算法设计与分析(2024年春季学期)

Graph Algorithms

授课老师: 栗师
南京大学计算机科学与技术系
Outline

1 Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2 Single Source Shortest Paths
   - Dijkstra’s Algorithm

3 Shortest Paths in Graphs with Negative Weights

4 All-Pair Shortest Paths and Floyd-Warshall

5 Minimum Cost Arborescence
Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of $G$ is a sub-graph of $G$ that is a tree including all vertices $V$. 
Lemma  Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- $T$ is a spanning tree of $G$;
- $T$ is acyclic and connected;
- $T$ is connected and has $n - 1$ edges;
- $T$ is acyclic and has $n - 1$ edges;
- $T$ is minimally connected: removal of any edge disconnects it;
- $T$ is maximally acyclic: addition of any edge creates a cycle;
- $T$ has a unique simple path between every pair of nodes.
Minimum Spanning Tree (MST) Problem

**Input:** Graph \( G = (V, E) \) and edge weights \( w : E \rightarrow \mathbb{R} \)

**Output:** the spanning tree \( T \) of \( G \) with the minimum total weight
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Recall: Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice
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Two Classic Greedy Algorithms for MST

- Kruskal’s Algorithm
- Prim’s Algorithm
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Q: Which edge can be safely included in the MST?
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A: The edge with the smallest weight (lightest edge).
**Lemma**  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof. Take a minimum spanning tree $T$. Assume the lightest edge $e^*$ is not in $T$. There is a unique path in $T$ connecting $u$ and $v$. Remove any edge $e$ in the path to obtain tree $T'$. $w(e^*) \leq w(e) \Rightarrow w(T') \leq w(T)$: $T'$ is also a MST.
Lemma  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree $T$
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\[ w(e^*) \leq w(e) \Rightarrow w(T') \leq w(T) : T' \text{ is also a MST} \]
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- Take a minimum spanning tree $T$
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- Take a minimum spanning tree $T$
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- Remove any edge $e$ in the path to obtain tree $T'$
- $w(e^*) \leq w(e) \implies w(T') \leq w(T)$: $T'$ is also a MST
Is the Residual Problem Still a MST Problem?

Residual problem: find the minimum spanning tree that contains edge \((g, h)\).

Contract the edge \((g, h)\).

Residual problem: find the minimum spanning tree in the contracted graph.
Is the Residual Problem Still a MST Problem?

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Is the Residual Problem Still a MST Problem?

Residual problem: find the minimum spanning tree that contains edge \((g, h)\)

Contract the edge \((g, h)\)

Residual problem: find the minimum spanning tree in the contracted graph
Contraction of an Edge \((u, v)\)

Remove \(u\) and \(v\) from the graph, and add a new vertex \(u^*\).

Remove all edges \((u, v)\) from \(E\).

For every edge \((u, w)\) \(\in E\), \(w \neq v\), change it to \((u^*, w)\).

For every edge \((v, w)\) \(\in E\), \(w \neq u\), change it to \((u^*, w)\).

May create parallel edges! E.g.: two edges \((i, g^*)\).
Contraction of an Edge \((u, v)\)

- Remove \(u\) and \(v\) from the graph, and add a new vertex \(u^*\)
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- May create parallel edges! E.g. : two edges \((i, g^*)\)
Greedy Algorithm

Repeat the following step until $G$ contains only one vertex:

1. Choose the lightest edge $e^*$, add $e^*$ to the spanning tree
2. Contract $e^*$ and update $G$ be the contracted graph

Q: What edges are removed due to contractions?
A: Edge $(u, v)$ is removed if and only if there is a path connecting $u$ and $v$ formed by edges we selected
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Greedy Algorithm

MST-Greedy($G, w$)

1: $F \leftarrow \emptyset$
2: sort edges in $E$ in non-decreasing order of weights $w$
3: for each edge $(u, v)$ in the order do
4: if $u$ and $v$ are not connected by a path of edges in $F$ then
5: $F \leftarrow F \cup \{(u, v)\}$
6: return $(V, F)$
Sets:  \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}
Kruskal’s Algorithm: Example

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Sets: \( \{a, b\} \), \( \{c, i, f, g, h\} \), \( \{d\} \), \( \{e\} \)
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Kruskal’s Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal\((G, w)\)

1: \(F \leftarrow \emptyset\)
2: \(S \leftarrow \{\{v\} : v \in V\}\)
3: sort the edges of \(E\) in non-decreasing order of weights \(w\)
4: for each edge \((u, v) \in E\) in the order do
5: \(S_u \leftarrow\) the set in \(S\) containing \(u\)
6: \(S_v \leftarrow\) the set in \(S\) containing \(v\)
7: if \(S_u \neq S_v\) then
8: \(F \leftarrow F \cup \{(u, v)\}\)
9: \(S \leftarrow S \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}\)
10: return \((V, F)\)
Running Time of Kruskal’s Algorithm

**MST-Kruskal**$(G, w)$

1. $F \leftarrow \emptyset$
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3. sort the edges of $E$ in non-decreasing order of weights $w$
4. **for** each edge $(u, v) \in E$ in the order **do**
5. \hspace{1em} $S_u \leftarrow$ the set in $S$ containing $u$
6. \hspace{1em} $S_v \leftarrow$ the set in $S$ containing $v$
7. \hspace{1em} **if** $S_u \neq S_v$ **then**
8. \hspace{2em} $F \leftarrow F \cup \{(u, v)\}$
9. \hspace{2em} $S \leftarrow S \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10. **return** $(V, F)$

Use **union-find** data structure to support 2, 5, 6, 7, 9.
Union-Find Data Structure

- $V$: ground set

We need to maintain a partition of $V$ and support following operations:
- Check if $u$ and $v$ are in the same set of the partition
- Merge two sets in partition
- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$

- $par[i]$: parent of $i$, ($par[i] = \bot$ if $i$ is a root).
Union-Find Data Structure

Q: how can we check if \( u \) and \( v \) are in the same set?

A: Check if \( \text{root}(u) = \text{root}(v) \).

\( \text{root}(u) \): the root of the tree containing \( u \)

Merge the trees with root \( r \) and \( r' \):

\( \text{par}[r] \leftarrow r' \).
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$\text{root}(u)$: the root of the tree containing $u$

Merge the trees with root $r$ and $r'$: $\text{par}[r] \leftarrow r'$. 
Union-Find Data Structure

**root(v)**

1. if $par[v] = \bot$ then
2. return $v$
3. else
4. return root($par[v]$)

Problem: the tree might be too deep; running time might be large.

