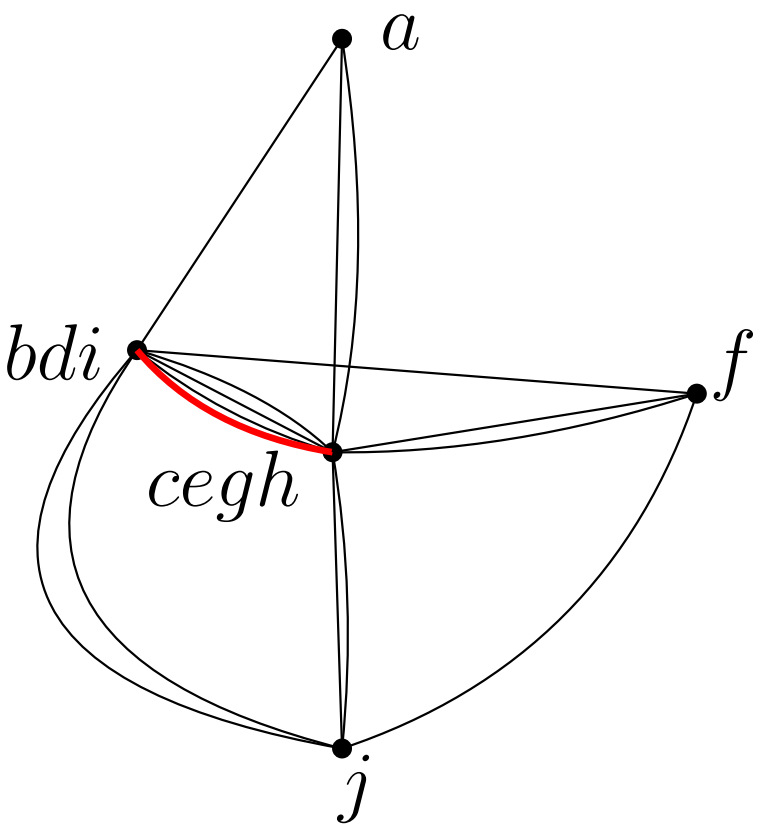

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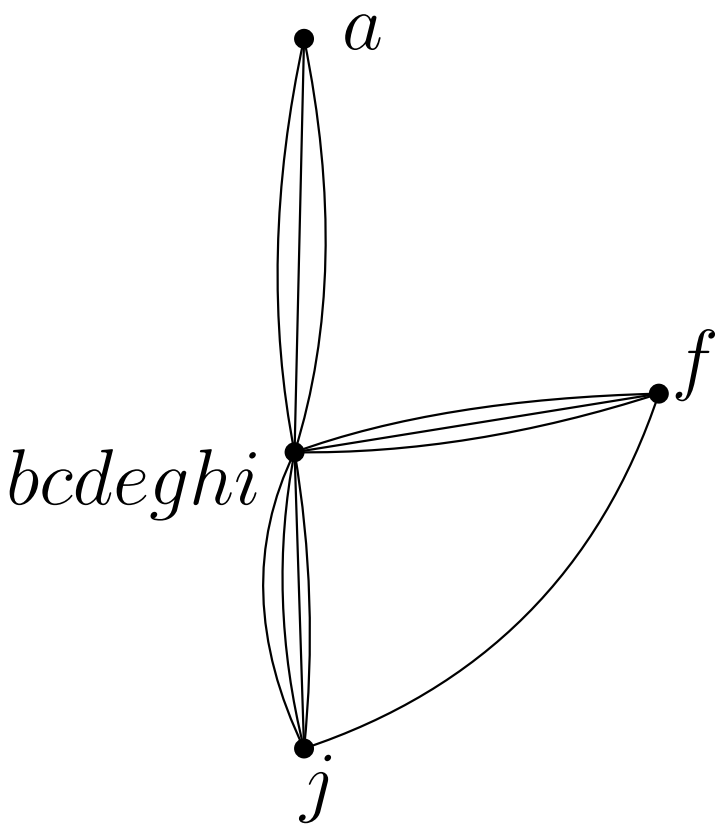

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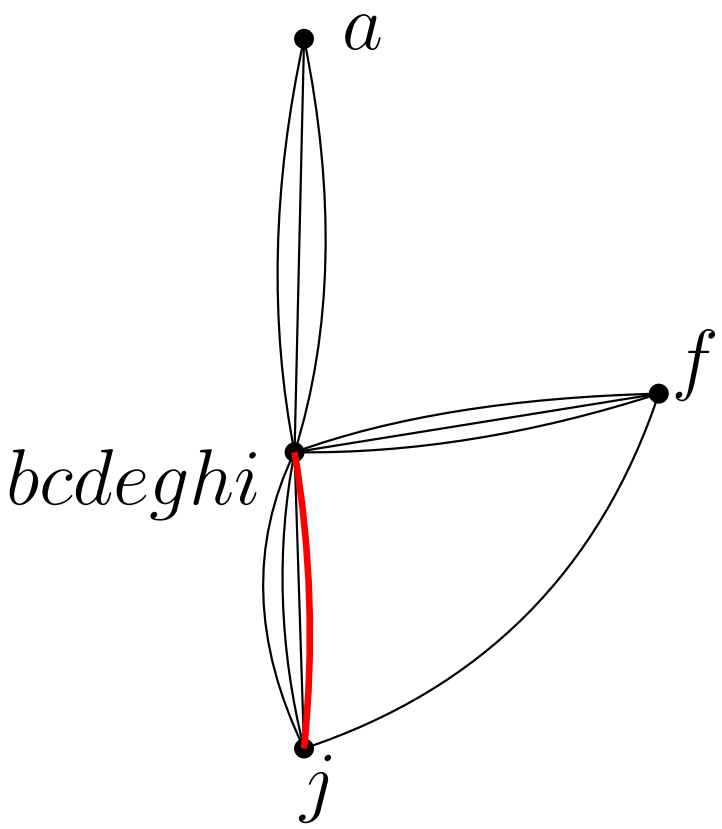

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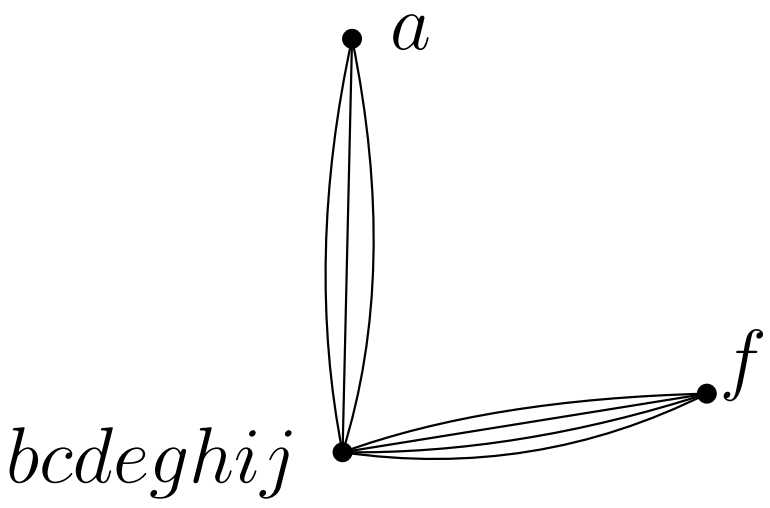

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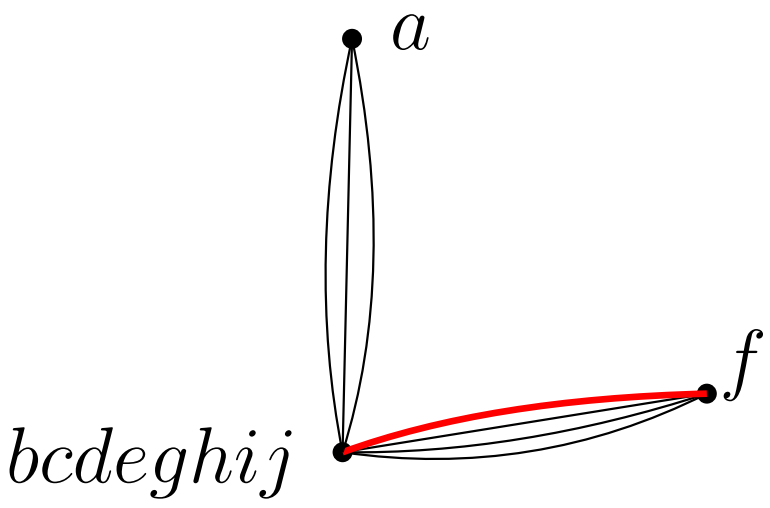

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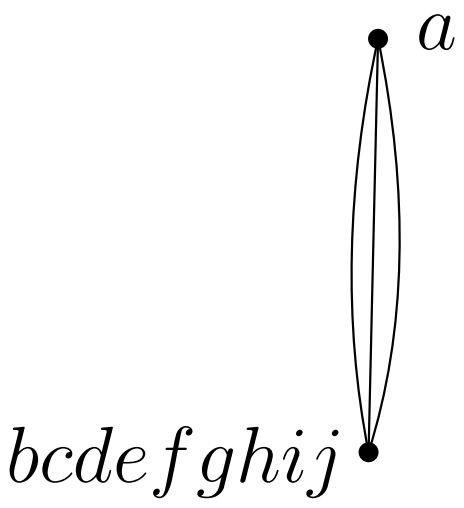

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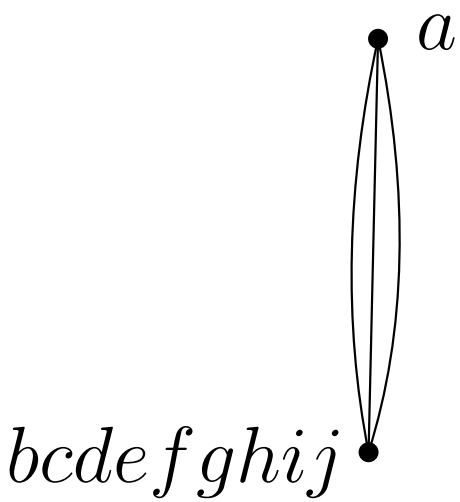

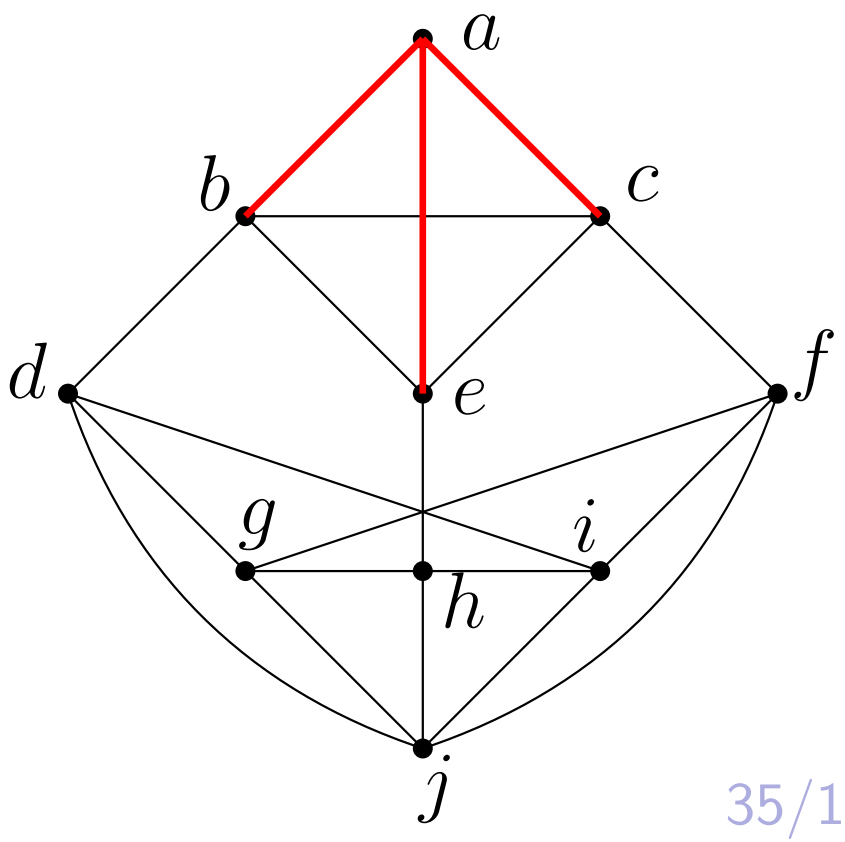

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Coro. Focus on some iteration where we have the graph $G' = (V', E')$ with $n' = |V'|$ at the beginning. Suppose all previous iterations succeed. Then the probability this iteration fails is at most $\frac{c}{n'c/2} = \frac{2}{n'}$.

- The probability that the algorithm succeeds is at least

$$\left(1-\frac{2}{n}\right)\left(1-\frac{2}{n-1}\right)\left(1-\frac{2}{n-2}\right)\cdots\left(1-\frac{2}{3}\right)$$
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- To get a success probability of $1-\delta$, run the algorithm for $O(n^2 \log \frac{1}{\delta})$ times.

Equivalent Algorithm

1: give every edge a weight in $[0, 1]$ uniformly at random.
2: solve the MST on the graph G with the weights, using either Kruskal or Prim's algorithm
3: remove the heaviest edge in the MST,
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- run it once: time $= O(m + n \log n)$
- to get success probability $1 - \delta$: time $= O(n^2(m + n \log n) \log \frac{1}{\delta})$

Karger-Stein: A Faster Algorithm

Karger-Stein$(G = (V, E))$

1: **if** $|V| \leq 6$ **then return** min cut of G directly

2: **repeat** twice and return the smaller cut:

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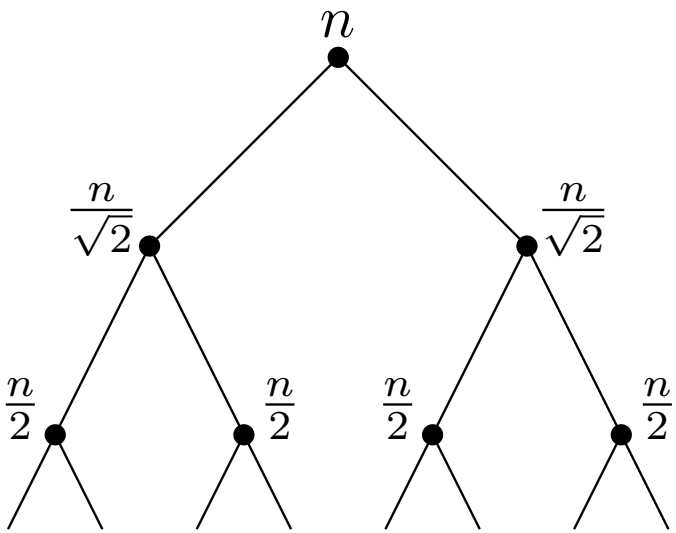

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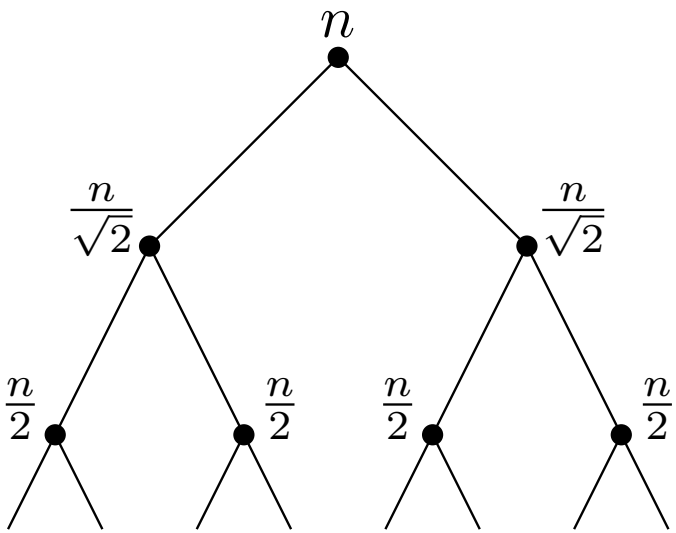

- Running time:
  $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2)$
- $T(n) = O(n^2 \log n)$

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Analysis of Probability of Success

- running Karger(G) down to $\lceil n/\sqrt{2} \rceil + 1$ vertices, success probability is at least

$$\frac{n-2}{n} \times \frac{n-3}{n-1} \times \cdots \times \frac{\lceil n/\sqrt{2} \rceil}{\lceil n/\sqrt{2} \rceil + 2} = \frac{(\lceil n/\sqrt{2} \rceil + 1)\lceil n/\sqrt{2} \rceil}{n(n-1)}$$

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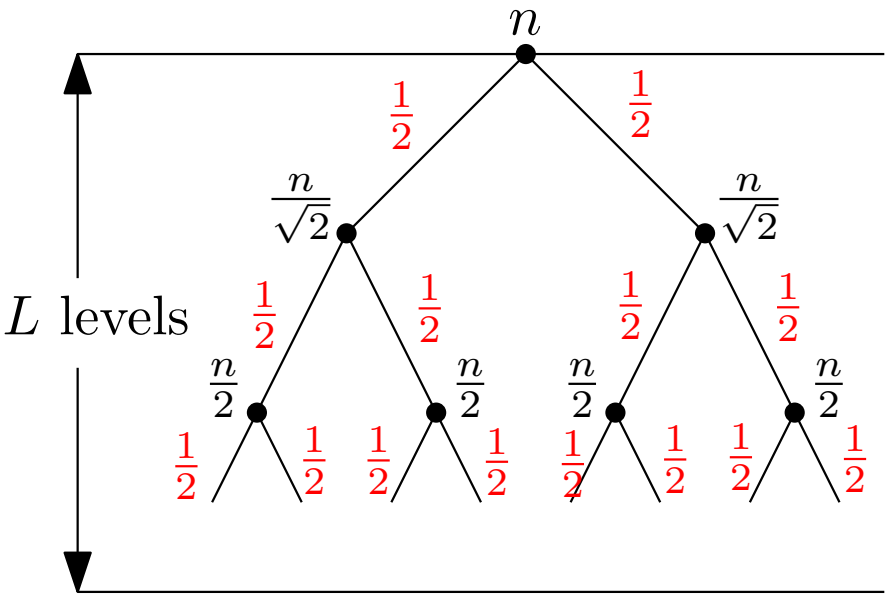
- recursion for Probability: $P(n) \geq 1 - \left(1 - \frac{1}{2}P(\frac{n}{\sqrt{2}})\right)^2$



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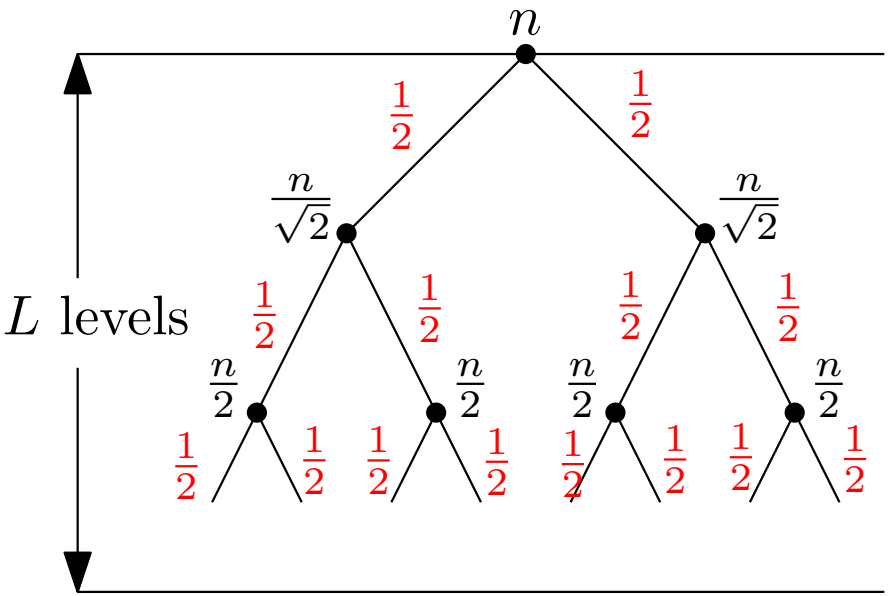
- what is the success probability in terms of L?



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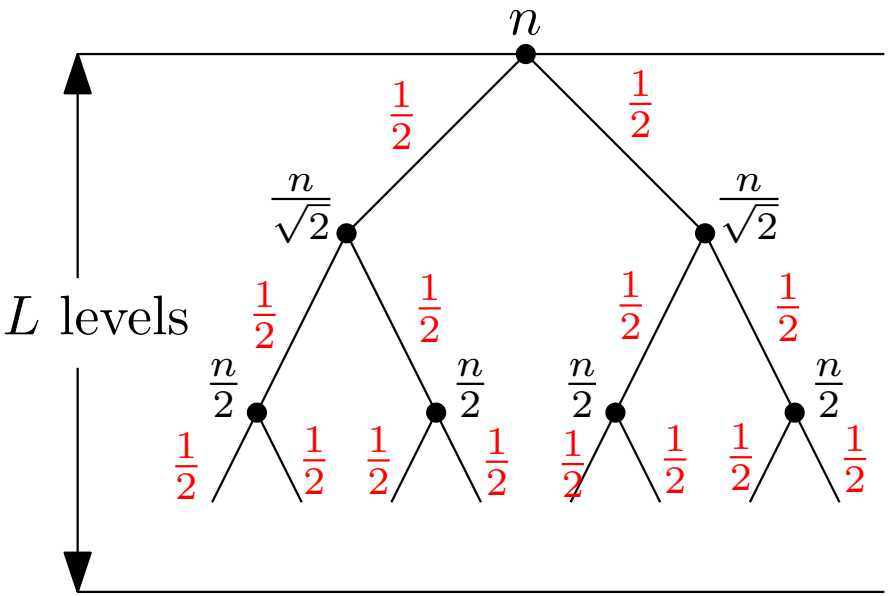

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Proof.

- $L = 0$: a singleton, holds trivially.
- induction:

$$P_L = 1 - \left(1 - \frac{1}{2}P_{L-1}\right)^2 \geq 1 - \left(1 - \frac{1}{2L}\right)^2 = \frac{1}{L} - \frac{1}{4L^2}$$
$$= \frac{4L-1}{4L^2} \geq \frac{1}{L+1}$$

□

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- Repeat $O(\log n)$ times can increase the probability to a constant

Outline

Approximation Algorithms

An algorithm for an optimization problem is an α-approximation algorithm, if it runs in polynomial time, and for any instance to the problem, it outputs a solution whose cost (or value) is within an α-factor of the cost (or value) of the optimum solution.

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- For minimization problems:
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- For maximization problems, there are two conventions:
  - $\alpha \leq 1$ and we require sol $\geq \alpha \cdot$ opt
  - $\alpha \geq 1$ and we require sol $\geq$ opt$/\alpha$

Max 3-SAT

Input: n boolean variables $x_1, x_2, \cdots, x_n$

m clauses, each clause is a disjunction of 3 literals from 3 distinct variables

Output: an assignment so as to satisfy as many clauses as possible

Example:

- clauses: $x_2 \vee \neg x_3 \vee \neg x_4, \quad x_2 \vee x_3 \vee \neg x_4,$
  $\neg x_1 \vee x_2 \vee x_4, \quad x_1 \vee \neg x_2 \vee x_3, \quad \neg x_1 \vee \neg x_2 \vee \neg x_4$
- We can satisfy all the 5 clauses: $x = (1, 1, 1, 0, 1)$

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- $\mathbb{E}[Z] = \mathbb{E}\left[\sum_{j=1}^{m} Z_j\right] = \sum_{j=1}^{m} \mathbb{E}[Z_j] = \sum_{j=1}^{m} \frac{7}{8} = \frac{7}{8}m$, by linearity of expectation. $\square$

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Theorem ([Hastad 97]) Unless P $=$ NP, there is no ρ-approximation algorithm for MAX-3-SAT for any $\rho > 7/8$.

