算法设计与分析(2024年春季学期)

Dynamic Programming

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Paradigms for Designing Algorithms

Greedy algorithm

- Make a greedy choice
- Prove that the greedy choice is safe
- Reduce the problem to a sub-problem and solve it iteratively
- Usually for optimization problems

Divide-and-conquer

- Break a problem into many independent sub-problems
- Solve each sub-problem separately
- Combine solutions for sub-problems to form a solution for the original one
- Usually used to design more efficient algorithms

Paradigms for Designing Algorithms

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Recall: Computing the n-th Fibonacci Number

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

Fib(n)

$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
for $i \leftarrow 2$ to n **do**
    $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$

- Store each $F[i]$ for future use.

Outline

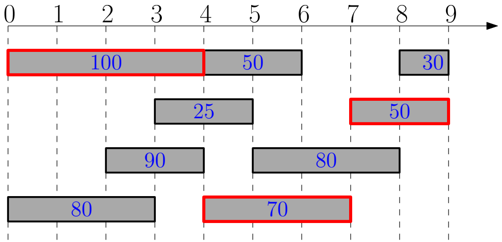
Recall: Interval Schduling

Input: n jobs, job i with start time s_i and finish time f_i

each job has a weight (or value) $v_i > 0$

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-size subset of mutually compatible jobs



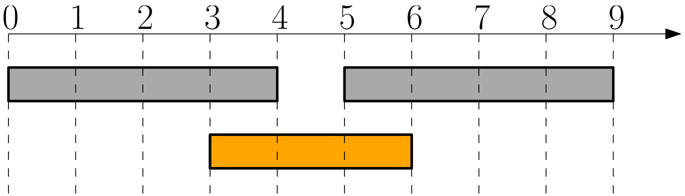
Optimum value $= 220$

Hard to Design a Greedy Algorithm

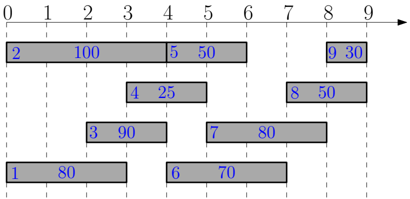
Q: Which job is safe to schedule?

- Job with the earliest finish time? No, we are ignoring weights
- Job with the largest weight? No, we are ignoring times
- Job with the largest $\frac{\text{weight}}{\text{length}}$?

  No, when weights are equal, this is the shortest job



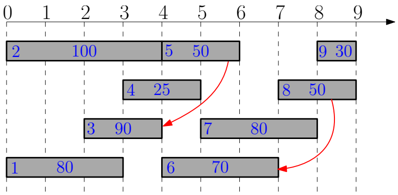
Designing a Dynamic Programming Algorithm



- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220

Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \leq s_i$

Designing a Dynamic Programming Algorithm

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \le s_i$

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

Designing a Dynamic Programming Algorithm

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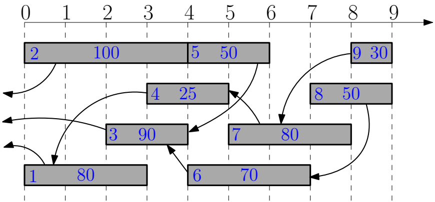
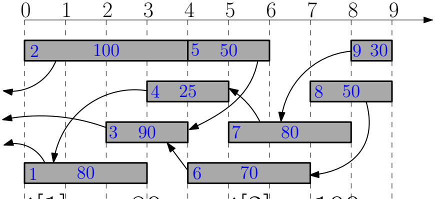


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\} = 150$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$



- $opt[0] = 0$, $opt[1] = 80$, $opt[2] = 100$
- $opt[3] = 100$, $opt[4] = 105$, $opt[5] = 150$
- $opt[6] = \max\{opt[5], 70 + opt[3]\} = 170$
- $opt[7] = \max\{opt[6], 80 + opt[4]\} = 185$
- $opt[8] = \max\{opt[7], 50 + opt[6]\} = 220$
- $opt[9] = \max\{opt[8], 30 + opt[7]\} = 220$

Dynamic Programming

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $opt[i] \leftarrow \max\{opt[i-1], v_i + opt[p_i]\}$

- Running time sorting: $O(n \lg n)$
- Running time for computing p: $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

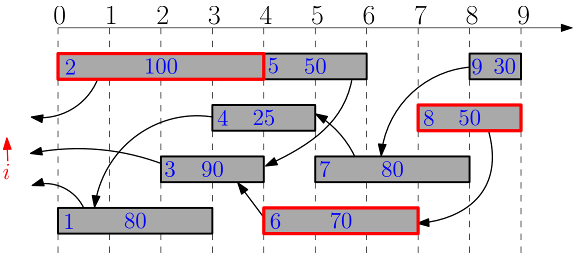
How Can We Recover the Optimum Schedule?