Improvement: all vertices in the path directly point to the root, saving time in the future.
Union-Find Data Structure

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root(\(v\))

1: \textbf{if} \(par[v] = \bot\) \textbf{then}
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Union-Find Data Structure

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root(v)
1: if \( \text{par}[v] = \perp \) then
2: return \( v \)
3: else
4: return \( \text{root}(\text{par}[v]) \)
root(\nu\)\\
1: \textbf{if} \ par[\nu] = \perp \ \textbf{then} \\
2: \ \textbf{return} \ \nu \\
3: \ \textbf{else} \\
4: \ par[\nu] \leftarrow \text{root}(par[\nu]) \\
5: \ \textbf{return} \ par[\nu]

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root(v)

1: if $par[v] = \perp$ then
2: return $v$
3: else
4: $par[v] \leftarrow \text{root}(par[v])$
5: return $par[v]$

---

![Diagram](image-url)
MST-Kruskal($G, w$)

1. $F \leftarrow \emptyset$
2. $S \leftarrow \{\{v\} : v \in V\}$
3. sort the edges of $E$ in non-decreasing order of weights $w$
4. for each edge $(u, v) \in E$ in the order do
5. \quad $S_u \leftarrow$ the set in $S$ containing $u$
6. \quad $S_v \leftarrow$ the set in $S$ containing $v$
7. if $S_u \neq S_v$ then
8. \quad $F \leftarrow F \cup \{(u, v)\}$
9. \quad $S \leftarrow S \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10. return $(V, F)$
MST-Kruskal($G, w$)

1: $F \leftarrow \emptyset$
2: for every $v \in V$ do: $par[v] \leftarrow \bot$
3: sort the edges of $E$ in non-decreasing order of weights $w$
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5: $u' \leftarrow \text{root}(u)$
6: $v' \leftarrow \text{root}(v)$
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10: return $(V, F)$

- \textbf{2, 5, 6, 7, 9} takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$. 
**MST-Kruskal**($G, w$)

1. $F \leftarrow \emptyset$
2. **for** every $v \in V$ **do**: $\text{par}[v] \leftarrow \bot$
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   5. $u' \leftarrow \text{root}(u)$
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   7. **if** $u' \neq v'$ **then**
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- **2, 5, 6, 7, 9** takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time $= \text{time for }$ **3** $= O(m \log n)$. 
**Assumption**  Assume all edge weights are different.

**Lemma**  An edge $e \in E$ is not in the MST, if and only if there is cycle $C$ in $G$ in which $e$ is the heaviest edge.
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- $(i, g)$ is not in the MST because of cycle $(i, c, f, g)$
**Assumption**  Assume all edge weights are different.

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- $(i, g)$ is not in the MST because of cycle $(i, c, f, g)$
- $(e, f)$ is in the MST because no such cycle exists
Outline

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5. **Minimum Cost Arborescence**
Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree.
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2. Start from $F \leftarrow E$, and remove edges from $F$ one by one until we obtain a spanning tree.

Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def.: A bridge is an edge whose removal disconnects the graph.
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Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.
**Lemma**  It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.
Reverse Kruskal’s Algorithm

MST-Greedy\((G, w)\)

1: \(F \leftarrow E\)
2: sort \(E\) in non-increasing order of weights
3: for every \(e\) in this order do
4: \hspace{1em} if \((V, F \setminus \{e\})\) is connected then
5: \hspace{2em} \(F \leftarrow F \setminus \{e\}\)
6: return \((V, F)\)
Reverse Kruskal’s Algorithm: Example
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The diagram shows a graph with labeled nodes and edges representing weights. The graph includes nodes labeled as follows: a, b, c, d, e, f, g, h, and i. The edges and their weights are as follows:

- a to b: 5
- a to h: 7
- b to c: 8
- b to i: 2
- c to f: 4
- c to i: 6
- d to e: 9
- f to g: 3
- g to i: 1
- h to i: 1
- h to g: 3

The graph is displayed with the weights associated with each edge, demonstrating the structure used in Reverse Kruskal’s Algorithm.
Reverse Kruskal’s Algorithm: Example
Reverse Kruskal’s Algorithm: Example
Reverse Kruskal’s Algorithm: Example
Reverse Kruskal’s Algorithm: Example
Reverse Kruskal’s Algorithm: Example
Outline

1. **Minimum Spanning Tree**
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2. **Single Source Shortest Paths**
   - Dijkstra’s Algorithm

3. **Shortest Paths in Graphs with Negative Weights**

4. **All-Pair Shortest Paths and Floyd-Warshall**

5. **Minimum Cost Arborescence**
Recall the greedy strategy for Kruskal’s algorithm: choose the edge with the smallest weight.
Recall the greedy strategy for Kruskal’s algorithm: choose the edge with the smallest weight.

Greedy strategy for Prim’s algorithm: choose the lightest edge incident to $a$. 
**Design Greedy Strategy for MST**

- Recall the greedy strategy for Kruskal’s algorithm: choose the edge with the smallest weight.

- Greedy strategy for Prim’s algorithm: choose the lightest edge incident to \( a \).
Lemma  It is safe to include the lightest edge incident to $a$. 

Proof. Let $T$ be a MST. Consider all components obtained by removing $a$ from $T$. Let $e^*$ be the lightest edge incident to $a$ and $e^*$ connects $a$ to component $C$. Let $e$ be the edge in $T$ connecting $a$ to $C$. Then $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$. 

Lemma It is safe to include the lightest edge incident to $a$.

Proof.

- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
Lemma  It is safe to include the lightest edge incident to $a$.

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- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
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Proof.

- Let \( T \) be a MST.
- Consider all components obtained by removing \( a \) from \( T \).
- Let \( e^* \) be the lightest edge incident to \( a \) and \( e^* \) connects \( a \) to component \( C \).
- Let \( e \) be the edge in \( T \) connecting \( a \) to \( C \).
Lemma  It is safe to include the lightest edge incident to $a$.

Proof.

- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
- Let $e^*$ be the lightest edge incident to $a$ and $e^*$ connects $a$ to component $C$
- Let $e$ be the edge in $T$ connecting $a$ to $C$
- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$
Prim’s Algorithm: Example
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Prim’s Algorithm: Example
Greedy Algorithm

MST-Greedy1\((G, w)\)

1: \( S \leftarrow \{s\} \), where \( s \) is arbitrary vertex in \( V \)
2: \( F \leftarrow \emptyset \)
3: while \( S \neq V \) do
4: \((u, v) \leftarrow \) lightest edge between \( S \) and \( V \setminus S \),
    where \( u \in S \) and \( v \in V \setminus S \)
5: \( S \leftarrow S \cup \{v\} \)
6: \( F \leftarrow F \cup \{(u, v)\} \)
7: return \((V, F)\)

Running time of naive implementation: \(O\left(nm\right)\)
Greedy Algorithm

**MST-Greedy1(G, w)**

1. $S \leftarrow \{s\}$, where $s$ is arbitrary vertex in $V$
2. $F \leftarrow \emptyset$
3. while $S \neq V$ do
4.   $(u, v) \leftarrow$ lightest edge between $S$ and $V \setminus S$, where $u \in S$ and $v \in V \setminus S$
5.   $S \leftarrow S \cup \{v\}$
6.   $F \leftarrow F \cup \{(u, v)\}$
7. return $(V, F)$

**Running time of naive implementation:** $O(nm)$
Prim’s Algorithm: Efficient Implementation of Greedy Algorithm

For every \( v \in V \setminus S \) maintain

- \( d[v] = \min_{u \in S: (u, v) \in E} w(u, v) \): the weight of the lightest edge between \( v \) and \( S \)
- \( \pi[v] = \arg \min_{u \in S: (u, v) \in E} w(u, v) \): \((\pi[v], v)\) is the lightest edge between \( v \) and \( S \)
Prim’s Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S : (u,v) \in E} w(u,v)$: the weight of the lightest edge between $v$ and $S$
- $\pi[v] = \arg \min_{u \in S : (u,v) \in E} w(u,v)$: $(\pi[v], v)$ is the lightest edge between $v$ and $S$

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to $F$
- Add $u$ to $S$, update $d$ and $\pi$ values.
Prim’s Algorithm