Outline

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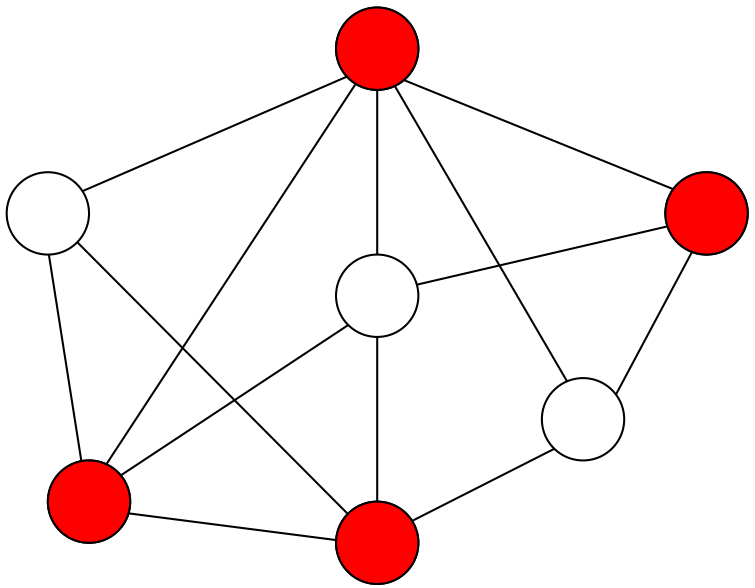
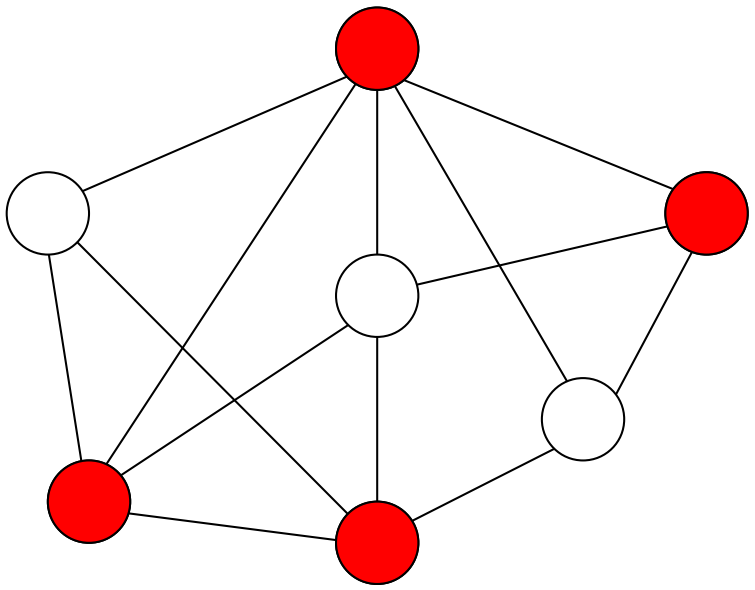
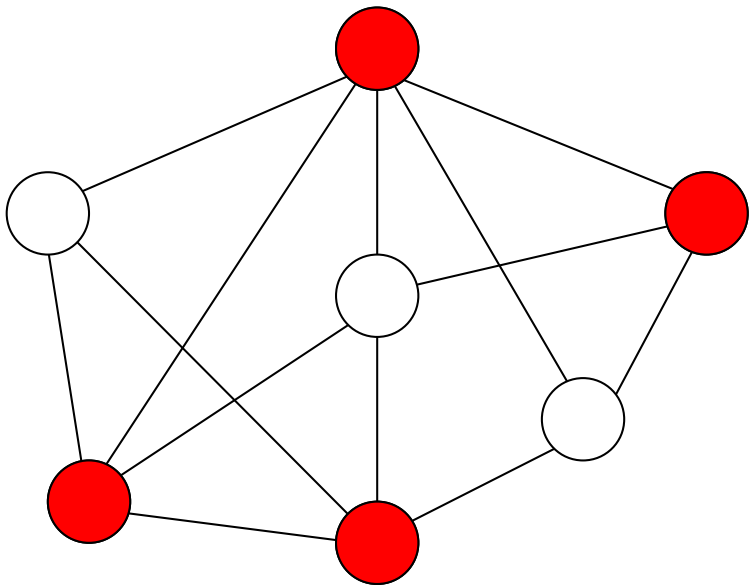




Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$



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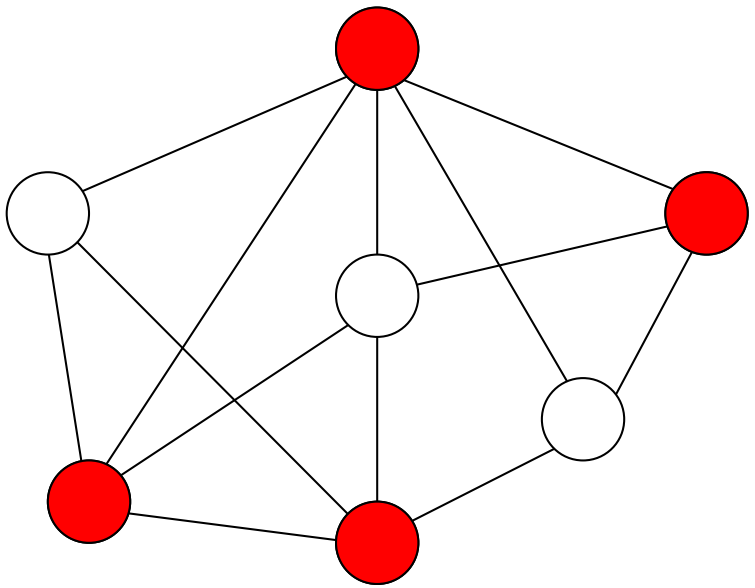

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- Remark: m does not appear in the running time. Indeed, if $m > kn$, then there is no vertex cover of size k.

Vertex-Cover$(G' = (V', E'), k)$

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if |E'| = ∅ then return true
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Def. An problem is fixed parameterized tractable (FPT) with respect to a parameter k, if it can be solved in $f(k) \cdot \text{poly}(n)$ time, where n is the size of its input and $\text{poly}(n) = \bigcup_{t=0}^{\infty} O(n^t)$.

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- Vertex cover is fixed parameterized tractable with respect to the size k of the optimum solution.

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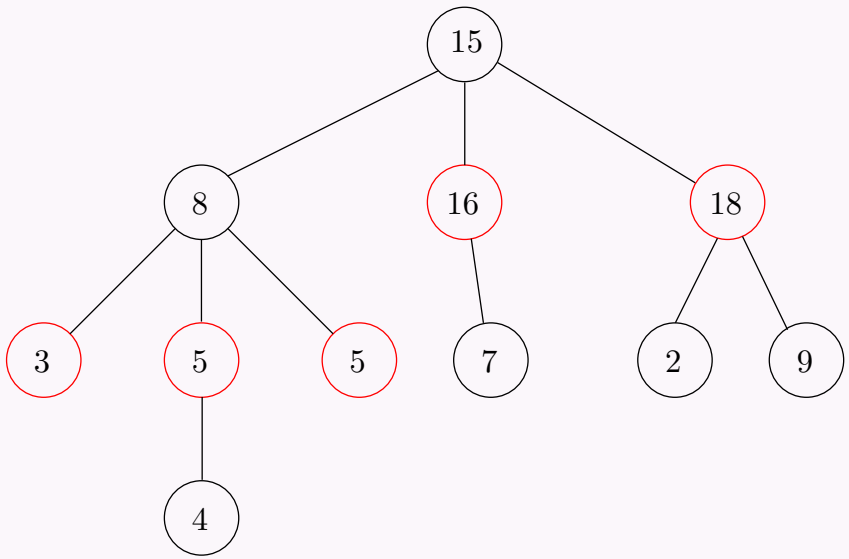

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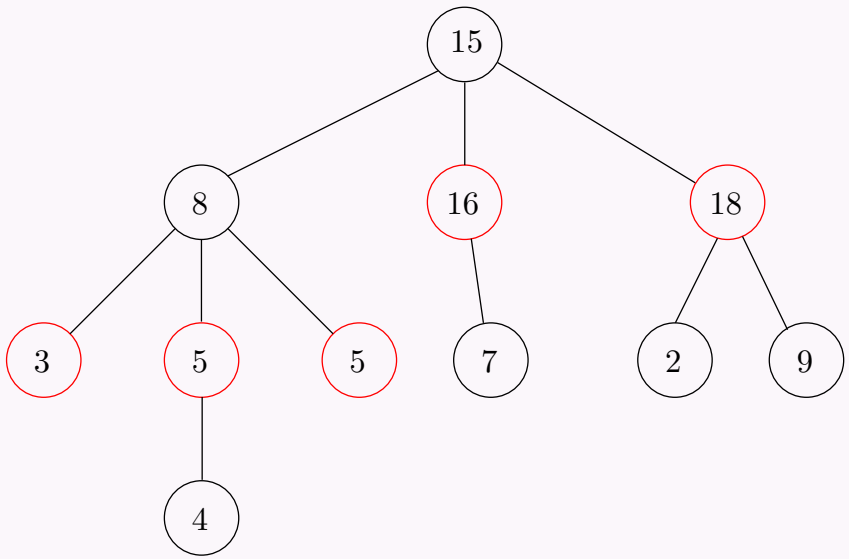

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- Reason why many problems can be solved using DP on trees: the child-trees of a vertex i are only connected through i.

Bounded-Tree-Width Graphs

Def. A tree decomposition of a graph $G = (V, E)$ consists of

- a tree T with node set U, and
- a subset $V_t \subseteq V$ for every $t \in U$, which we call the bag for t,

satisfying the following properties:

- (Vertex Coverage) Every $v \in V$ appears in at least one bag.
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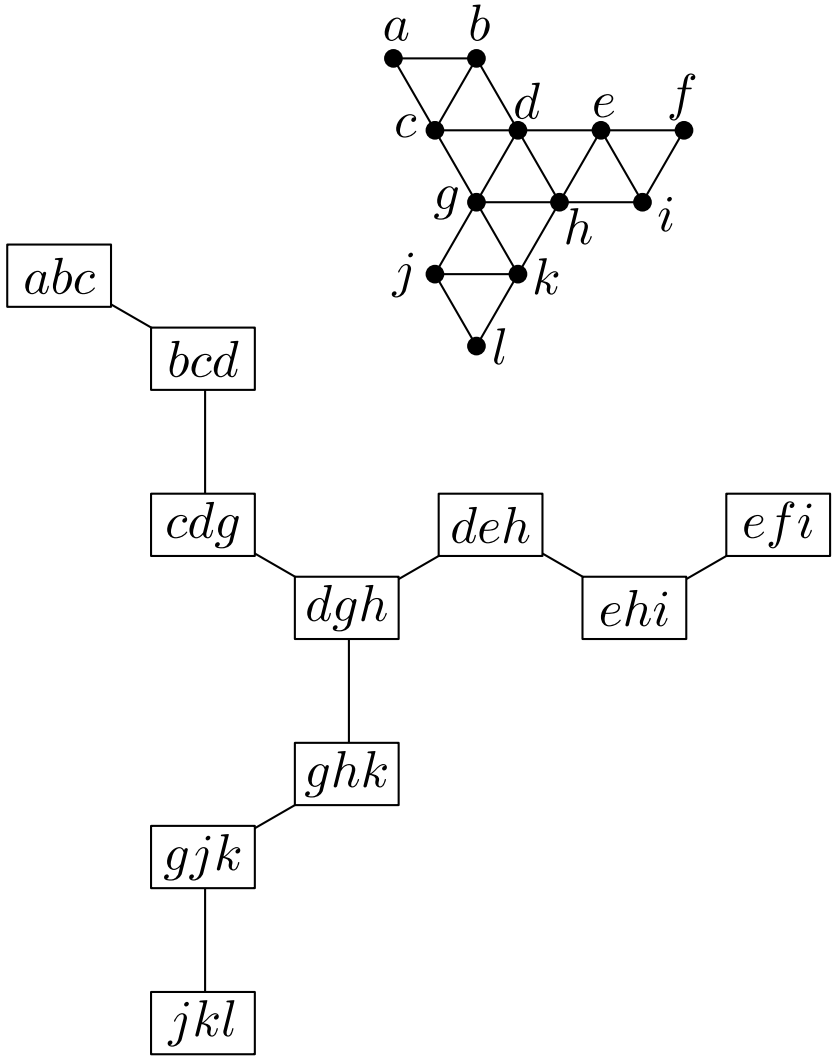

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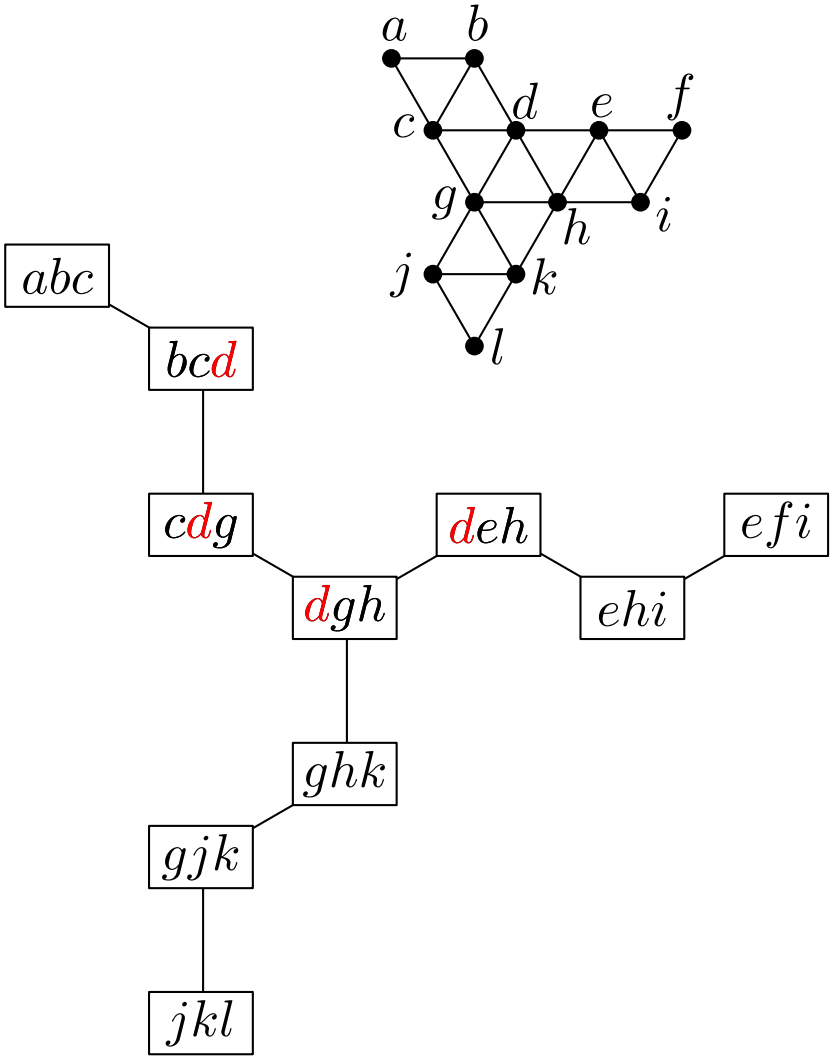

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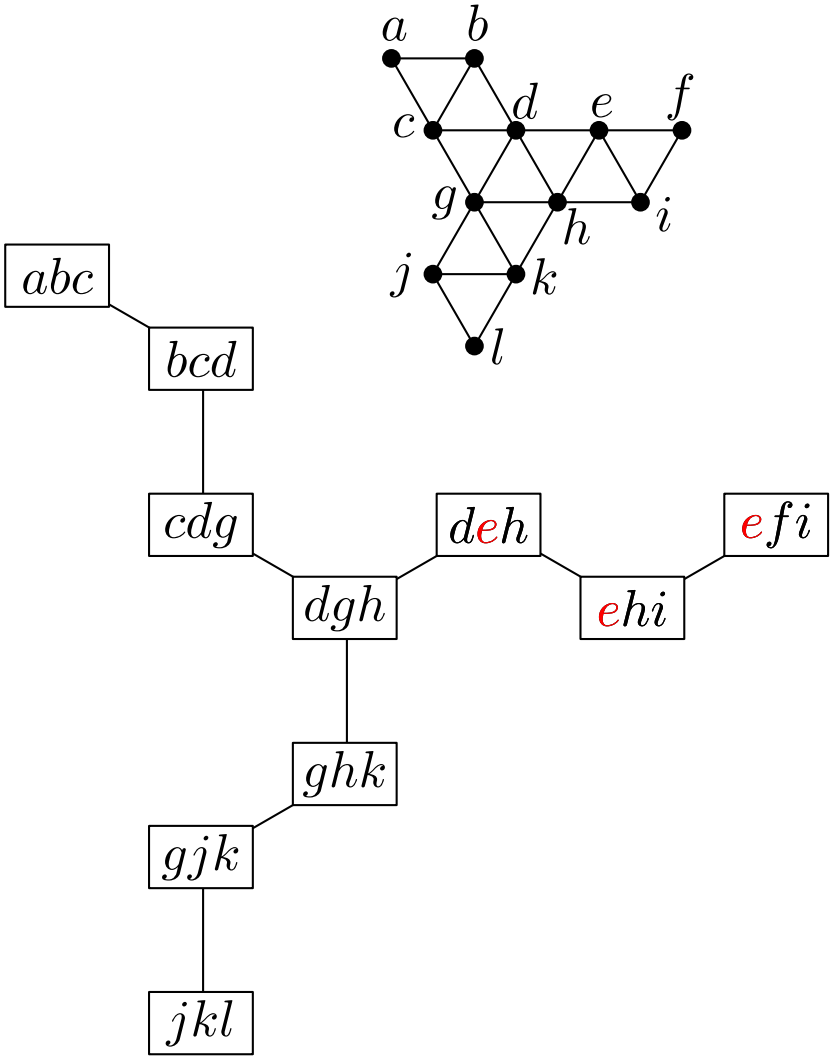

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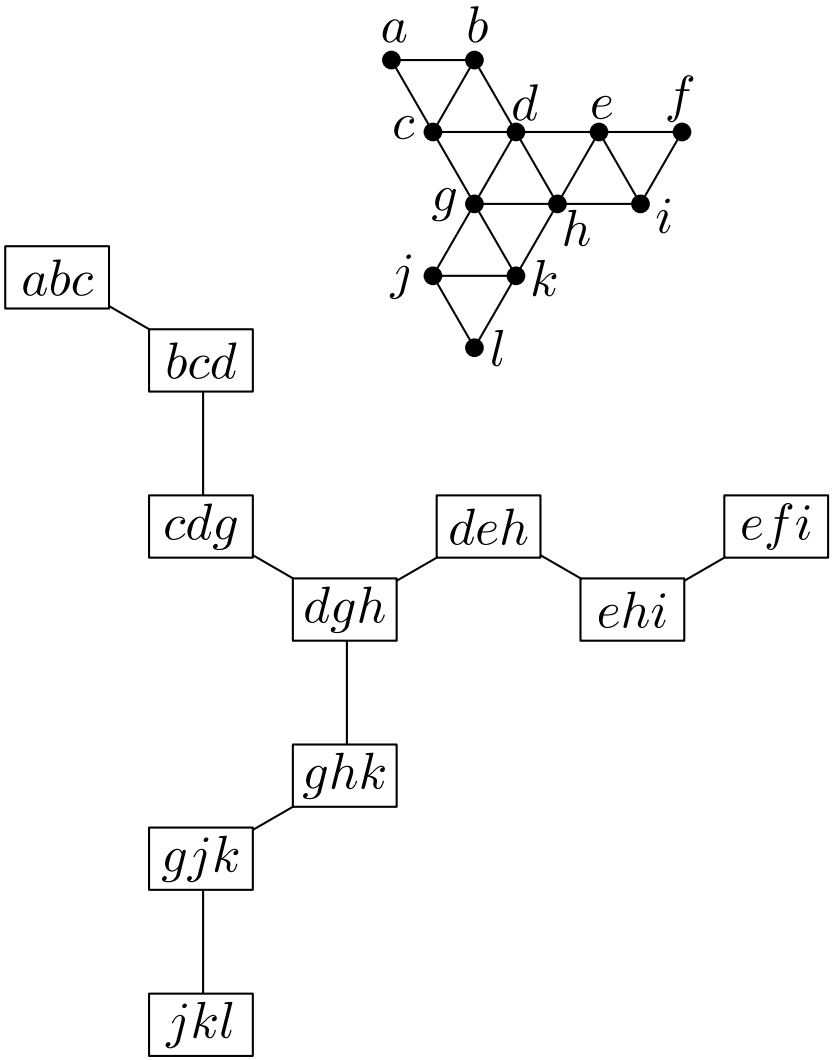

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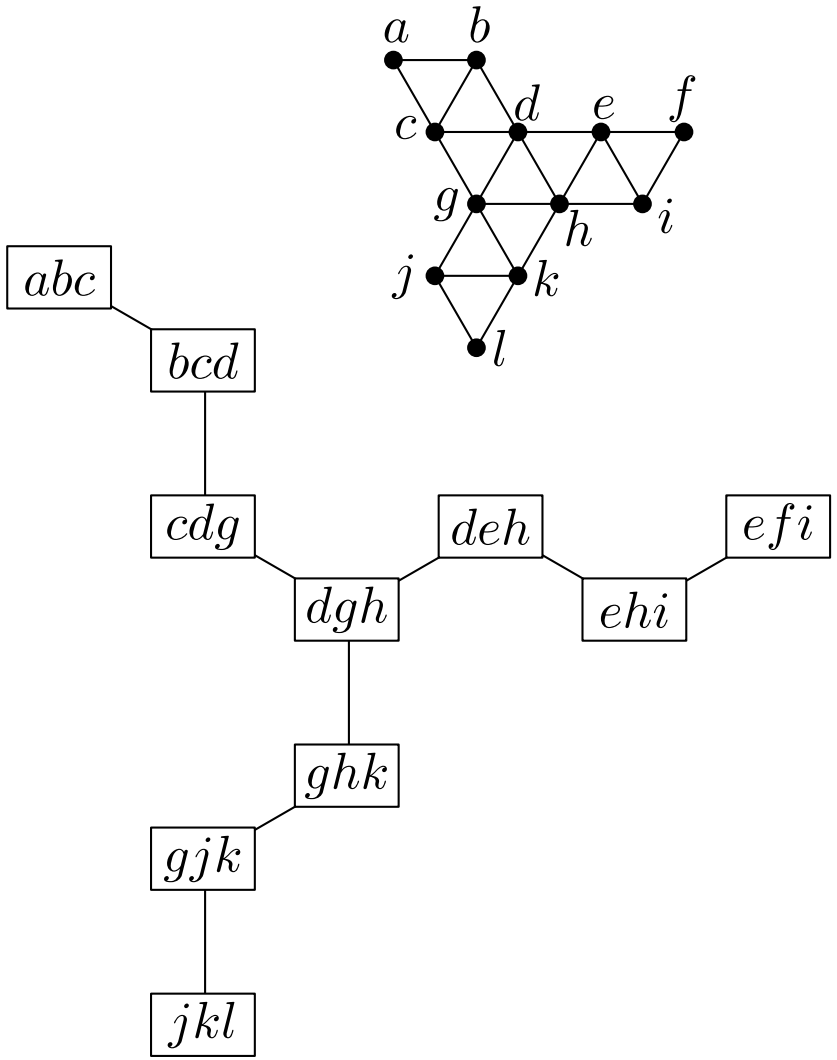

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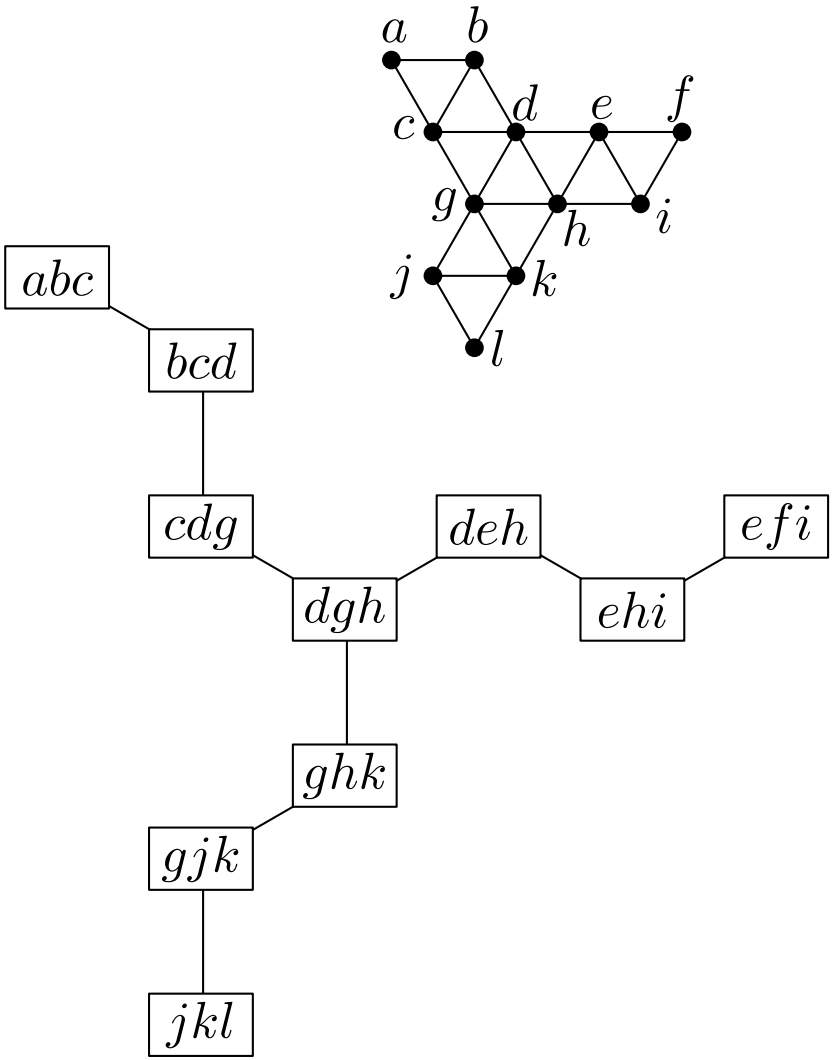
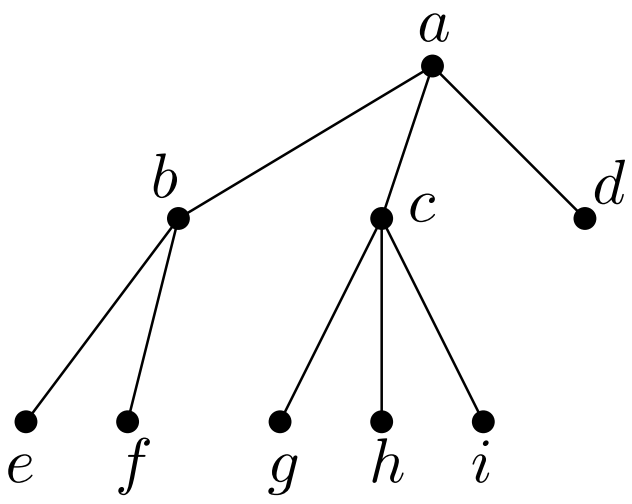
- The graph on the top right has tree-width 2.



Obs. A (non-empty) tree has tree-width 1.