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    if $opt[i-1] \geq v_i + opt[p_i]$ **then**
        $opt[i] \leftarrow opt[i-1]$
        $b[i] \leftarrow$ N
    else
        $opt[i] \leftarrow v_i + opt[p_i]$
        $b[i] \leftarrow$ Y

$i \leftarrow n, S \leftarrow \emptyset$
while $i \neq 0$ **do**
    if $b[i] =$ N **then**
        $i \leftarrow i - 1$
    else
        $S \leftarrow S \cup \{i\}$
        $i \leftarrow p_i$
return S

Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



Dynamic Programming

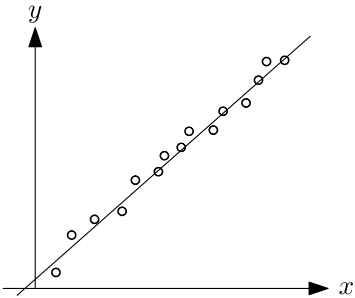
- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Outline

Linear Regression

- $P = \{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}, x_1 < x_2 < \cdots < x_n$
- $L : y = ax + b$

$$\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$

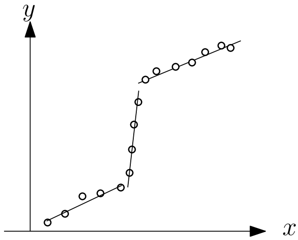


Linear Regression

- find L, minimize $\text{Error}(L, P)$

$$a := \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}$$

$$b := \frac{\sum_i y_i - a \sum_i x_i}{n}$$



- Data may come from multiple line-segments. One line may have a large error.
- Solution: using segments

Segmented Least Squares

Input: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n), x_1 < x_2 < \cdots < x_n$

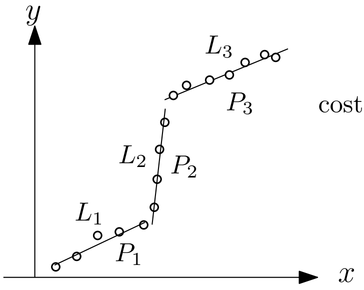
penalty parameter $C > 0$

Output: partition into $k \geq 1$ (k unknown) segments,

minimize cost := error + penalty

error: sum of squared error over all the k segments

penalty: kC

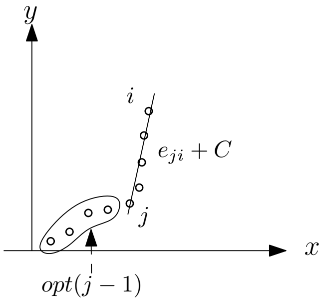


$$\begin{aligned} \text{cost} = & \ \text{error}(L_1, P_1) \\ & + \text{error}(L_2, P_2) \\ & + \text{error}(L_3, P_3) \\ & + 3C \end{aligned}$$

Dynamic Programming

- $e_{ji}, 1 \leq j \leq i \leq n$: minimum error for $(x_j, y_j), \cdots, (x_i, y_i)$ using 1 line
- $opt[i]$: minimum cost for the instance with first i points

$$opt[i] = \begin{cases} 0 & \text{if } i = 0 \\ \min_{j:1 \leq j \leq i}(opt[j-1] + e_{ji}) + C & \text{if } i \geq 1 \end{cases}$$



- running time $= O(n^2)$.

Outline

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} w_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$
- Optimum: $S = \{1, 2, 4\}$ and $14 + 9 + 10 = 33$

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Q: Does candidate algorithm always produce optimal solutions?

A: No. $W = 100, n = 3, w = (51, 50, 50)$.

Q: What if we change "non-increasing" to "non-decreasing"?

A: No. $W = 100, n = 3, w = (1, 50, 50)$

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

A: $opt[i-1, W']$

Q: The value of the optimum solution that contains i?