**MST-Prim**(\(G, w\))

1: \(s \leftarrow \text{arbitrary vertex in } G\)
2: \(S \leftarrow \emptyset, d(s) \leftarrow 0 \text{ and } d[v] \leftarrow \infty \text{ for every } v \in V \setminus \{s\}\)
3: \(\textbf{while } S \neq V \textbf{ do}\)
4: \(u \leftarrow \text{vertex in } V \setminus S \text{ with the minimum } d[u]\)
5: \(S \leftarrow S \cup \{u\}\)
6: \(\textbf{for each } v \in V \setminus S \text{ such that } (u, v) \in E \textbf{ do}\)
7: \(\textbf{if } w(u, v) < d[v] \textbf{ then}\)
8: \(d[v] \leftarrow w(u, v)\)
9: \(\pi[v] \leftarrow u\)
10: \(\textbf{return } \{(u, \pi[u]) | u \in V \setminus \{s\}\}\)
Example
Example
Example
Example

(5, a)

a

b

c

d

e

(12, a)

h

i

f

g

5

8

13

2

7

11

1

6

4

3

9

10

14

12

11

14

9

10

12

5

7

6

2

4


Example

(5, a)

(12, a)
Example

\begin{center}
\begin{tikzpicture}
\node[shape=circle,draw=black] (a) at (1,8) {$a$};
\node[shape=circle,draw=black] (b) at (1,6) {$b$};
\node[shape=circle,draw=black] (i) at (2.5,4) {$i$};
\node[shape=circle,draw=black] (c) at (4,6) {$c$};
\node[shape=circle,draw=black] (d) at (7,6) {$d$};
\node[shape=circle,draw=black] (f) at (5.5,4) {$f$};
\node[shape=circle,draw=black] (g) at (4,2) {$g$};
\node[shape=circle,draw=black] (h) at (1.5,2) {$h$};
\node[shape=circle,draw=black] (e) at (8,4) {$e$};

\path[draw=black]
(a) edge node [above left] {$5$} (b)
(b) edge node [above right] {$8$} (c)
(h) edge node [left] {$11$} (i)
(b) edge node [above right] {$8$} (c)
(c) edge node [right] {$13$} (d)
(c) edge node [right] {$4$} (i)
(i) edge node [right] {$6$} (g)
(i) edge node [right] {$7$} (h)
(h) edge node [left] {$1$} (g)
(g) edge node [right] {$3$} (f)
(d) edge node [right] {$14$} (e)
(f) edge node [right] {$10$} (e);
\end{tikzpicture}
\end{center}

\begin{itemize}
\item \((8, b)\)
\item \((11, b)\)
\end{itemize}
Example
Example
Example
Example
Example
Example

\[ (7, i) \]

\[ (6, i) \]

\[ (4, c) \]

\[ (13, c) \]
Example

\[
\begin{align*}
\text{(13, c)} & \quad (7, i) & \quad (4, c) \\
1 & \quad 2 & \quad 3 \\
\end{align*}
\]
Example

```
(13, c)
(7, i)
(4, c)
(6, i)
(7, i)
(6, i)
(4, c)
```
Example
Example
Example
Example
Example

\[
\begin{array}{ccccccc}
\text{a} & \text{i} & \text{b} & \text{h} & \text{g} & \text{c} & \text{d} \\
5 & 8 & 13 & 1 & 6 & 4 & 3 \\
\end{array}
\]

(13, c) （10, f)
Example
Example
Example

![Graph Image]

- Vertices: a, b, c, d, e, f, g, h, i
- Edges with weights: 1, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14
- Edges highlighted with red color
- Edge labels: (10, f), (13, c)
Example

\[
(9, e)
\]
Example

![Graph](image-url)
Example
Example
Prim’s Algorithm

For every \( v \in V \setminus S \) maintain

- \( d[v] = \min_{u \in S: (u,v) \in E} w(u, v) \): the weight of the lightest edge between \( v \) and \( S \)
- \( \pi[v] = \arg \min_{u \in S: (u,v) \in E} w(u, v) \): \( (\pi[v], v) \) is the lightest edge between \( v \) and \( S \)

In every iteration

- Pick \( u \in V \setminus S \) with the smallest \( d[u] \) value
- Add \( (\pi[u], u) \) to \( F \)
- Add \( u \) to \( S \), update \( d \) and \( \pi \) values.
Prim’s Algorithm

For every $v \in V \setminus S$ maintain
- $d[v] = \min_{u \in S: (u,v) \in E} w(u, v)$: the weight of the lightest edge between $v$ and $S$
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  $(\pi[v], v)$ is the lightest edge between $v$ and $S$

In every iteration
- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to $F$
- Add $u$ to $S$, update $d$ and $\pi$ values.

Use a priority queue to support the operations
Def. A priority queue is an abstract data structure that maintains a set $U$ of elements, each with an associated key value, and supports the following operations:

- $\text{insert}(v, key\_value)$: insert an element $v$, whose associated key value is $key\_value$.
- $\text{decrease\_key}(v, new\_key\_value)$: decrease the key value of an element $v$ in queue to $new\_key\_value$.
- $\text{extract\_min}()$: return and remove the element in queue with the smallest key value.

...
Prim’s Algorithm

**MST-Prim**\((G, w)\)

1: \(s \leftarrow\) arbitrary vertex in \(G\)
2: \(S \leftarrow \emptyset\), \(d(s) \leftarrow 0\) and \(d[v] \leftarrow \infty\) for every \(v \in V \setminus \{s\}\)
3: 
4: \textbf{while } S \neq V \textbf{ do}
5: \quad u \leftarrow\) vertex in \(V \setminus S\) with the minimum \(d[u]\)
6: \quad S \leftarrow S \cup \{u\}
7: \quad\textbf{for each } v \in V \setminus S\) such that \((u, v) \in E\) \textbf{ do}
8: \quad\quad \textbf{if } w(u, v) < d[v] \textbf{ then}
9: \quad\quad\quad d[v] \leftarrow w(u, v)
10: \quad\quad\quad \pi[v] \leftarrow u
11: \textbf{return } \{(u, \pi[u])|u \in V \setminus \{s\}\}
Prim’s Algorithm Using Priority Queue

MST-Prim\((G, w)\)

1: \(s \leftarrow\) arbitrary vertex in \(G\)
2: \(S \leftarrow \emptyset, d(s) \leftarrow 0\) and \(d[v] \leftarrow \infty\) for every \(v \in V \setminus \{s\}\)
3: \(Q \leftarrow\) empty queue, for each \(v \in V: Q.\text{insert}(v, d[v])\)
4: \textbf{while} \(S \neq V\) \textbf{do}
5: \(u \leftarrow Q.\text{extract}\_\text{min}()\)
6: \(S \leftarrow S \cup \{u\}\)
7: \textbf{for} each \(v \in V \setminus S\) such that \((u, v) \in E\) \textbf{do}
8: \hspace{1em} \textbf{if} \(w(u, v) < d[v]\) \textbf{then}
9: \hspace{2em} \(d[v] \leftarrow w(u, v), Q.\text{decrease\_key}(v, d[v])\)
10: \hspace{1em} \(\pi[v] \leftarrow u\)
11: \textbf{return} \((u, \pi[u])|u \in V \setminus \{s\}\)
Running Time of Prim’s Algorithm Using Priority Queue

\[ O(n) \times \text{(time for extract\_min)} + O(m) \times \text{(time for decrease\_key)} \]
Running Time of Prim’s Algorithm Using Priority Queue

\[ O(n) \times (\text{time for extract\_min}) + O(m) \times (\text{time for decrease\_key}) \]

<table>
<thead>
<tr>
<th>concrete DS</th>
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<th>decrease_key</th>
<th>overall time</th>
</tr>
</thead>
<tbody>
<tr>
<td>heap</td>
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<td>(O(m \log n))</td>
</tr>
<tr>
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Running Time of Prim’s Algorithm Using Priority Queue

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**Assumption**  Assume all edge weights are different.