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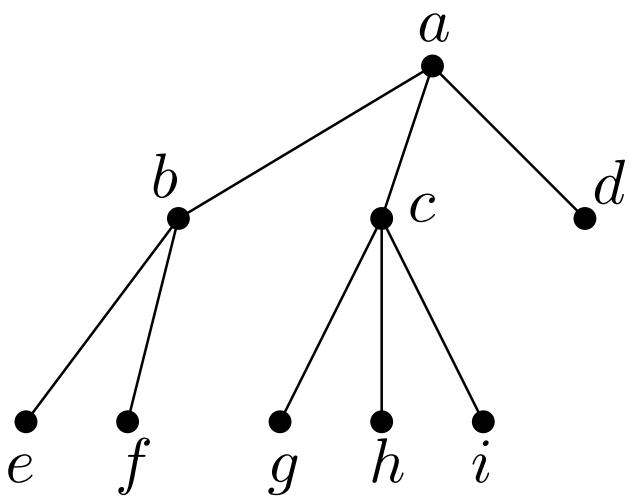

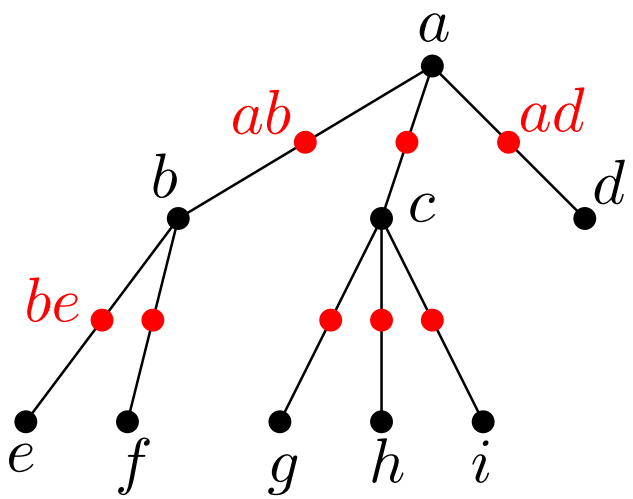

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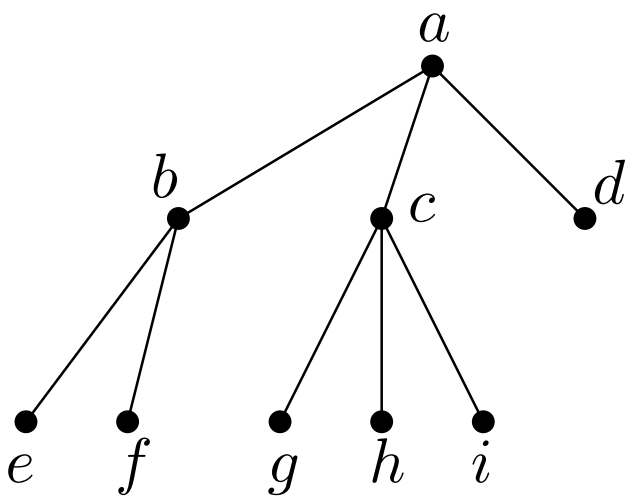

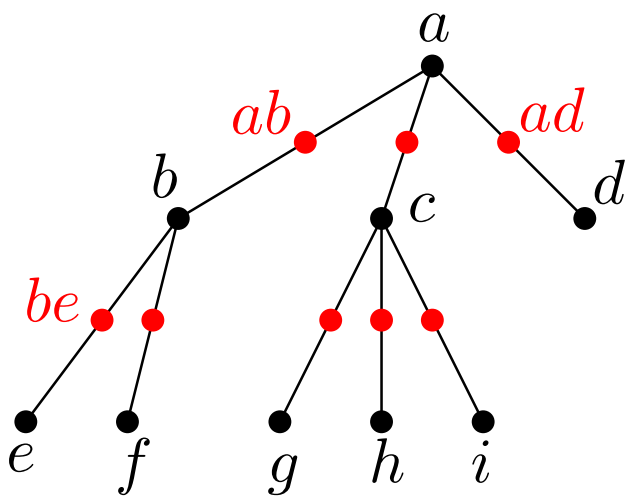

Lemma A graph has tree-width 1 if and only if it is a (non-empty) forest.

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- given $G = (V, E)$, a tree-decomposition $(T, (V_t)_{t \in U})$ of G with tree-width tw.
- vertex weights $w \in \mathbb{R}^V_{>0}$.
- find an independent set S of G with the maximum total weight.

Assumption: every node in T has at most 2 children. Moreover, every internal nodes in T is one of the following types:

- Splitter: a node t with two children t' and t'', $V_t = V_{t'} = V_{t''}$
- Insertion node: a node t with one child t', $\exists u \notin V_t, V_{t'} = V_t \cup \{u\}$
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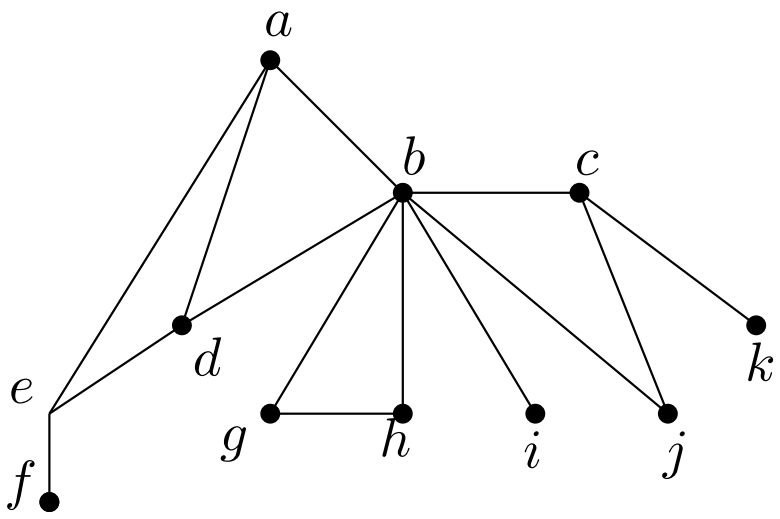

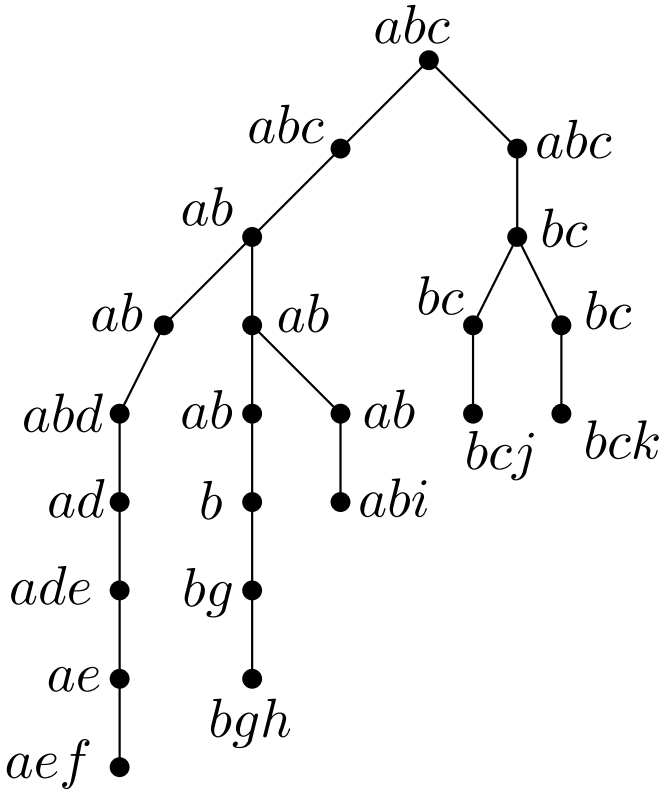

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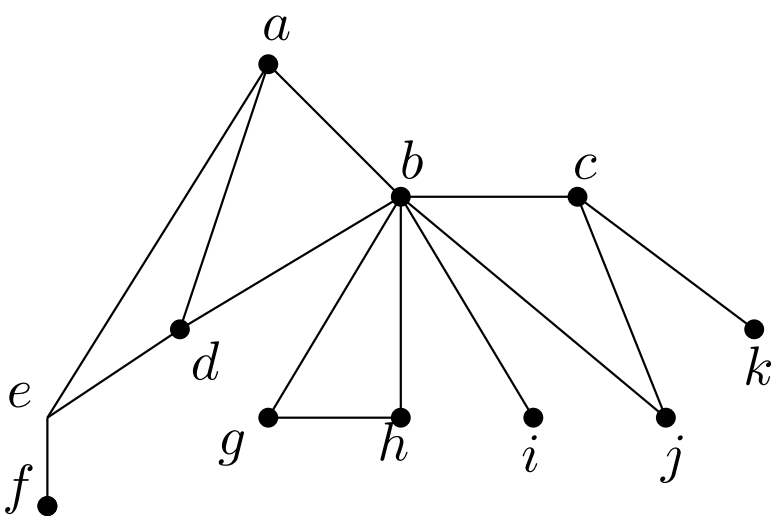
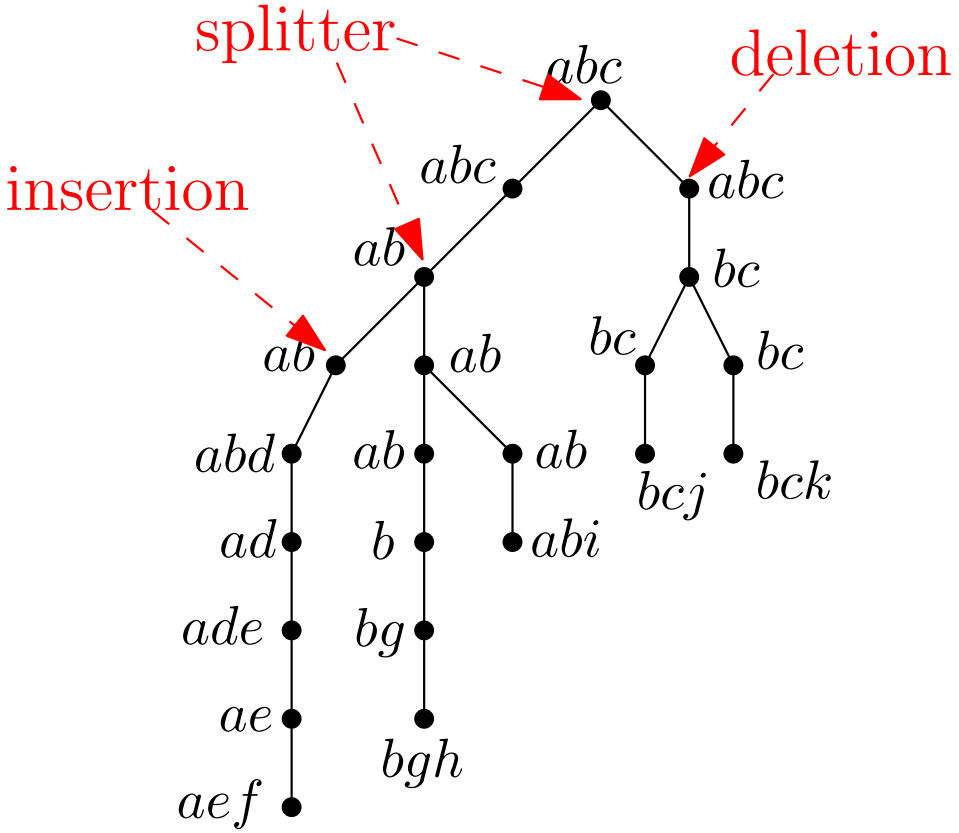
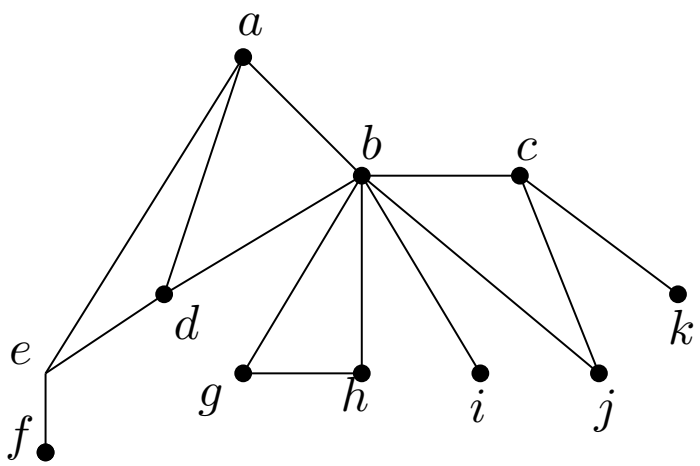
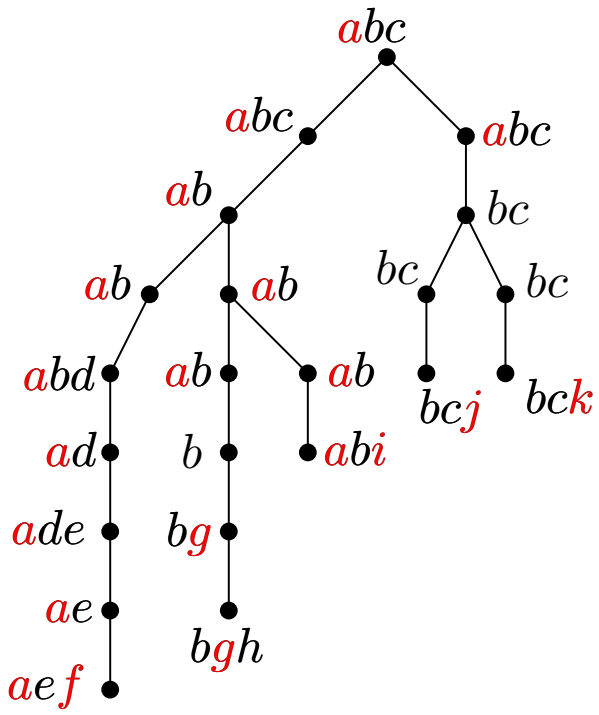




Def. Given a graph $G = (V, E)$, and a tree decomposition $(T, (V_t)_{t \in U})$, a valid labeling of T is a vector $(R_t)_{t \in U}$ of sets, one for every node t, such that the following conditions hold.

- $R_t \subseteq V_t, \forall t \in U$, and R_t is an independent set in G
- $R_t = R_{t'} = R_{t''}$ for a S-node t, and its two children t', t''.
- $R_{t'} \setminus \{u\} = R_t$ for an I-node t and its child t' with $V_{t'} = V_t \cup \{u\}$.
- $R_{t'} = R_t \setminus \{u\}$ for a D-node t and its child t' with $V_{t'} = V_t \setminus \{u\}$.





Lemma If S is an IS of G, then $(R_t := S \cap V_t)_{t \in U}$ is a valid labeling.

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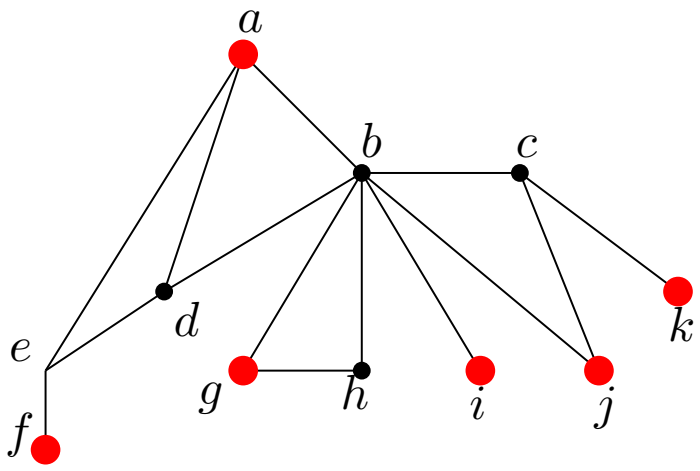

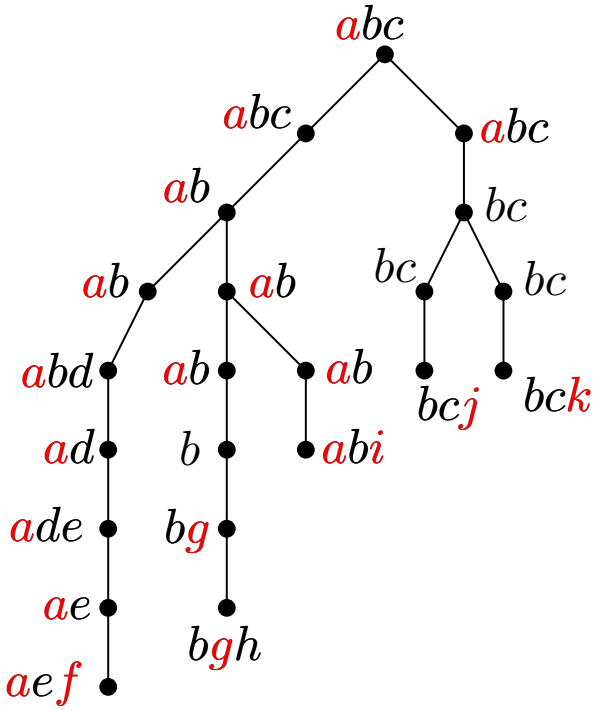

Lemma If S is an IS of G, then $(R_t := S \cap V_t)_{t \in U}$ is a valid labeling.

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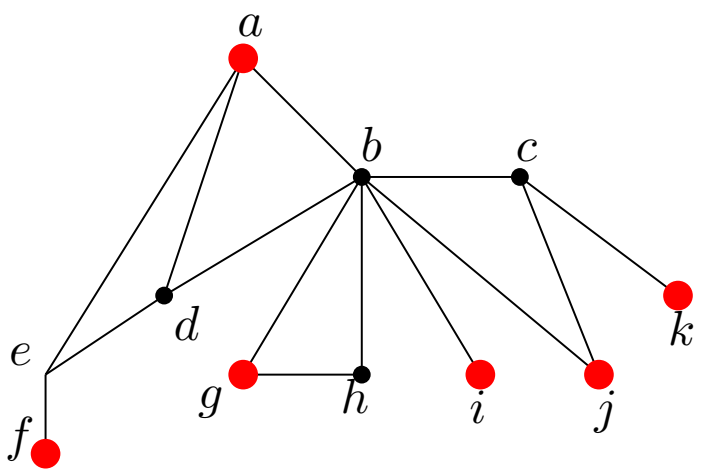
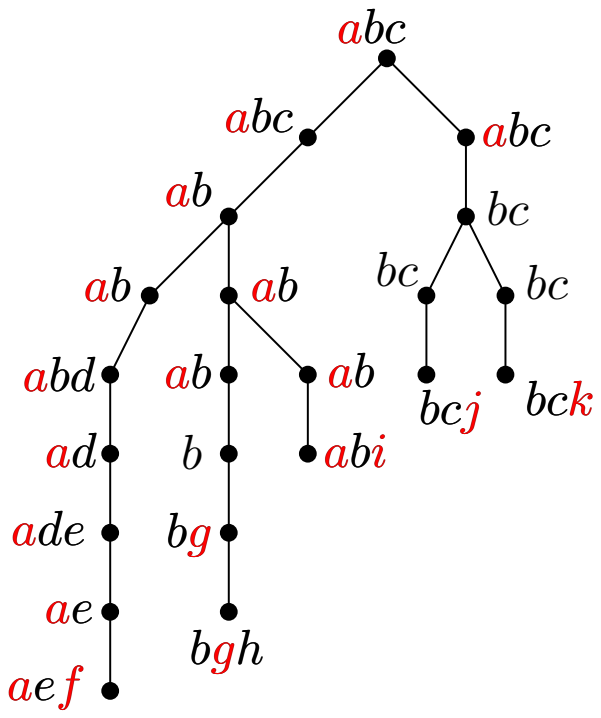
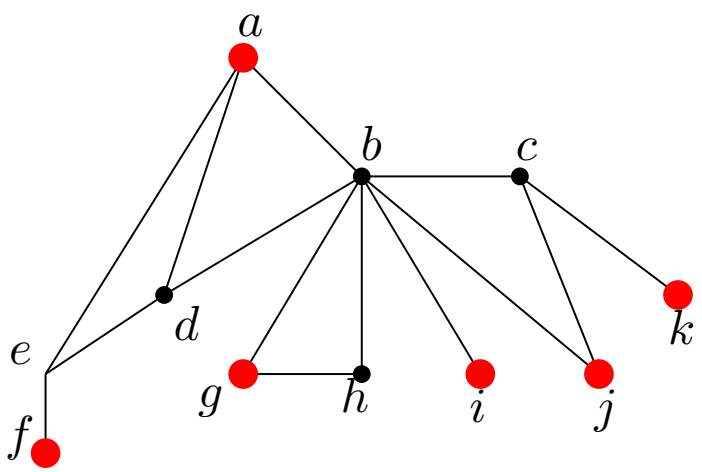
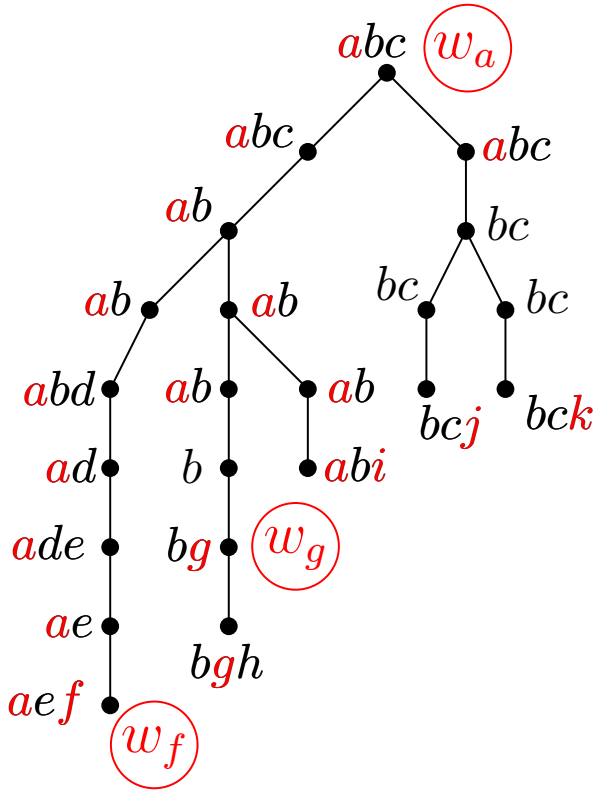




- Therefore, there is an one-to-one mapping between independent sets and valid labelings.

- For every $t \in U$, every $R \subseteq V_t$ that is an IS in G (we call R a label for t), we define a weight $w_t(R)$.
- for the root t and a label R for t, $w_t(R) = \sum_{v \in R} w_r$.
- for an insertion node t with the child t' such that $V_{t'} = V_t \cup \{u\}$, a label R for t', we define $w_{t'}(R) = w_u$ if $u \in R$ and 0 otherwise.
- For all other cases, the weights are defined as 0





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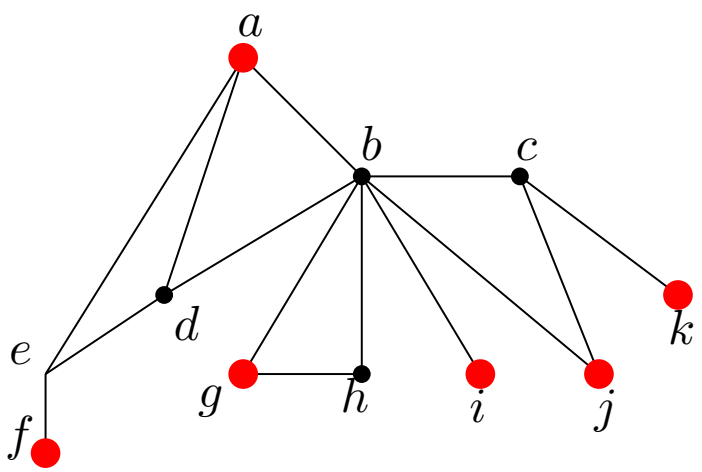

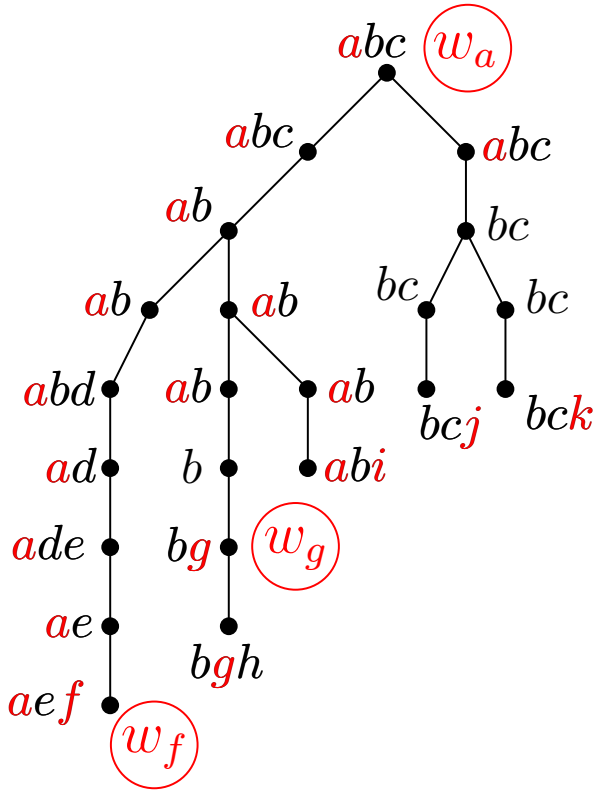

- Problem: find a valid labeling for T with maximum weight

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Dynamic Programming

- $\forall t \in U$, a label R for t: let $f(t, R)$ be the maximum weight of a valid (partial) labeling for the sub-tree of T rooted at t.

$$f(t,R) := \begin{cases} w_t(R) & t \text{ is a leaf} \\ w_t(R) + f(t', R) + f(t'', R) & \\ & t \text{ is an S-node with children } t' \text{ and } t'' \\ w_t(R) + \max\{f(t', R), f(t', R \cup \{u\})\} & \\ & t \text{ is I-node w. child } t',\ V_{t'} = V_t \cup \{u\} \\ w_t(R) + f(t', R \setminus \{u\}) & \\ & t \text{ is D-node w. child } t',\ V_{t'} = V_t \setminus \{u\} \end{cases}$$

- In I-node case, if $R \cup \{u\}$ is an invalid label, then $f(t, R \cup \{u\}) = -\infty$.

- The running time of the dynamic programming: $O\big(2^{\mathsf{tw}} \cdot \mathsf{tw} \cdot n\big)$.
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- This is an NP-hard problem.
- We can achieve a weaker goal: find a tree-decomposition of with at most $4\mathsf{tw}$ in time $f(\mathsf{tw}) \cdot \mathrm{poly}(n)$, where $f(\mathsf{tw})$ is a function of tw.
- If $\mathsf{tw} = O(1)$, the algorithm runs in polynomial time.
- The constant 4 is acceptable.