A: $opt[i-1, W'-w_i] + w_i$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max\left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W'-w_i] + w_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

Dynamic Programming

for $W' \leftarrow 0$ to W **do**
  $opt[0, W'] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
  for $W' \leftarrow 0$ to W **do**
    $opt[i, W'] \leftarrow opt[i-1, W']$
    if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ **then**
      $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
return $opt[n, W]$

Recover the Optimum Set

for $W' \leftarrow 0$ to W **do**
  $opt[0, W'] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
  for $W' \leftarrow 0$ to W **do**
    $opt[i, W'] \leftarrow opt[i-1, W']$
    $b[i, W'] \leftarrow$ N
    if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ **then**
      $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
      $b[i, W'] \leftarrow$ Y
return $opt[n, W]$

Recover the Optimum Set

```
i ← n, W' ← W, S ← ∅
while i > 0 do
    if b[i, W'] = Y then
        W' ← W' − w_i
        S ← S ∪ {i}
    i ← i − 1
return S
```

Running Time of Algorithm

for $W' \leftarrow 0$ to W **do**
    $opt[0, W'] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    for $W' \leftarrow 0$ to W **do**
        $opt[i, W'] \leftarrow opt[i-1, W']$
        if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ **then**
            $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
return $opt[n, W]$

- Running time is $O(nW)$
- Running time is pseudo-polynomial because it depends on value of the input integers.

Example

- $n = 4$, $w = (2, 3, 9, 8)$, $W = 14$

i, W'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	2	2	2	2	2	2	2	2	2	2	2	2	2
2	0	0	2	3	3	5	5	5	5	5	5	5	5	5	5
3	0	0	2	3	3	5	5	5	5	9	9	11	12	12	14
4	0	0	2	3	3	5	5	5	8	9	10	11	12	13	14

Avoiding Unncessary Computation and Memory Using Memoized Algorithm and Hash Map

compute-opt(i, W')

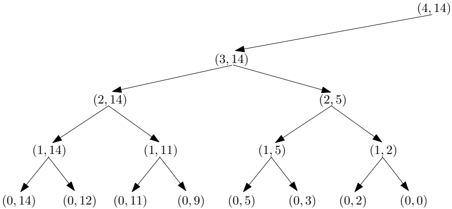
if $opt[i, W'] \neq \bot$ **then return** $opt[i, W']$
if $i = 0$ **then** $r \leftarrow 0$
else
    $r \leftarrow$ compute-opt$(i-1, W')$
    if $w_i \leq W'$ **then**
        $r' \leftarrow$ compute-opt$(i-1, W'-w_i) + w_i$
        if $r' > r$ **then** $r \leftarrow r'$
$opt[i, W'] \leftarrow r$
return r

- Use hash map for opt

Example Using Memoized Rounding

- $n = 4$, $w = (2, 3, 9, 8)$, $W = 14$

i, W'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0		0	0		0				0		0	0		0
1			2			2						2			2
2						5									5
3															14
4															



Outline

Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W, and want to buy a subset of items of maximum total value

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.
- If $i = 0$, $opt[i, W'] = 0$ for every $W' = 0, 1, 2, \cdots, W$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max\left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W'-w_i] + v_i \end{array} \right\} & i > 0, w_i \le W' \end{cases}$$

Exercise: Items with 3 Parameters

Input: integer bounds $W > 0, Z > 0$,

a set of n items, each with an integer weight $w_i > 0$

a size $z_i > 0$ for each item i

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} v_i \qquad \text{s.t.}$$

$$\sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} z_i \leq Z$$

Outline

Subsequence

- $A = bacdca$
- $C = adca$
- C is a subsequence of A

Def. Given two sequences $A[1 .. n]$ and $C[1 .. t]$ of letters, C is called a subsequence of A if there exists integers $1 \leq i_1 < i_2 < i_3 < \ldots < i_t \leq n$ such that $A[i_j] = C[j]$ for every $j = 1, 2, 3, \cdots, t$.

- Exercise: how to check if sequence C is a subsequence of A?

Longest Common Subsequence

Input: $A[1..n]$ and $B[1..m]$

Output: the longest common subsequence of A and B

Example:

- $A = \text{`}bacdca\text{'}$
- $B = \text{`}adbcda\text{'}$
- $\mathrm{LCS}(A, B) = \text{`}adca\text{'}$

- Applications: edit distance (diff), similarity of DNAs

Matching View of LCS

$$b \quad a \quad c \quad d \quad c \quad a$$

$$a \quad d \quad b \quad c \quad d \quad a$$

- Goal of LCS: find a maximum-size non-crossing matching between letters in A and letters in B.