**Lemma**  \((u, v)\) is in MST, if and only if there exists a cut \((U, V \setminus U)\), such that \((u, v)\) is the lightest edge between \(U\) and \(V \setminus U\).
**Assumption**  Assume all edge weights are different.

**Lemma**  

\((u, v)\) is in MST, if and only if there exists a **cut** \((U, V \setminus U)\), such that \((u, v)\) is the lightest edge between \(U\) and \(V \setminus U\).

\[
\begin{align*}
(c, f) & \text{ is in MST because of cut } \left(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\right)
\end{align*}
\]
**Assumption** Assume all edge weights are different.

**Lemma** \((u, v)\) is in MST, if and only if there exists a cut \((U, V \setminus U)\), such that \((u, v)\) is the lightest edge between \(U\) and \(V \setminus U\).

- \((c, f)\) is in MST because of cut \(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\)
- \((i, g)\) is not in MST because no such cut exists
“Evidence” for $e \in \text{MST}$ or $e \notin \text{MST}$

**Assumption**  Assume all edge weights are different.

- $e \in \text{MST} \leftrightarrow$ there is a cut in which $e$ is the lightest edge
- $e \notin \text{MST} \leftrightarrow$ there is a cycle in which $e$ is the heaviest edge
Assumption Assume all edge weights are different.

- $e \in \text{MST} \iff$ there is a cut in which $e$ is the lightest edge
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Exactly one of the following is true:

- There is a cut in which $e$ is the lightest edge
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Exactly one of the following is true:
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Thus, the minimum spanning tree is unique with assumption.
Outline

1 Minimum Spanning Tree
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   - Reverse-Kruskal’s Algorithm
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2 Single Source Shortest Paths
   - Dijkstra’s Algorithm

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4 All-Pair Shortest Paths and Floyd-Warshall

5 Minimum Cost Arborescence
<table>
<thead>
<tr>
<th>algorithm</th>
<th>graph</th>
<th>weights</th>
<th>SS?</th>
<th>running time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple DP</td>
<td>DAG</td>
<td>$\mathbb{R}$</td>
<td>SS</td>
<td>$O(n + m)$</td>
</tr>
<tr>
<td>Dijkstra</td>
<td>U/D</td>
<td>$\mathbb{R}_{\geq 0}$</td>
<td>SS</td>
<td>$O(n \log n + m)$</td>
</tr>
<tr>
<td>Bellman-Ford</td>
<td>U/D</td>
<td>$\mathbb{R}$</td>
<td>SS</td>
<td>$O(nm)$</td>
</tr>
<tr>
<td>Floyd-Warshall</td>
<td>U/D</td>
<td>$\mathbb{R}$</td>
<td>AP</td>
<td>$O(n^3)$</td>
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- DAG = directed acyclic graph
- U = undirected
- D = directed
- SS = single source
- AP = all pairs
\textbf{s-t Shortest Paths}

**Input:** (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

**Output:** shortest path from $s$ to $t$
**s-t Shortest Paths**

**Input:** (directed or undirected) graph $G = (V, E)$, $s, t \in V$

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\[ s-t \text{ Shortest Paths} \]

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\( w : E \rightarrow \mathbb{R}_{\geq 0} \)

**Output:** shortest path from \( s \) to \( t \)

---

![Graph with labels](image)
Single Source Shortest Paths

**Input:** (directed or undirected) graph $G = (V, E)$, $s \in V$

$$w : E \rightarrow \mathbb{R}_{\geq 0}$$

**Output:** shortest paths from $s$ to all other vertices $v \in V$
Single Source Shortest Paths

**Input:** (directed or undirected) graph $G = (V, E)$, $s \in V$

\[ w : E \rightarrow \mathbb{R}_{\geq 0} \]

**Output:** shortest paths from $s$ to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve $s$-$t$ shortest path problem more efficiently than solving single source shortest path problem
**Single Source Shortest Paths**

**Input:** (directed or undirected) graph $G = (V, E)$, $s \in V$

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- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight
Single Source Shortest Paths

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Single Source Shortest Paths

**Input:** directed graph $G = (V, E)$, $s \in V$

$$w : E \rightarrow \mathbb{R}_{\geq 0}$$

**Output:**
- $\pi[v], v \in V \setminus s$: the parent of $v$ in shortest path tree
- $d[v], v \in V \setminus s$: the length of shortest path from $s$ to $v$
Q: How to compute shortest paths from $s$ when all edges have weight 1?
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A: Breadth first search (BFS) from source $s$
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Assumption  Weights $w(u, v)$ are integers (w.l.o.g.).
**Assumption**  Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges.

```
          4
u -------v
          1  1  1  1  1
u -----> v
```
Assumption  Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges.

![Diagram](image)

Shortest Path Algorithm by Running BFS

1: replace $(u, v)$ of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2: run BFS
3: $\pi[v] \leftarrow$ vertex from which $v$ is visited
4: $d[v] \leftarrow$ index of the level containing $v$
Assumption  Weights $w(u, v)$ are integers (w.l.o.g.).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges.

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- Problem: $w(u, v)$ may be too large!
Assumption  Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges

Shortest Path Algorithm by Running BFS

1. replace $(u, v)$ of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2. run BFS virtually
3. $\pi[v] \leftarrow$ vertex from which $v$ is visited
4. $d[v] \leftarrow$ index of the level containing $v$

- Problem: $w(u, v)$ may be too large!
Shortest Path Algorithm by Running BFS Virtually

1: \( S \leftarrow \{s\}, d(s) \leftarrow 0 \)
2: \textbf{while} \(|S| \leq n\) \textbf{do}
3: \quad \textbf{find a} \( v \notin S \) \textbf{that minimizing} \( \min_{u \in S: (u,v) \in E} \{d[u] + w(u, v)\} \)
4: \quad \( S \leftarrow S \cup \{v\} \)
5: \quad \( d[v] \leftarrow \min_{u \in S: (u,v) \in E} \{d[u] + w(u, v)\} \)
Virtual BFS: Example
Virtual BFS: Example

Time 0
Virtual BFS: Example

Time 2
Virtual BFS: Example

Time 4
Virtual BFS: Example

Time 7
Virtual BFS: Example

Time 9
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Dijkstra’s Algorithm

Dijkstra\((G, w, s)\)

1: \( S \leftarrow \emptyset, d(s) \leftarrow 0 \) and \( d[v] \leftarrow \infty \) for every \( v \in V \setminus \{s\} \)
2: while \( S \neq V \) do
3: \( u \leftarrow \) vertex in \( V \setminus S \) with the minimum \( d[u] \)
4: add \( u \) to \( S \)
5: for each \( v \in V \setminus S \) such that \((u, v) \in E\) do
6: if \( d[u] + w(u, v) < d[v] \) then
7: \( d[v] \leftarrow d[u] + w(u, v) \)
8: \( \pi[v] \leftarrow u \)
9: return \((d, \pi)\)

Running time = \(O(n^2)\)
Dijkstra’s Algorithm

\[\text{Dijkstra}(G, w, s)\]

1: \( S \leftarrow \emptyset, d(s) \leftarrow 0 \) and \( d[v] \leftarrow \infty \) for every \( v \in V \setminus \{s\} \)
2: while \( S \neq V \) do
3: \( u \leftarrow \text{vertex in } V \setminus S \text{ with the minimum } d[u] \)
4: add \( u \) to \( S \)
5: for each \( v \in V \setminus S \text{ such that } (u, v) \in E \) do
6: \( \text{if } d[u] + w(u, v) < d[v] \) then
7: \( d[v] \leftarrow d[u] + w(u, v) \)
8: \( \pi[v] \leftarrow u \)
9: return \((d, \pi)\)