Outline

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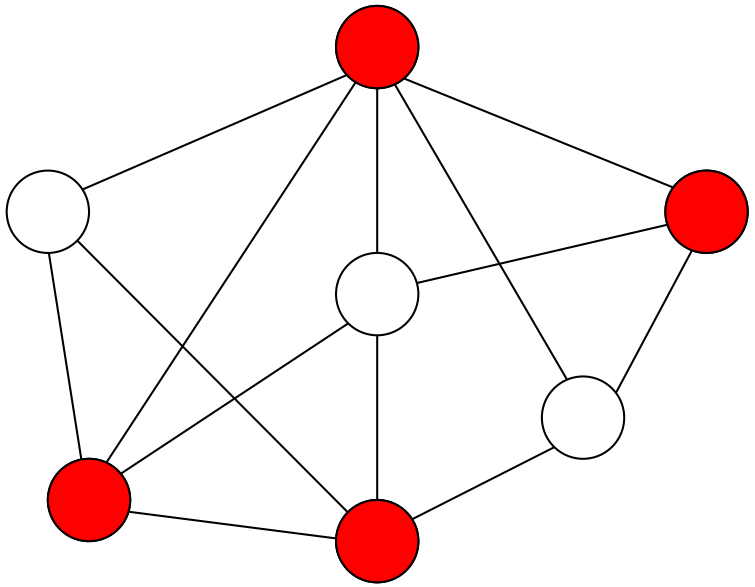
Vertex Cover Problem

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $C \subseteq V$ such that for every $(u, v) \in E$ then $u \in C$ or $v \in C$.



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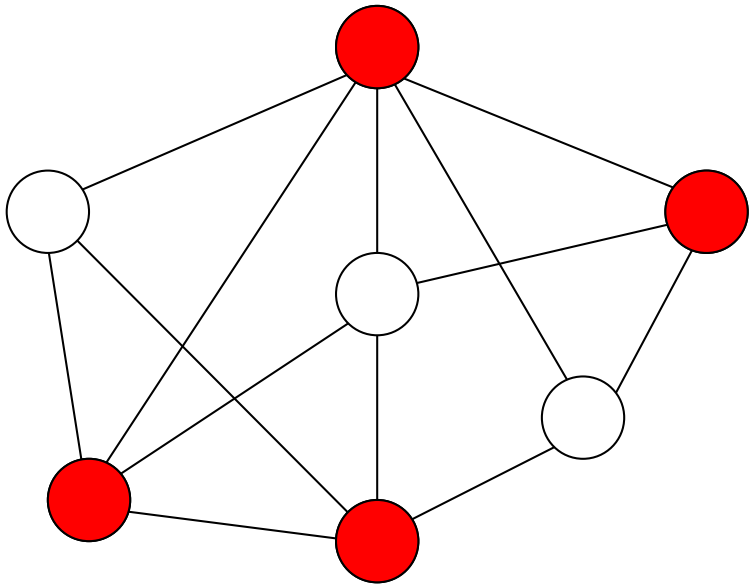

Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover C with minimum $|C|$

First Try: A "Natural" Greedy Algorithm

Natural Greedy Algorithm for Vertex-Cover

```
E' ← E, C ← ∅
while E' ≠ ∅ do
    let v be the vertex of the maximum degree in (V, E')
    C ← C ∪ {v},
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return C
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- The logarithmic factor is tight for this algorithm

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Theorem The algorithm is a 2-approximation algorithm for vertex-cover.

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- To cover E', the optimum solution needs $|E'|$ vertices □

Outline

Set Cover

Input: $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

Output: minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

Set Cover with Bounded Frequency f

Input: $U, |U| = n$: ground set

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Vertex Cover = Set Cover with Frequency 2

- edges $\Leftrightarrow$ elements
- vertices $\Leftrightarrow$ sets
- every edge (element) can be covered by 2 vertices (sets)

f-Approximation Algorithm for Set Cover with Frequency f

```
C ← ∅
while ⋃_{i∈C} S_i ≠ U do
    let e be any element in U \ ⋃_{i∈C} S_i
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```

Theorem The algorithm is a f-approximation algorithm.

f-Approximation Algorithm for Set Cover with Frequency f

1: $C \leftarrow \emptyset$
2: **while** $\bigcup_{i \in C} S_i \neq U$ **do**
3: let e be any element in $U \setminus \bigcup_{i \in C} S_i$
4: $C \leftarrow C \cup \{i \in [m] : e \in S_i\}$
5: **return** C

Theorem The algorithm is a f-approximation algorithm.

Proof.

- Let U' be the set of all elements e considered in Step 3
- Observation: no set S_i contains two elements in U'
- To cover U', the optimum solution needs $|U'|$ sets
- $C \leq f \cdot |U'|$ $\square$

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4: $C \leftarrow C \cup \{i\}, U' \leftarrow U' \setminus S_i$
5: **return** C

- g: minimum number of sets needed to cover U

Lemma Let $u_t, t \in \mathbb{Z}_{\geq 0}$ be the number of uncovered elements after t steps. Then for every $t \geq 1$, we have

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- $u_t \leq u_{t-1} - \frac{u_{t-1}}{g} = \left(1 - \frac{1}{g}\right) u_{t-1}$. $\square$

Proof of $(\ln n + 1)$-approximation.

- Let $t = \lceil g \cdot \ln n \rceil$. $u_0 = n$. Then
$$u_t \leq \left(1 - \frac{1}{g}\right)^{g \cdot \ln n} \cdot n < e^{-\ln n} \cdot n = n \cdot \frac{1}{n} = 1.$$
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- A more careful analysis gives a H_n-approximation, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the n-th harmonic number.
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$(1-c)\ln n$-hardness for any $c = \Omega(1)$

Let $c > 0$ be any constant. There is no polynomial-time $(1-c)\ln n$-approximation algorithm for set-cover, unless

- NP $\subseteq$ quasi-poly-time, [Lund, Yannakakis 1994; Feige 1998]
- P $=$ NP. [Dinur, Steuer 2014]

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- set cover: use smallest number of sets to cover all elements.
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Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

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- Motivation: you have budget W, and want to buy a subset of items of maximum total value

Greedy Algorithm

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- Bad example: $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$.

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```

- Bad example: $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$.
- Optimum takes item 2 and greedy takes item 1.

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W' - w_i] + v_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

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Q: Is this a polynomial time?

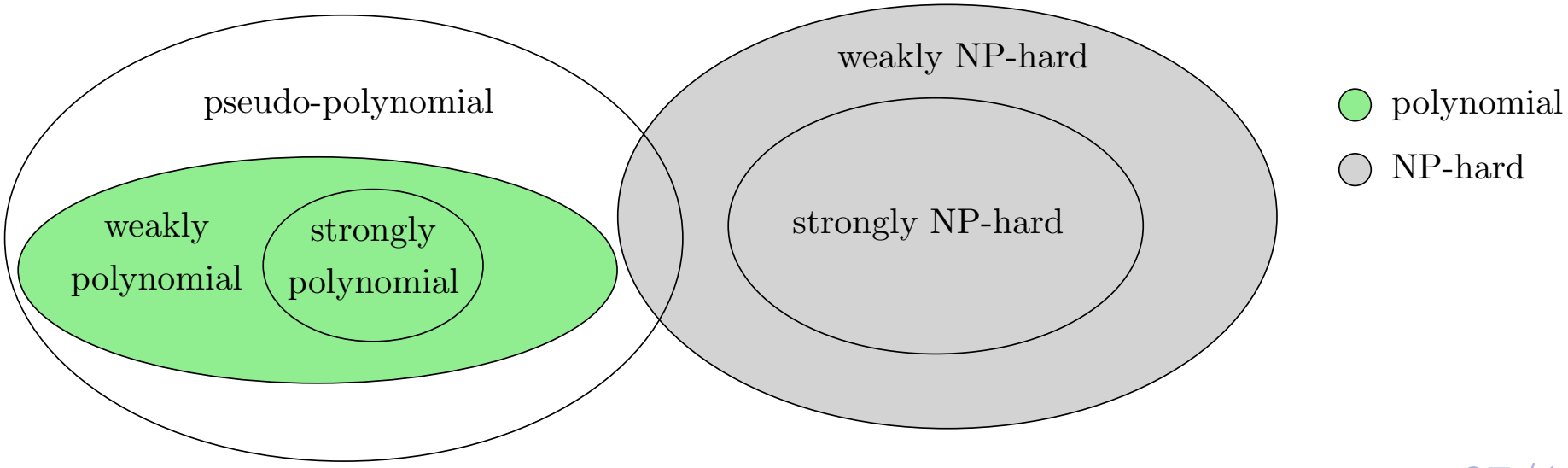
A: No.

- The input size is polynomial in n and $\log W$; running time is polynomial in n and W.
- The running time is pseudo-polynomial.

- n: number of integers W: maximum value of all integers

- pseudo-polynomial time: $\mathrm{poly}(n, W)$ (e.g., DP for Knapsack)
- weakly polynomial time: $\mathrm{poly}(n, \log W)$ (e.g., Euclidean Algorithm for Greatest Common Divisor)
- strongly polynomial time: $\mathrm{poly}(n)$ time, assuming basic operations on integers taking $O(1)$ time (e.g., Kruskal's)

- weakly NP-hard: NP-hard to solve in time $\mathrm{poly}(n, \log W)$
- strongly NP-hard: NP-hard even if $W = \mathrm{poly}(n)$



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- If we make weights upper bounded by $\mathrm{poly}(n)$, then pseudo-polynomial time becomes polynomial time
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weight budget constraint	:	hard
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- In the DP, we use values as parameters

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- Output A times the largest V' such that $f[n, V'] \leq W$.

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Theorem There is a $(1+\epsilon)$-approximation for the knapsack problem in time $O(\frac{n^3}{\epsilon})$.

Def. A polynomial-time approximation scheme (PTAS) is a family of algorithms A_ϵ, where A_ϵ for every $\epsilon > 0$ is a (polynomial-time) $(1 \pm \epsilon)$-approximation algorithm.

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- Vertex cover? Maximum independent set?

Outline

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Input: n jobs index as $[n]$

each job $j \in [n]$ has a processing time $p_j \in \mathbb{Z}_{>0}$

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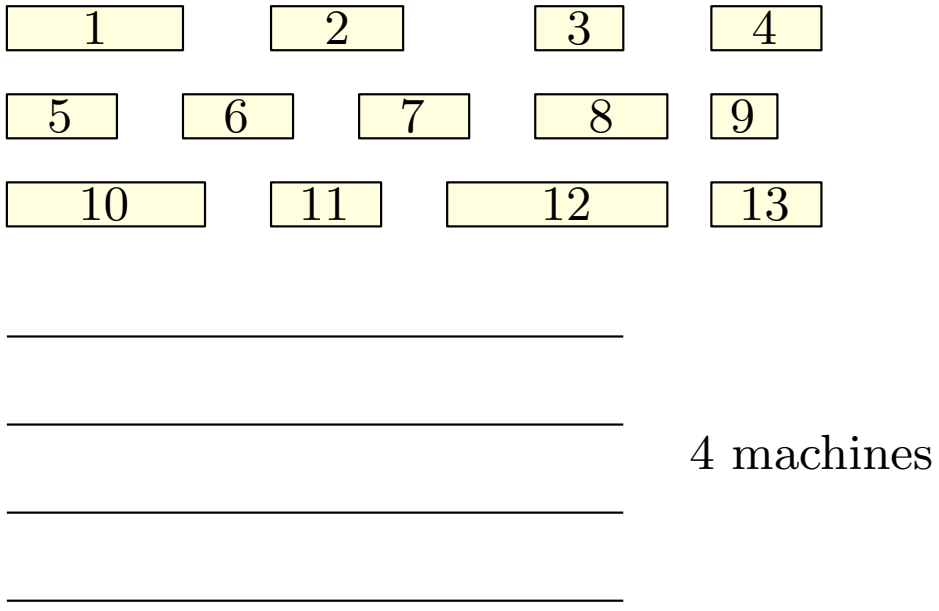

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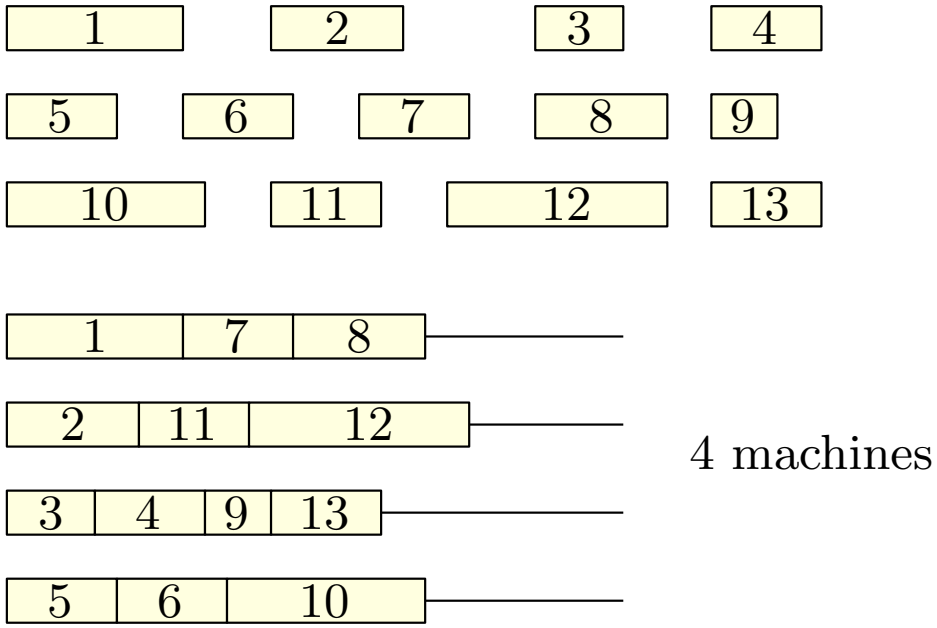

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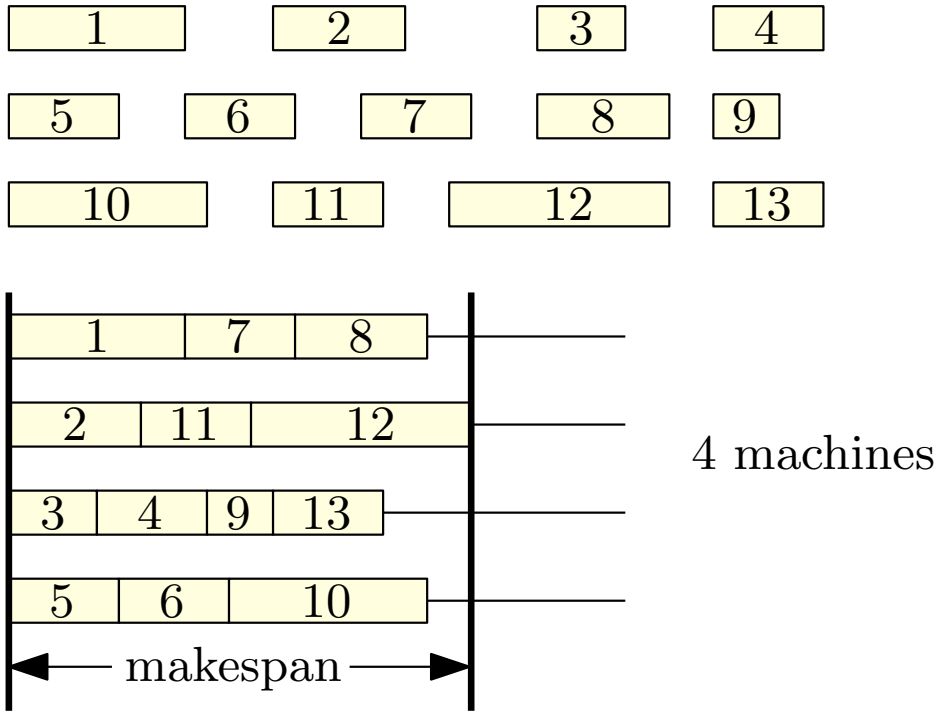

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- Goal: find a path from $(0, \cdots, 0)$ to $(n_1, \cdots, n_k)$ of at most m edges, so as to minimize the maximum weight on the path.

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  - reducing instance $(b_1, \cdots b_k)$ to $(a_1, \cdots, a_k)$ requires 1 machine of load $\sum_{i=1}^{k}(b_i - a_i)q_i$

- Goal: find a path from $(0, \cdots, 0)$ to $(n_1, \cdots, n_k)$ of at most m edges, so as to minimize the maximum weight on the path.

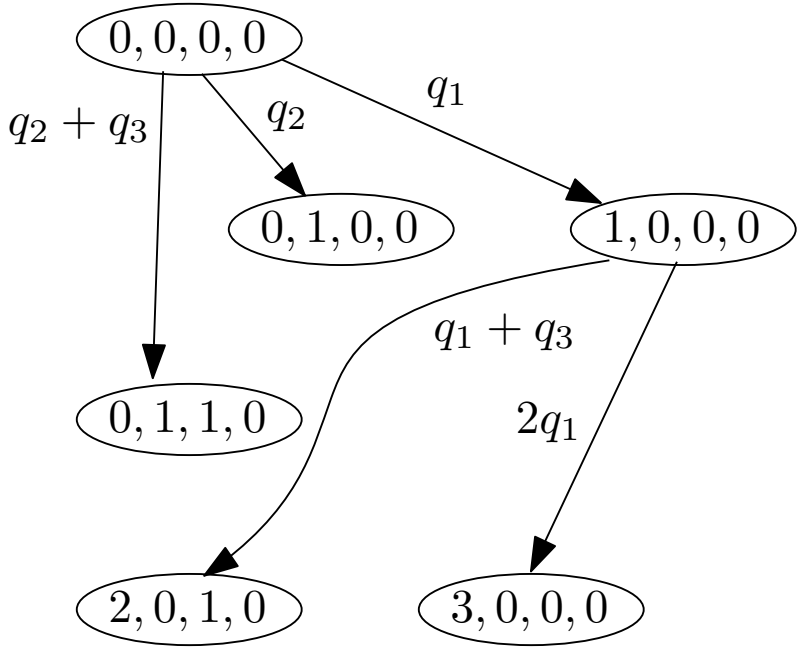
- problem can be solved in $O(m \cdot |E|)$ time using DP
- $O(m \cdot |E|) = O(m \cdot n^{2k}) = n^{O\left(\frac{1}{\epsilon} \cdot \log \frac{1}{\epsilon}\right)}$.

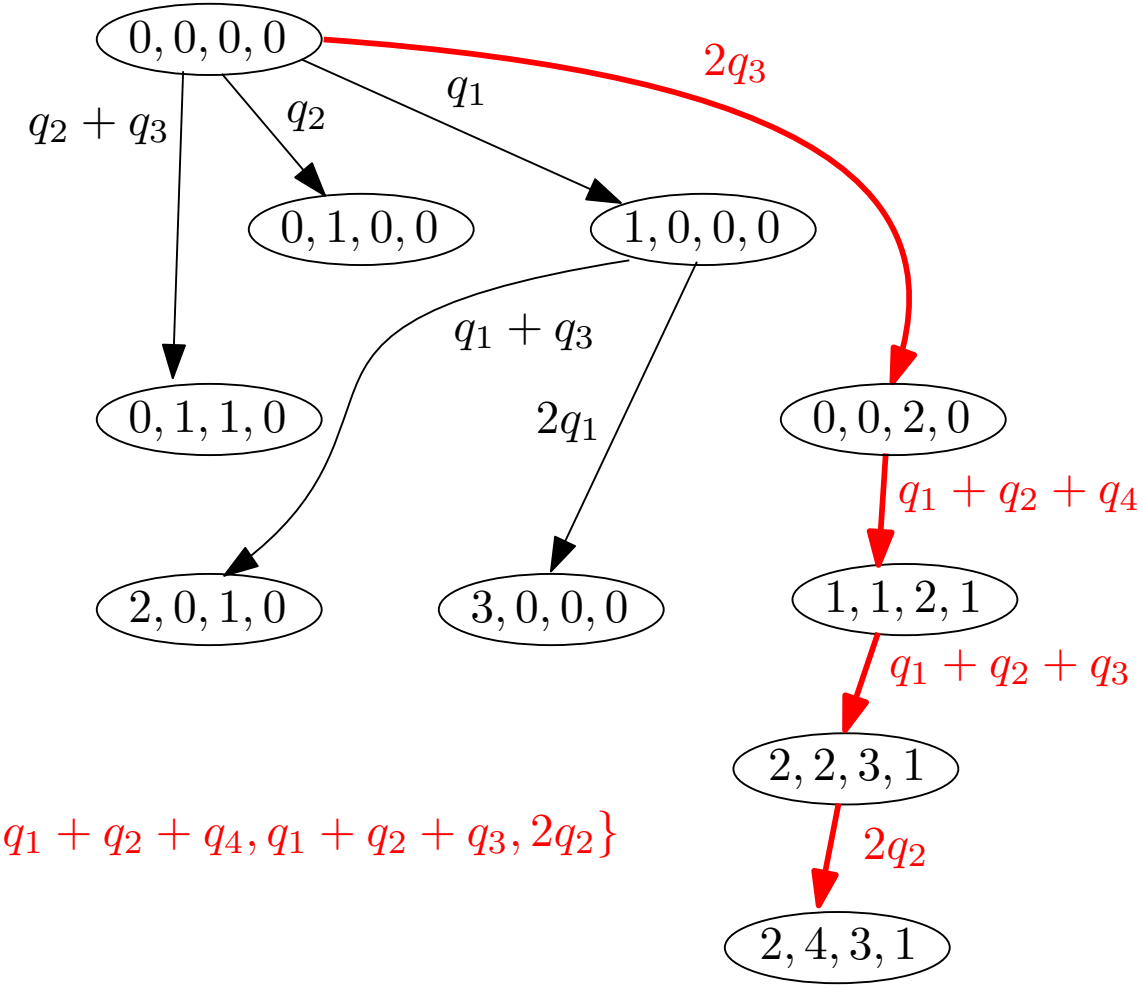
$0, 0, 0, 0$

$0, 1, 0, 0$ $\quad$ $1, 0, 0, 0$

$0, 1, 1, 0$

$2, 0, 1, 0$ $\quad$ $3, 0, 0, 0$


0, 0, 0, 0
$q_2 + q_3$
q_2
q_1
0, 1, 0, 0
1, 0, 0, 0
$q_1 + q_3$
0, 1, 1, 0
$2q_1$
2, 0, 1, 0
3, 0, 0, 0



$$\text{cost} = \max\{2q_3, q_1 + q_2 + q_4, q_1 + q_2 + q_3, 2q_2\}$$

Analysis of Algorithm for Big Jobs

- $\mathcal{I}_B$: instance $(p_j)_{j \in B}$ $\quad \mathrm{opt}_B$: its optimum makespan
- $\mathcal{I}'_B$: instance $(p'_j)_{j \in B}$ $\quad \mathrm{opt}'_B$: its optimum makespan

Analysis of Algorithm for Big Jobs

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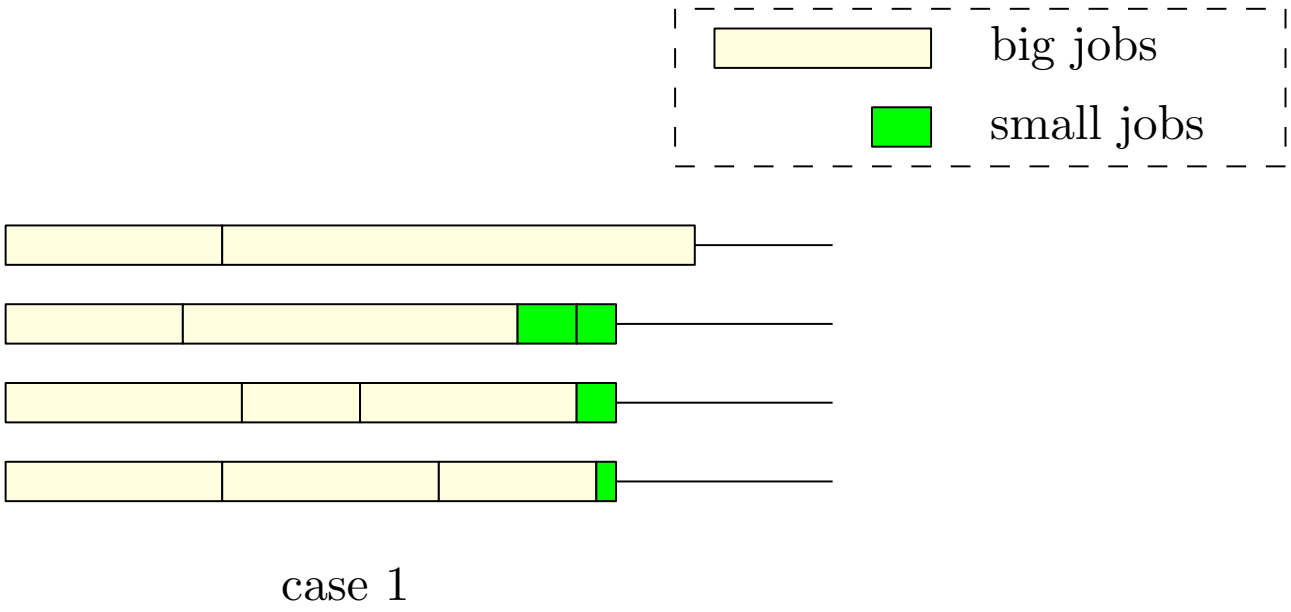
Adding small jobs to schedule

```
starting from the schedule for big jobs
for every small job j do
    add j to the machine with the smallest load
```