Reduce to Subproblems

- $A = `bacdca'$
- $B = `adbcda'$

- either the last letter of A is not matched:
- need to compute LCS($`bacd'$, $`adbcd'$)
- or the last letter of B is not matched:
- need to compute LCS($`bacdc'$, $`adbc'$)

Dynamic Programming for LCS

- $opt[i, j], 0 \le i \le n, 0 \le j \le m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ or $j = 0$, then $opt[i, j] = 0$.
- if $i > 0, j > 0$, then

$$opt[i,j] = \begin{cases} opt[i-1, j-1] + 1 & \text{if } A[i] = B[j] \\ \max \begin{cases} opt[i-1, j] \\ opt[i, j-1] \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Dynamic Programming for LCS

for $j \leftarrow 0$ to m **do**
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $opt[i, 0] \leftarrow 0$
    for $j \leftarrow 1$ to m **do**
        if $A[i] = B[j]$ **then**
            $opt[i, j] \leftarrow opt[i-1, j-1] + 1$, $\pi[i, j] \leftarrow$ "$\nwarrow$"
        else if $opt[i, j-1] \geq opt[i-1, j]$ **then**
            $opt[i, j] \leftarrow opt[i, j-1]$, $\pi[i, j] \leftarrow$ "$\leftarrow$"
        else
            $opt[i, j] \leftarrow opt[i-1, j]$, $\pi[i, j] \leftarrow$ "$\uparrow$"

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$	3 $\uparrow$	3 $\leftarrow$	4 $\nwarrow$

Find Common Subsequence

$i \leftarrow n, j \leftarrow m, S \leftarrow ()$
while $i > 0$ and $j > 0$ **do**
    if $\pi[i, j] =$ "$\nwarrow$" **then**
        add $A[i]$ to beginning of S, $i \leftarrow i - 1, j \leftarrow j - 1$
    else if $\pi[i, j] =$ "$\uparrow$" **then**
        $i \leftarrow i - 1$
    else
        $j \leftarrow j - 1$
return S

Variants of Problem

Edit Distance with Insertions and Deletions

Input: a string A

each time we can delete a letter from A or insert a letter to A

Output: minimum number of operations (insertions or deletions) we need to change A to B?

Example:

- $A =$ ocurrance, $B =$ occurrence
- 3 operations: insert 'c', remove 'a' and insert 'e'

Obs. $\#\text{OPs} = \text{length}(A) + \text{length}(B)$ - $2 \cdot \text{length}(\text{LCS}(A, B))$

Variants of Problem

Edit Distance with Insertions, Deletions and Replacing

Input: a string A,

each time we can delete a letter from A, insert a letter to A or change a letter

Output: how many operations do we need to change A to B?

Example:

- $A =$ ocurrance, $B =$ occurrence.
- 2 operations: insert 'c', change 'a' to 'e'

- Not related to LCS any more

Edit Distance with Replacing: Reduction to a Variant of LCS

- Need to match letters in A and B, every letter is matched at most once and there should be no crosses.
- However, we can match two different letters: Matching a same letter gives score 2, matching two different letters gives score 1.
- Need to maximize the score.
- DP recursion for the case $i > 0$ and $j > 0$:

$$opt[i,j] = \begin{cases} opt[i-1,j-1] + 2 & \text{if } A[i] = B[j] \\ \max \begin{cases} opt[i-1,j] \\ opt[i,j-1] \\ opt[i-1,j-1] + 1 \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

- Relation : #OPs $=$ length(A) $+$ length(B) - max_score

Edit Distance (with Replacing): using DP directly

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: edit distance between $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ then $opt[i, j] = j$; if $j = 0$ then $opt[i, j] = i$.
- if $i > 0, j > 0$, then

$$opt[i, j] = \begin{cases} opt[i-1, j-1] & \text{if } A[i] = B[j] \\ \min \begin{cases} opt[i-1, j] + 1 \\ opt[i, j-1] + 1 \\ opt[i-1, j-1] + 1 \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Exercise: Longest Palindrome

Def. A palindrome is a string which reads the same backward or forward.

- example: "racecar", "wasitacaroracatisaw", "putitup"

Longest Palindrome Subsequence

Input: a sequence A

Output: the longest subsequence C of A that is a palindrome.

Example:

- Input: acbcedeacab
- Output: acedeca

Outline

Computing the Length of LCS

for $j \leftarrow 0$ to m **do**
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $opt[i, 0] \leftarrow 0$
    for $j \leftarrow 1$ to m **do**
        if $A[i] = B[j]$ **then**
            $opt[i, j] \leftarrow opt[i-1, j-1] + 1$
        else if $opt[i, j-1] \geq opt[i-1, j]$ **then**
            $opt[i, j] \leftarrow opt[i, j-1]$
        else
            $opt[i, j] \leftarrow opt[i-1, j]$

Obs. The i-th row of table only depends on $(i-1)$-th row.