- Running time = \(O(n^2)\)
Improved Running Time using Priority Queue

**Dijkstra**$(G, w, s)$

1: $s \leftarrow$ arbitrary vertex in $G$
2: $S \leftarrow \emptyset$, $d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
3: $Q \leftarrow$ empty queue, for each $v \in V$: $Q$.insert($v, d[v]$)
4: **while** $S \neq V$ **do**
5: \hspace{1em} $u \leftarrow Q$.extract\_min()
6: \hspace{1em} $S \leftarrow S \cup \{u\}$
7: \hspace{1em} **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
8: \hspace{2em} **if** $d[u] + w(u, v) < d[v]$ **then**
9: \hspace{3em} $d[v] \leftarrow d[u] + w(u, v)$, $Q$.decrease\_key($v, d[v]$)
10: \hspace{2em}\hspace{1em} $\pi[v] \leftarrow u$
11: **return** $(\pi, d)$
Recall: Prim’s Algorithm for MST

**MST-Prim(\(G, w\))**

1. \(s \leftarrow \text{arbitrary vertex in } G\)
2. \(S \leftarrow \emptyset, d(s) \leftarrow 0 \text{ and } d[v] \leftarrow \infty \text{ for every } v \in V \setminus \{s\}\)
3. \(Q \leftarrow \text{empty queue, for each } v \in V: Q.\text{insert}(v, d[v])\)
4. while \(S \neq V\) do
5. \(u \leftarrow Q.\text{extract\_min()}\)
6. \(S \leftarrow S \cup \{u\}\)
7. for each \(v \in V \setminus S\) such that \((u, v) \in E\) do
8. if \(w(u, v) < d[v]\) then
9. \(d[v] \leftarrow w(u, v), Q.\text{decrease\_key}(v, d[v])\)
10. \(\pi[v] \leftarrow u\)
11. return \(\{(u, \pi[u]) | u \in V \setminus \{s\}\}\)
Improved Running Time

Running time:
\[ O(n) \times \text{(time for extract\_min)} + O(m) \times \text{(time for decrease\_key)} \]

<table>
<thead>
<tr>
<th>Priority-Queue</th>
<th>extract_min</th>
<th>decrease_key</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heap</td>
<td>( O(\log n) )</td>
<td>( O(\log n) )</td>
<td>( O(m \log n) )</td>
</tr>
<tr>
<td>Fibonacci Heap</td>
<td>( O(\log n) )</td>
<td>( O(1) )</td>
<td>( O(n \log n + m) )</td>
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Outline

1 Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2 Single Source Shortest Paths
   - Dijkstra’s Algorithm

3 Shortest Paths in Graphs with Negative Weights

4 All-Pair Shortest Paths and Floyd-Warshall

5 Minimum Cost Arborescence
**Input:** directed graph $G = (V, E)$, $s \in V$
assumed all vertices are reachable from $s$

$$w : E \rightarrow \mathbb{R}$$

**Output:** shortest paths from $s$ to all other vertices $v \in V$
Single Source Shortest Paths, Weights May be Negative

**Input:** directed graph $G = (V, E)$, $s \in V$

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- In transition graphs, negative weights make sense
**Input**: directed graph $G = (V, E)$, $s \in V$

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- If we sell a item: ‘having the item’ $\rightarrow$ ‘not having the item’, weight is negative (we gain money)
Single Source Shortest Paths, Weights May be Negative

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- In transition graphs, negative weights make sense
- If we sell a item: ‘having the item’ \( \rightarrow \) ‘not having the item’, weight is negative (we gain money)
- Dijkstra’s algorithm does not work any more!
Dijkstra’s Algorithm Fails if We Have Negative Weights
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Dijkstra’s Algorithm Fails if We Have Negative Weights
Q: What is the length of the shortest path from $s$ to $d$?

A: $-\infty$

Def. A negative cycle is a cycle in which the total weight of edges is negative.

Q: What is the length of the shortest simple path from $s$ to $d$?

A: 1
Q: What is the length of the shortest path from $s$ to $d$?
Q: What is the length of the shortest path from $s$ to $d$?

A: $-\infty$
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**Def.** A negative cycle is a cycle in which the total weight of edges is negative.

Q: What is the length of the shortest **simple** path from \( s \) to \( d \)?

A: 1
Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.

Dealing with Negative Cycles

We need to compute the shortest paths, among both simple and complex paths.

Hardest: output $-\infty$ as a distance

Easier: if negative cycle exists, allow algorithm to report "negative cycle exists" without computing distances

Easiest: assume negative cycles do not exist; all shortest paths are automatically simple paths
Unfortunately, computing the shortest simple path between two vertices is an **NP-hard** problem.
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<table>
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<th>graph</th>
<th>weights</th>
<th>SS?</th>
<th>running time</th>
</tr>
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<tr>
<td>Simple DP</td>
<td>DAG</td>
<td>$\mathbb{R}$</td>
<td>SS</td>
<td>$O(n + m)$</td>
</tr>
<tr>
<td>Dijkstra</td>
<td>U/D</td>
<td>$\mathbb{R}_{\geq 0}$</td>
<td>SS</td>
<td>$O(n \log n + m)$</td>
</tr>
<tr>
<td>Bellman-Ford</td>
<td>U/D</td>
<td>$\mathbb{R}$</td>
<td>SS</td>
<td>$O(nm)$</td>
</tr>
<tr>
<td>Floyd-Warshall</td>
<td>U/D</td>
<td>$\mathbb{R}$</td>
<td>AP</td>
<td>$O(n^3)$</td>
</tr>
</tbody>
</table>

- DAG = directed acyclic graph
- U = undirected
- D = directed
- SS = single source
- AP = all pairs
Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

**Input:** directed graph $G = (V, E)$, $s \in V$
- assume all vertices are reachable from $s$
- $w : E \rightarrow \mathbb{R}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$
## Single Source Shortest Paths, Weights May be Negative

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- first try: $f[v]$: length of shortest path from $s$ to $v$
Single Source Shortest Paths, Weights May be Negative

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- issue: do not know in which order we compute \( f[v]'s \)
Single Source Shortest Paths, Weights May be Negative

**Input:** directed graph $G = (V, E)$, $s \in V$
assume all vertices are reachable from $s$

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**Output:** shortest paths from $s$ to all other vertices $v \in V$

- first try: $f[v]$: length of shortest path from $s$ to $v$
- issue: do not know in which order we compute $f[v]$’s
- $f^\ell[v], \ell \in \{0, 1, 2, 3 \ldots, n - 1\}, v \in V$ : length of shortest path from $s$ to $v$ that uses at most $\ell$ edges
• $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n - 1\}$, $v \in V$ : length of shortest path from $s$ to $v$ that uses at most $\ell$ edges
\[ f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, v \in V: \]
length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

\[ f^2[a] = \]
\[ f^\ell[v], \quad \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, \quad v \in V: \]
length of shortest path from \( s \) to \( v \) that uses
at most \( \ell \) edges

\[ f^2[a] = 6 \]
\( f^\ell[v] \), \( \ell \in \{0, 1, 2, 3 \cdots, n - 1\} \), \( v \in V \): length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

- \( f^2[a] = 6 \)
- \( f^3[a] = \)
\[ f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, \ v \in V : \]

length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

- \( f^2[a] = 6 \)
- \( f^3[a] = 2 \)
\[ f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, v \in V : \text{length of shortest path from } s \text{ to } v \text{ that uses at most } \ell \text{ edges} \]