Analysis of the Final Algorithm



- Case 1: makespan is not increased by small jobs

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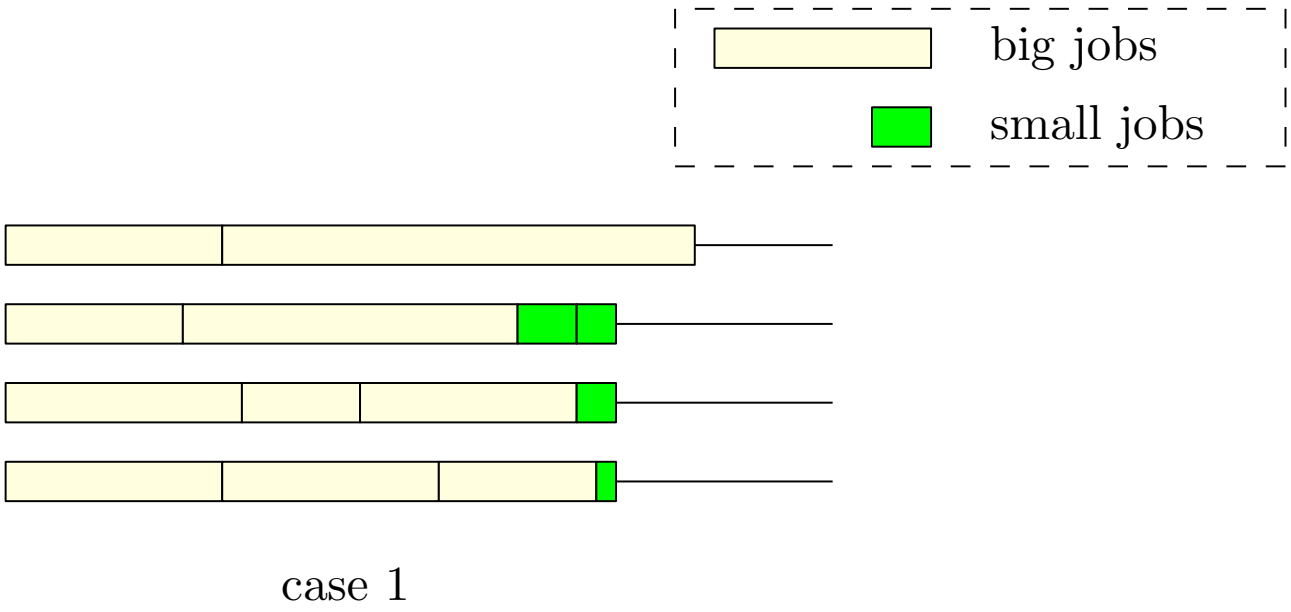

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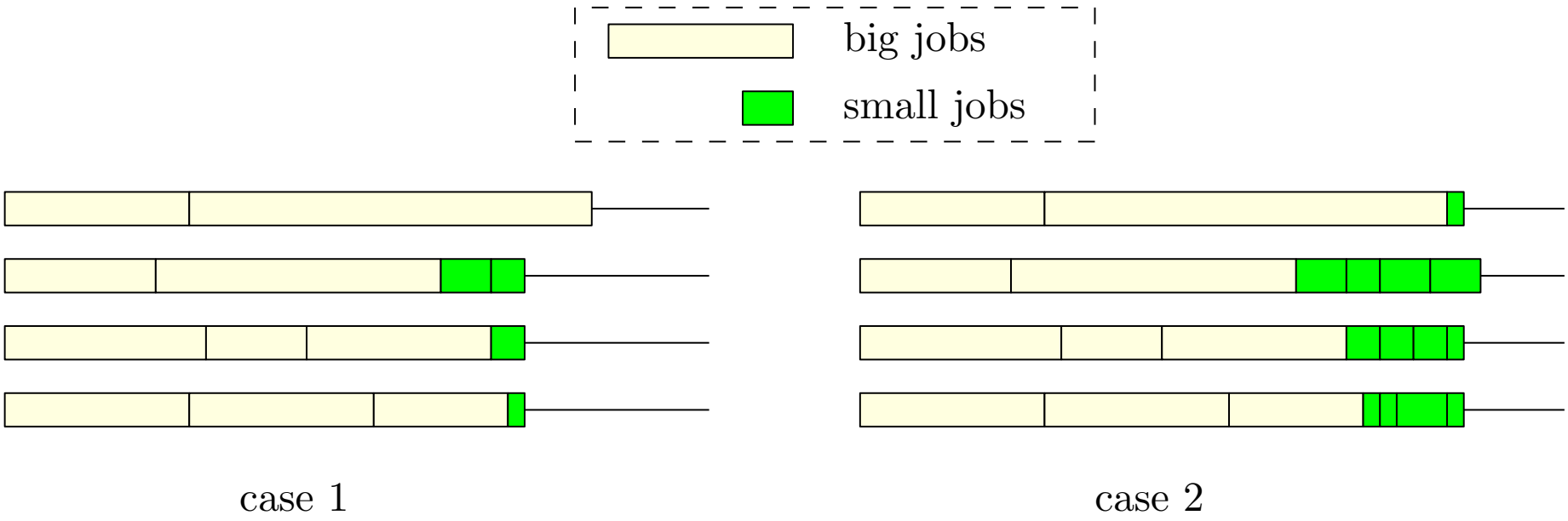

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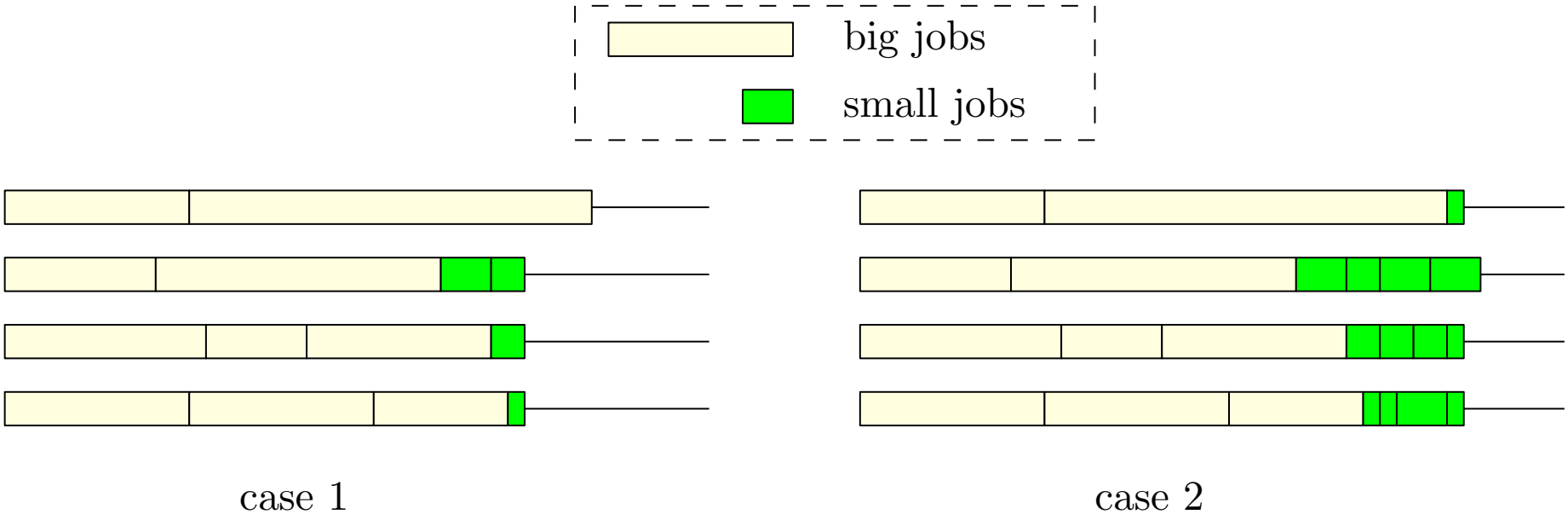

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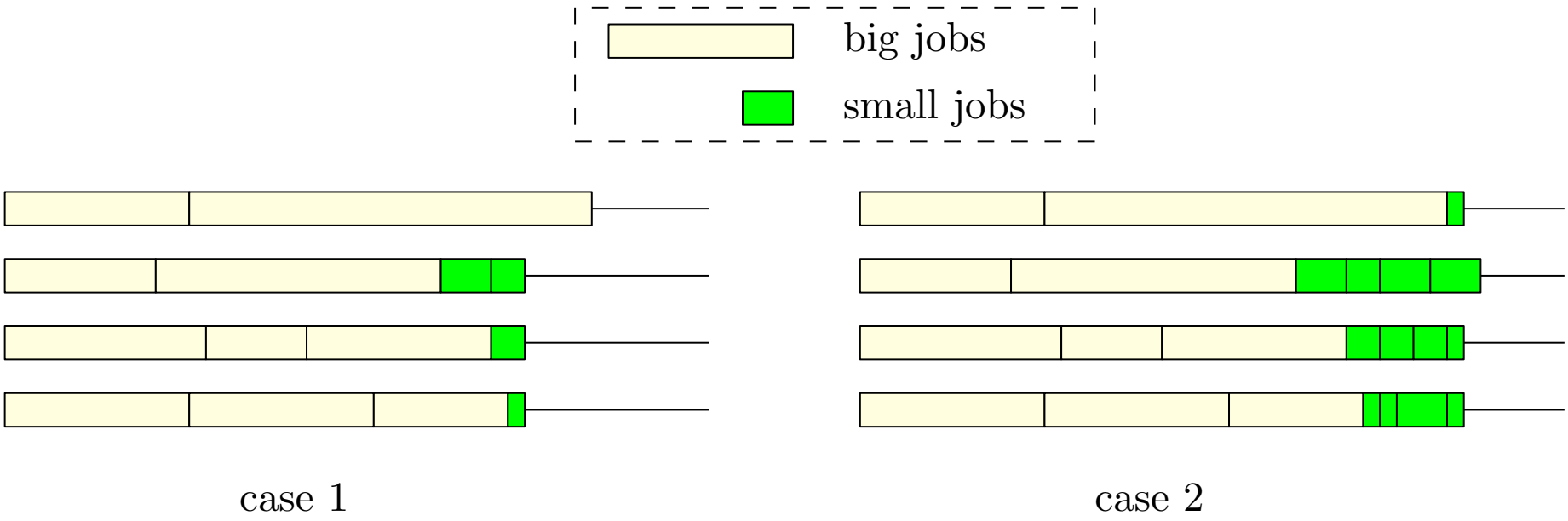

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$$\mathrm{alg} \leq \epsilon \cdot p_{\max} + \frac{1}{m}\sum_{j\in[n]} p_j \leq \epsilon \cdot \mathrm{opt} + \mathrm{opt} = (1+\epsilon)\cdot \mathrm{opt}.$$

Outline

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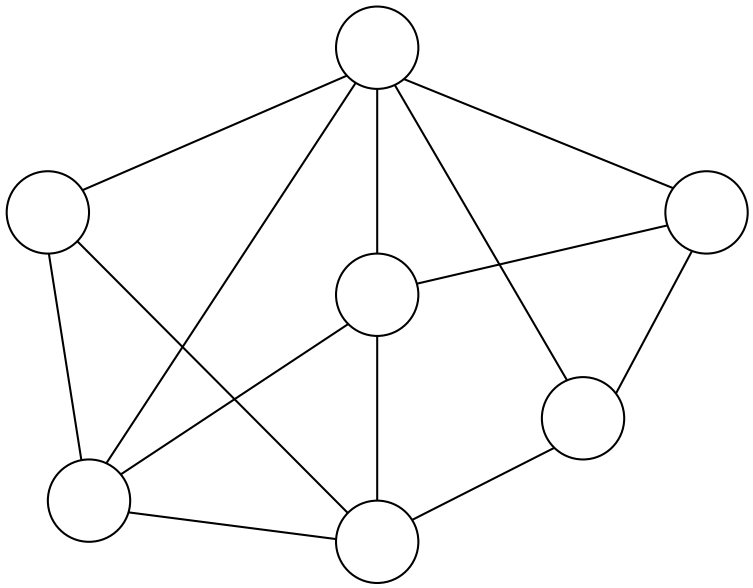
Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



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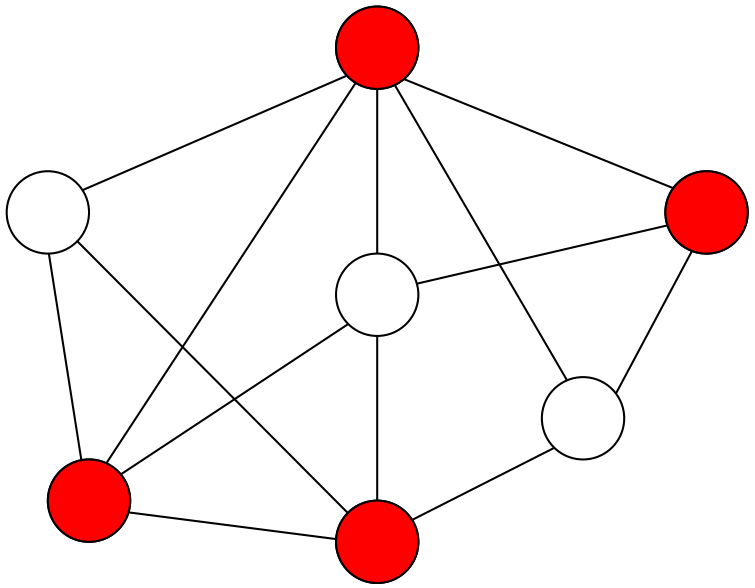

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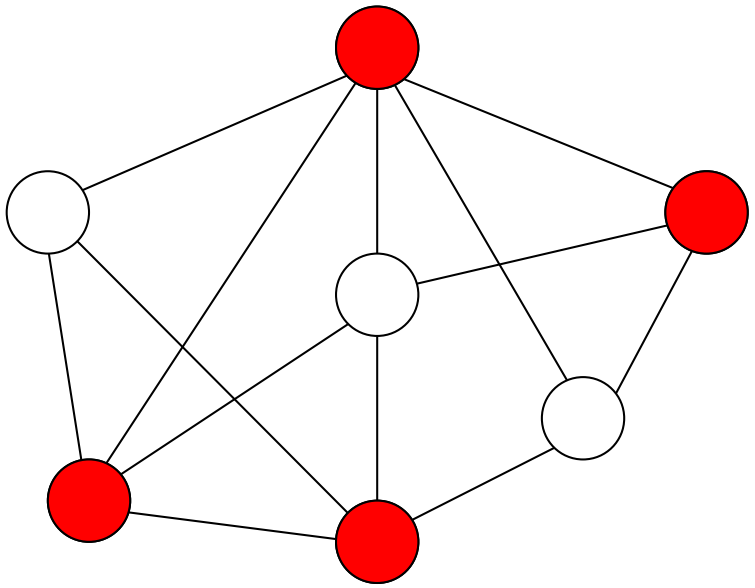

Weighted Vertex-Cover Problem

Input: $G = (V, E)$ with vertex weights $\{w_v\}_{v \in V}$

Output: a vertex cover S with minimum $\sum_{v \in S} w_v$

Integer Programming for Weighted Vertex Cover

- For every $v \in V$, let $x_v \in \{0, 1\}$ indicate whether we select v in the vertex cover S
- The integer programming for weighted vertex cover:

$$
\begin{aligned}
(\mathsf{IP}_{\mathsf{WVC}}) \qquad \min \quad & \sum_{v \in V} w_v x_v \qquad \text{s.t.} \\
& x_u + x_v \geq 1 \qquad \forall (u, v) \in E \\
& x_v \in \{0, 1\} \qquad \forall v \in V
\end{aligned}
$$

- $(\mathsf{IP}_{\mathsf{WVC}}) \Leftrightarrow$ weighted vertex cover
- Thus it is NP-hard to solve integer programmings in general

- Integer programming for WVC:

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- let IP = value of $(\mathsf{IP}_{\mathsf{WVC}})$, LP = value of $(\mathsf{LP}_{\mathsf{WVC}})$
- Then, LP $\leq$ IP

Algorithm for Weighted Vertex Cover

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1: Solving ($\mathsf{LP}_{\mathsf{WVC}}$) to obtain a solution $\{x_u^*\}_{u \in V}$

2:

3:

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$$\begin{aligned}\mathsf{cost}(S) &= \sum_{u \in S} w_u \leq \sum_{u \in S} w_u \cdot 2x_u^* = 2 \sum_{u \in S} w_u \cdot x_u^* \\ &\leq 2 \sum_{u \in V} w_u \cdot x_u^* = 2 \cdot \mathsf{LP}.\end{aligned}$$

□

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Proof.

$\mathsf{cost}(S) \leq 2 \cdot \mathsf{LP} \leq 2 \cdot \mathsf{IP} = 2 \cdot \mathsf{cost}(\text{best vertex cover})$. □

Outline

LP Relaxation

$$\min \quad \sum_{v \in V} w_v x_v$$

$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$

$$x_v \geq 0 \qquad \forall v \in V$$

Dual LP

$$\max \quad \sum_{e \in E} y_e$$

$$\sum_{e \in \delta(v)} y_e \leq w_v \qquad \forall v \in V$$

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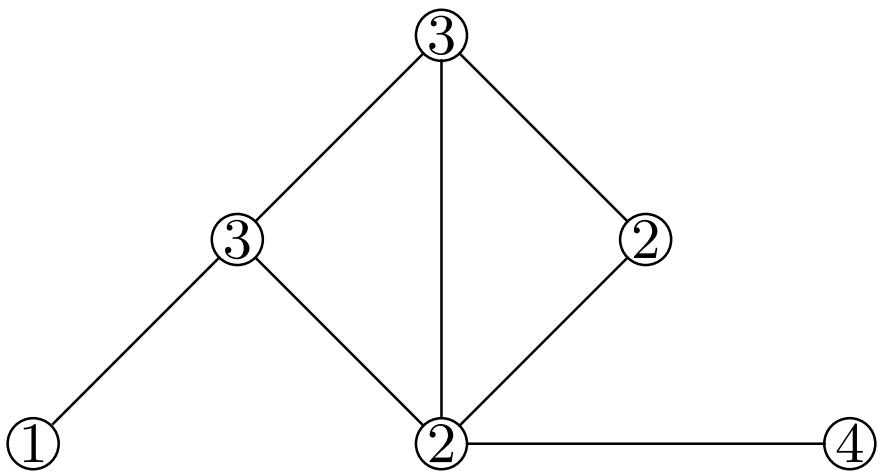
- Algorithm constructs integral primal solution x and dual solution y simultaneously.

Primal-Dual Algorithm for Weighted Vertex Cover Problem

1: $x \leftarrow 0, y \leftarrow 0$, all edges said to be uncovered
2: **while** there exists at least one uncovered edge **do**
3: take such an edge e arbitrarily
4: increasing y_e until the dual constraint for one end-vertex v of e becomes tight
5: $x_v \leftarrow 1$, claim all edges incident to v are covered
6: **return** x



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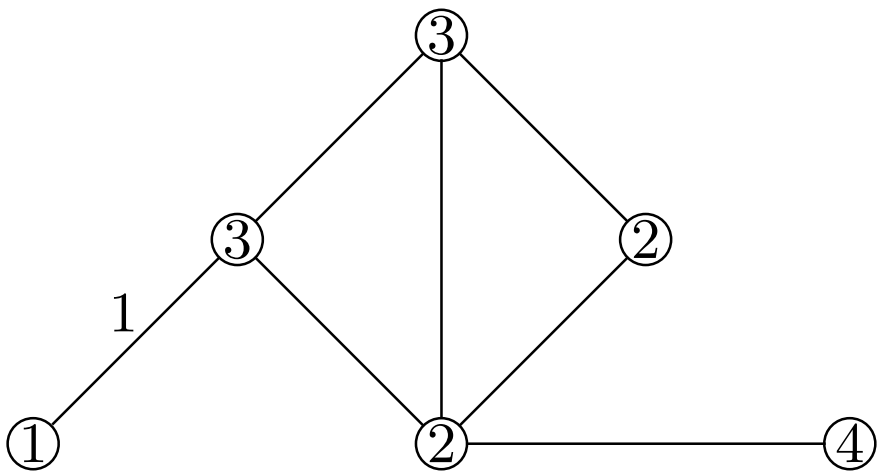

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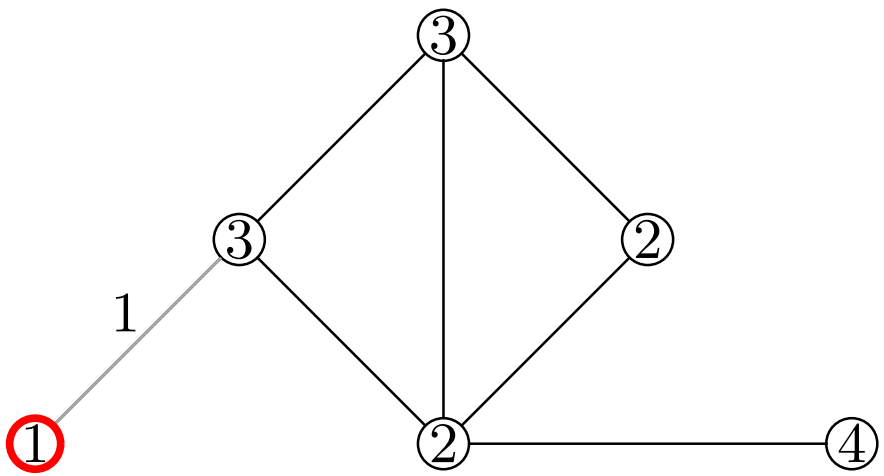

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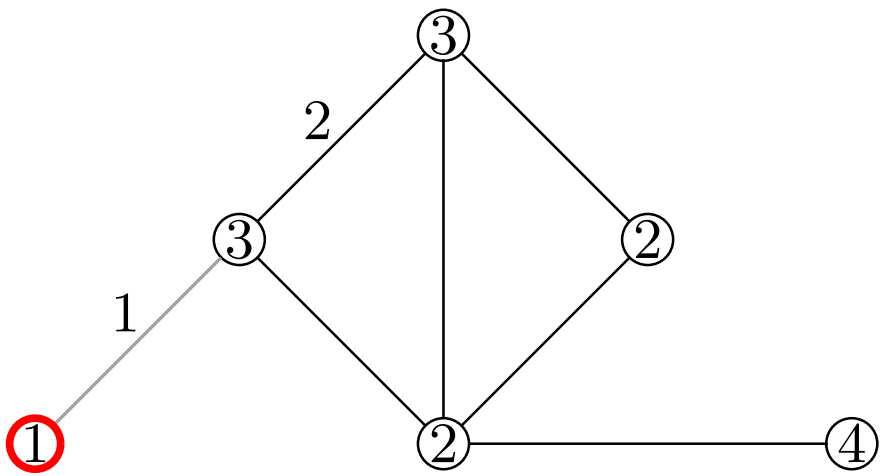

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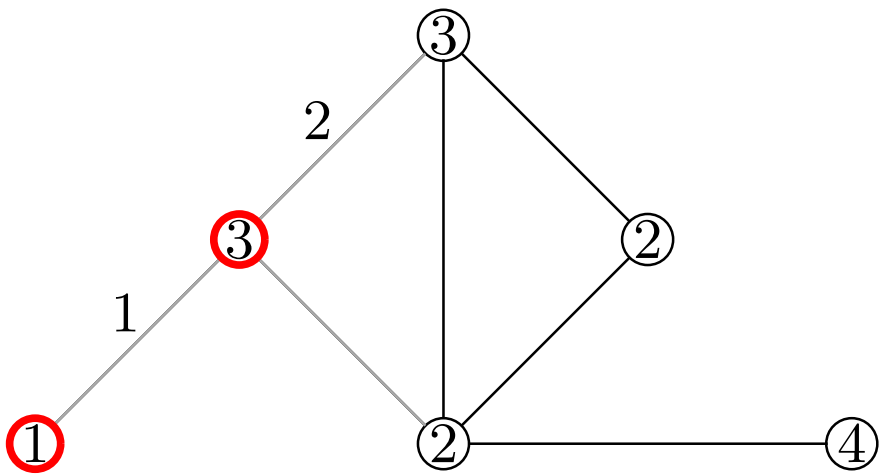

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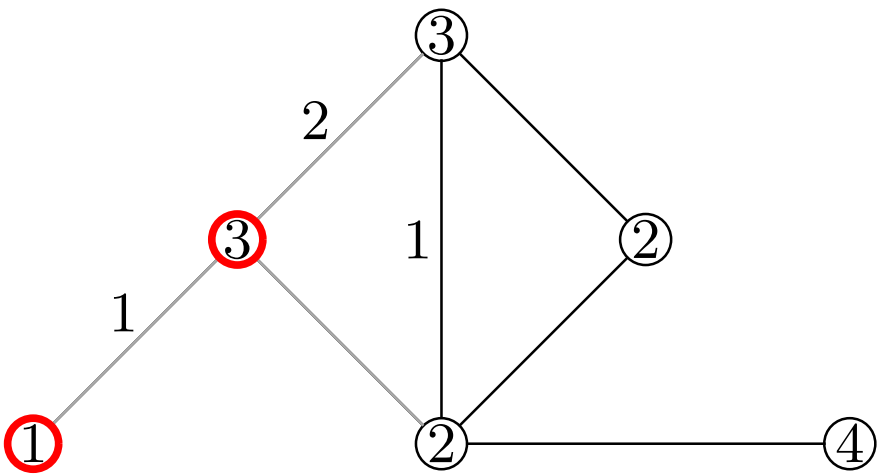

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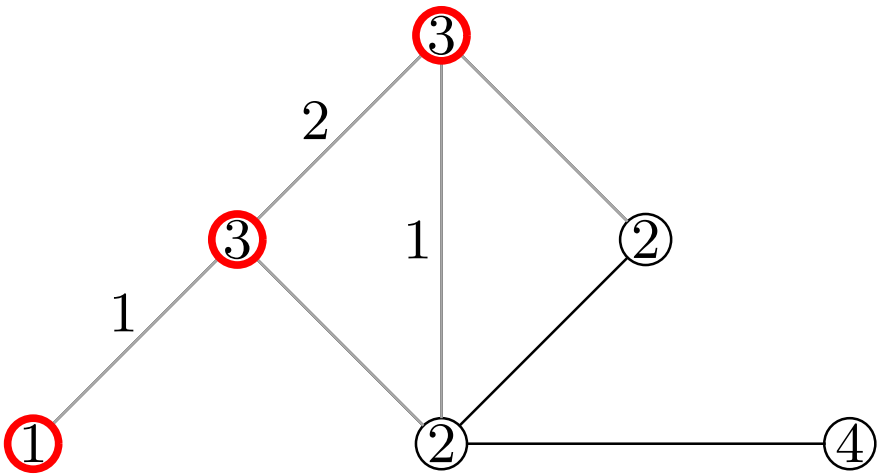