Reducing Space to $O(n+m)$

Obs. The i-th row of table only depends on $(i-1)$-th row.

Q: How to use this observation to reduce space?

A: We only keep two rows: the $(i-1)$-th row and the i-th row.

Linear Space Algorithm to Compute Length of LCS

```
for j ← 0 to m do
    opt[0, j] ← 0
for i ← 1 to n do
    opt[i mod 2, 0] ← 0
    for j ← 1 to m do
        if A[i] = B[j] then
            opt[i mod 2, j] ← opt[i − 1 mod 2, j − 1] + 1
        else if opt[i mod 2, j − 1] ≥ opt[i − 1 mod 2, j] then
            opt[i mod 2, j] ← opt[i mod 2, j − 1]
        else
            opt[i mod 2, j] ← opt[i − 1 mod 2, j]
return opt[n mod 2, m]
```

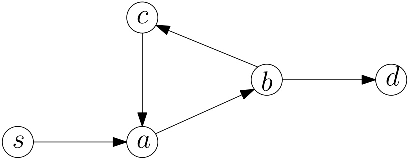
How to Recover LCS Using Linear Space?

- Only keep the last two rows: only know how to match $A[n]$
- Can recover the LCS using n rounds: time $= O(n^2 m)$
- Using Divide and Conquer + Dynamic Programming:
  - Space: $O(m+n)$
  - Time: $O(nm)$

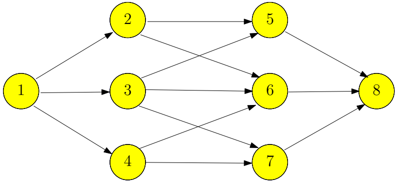
Outline

Directed Acyclic Graphs

Def. A directed acyclic graph (DAG) is a directed graph without (directed) cycles.



not a DAG



a DAG

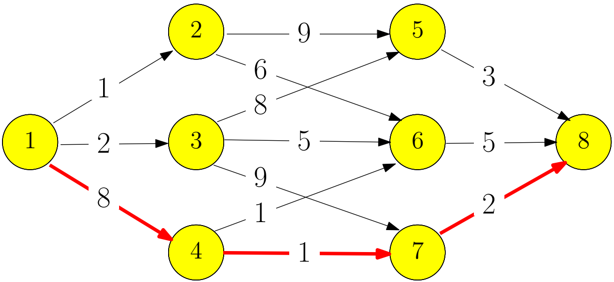
Lemma A directed graph is a DAG if and only its vertices can be topologically sorted.

Shortest Paths in DAG

Input: directed acyclic graph $G = (V, E)$ and $w : E \to \mathbb{R}$.

Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the shortest path from 1 to i, for every $i \in V$



Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \min_{j:(j,i)\in E} \{f(j) + w(j,i)\} & i = 2, 3, \cdots, n \end{cases}$$

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

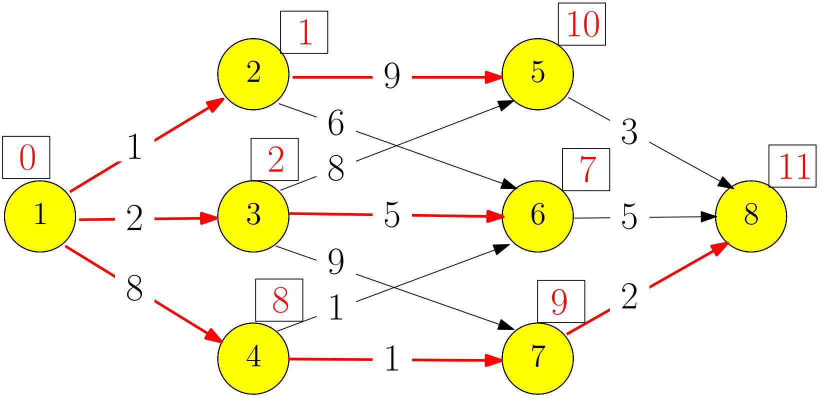
Shortest Paths in DAG

$f[1] \leftarrow 0$
for $i \leftarrow 2$ to n **do**
    $f[i] \leftarrow \infty$
    for each incoming edge (j, i) of i **do**
        if $f[j] + w(j, i) < f[i]$ **then**
            $f[i] \leftarrow f[j] + w(j, i)$
            $\pi(i) \leftarrow j$

print-path(t)

if $t = 1$ **then**
    print(1)
    return
print-path$(\pi(t))$
print(“,”, t)

Example



Variant: Heavist Path in a Directed Acyclic Graph

Heaviest Path in a Directed Acyclic Graph

Input: directed acyclic graph $G = (V, E)$ and $w : E \to \mathbb{R}$.
Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the path with the largest weight (the heaviest path) from 1 to n.