- \( f^2[a] = 6 \)
- \( f^3[a] = 2 \)

\[
f^\ell[v] = \begin{cases} 
0 & \ell = 0, v = s \\
\infty & \ell = 0, v \neq s \\
\min\left( f^\ell - 1[u] + w(u, v) \right) & \ell > 0 
\end{cases}
\]
\( f^\ell[v], \ell \in \{0, 1, 2, 3 \ldots, n - 1\}, v \in V \):

- length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

- \( f^2[a] = 6 \)

- \( f^3[a] = 2 \)

\[
f^\ell[v] = \begin{cases} 
0 & \ell = 0, v = s \\
\ell = 0, v \neq s \\
\ell > 0 
\end{cases}
\]
\[ f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, v \in V: \]

length of shortest path from \(s\) to \(v\) that uses at most \(\ell\) edges

- \(f^2[a] = 6\)
- \(f^3[a] = 2\)

\[
f^\ell[v] = \begin{cases} 
0 & \text{if } \ell = 0, v = s \\
\infty & \text{if } \ell = 0, v \neq s \\
\ell > 0 & \text{otherwise}
\end{cases}
\]
\[ f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots , n - 1\}, v \in V : \text{length of shortest path from } s \text{ to } v \text{ that uses at most } \ell \text{ edges} \]

\[ f^2[a] = 6 \]

\[ f^3[a] = 2 \]

\[ f^\ell[v] = \begin{cases} 
0 & \ell = 0, v = s \\
\infty & \ell = 0, v \neq s \\
\min & \ell > 0
\end{cases} \]
\( f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, v \in V : \) length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

- \( f^2[a] = 6 \)
- \( f^3[a] = 2 \)

\[
f^\ell[v] = \begin{cases} 
0 & \quad \ell = 0, v = s \\
\infty & \quad \ell = 0, v \neq s \\
\min \{ f^{\ell-1}[v] \} & \quad \ell > 0
\end{cases}
\]
\[ f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots , n - 1\}, v \in V : \text{length of shortest path from } s \text{ to } v \text{ that uses at most } \ell \text{ edges} \]

- \( f^2[a] = 6 \)
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\[ f^\ell[v] = \begin{cases} 
0 & \ell = 0, v = s \\
\infty & \ell = 0, v \neq s \\
\min \left\{ \min_{u:(u,v) \in E} \left( f^{\ell-1}[u] + w(u,v) \right) \right\} & \ell > 0 
\end{cases} \]
Dynamic Programming: Example

\[
\begin{array}{ccccccc}
 & s & & & & & \\
f^0 & 0 & a & b & c & d \\
\hline
s & & & & & & \\
b & 7 & 6 & & & & \\
a & & & 8 & & & \\
c & -4 & -3 & & -2 & & \\
d & & 7 & & & & \\
\end{array}
\]
Dynamic Programming: Example

\[ f^0 \]
\[ f^1 \]

\[ \begin{array}{c}
    s \\
    a \\
    b \\
    c \\
    d
\end{array} \]

\[ \begin{array}{cccccc}
    & 6 & 7 & 8 & -2 & -3 \\
0 & \infty & \infty & \infty & \infty & \infty \\
\end{array} \]
Dynamic Programming: Example

\[\begin{array}{c}
\begin{array}{c}
 s \\
 \downarrow 7 \\
 b \\
 \downarrow 8 \\
 a \\
 \downarrow 8 \\
 c \\
 \downarrow 7 \\
 d \\
 \end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
 f^0 \\
 s \\
 \downarrow 6 \\
 a \\
 \downarrow 7 \\
 b \\
 \downarrow 8 \\
 c \\
 \downarrow -4 \\
 d \\
 \end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
 f^1 \\
 s \\
 \downarrow 8 \\
 a \\
 \downarrow 8 \\
 b \\
 \downarrow 7 \\
 c \\
 \downarrow 7 \\
 d \\
 \end{array}
\end{array}\]

length-0 edge
Dynamic Programming: Example

![Graph diagram]
Dynamic Programming: Example

\begin{itemize}
  \item \textbf{Graph:}
  \begin{itemize}
    \item Vertices: \{s, a, b, c, d\}
    \item Edges:
      \begin{align*}
        &s \rightarrow a: 6, \quad s \rightarrow b: 7, \\
        &a \rightarrow b: 8, \quad a \rightarrow c: 7, \\
        &b \rightarrow c: -2, \quad a \rightarrow d: -3
      \end{align*}
  \end{itemize}
  \item Length-0 edge: \{s\} \rightarrow s
\end{itemize}

\begin{itemize}
  \item Depth-first search:
    \begin{itemize}
      \item \textbf{Root:} s
      \item \textbf{Children:}
        \begin{itemize}
          \item s \rightarrow a: (0, \infty)
          \item s \rightarrow b: (7, \infty)
        \end{itemize}
      \item \textbf{Children of a:}
        \begin{itemize}
          \item a \rightarrow b: (6, 7)
          \item a \rightarrow c: (8, -3)
        \end{itemize}
      \item \textbf{Children of b:}
        \begin{itemize}
          \item b \rightarrow c: (8, -2)
        \end{itemize}
    \end{itemize}
\end{itemize}
Dynamic Programming: Example

\[
\begin{align*}
\text{length-0 edge} & \\
0 & \\
\end{align*}
\]
Dynamic Programming: Example
Dynamic Programming: Example

-2 6 7
-4 6 7
-3 8
-2

length-0 edge

---

\[ f^0 \]

\[
\begin{array}{c}
0 & a & b & c & d \\
0 & \infty & \infty & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
\end{array}
\]

\[ f^1 \]

\[
\begin{array}{c}
0 & a & b & c & d \\
0 & \infty & \infty & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
\end{array}
\]

\[ f^2 \]

\[
\begin{array}{c}
0 & a & b & c & d \\
0 & \infty & \infty & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
6 & 7 & 8 & \infty & \infty \\
\end{array}
\]
Dynamic Programming: Example

length-0 edge
Dynamic Programming: Example

\[ f^0 \]

\[ f^1 \]

\[ f^2 \]

length-0 edge
Dynamic Programming: Example
Dynamic Programming: Example

- \( f^0 \)
- \( f^1 \)
- \( f^2 \)

Length-0 edge

\( s \) \( a \) \( b \) \( c \) \( d \)
Dynamic Programming: Example

\[\begin{align*}
&f^0 \\
&s: 0 & a: \infty & b: \infty & c: \infty & d: \infty \\
&f^1 \\
&0: 6 & 6: 7 & 8: -4 & 7: -3 & 2: -2 & 4: -2 \\
&f^2 \\
&0: 6 & 6: 7 & 8: -4 & 7: -3 & 2: -2 & 4: -2 \\
&f^3 \\
&0: 6 & 6: 7 & 8: -4 & 7: -3 & 2: -2 & 4: -2 \\
\end{align*}\]

length-0 edge
Dynamic Programming: Example

\[
\begin{align*}
  &f^0 \\
  &f^1 \\
  &f^2 \\
  &f^3 \\
\end{align*}
\]
Dynamic Programming: Example

\[ \begin{array}{ccc}
  & f^0 & \\
 s & 0 & \infty \\
 a & 6 & 7 \\
 b & 8 & 6 \\
 c & 7 & \infty \\
 d & \infty & \infty \\
\end{array} \]