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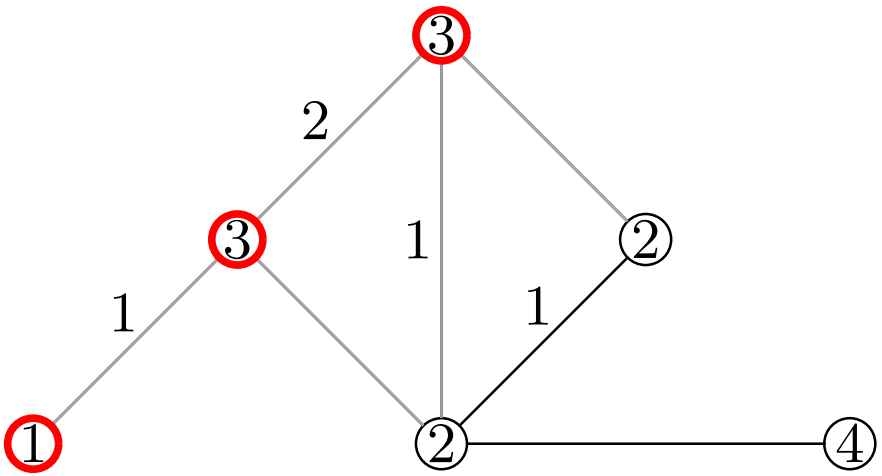

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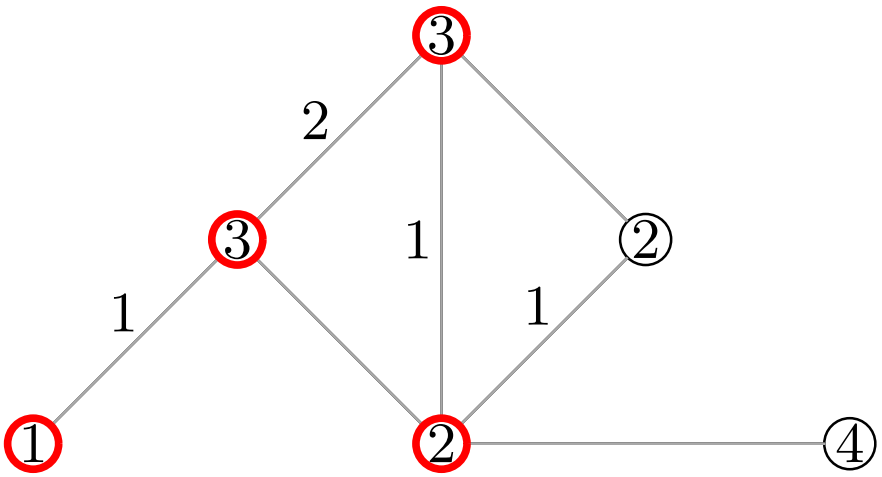

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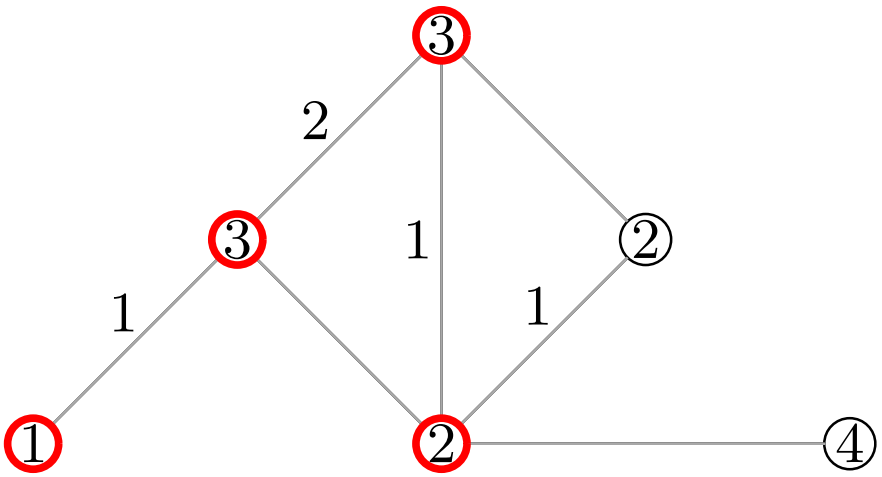

Lemma

1. x satisfies all primal constraints
2. y satisfies all dual constraints
3. $P \leq 2D \leq 2D^* \leq 2 \cdot \text{opt}$

   $P := \sum_{v \in V} x_v$: value of x

   $D := \sum_{e \in E} y_e$: value of y

   D^*: dual LP value

Proof of $P \leq 2D$.

$$P = \sum_{v \in V} w_v x_v \leq \sum_{v \in V} x_v \sum_{e \in \delta(v)} y_e = \sum_{(u,v) \in E} y_{(u,v)}(x_u + x_v)$$

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- primal-dual algorithms do not need to solve LPs
- LPs are used in analysis only
- faster than LP-rounding algorithm in general

Outline

Unrelated Machine Scheduling

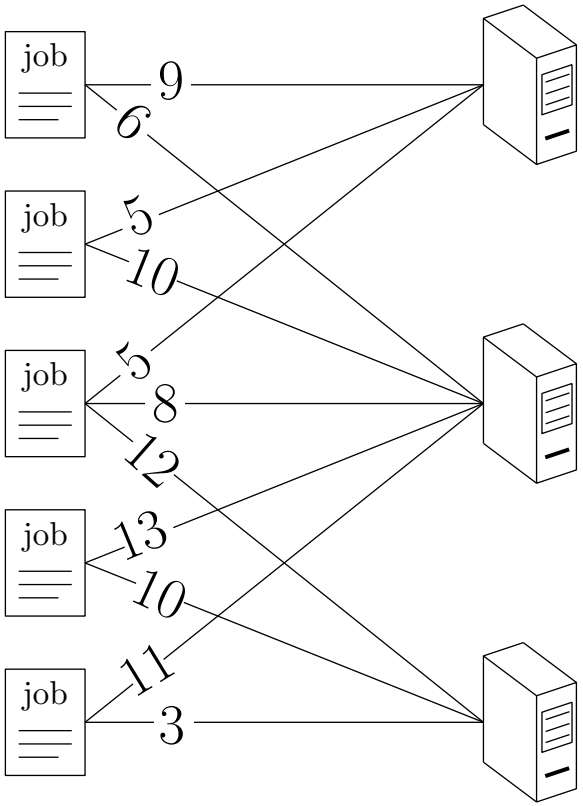
Input: $J, |J| = n$: jobs

$M, |M| = m$: machines

p_{ij}: processing time of job j on machine i

Output: assignment $\sigma : J \mapsto M$:, so as to minimize makespan:

$$\max_{i \in M} \sum_{j \in \sigma^{-1}(i)} p_{ij}$$



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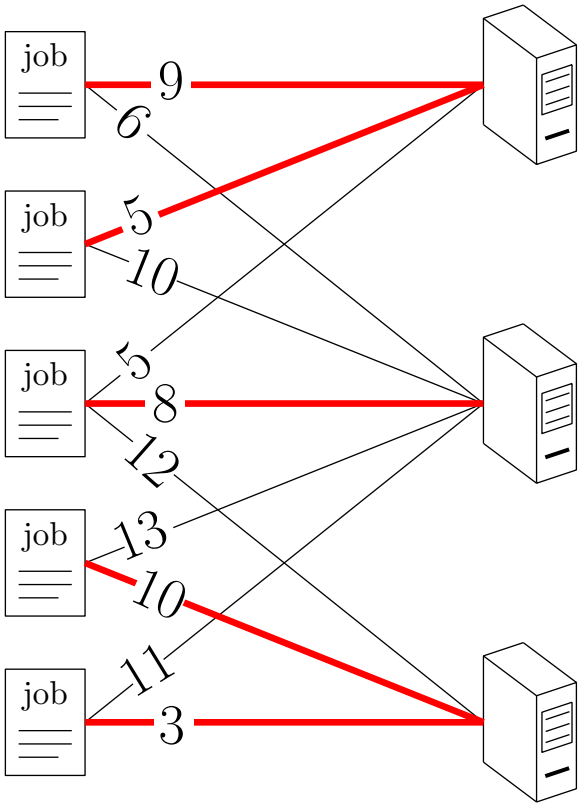

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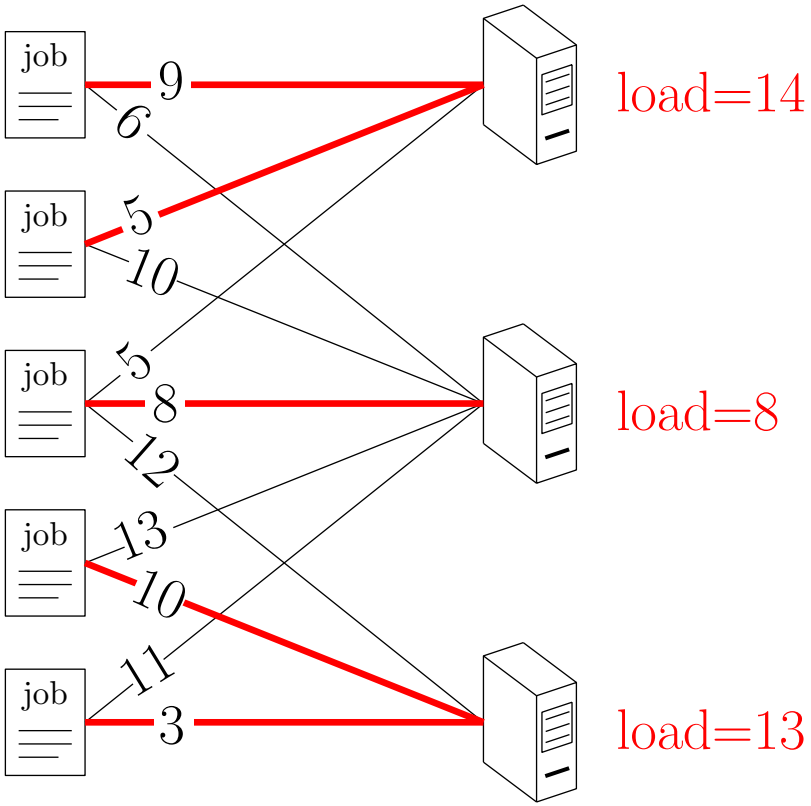

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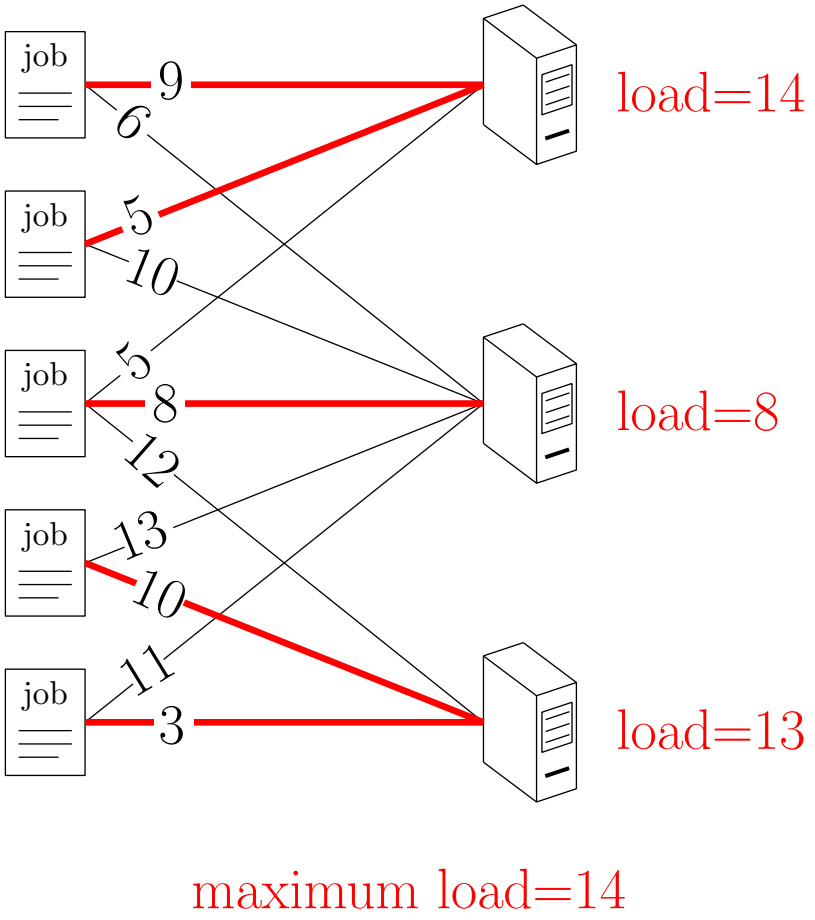

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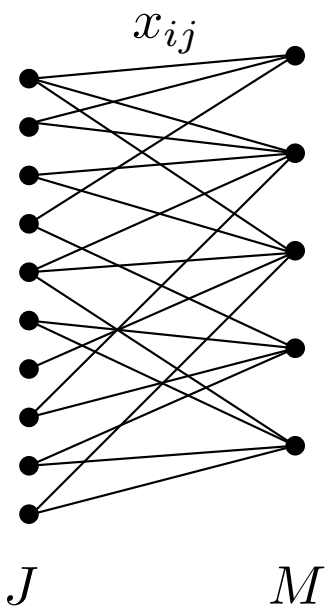

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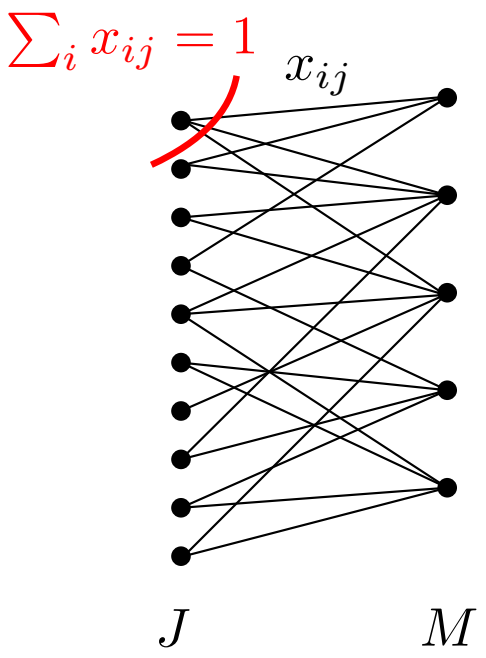

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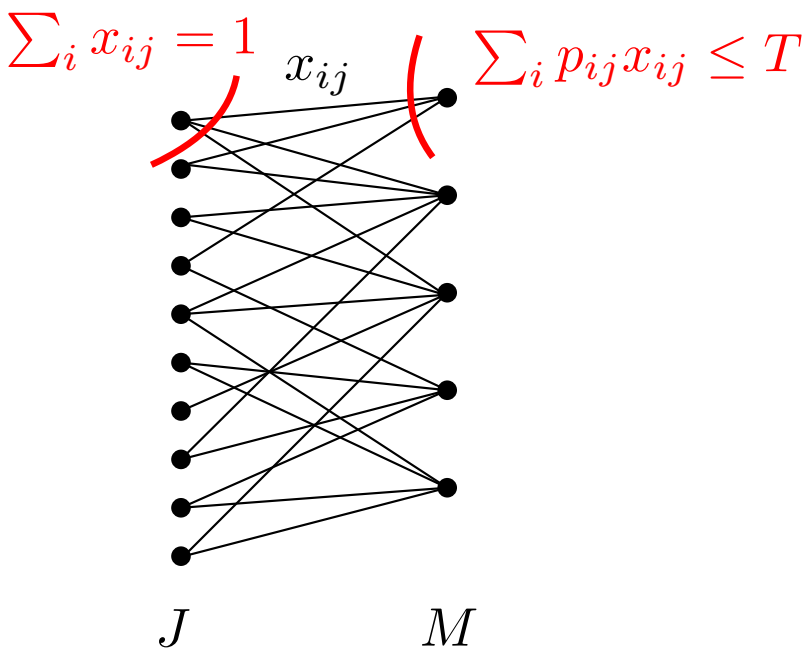

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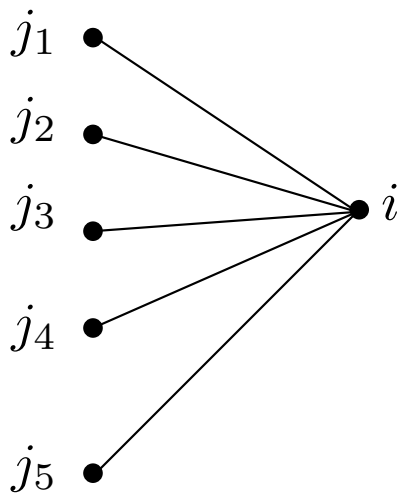

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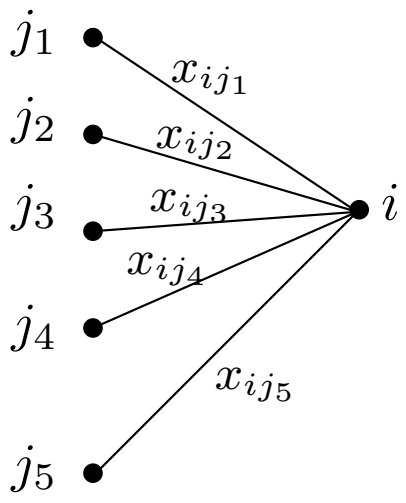

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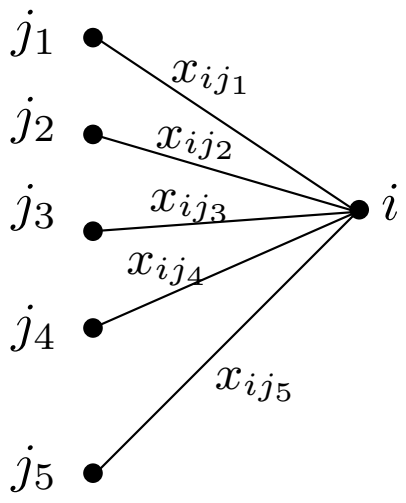

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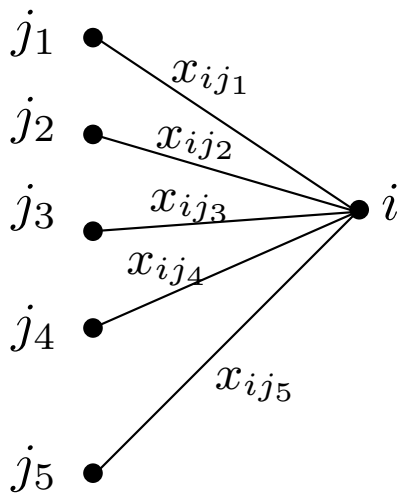

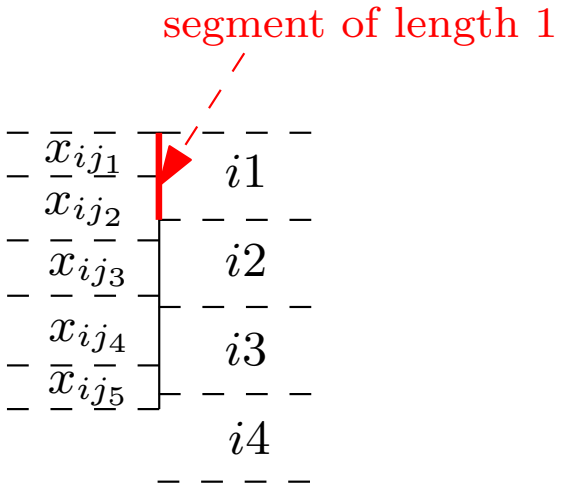

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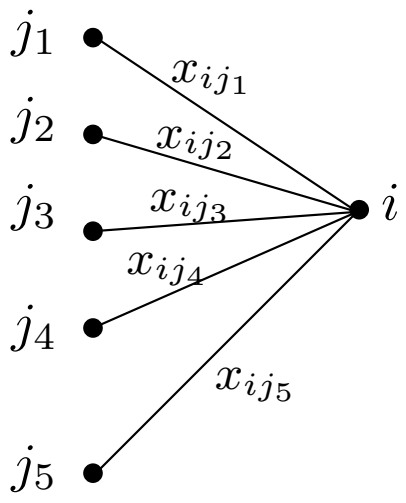

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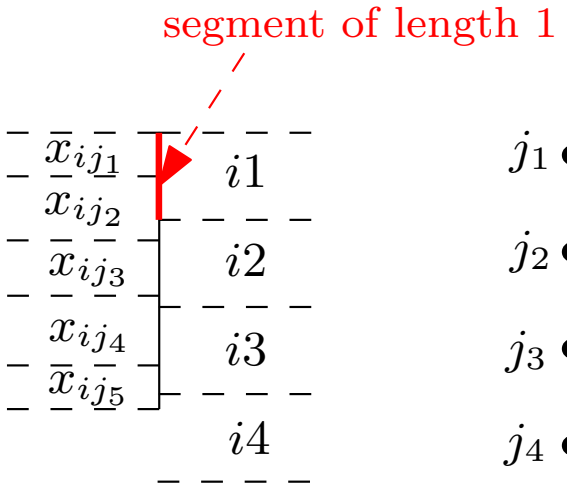

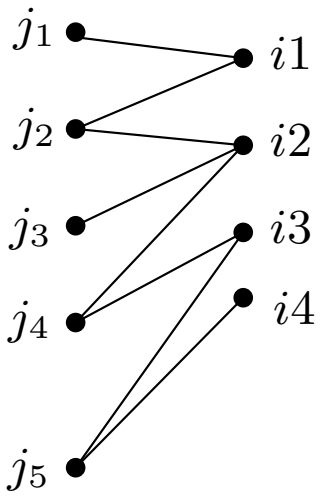

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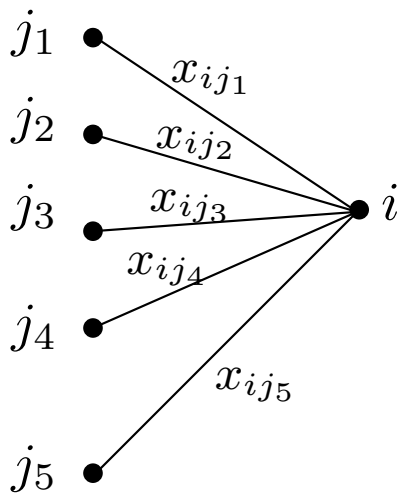

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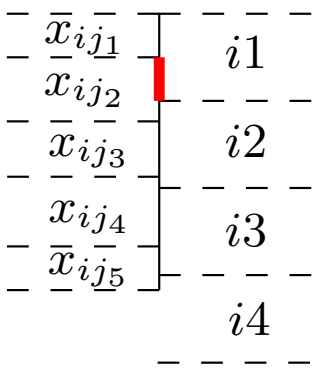

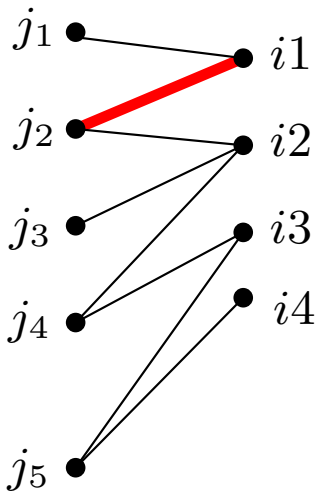

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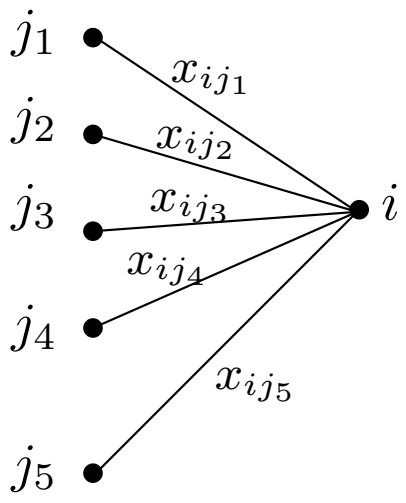

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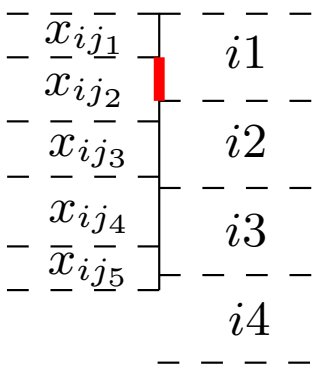

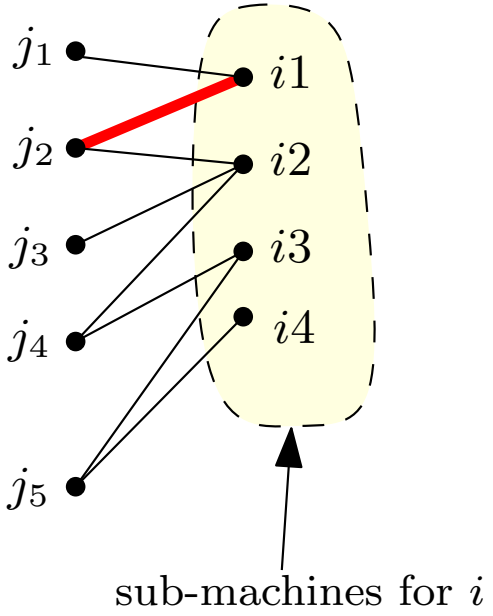

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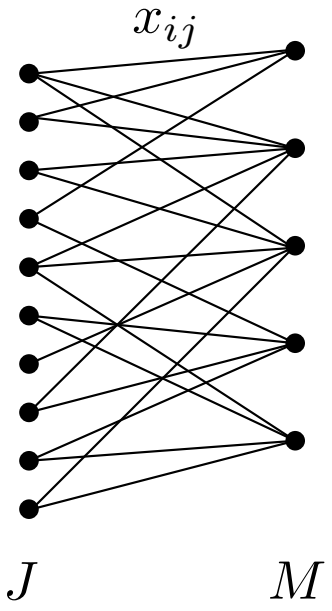