- $f[i]$: weight of the heaviest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \max_{j:(j,i)\in E} \{f(j) + w(j, i)\} & i = 2, 3, \cdots, n \end{cases}$$

Outline

Matrix Chain Multiplication

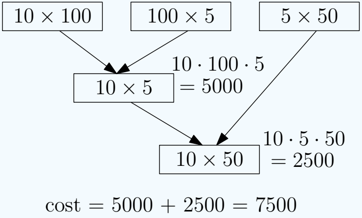
Input: n matrices $A_1, A_2, \cdots, A_n$ of sizes $r_1 \times c_1, r_2 \times c_2, \cdots, r_n \times c_n$, such that $c_i = r_{i+1}$ for every $i = 1, 2, \cdots, n-1$.

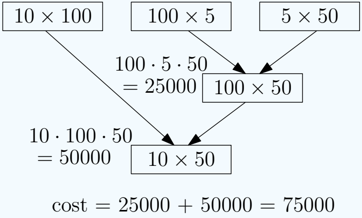
Output: the order of computing $A_1 A_2 \cdots A_n$ with the minimum number of multiplications

Fact Multiplying two matrices of size $r \times k$ and $k \times c$ takes $r \times k \times c$ multiplications.

Example:

- $A_1 : 10 \times 100, \quad A_2 : 100 \times 5, \quad A_3 : 5 \times 50$





- $(A_1A_2)A_3$: $10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1(A_2A_3)$: $100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

Matrix Chain Multiplication: Design DP

- Assume the last step is $(A_1 A_2 \cdots A_i)(A_{i+1} A_{i+2} \cdots A_n)$
- Cost of last step: $r_1 \times c_i \times c_n$
- Optimality for sub-instances: we need to compute $A_1 A_2 \cdots A_i$ and $A_{i+1} A_{i+2} \cdots A_n$ optimally
- $opt[i, j]$: the minimum cost of computing $A_i A_{i+1} \cdots A_j$

$$opt[i,j] = \begin{cases} 0 & i = j \\ \min_{k: i \le k < j} \left(opt[i,k] + opt[k+1,j] + r_i c_k c_j \right) & i < j \end{cases}$$

Matrix Chain Multiplication: Design DP

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

let $opt[i,i] \leftarrow 0$ for every $i = 1, 2, \cdots, n$
for $\ell \leftarrow 2$ to n **do**
    for $i \leftarrow 1$ to $n - \ell + 1$ **do**
        $j \leftarrow i + \ell - 1$
        $opt[i,j] \leftarrow \infty$
        for $k \leftarrow i$ to $j - 1$ **do**
            if $opt[i,k] + opt[k+1,j] + r_i c_k c_j < opt[i,j]$ **then**
                $opt[i,j] \leftarrow opt[i,k] + opt[k+1,j] + r_i c_k c_j$
                $\pi[i,j] \leftarrow k$
return $opt[1,n]$

Constructing Optimal Solution

Print-Optimal-Order(i, j)

```
if i = j then
    print("A"_i)
else
    print("(")
    Print-Optimal-Order(i, π[i, j])
    Print-Optimal-Order(π[i, j] + 1, j)
    print(")")
```