\[ \begin{array}{ccc}
  & f^1 & \\
 s & 0 & 6 \\
 a & 6 & 7 \\
 b & 8 & 6 \\
 c & 7 & \infty \\
 d & \infty & \infty \\
\end{array} \]

\[ \begin{array}{ccc}
  & f^2 & \\
 s & 0 & 6 \\
 a & 6 & 7 \\
 b & 8 & 6 \\
 c & 7 & 2 \\
 d & \infty & \infty \\
\end{array} \]

\[ \begin{array}{ccc}
  & f^3 & \\
 s & 0 & 6 \\
 a & 6 & 7 \\
 b & 8 & 6 \\
 c & 7 & 2 \\
 d & \infty & \infty \\
\end{array} \]

length-0 edge
Dynamic Programming: Example

\[ \begin{array}{cccc}
    & s & a & d \\
    b & 7 & 8 & -2 \\
    c & 8 & -3 & 7 \\
\end{array} \]

\[ \begin{array}{cccc}
    f^0 & 0 & \infty & \infty \\
    & 6 & 7 & -4 \\
    & 8 & -3 & -2 \\
    & \infty & \infty & \infty \\
\end{array} \]

length-0 edge
Dynamic Programming: Example

[Diagram of a graph with nodes labeled s, a, b, c, d and edges labeled with lengths 7, 6, 8, -4, -3, 7, and 2, and annotations for f^0, f^1, f^2, f^3 with values 0, 6, 6, 2, 2, 6, 6, and arrows indicating the flow of values.]
Dynamic Programming: Example

\[ f^0 \]

\[ f^1 \]

\[ f^2 \]

\[ f^3 \]

\begin{align*}
    s & \quad 0 \quad ∞ \quad ∞ \quad ∞ \\
    a & \quad 6 \quad 7 \quad 8 \quad -4 \quad -3 \\
    b & \quad 6 \quad 7 \quad 8 \quad -4 \quad -3 \\
    c & \quad 6 \quad 7 \quad 8 \quad -4 \quad -3 \\
    d & \quad 6 \quad 7 \quad 8 \quad -4 \quad -3 \\
\end{align*}

length-0 edge
Dynamic Programming: Example

\begin{figure}
\centering
\begin{tikzpicture}
\node (s) at (0,0) {$s$};
\node (a) at (1,1) {$a$};
\node (b) at (-1,1) {$b$};
\node (c) at (0,-1) {$c$};
\node (d) at (1,-1) {$d$};
\draw[->] (s) -- (a) node[midway, above] {7};
\draw[->] (s) -- (b) node[midway, left] {6};
\draw[->] (a) -- (b) node[midway, above] {8};
\draw[->] (a) -- (c) node[midway, right] {8};
\draw[->] (a) -- (d) node[midway, above] {7};
\draw[->] (b) -- (c) node[midway, right] {7};
\draw[->] (b) -- (d) node[midway, above] {-4};
\draw[->] (c) -- (d) node[midway, right] {-3};
\node [below] at (-1.5,0) {length-0 edge};
\end{tikzpicture}
\end{figure}
Dynamic Programming: Example

![Diagram of a graph with nodes labeled s, a, b, c, d and edges with weights 7, 6, 8, -2, -3, 7, and a length-0 edge.]

- $f^0$:
  - $s$: 0, $a$: 6, $b$: 8, $c$: $\infty$, $d$: $\infty$

- $f^1$:
  - $s$: 6, $a$: 7, $b$: 8, $c$: $\infty$, $d$: 7

- $f^2$:
  - $s$: 6, $a$: 7, $b$: 8, $c$: 2, $d$: 4

- $f^3$:
  - $s$: 6, $a$: 7, $b$: 8, $c$: 2, $d$: 4

- $f^4$:
  - $s$: 6, $a$: 7, $b$: 8, $c$: 2, $d$: 4
dynamic-programming$(G, w, s)$

1: $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: for $\ell \leftarrow 1$ to $n - 1$ do
3:   copy $f^{\ell - 1} \rightarrow f^\ell$
4: for each $(u, v) \in E$ do
5:   if $f^{\ell - 1}[u] + w(u, v) < f^\ell[v]$ then
6:     $f^\ell[v] \leftarrow f^{\ell - 1}[u] + w(u, v)$
7: return $(f^{n - 1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges.
Proof. If there is a path containing at least $n$ edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length.
dynamic-programming\((G, w, s)\)

1: \( f^0[s] \leftarrow 0 \) and \( f^0[v] \leftarrow \infty \) for any \( v \in V \setminus \{s\} \)
2: for \( \ell \leftarrow 1 \) to \( n - 1 \) do
3: copy \( f^{\ell-1} \rightarrow f^\ell \)
4: for each \( (u, v) \in E \) do
5: if \( f^{\ell-1}[u] + w(u, v) < f^\ell[v] \) then
6: \( f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v) \)
7: return \( (f^{n-1}[v])_{v \in V} \)

Obs. Assuming there are no negative cycles, then a shortest path contains at most \( n - 1 \) edges
**dynamic-programming**$(G, w, s)$

1. $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n - 1$ do
3. \hspace{1em} copy $f^{\ell-1} \rightarrow f^\ell$
4. for each $(u, v) \in E$ do
5. \hspace{1em} if $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$ then
6. \hspace{2em} $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
7. return $(f^{n-1}[v])_{v \in V}$

**Obs.** Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges

**Proof.**

If there is a path containing at least $n$ edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length.
**Dynamic Programming with Better Space Usage**

**dynamic-programming**$(G, w, s)$

1. $f^{\text{old}}[s] \leftarrow 0$ and $f^{\text{old}}[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. **for** $\ell \leftarrow 1$ **to** $n - 1$ **do**
3. copy $f^{\text{old}} \rightarrow f^{\text{new}}$
4. **for** each $(u, v) \in E$ **do**
   5. **if** $f^{\text{old}}[u] + w(u, v) < f^{\text{new}}[v]$ **then**
   6. $f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v)$
7. copy $f^{\text{new}} \rightarrow f^{\text{old}}$
8. **return** $f^{\text{old}}$

- $f^\ell$ only depends on $f^{\ell-1}$: only need 2 vectors
Dynamic Programming with Better Space Usage

**dynamic-programming**$(G, w, s)$

1. $f^{\text{old}}[s] \leftarrow 0$ and $f^{\text{old}}[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. **for** $\ell \leftarrow 1$ **to** $n - 1$ **do**
3.   copy $f^{\text{old}} \rightarrow f^{\text{new}}$
4.   **for** each $(u, v) \in E$ **do**
5.     **if** $f^{\text{old}}[u] + w(u, v) < f^{\text{new}}[v]$ **then**
6.       $f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v)$
7.   copy $f^{\text{new}} \rightarrow f^{\text{old}}$
8. **return** $f^{\text{old}}$

- $f^\ell$ only depends on $f^{\ell-1}$: only need 2 vectors
- only need 1 vector!
Dynamic Programming with Better Space Usage

**dynamic-programming**\((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: \(\text{for } \ell \leftarrow 1\text{ to } n - 1\text{ do}\)
3: \(\text{copy } f \rightarrow f\)
4: \(\text{for each } (u, v) \in E\text{ do}\)
5: \(\text{if } f[u] + w(u, v) < f[v]\text{ then}\)
6: \(f[v] \leftarrow f[u] + w(u, v)\)
7: \(\text{copy } f \rightarrow f\)
8: \(\text{return } f\)