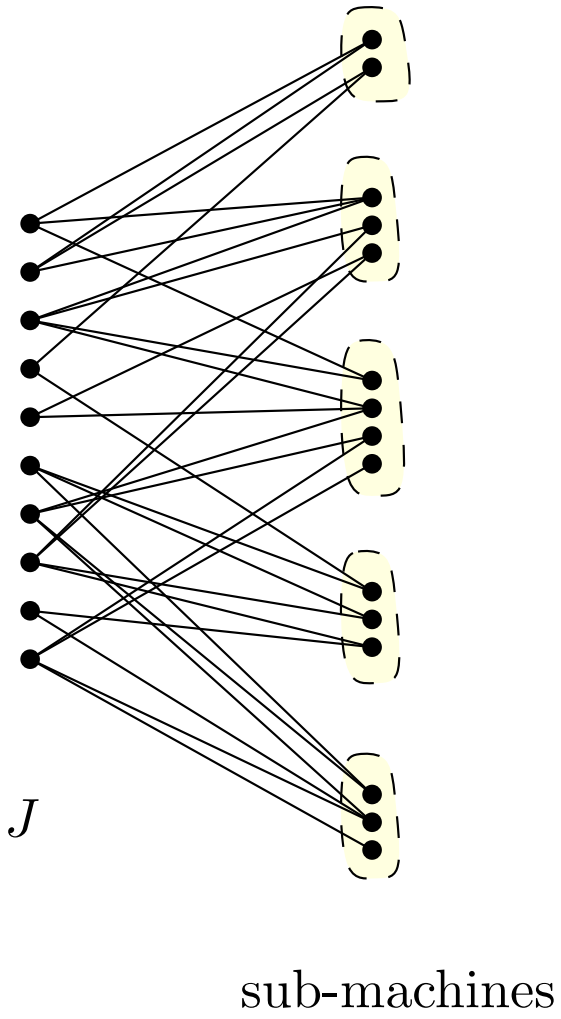

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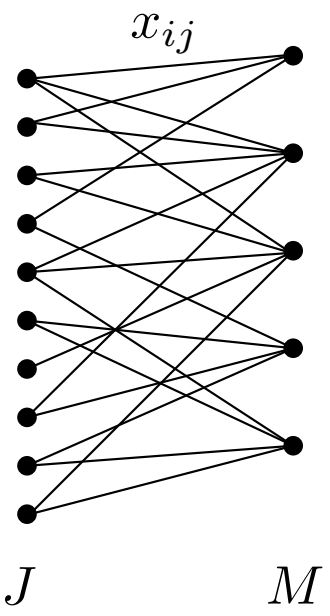

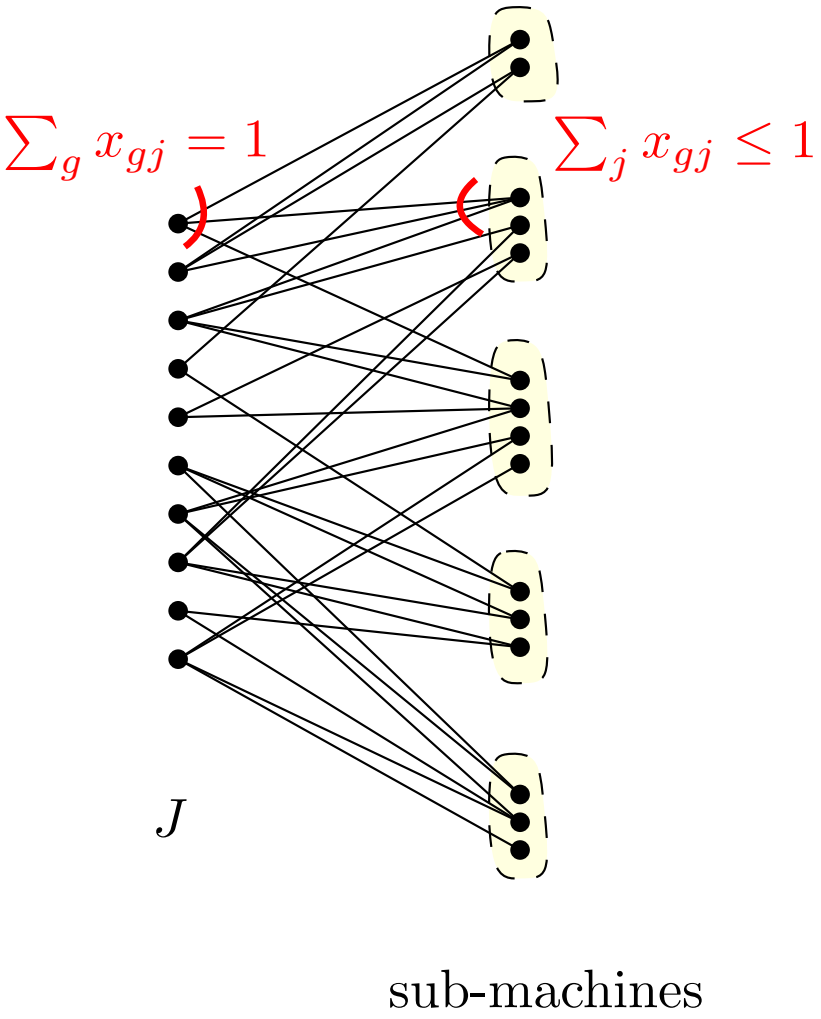

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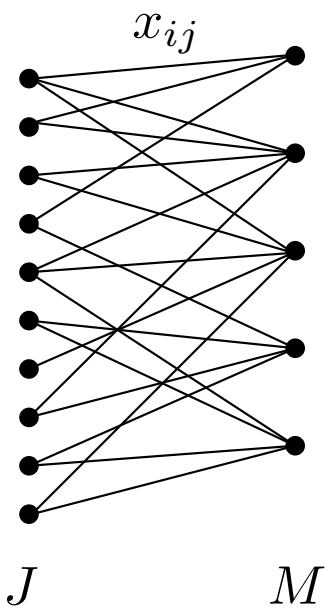

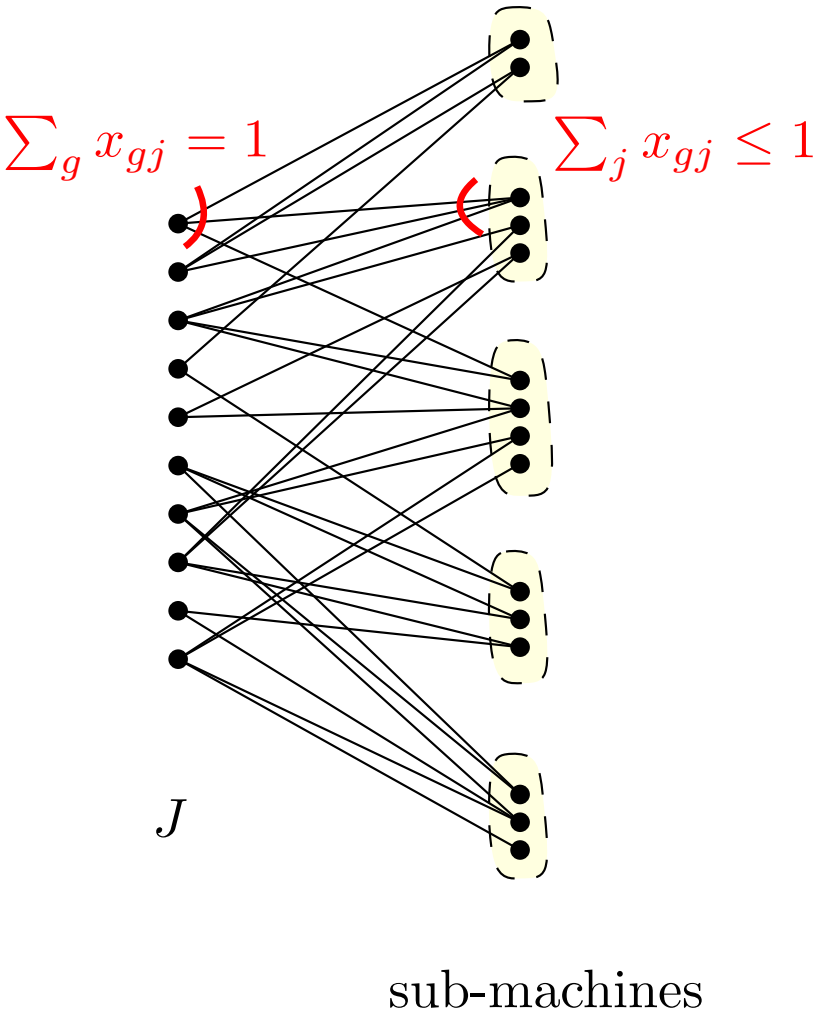

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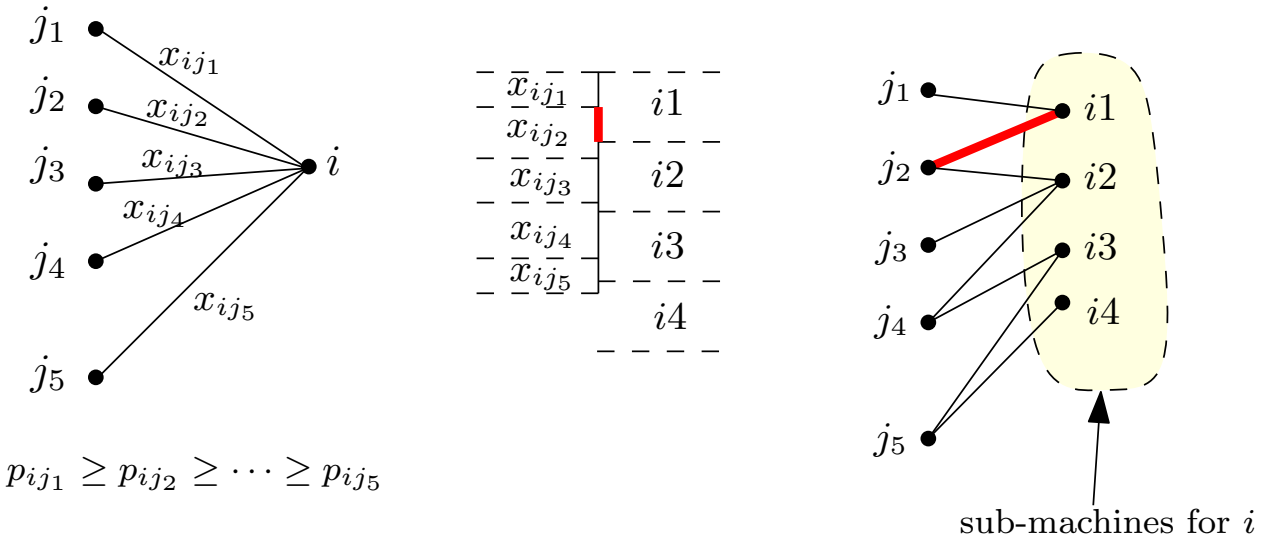

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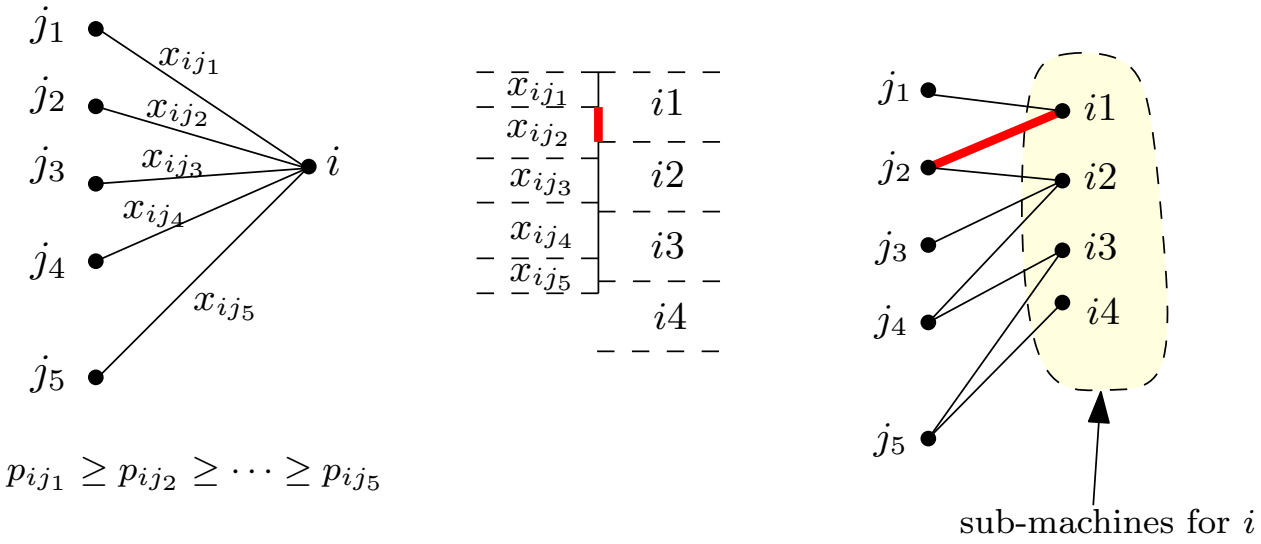

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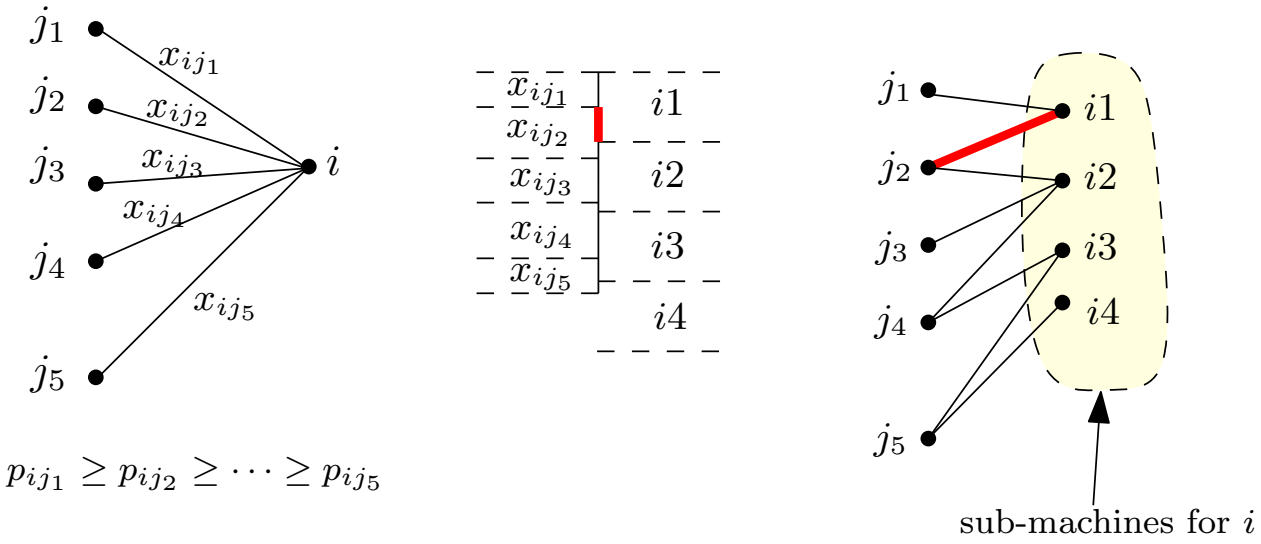

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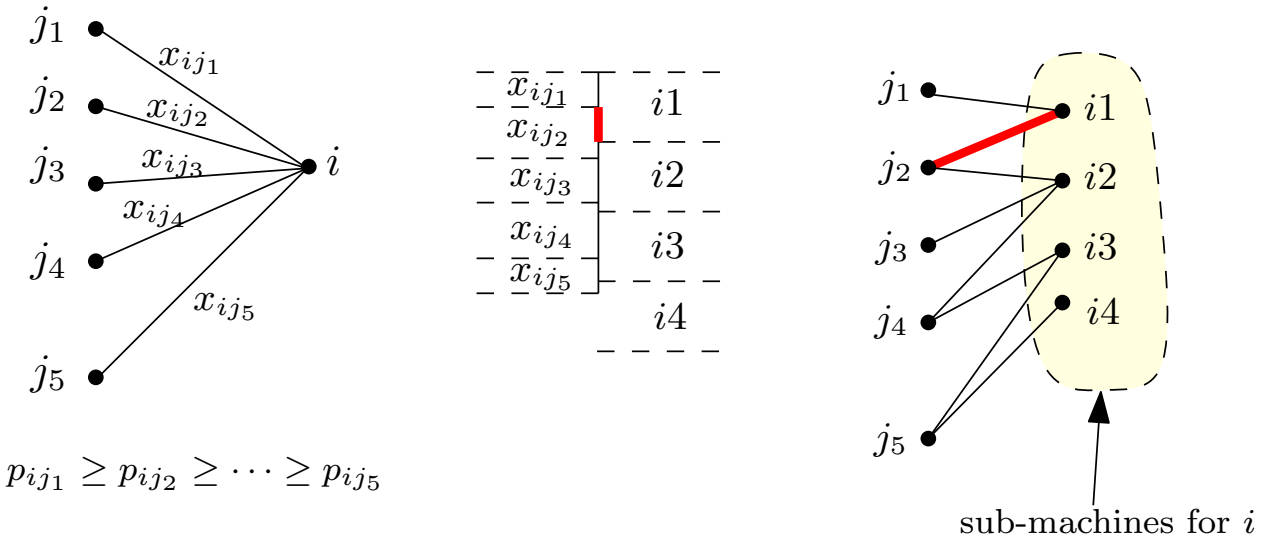

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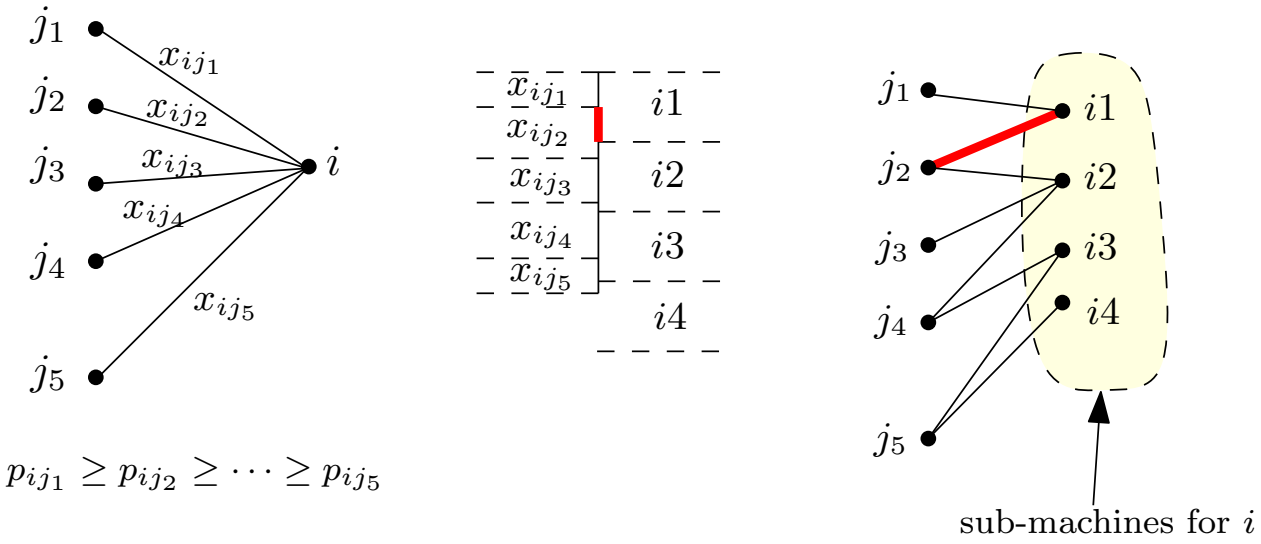

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$$(\text{load on } i) = \sum_{t=1}^{a} p_{ik_t} \leq p_{ik_1} + \sum_{t=2}^{a} \sum_{j} x_{i_{t-1}j} \cdot p_{ij}$$

$$\leq p_{ik_1} + \sum_{j} x_{ij} p_{ij} \leq T + T = 2T. \qquad \square$$

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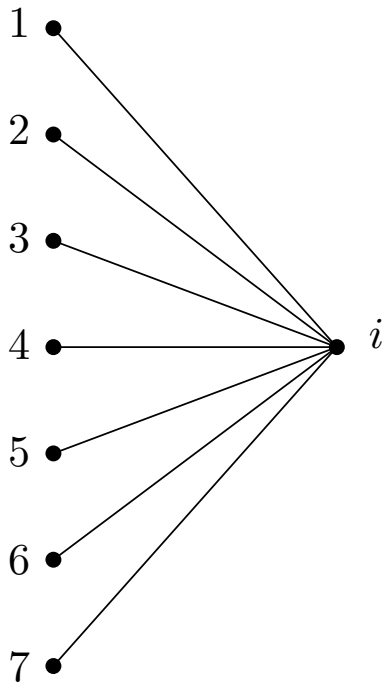

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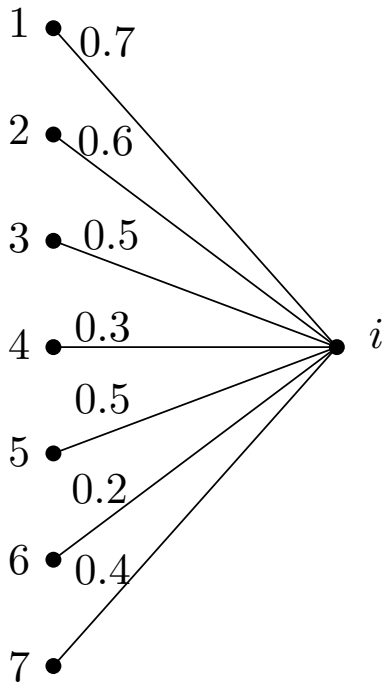

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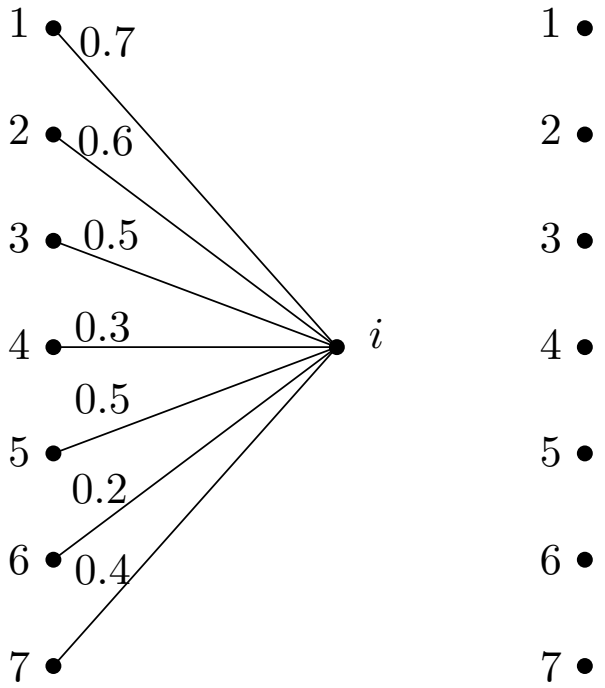

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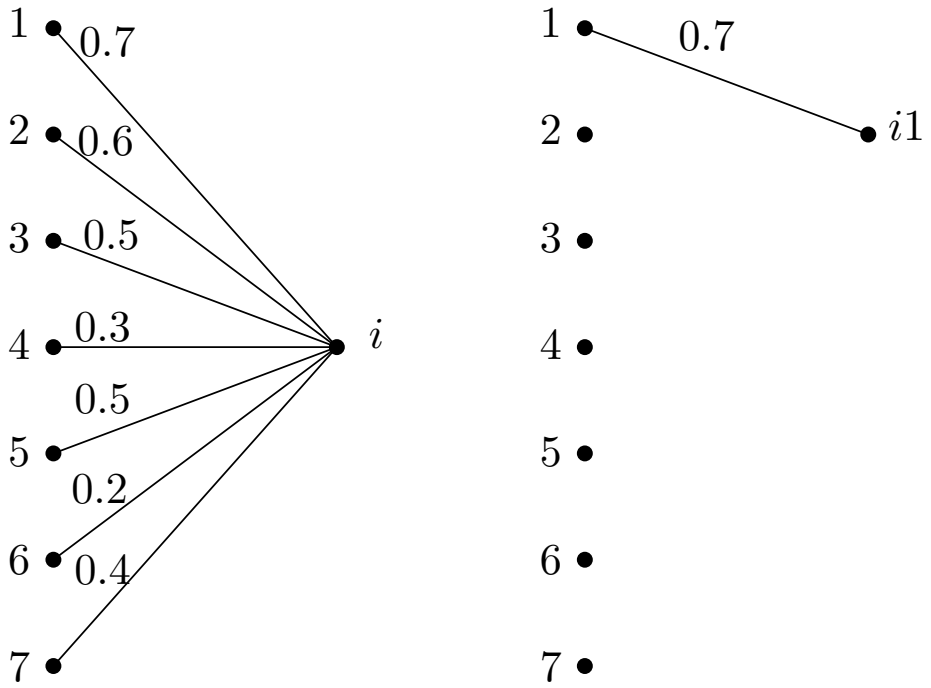

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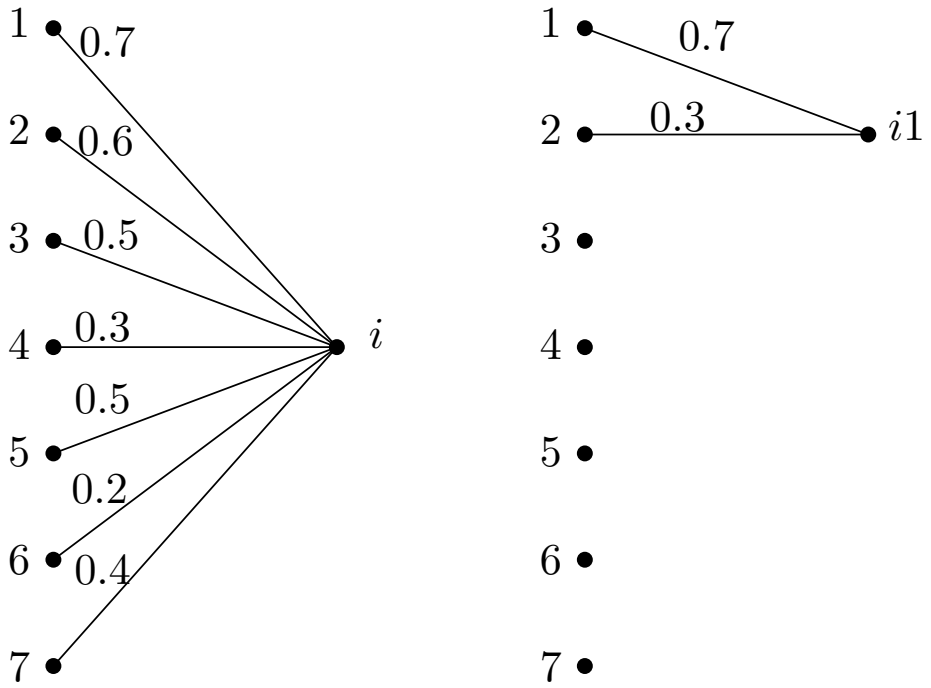