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$\begin{aligned}
opt[1,2] &= opt[1,1] + opt[2,2] + 3 \times 5 \times 2 = 30, && \pi[1,2] = 1 \\
opt[2,3] &= opt[2,2] + opt[3,3] + 5 \times 2 \times 6 = 60, && \pi[2,3] = 2 \\
opt[3,4] &= opt[3,3] + opt[4,4] + 2 \times 6 \times 9 = 108, && \pi[3,4] = 3 \\
opt[4,5] &= opt[4,4] + opt[5,5] + 6 \times 9 \times 4 = 216, && \pi[4,5] = 4 \\
opt[1,3] &= \min\{opt[1,1] + opt[2,3] + 3 \times 5 \times 6, \\
&\qquad\quad opt[1,2] + opt[3,3] + 3 \times 2 \times 6\} \\
&= \min\{0 + 60 + 90, 30 + 0 + 36\} = 66, && \pi[1,3] = 2 \\
opt[2,4] &= \min\{opt[2,2] + opt[3,4] + 5 \times 2 \times 9, \\
&\qquad\quad opt[2,3] + opt[4,4] + 5 \times 6 \times 9\} \\
&= \min\{0 + 108 + 90, 60 + 0 + 270\} = 198, && \pi[2,4] = 2,
\end{aligned}$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[3,5] &= \min\{opt[3,3] + opt[4,5] + 2 \times 6 \times 4, \\
&\qquad\quad opt[3,4] + opt[5,5] + 2 \times 9 \times 4\} \\
&= \min\{0 + 216 + 48, 108 + 0 + 72\} = 180, \\
\pi[3,5] &= 4, \\
opt[1,4] &= \min\{opt[1,1] + opt[2,4] + 3 \times 5 \times 9, \\
&\qquad\quad opt[1,2] + opt[3,4] + 3 \times 2 \times 9, \\
&\qquad\quad opt[1,3] + opt[4,4] + 3 \times 6 \times 9\} \\
&= \min\{0 + 198 + 135, 30 + 108 + 54, 66 + 0 + 162\} = 192, \\
\pi[1,4] &= 2,
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[2,5] &= \min\{opt[2,2] + opt[3,5] + 5 \times 2 \times 4, \\
&\qquad opt[2,3] + opt[4,5] + 5 \times 6 \times 4, \\
&\qquad opt[2,4] + opt[5,5] + 5 \times 9 \times 4\} \\
&= \min\{0 + 180 + 40, 60 + 216 + 120, 198 + 0 + 180\} = 220, \\
opt[1,5] &= \min\{opt[1,1] + opt[2,5] + 3 \times 5 \times 4, \\
&\qquad opt[1,2] + opt[3,5] + 3 \times 2 \times 4, \\
&\qquad opt[1,3] + opt[4,5] + 3 \times 6 \times 4, \\
&\qquad opt[1,4] + opt[5,5] + 3 \times 9 \times 4\} \\
&= \min\{0 + 220 + 60, 30 + 180 + 24, \\
&\qquad 66 + 216 + 72, 192 + 0 + 108\} \\
&= 234, \\
\pi[1,5] &= 2.
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

opt, π	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0, /	30, 1	66, 2	192, 2	234, 2
$i = 2$		0, /	60, 2	198, 2	220, 2
$i = 3$			0, /	108, 3	180, 4
$i = 4$				0, /	216, 4
$i = 5$					0, /

opt, π	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0, $/$	30, 1	66, 2	192, 2	234, 2
$i = 2$		0, $/$	60, 2	198, 2	220, 2
$i = 3$			0, $/$	108, 3	180, 4
$i = 4$				0, $/$	216, 4
$i = 5$					0, $/$

Print-Optimal-Order(1,5)
- Print-Optimal-Order(1, 2)
  - Print-Optimal-Order(1, 1)
  - Print-Optimal-Order(2, 2)
- Print-Optimal-Order(3, 5)
  - Print-Optimal-Order(3, 4)
    - Print-Optimal-Order(3, 3)
    - Print-Optimal-Order(4, 4)
  - Print-Optimal-Order(5, 5)

Optimum way for multiplication: $((A_1A_2)((A_3A_4)A_5))$

Outline

Optimum Binary Search Tree

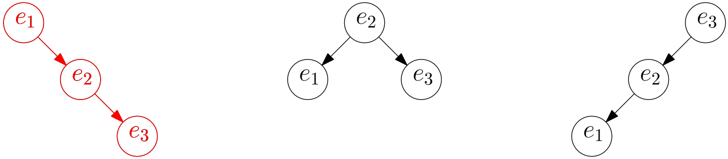
- n elements $e_1 < e_2 < e_3 < \cdots < e_n$
- e_i has frequency f_i
- goal: build a binary search tree for $\{e_1, e_2, \cdots, e_n\}$ with the minimum accessing cost:

$$\sum_{i=1}^{n} f_i \times (\text{depth of } e_i \text{ in the tree})$$

- motivation: the time to access e_i in the tree is linear in the depth of e_i

Optimum Binary Search Tree

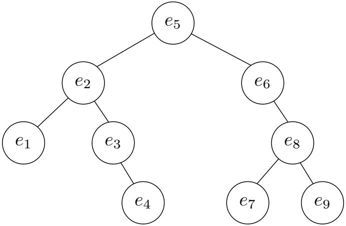
- Example: $f_1 = 10, f_2 = 5, f_3 = 3$



- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

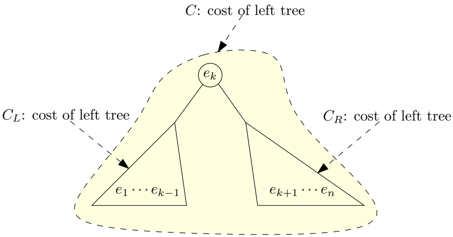
- suppose we decided to let e_k be the root
- $e_1, e_2, \cdots, e_{k-1}$ are on left sub-tree
- $e_{k+1}, e_{k+2}, \cdots, e_n$ are on right sub-tree