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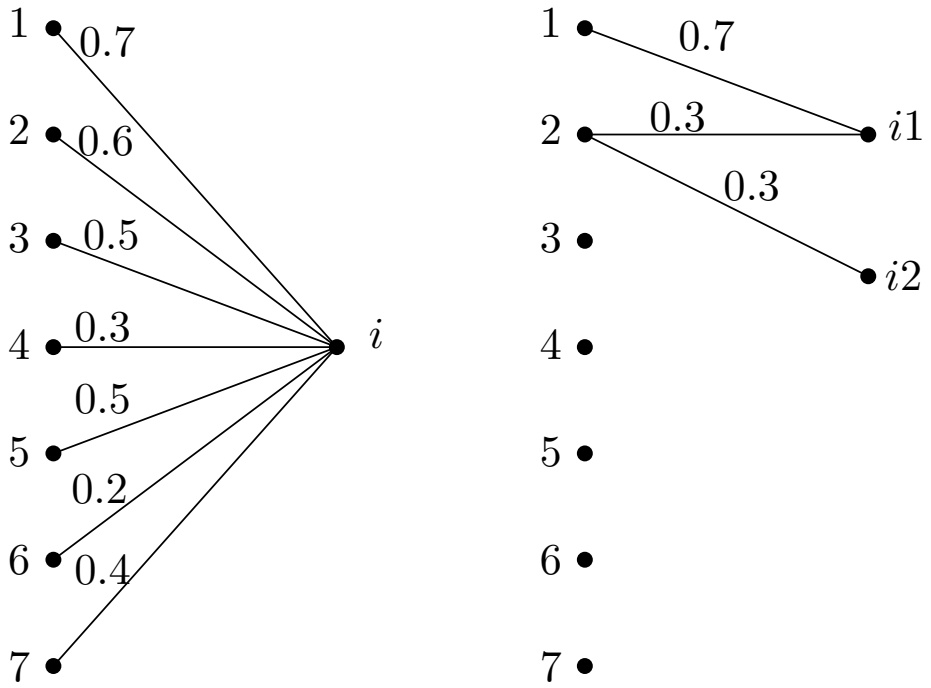

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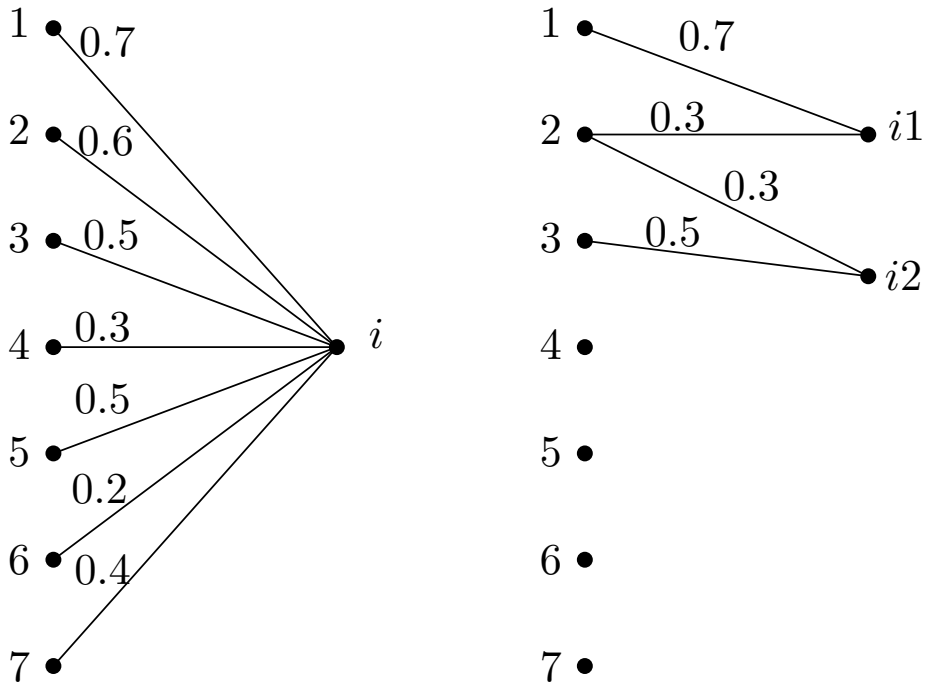

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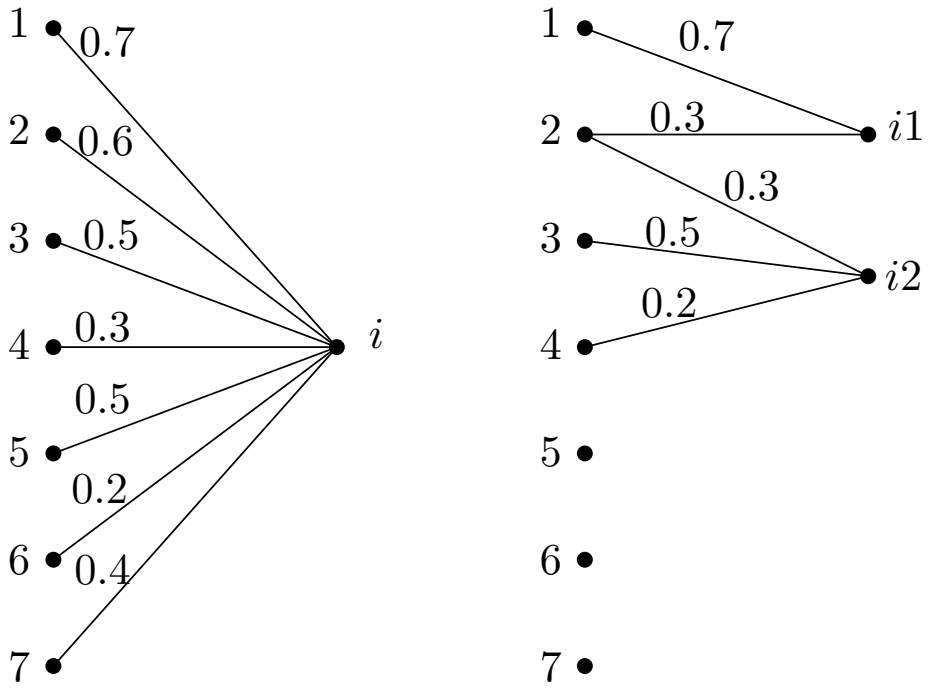
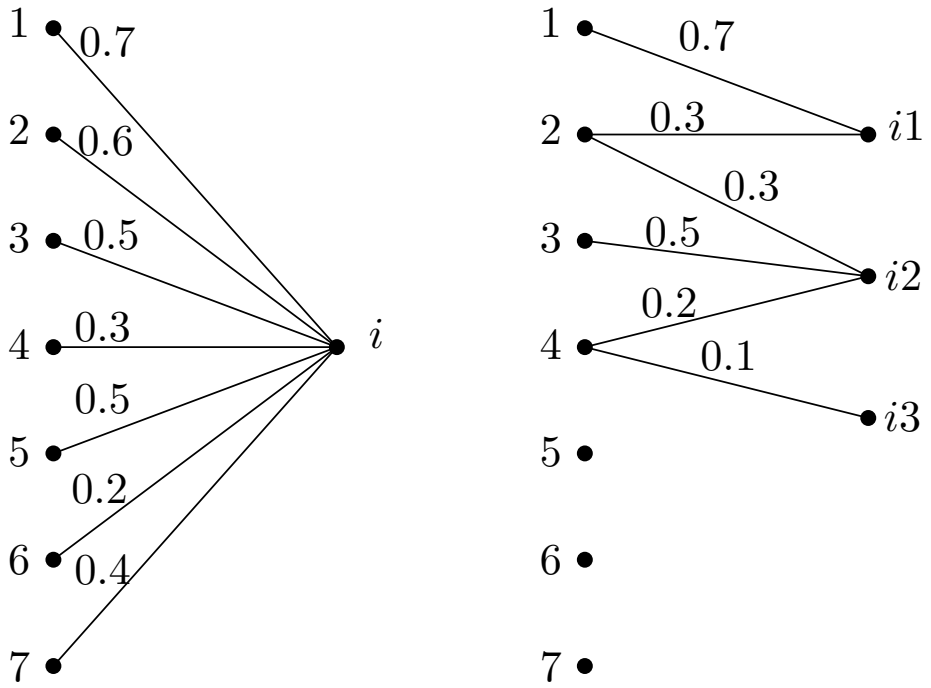
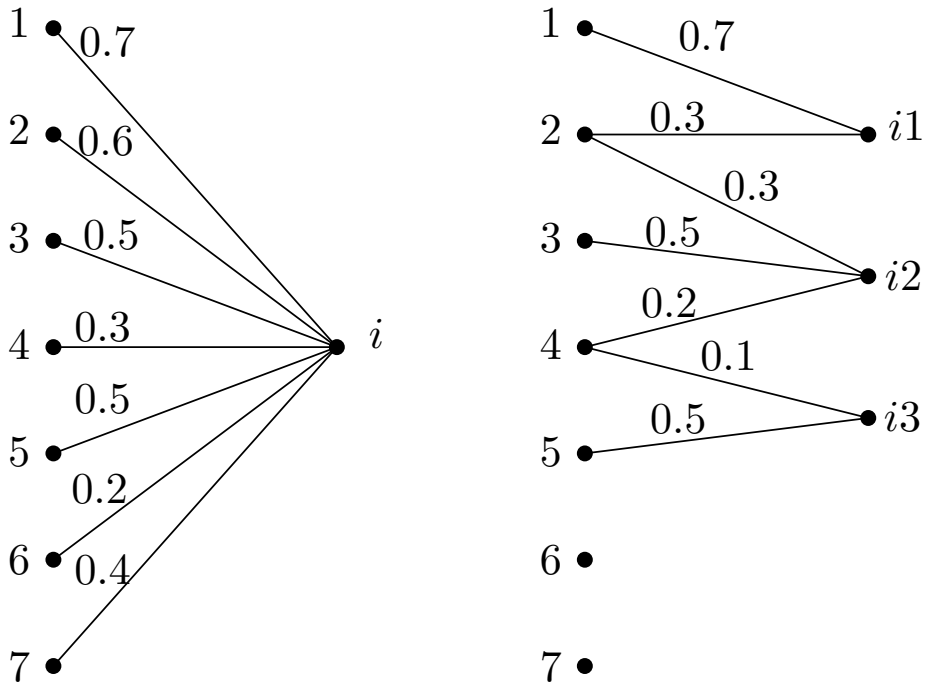


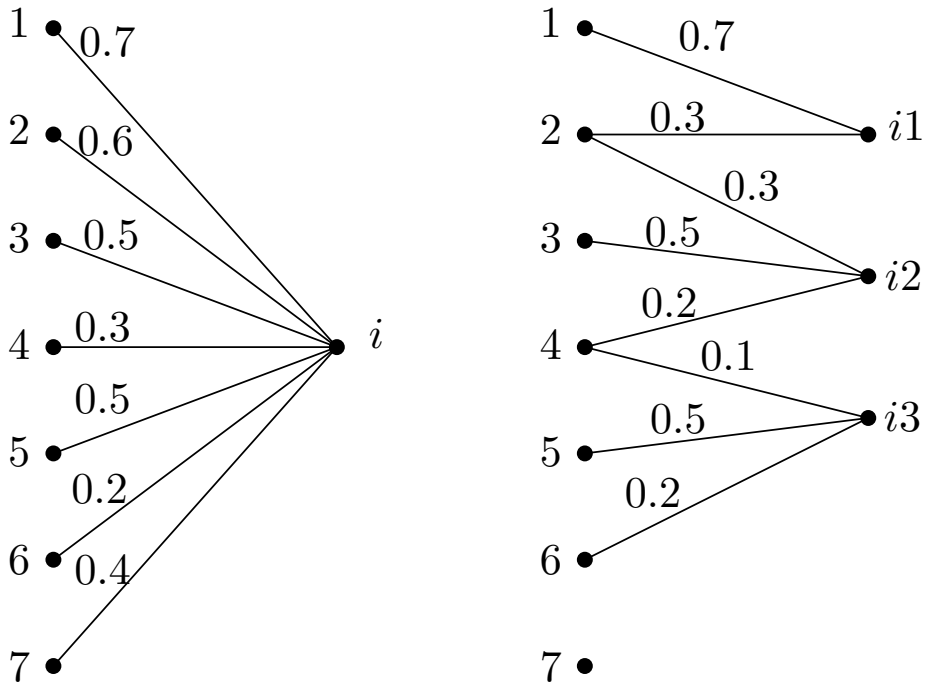
- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:



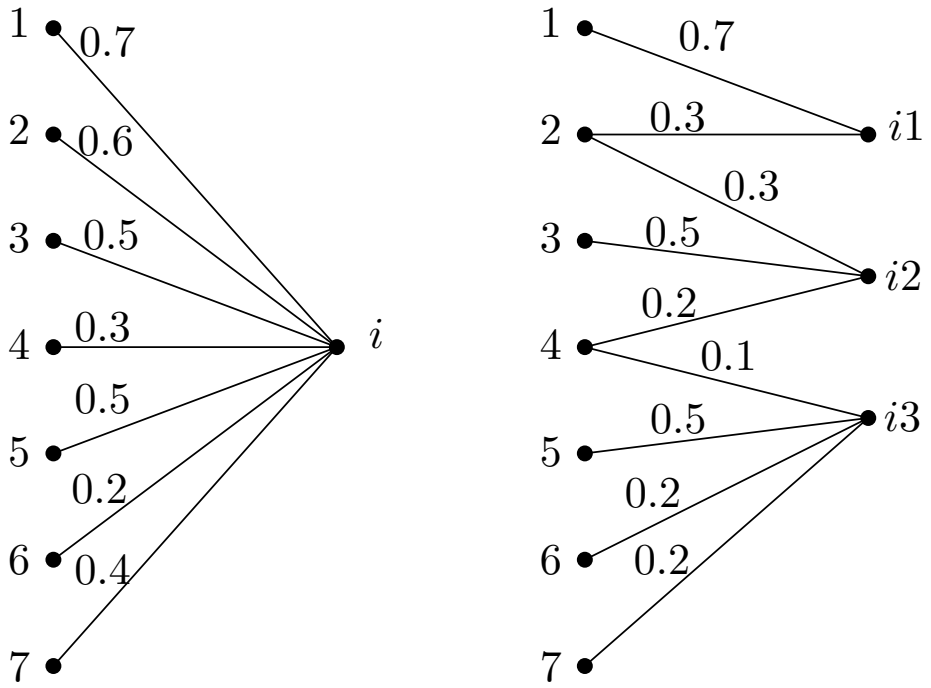
- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:



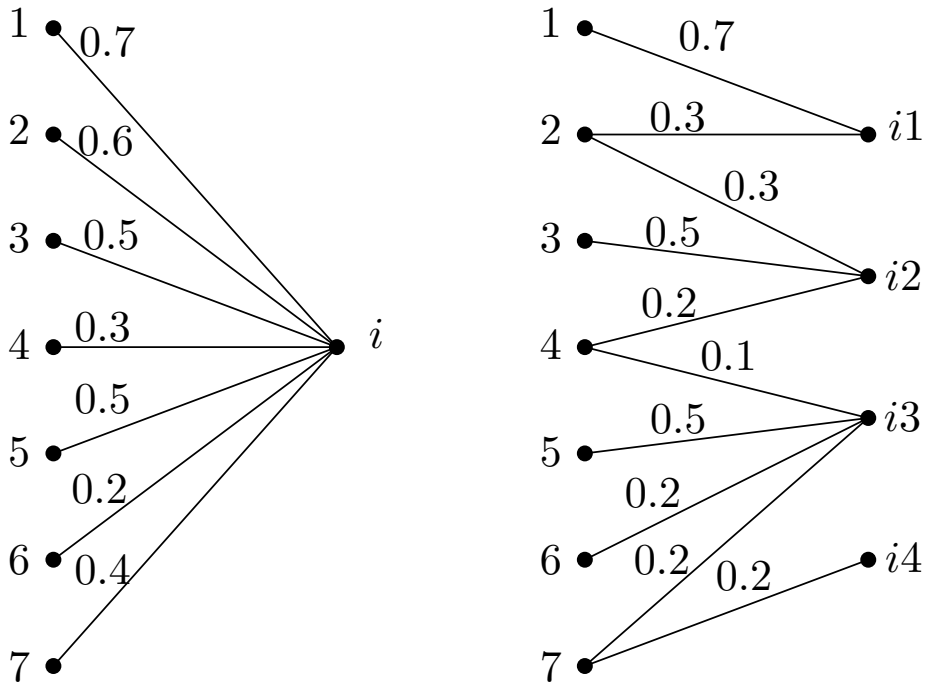
- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:



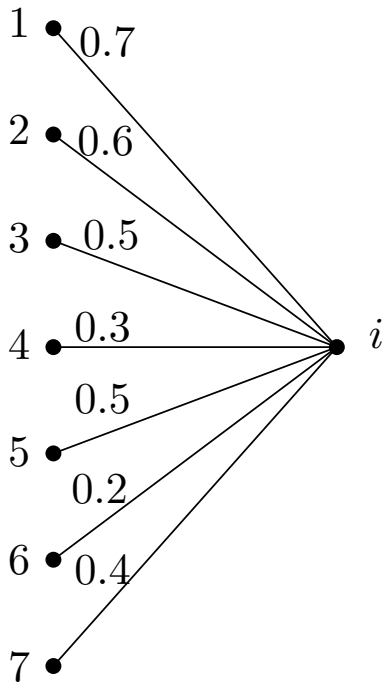
- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:

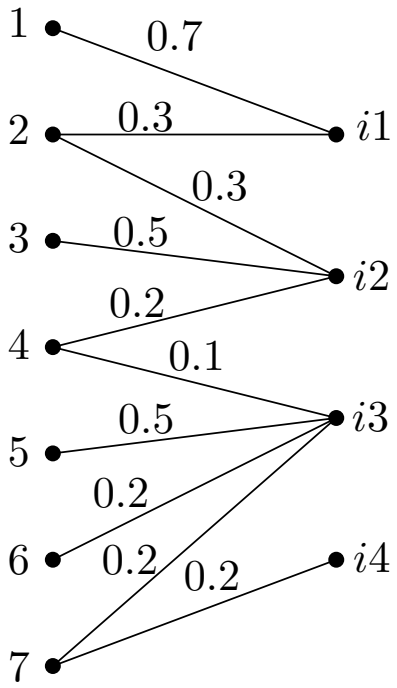


- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:

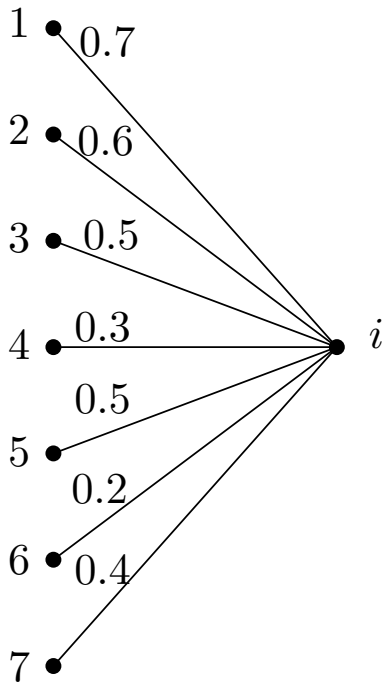


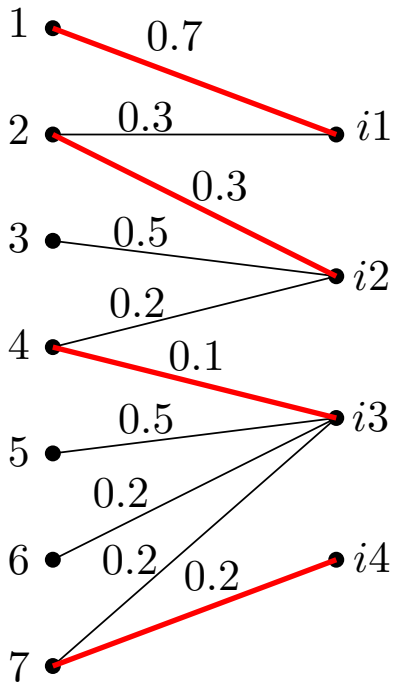
- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:
  - $1 \to i1, 2 \to i2$
  - $4 \to i3, 7 \to i4$





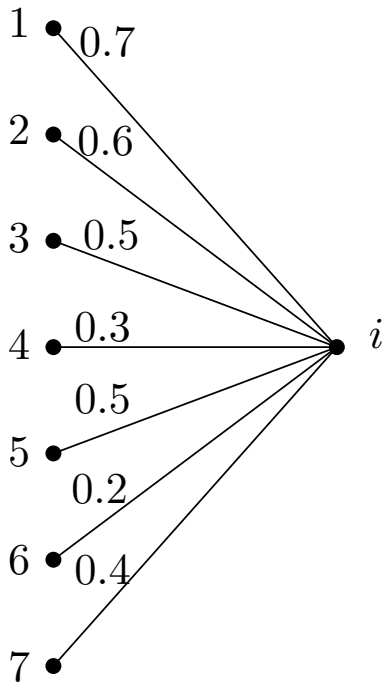
- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:
  - $1 \to i1, 2 \to i2$
  - $4 \to i3, 7 \to i4$

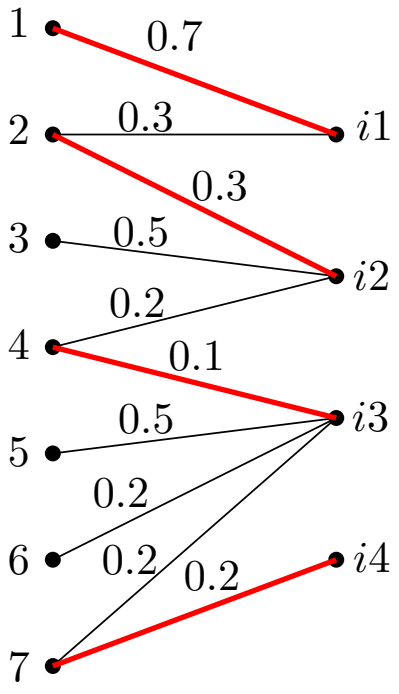




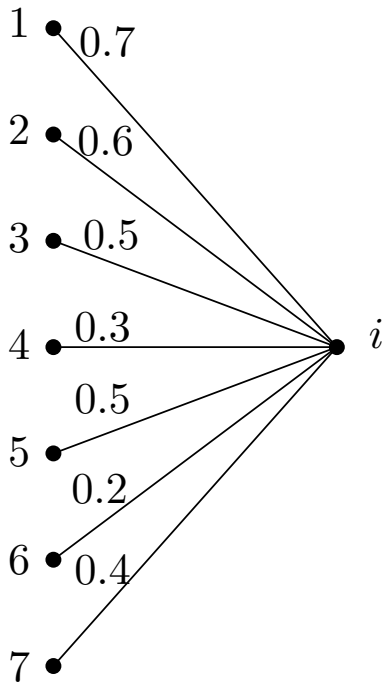
- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:
  - $1 \to i1, 2 \to i2$
  - $4 \to i3, 7 \to i4$

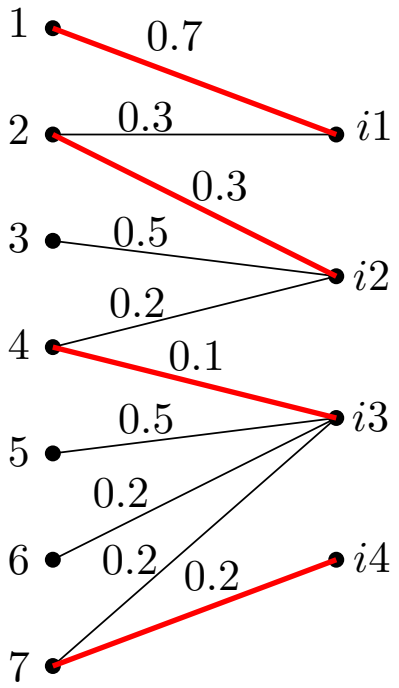
$$p_1 \leq T$$
$$p_2 \leq 0.7p_1 + 0.3p_2$$
$$p_4 \leq 0.3p_2 + 0.5p_3 + 0.2p_4$$
$$p_7 \leq 0.1p_4 + 0.5p_5 + 0.2p_6 + 0.2p_7$$





- fix i, use p_j for p_{ij}
- $p_1 \geq p_2 \geq \cdots \geq p_7$
- worst case:
  - $1 \to i1, 2 \to i2$
  - $4 \to i3, 7 \to i4$





$$p_1 \leq T$$
$$p_2 \leq 0.7p_1 + 0.3p_2$$
$$p_4 \leq 0.3p_2 + 0.5p_3 + 0.2p_4$$
$$p_7 \leq 0.1p_4 + 0.5p_5 + 0.2p_6 + 0.2p_7$$

$$\begin{aligned} p_1 + p_2 + p_4 + p_7 &\leq T + (0.7p_1 + 0.3p_2) + (0.3p_2 + 0.5p_3 + 0.2p_4) \\ &\quad + (0.1p_4 + 0.5p_5 + 0.2p_6 + 0.2p_7) \\ &\leq T + (0.7p_1 + 0.6p_2 + 0.5p_3 + 0.3p_4 + 0.5p_5 + 0.2p_6 + 0.4p_7) \\ &\leq T + T = 2T \end{aligned}$$