- d_j: depth of e_j in our tree
- C, C_L, C_R: cost of tree, left sub-tree and right sub-tree



- $d_1 = 3, d_2 = 2, d_3 = 3, d_4 = 4, d_5 = 1,$
- $d_6 = 2, d_7 = 4, d_8 = 3, d_9 = 4,$
- $C = 3f_1 + 2f_2 + 3f_3 + 4f_4 + f_5 + 2f_6 + 4f_7 + 3f_8 + 4f_9$
- $C_L = 2f_1 + f_2 + 2f_3 + 3f_4$
- $C_R = f_6 + 3f_7 + 2f_8 + 3f_9$

- $C = C_L + C_R + \sum_{j=1}^{9} f_j$



$$
\begin{aligned}
C = \sum_{\ell=1}^{n} f_\ell d_\ell &= \sum_{\ell=1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= \sum_{\ell=1}^{k-1} f_\ell (d_\ell - 1) + \sum_{\ell=k+1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= C_L + C_R + \sum_{\ell=1}^{n} f_\ell
\end{aligned}
$$

$$C = C_L + C_R + \sum_{\ell=1}^{n} f_\ell$$

- In order to minimize C, need to minimize C_L and C_R respectively
- $opt[i,j]$: the optimum cost for the instance $(f_i, f_{i+1}, \cdots, f_j)$

$$opt[1,n] = \min_{k:1\le k\le n} (opt[1,k-1] + opt[k+1,n]) + \sum_{\ell=1}^{n} f_\ell$$

- In general, $opt[i,j] =$

$$\begin{cases} 0 & \text{if } i = j+1 \\ \min_{k:i\le k\le j} \big(opt[i,k-1] + opt[k+1,j]\big) + \sum_{\ell=i}^{j} f_\ell & \text{if } i \le j \end{cases}$$

Optimum Binary Search Tree

$fsum[0] \leftarrow 0$

for $i \leftarrow 1$ to n **do** $fsum[i] \leftarrow fsum[i-1] + f_i$ $\quad \triangleright\ fsum[i] = \sum_{j=1}^{i} f_j$

for $i \leftarrow 0$ to n **do** $opt[i+1, i] \leftarrow 0$

for $\ell \leftarrow 1$ to n **do**

  for $i \leftarrow 1$ to $n - \ell + 1$ **do**

    $j \leftarrow i + \ell - 1$, $opt[i, j] \leftarrow \infty$

    for $k \leftarrow i$ to j **do**

      if $opt[i, k-1] + opt[k+1, j] < opt[i, j]$ **then**

        $opt[i, j] \leftarrow opt[i, k-1] + opt[k+1, j]$

        $\pi[i, j] \leftarrow k$

    $opt[i, j] \leftarrow opt[i, j] + fsum[j] - fsum[i-1]$

Printing the Tree

Print-Tree(i, j)

```
if i > j then
    return
else
    print('(')
    Print-Tree(i, π[i, j] − 1)
    print(π[i, j])
    Print-Tree(π[i, j] + 1, j)
    print(')')
```

Outline

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Comparison with greedy algorithms

- Greedy algorithm: each step is making a small progress towards constructing the solution
- Dynamic programming: the whole solution is constructed in the last step

Comparison with divide and conquer

- Divide and conquer: an instance is broken into many independent sub-instances, which are solved separately.
- Dynamic programming: the sub-instances we constructed are overlapping.

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i] =$ value of instance defined by jobs $\{1, 2, \cdots, i\}$
- Segmented Least Square: $opt[i] =$ cost of instance defined by first i points.
- Subset sum, knapsack: $opt[i, W'] =$ value of instance with items $\{1, 2, \cdots, i\}$ and budget W'
- Longest common subsequence: $opt[i, j] =$ value of instance defined by $A[1..i]$ and $B[1..j]$
- Shortest paths in DAG: $f[v] =$ length of shortest path from s to v
- Matrix chain multiplication, optimum binary search tree: $opt[i, j] =$ value of instances defined by matrices i to j