- \(f^\ell\) only depends on \(f^{\ell-1}\): only need 2 vectors
- only need 1 vector!
Dynamic Programming with Better Space Usage

dynamic-programming\((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: \textbf{for} \(\ell \leftarrow 1\) to \(n - 1\) \textbf{do}
3: \textbf{for each} \((u, v) \in E\) \textbf{do}
4: \textbf{if} \(f[u] + w(u, v) < f[v]\) \textbf{then}
5: \hspace{1em} \(f[v] \leftarrow f[u] + w(u, v)\)
6: \textbf{return} \(f\)

- \(f^\ell\) only depends on \(f^{\ell-1}\): only need 2 vectors
- only need 1 vector!
Bellman-Ford Algorithm

Bellman-Ford($G, w, s$)

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: for $\ell \leftarrow 1$ to $n - 1$ do
3:    for each $(u, v) \in E$ do
4:       if $f[u] + w(u, v) < f[v]$ then
5:          $f[v] \leftarrow f[u] + w(u, v)$
6: return $f$

- $f^\ell$ only depends on $f^{\ell-1}$: only need 2 vectors
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6: return $f$

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
Bellman-Ford Algorithm

Bellman-Ford($G, w, s$)

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
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6: return $f$

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only “accelerate” the process!
**Bellman-Ford Algorithm**

**Bellman-Ford**\( (G, w, s) \)

1. \( f[s] \leftarrow 0 \) and \( f[v] \leftarrow \infty \) for any \( v \in V \setminus \{s\} \)
2. for \( \ell \leftarrow 1 \) to \( n - 1 \) do
3. for each \((u, v) \in E\) do
4. \hspace{1em} if \( f[u] + w(u, v) < f[v] \) then
5. \hspace{2em} \( f[v] \leftarrow f[u] + w(u, v) \)
6. return \( f \)

- Issue: when we compute \( f[u] + w(u, v) \), \( f[u] \) may be changed since the end of last iteration.
- This is OK: it can only “accelerate” the process!
- After iteration \( \ell \), \( f[v] \) is at most the length of the shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges.
Bellman-Ford Algorithm

**Bellman-Ford**\((G,w,s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: **for** \(\ell \leftarrow 1\) to \(n - 1\) **do**
3: **for** each \((u,v) \in E\) **do**
4: **if** \(f[u] + w(u,v) < f[v]\) **then**
5: \(f[v] \leftarrow f[u] + w(u,v)\)
6: **return** \(f\)

- **Issue:** when we compute \(f[u] + w(u,v)\), \(f[u]\) may be changed since the end of last iteration.
- This is OK: it can only “accelerate” the process!
- After iteration \(\ell\), \(f[v]\) is at most the length of the shortest path from \(s\) to \(v\) that uses at most \(\ell\) edges.
- \(f[v]\) is always the length of some path from \(s\) to \(v\).
Bellman-Ford Algorithm

- After iteration $\ell$:

  length of shortest $s$-$v$ path
  \[ \leq f[v] \]
  \[ \leq \text{length of shortest } s$-$v$ path using at most $\ell$ edges \]
Bellman-Ford Algorithm

- After iteration $\ell$:

  length of shortest $s-v$ path
  
  $\leq f[v]$
  
  $\leq$ length of shortest $s-v$ path using at most $\ell$ edges

- Assuming there are no negative cycles:

  length of shortest $s-v$ path
  
  $= \text{length of shortest } s-v \text{ path using at most } n - 1 \text{ edges}$
Bellman-Ford Algorithm

- After iteration \( \ell \):
  
  \[
  \text{length of shortest } s-v \text{ path} \\
  \leq f[v] \\
  \leq \text{length of shortest } s-v \text{ path using at most } \ell \text{ edges}
  \]

- Assuming there are no negative cycles:
  
  \[
  \text{length of shortest } s-v \text{ path} \\
  = \text{length of shortest } s-v \text{ path using at most } n - 1 \text{ edges}
  \]

- So, assuming there are no negative cycles, after iteration \( n - 1 \):
  
  \[
  f[v] = \text{length of shortest } s-v \text{ path}
  \]
order in which we consider edges:

(\(s, a\)), (\(s, b\)), (\(a, b\)), (\(a, c\)), (\(b, d\)),

(\(c, d\)), (\(d, a\))

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<thead>
<tr>
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(s, a), (s, b), (a, b), (a, c), (b, d),
(c, d), (d, a)

vertices  s  a  b  c  d
          f
0  6  7  ∞  ∞
order in which we consider edges:
(s, a), (s, b), (a, b), (a, c), (b, d),
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\[(s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)\]

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order in which we consider edges:
(s, a), (s, b), (a, b), (a, c), (b, d),
(c, d), (d, a)

end of iteration 1: 0, 2, 7, 2, 4
end of iteration 2: 0, 2, 7, -2, 4
end of iteration 3: 0, 2, 7, -2, 4
Algorithm terminates in 3 iterations,
instead of 4.
order in which we consider edges:
(s, a), (s, b), (a, b), (a, c), (b, d),
(c, d), (d, a)

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end of iteration 1: 0, 2, 7, 2, 4
order in which we consider edges:

\((s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)\)

\[
\begin{array}{c|c|c|c|c|c}
\text{vertices} & s & a & b & c & d \\
\hline
f & 0 & 2 & 7 & 2 & 4 \\
\end{array}
\]

end of iteration 1: 0, 2, 7, 2, 4

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order in which we consider edges: 
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(c, d), (d, a)

vertices | s | a | b | c | d
---|---|---|---|---|---
f    | 0 | 2 | 7 | -2 | 4

end of iteration 1: 0, 2, 7, 2, 4
order in which we consider edges: 
(s, a), (s, b), (a, b), (a, c), (b, d), 
(c, d), (d, a) 

vertices | s | a | b | c | d 
---|---|---|---|---|--- 
 f | 0 | 2 | 7 | -2 | 4 

end of iteration 1: 0, 2, 7, 2, 4
order in which we consider edges:
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end of iteration 2: 0, 2, 7, -2, 4
order in which we consider edges:
(s, a), (s, b), (a, b), (a, c), (b, d),
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end of iteration 1: 0, 2, 7, 2, 4
end of iteration 2: 0, 2, 7, -2, 4
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vertices

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end of iteration 1: 0, 2, 7, 2, 4
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Algorithm terminates in 3 iterations, instead of 4.
Bellman-Ford Algorithm

Bellman-Ford\((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: for \(\ell \leftarrow 1\) to \(n\) do
3: \(\text{updated} \leftarrow \text{false}\)
4: for each \((u, v) \in E\) do
5: \(\text{if } f[u] + w(u, v) < f[v] \text{ then}\)
6: \(f[v] \leftarrow f[u] + w(u, v)\)
7: \(\text{updated} \leftarrow \text{true}\)
8: if not \text{updated}, then return \(f\)
9: output “negative cycle exists”
Bellman-Ford Algorithm

Bellman-Ford\((G, w, s)\)

1:  \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)  
2:  \textbf{for } \ell \leftarrow 1 \textbf{ to } n \textbf{ do}  
3:  \quad \textit{updated} \leftarrow \text{false}  
4:  \quad \textbf{for each } (u, v) \in E \textbf{ do}  
5:  \quad \quad \textbf{if } f[u] + w(u, v) < f[v] \textbf{ then}  
6:  \quad \quad \quad f[v] \leftarrow f[u] + w(u, v), \pi[v] \leftarrow u  
7:  \quad \text{updated} \leftarrow \text{true}  
8:  \quad \textbf{if not } \text{updated}, \text{ then return } f  
9:  \text{output “negative cycle exists”}

\(\pi[v]\): the parent of \(v\) in the shortest path tree
Bellman-Ford Algorithm

Bellman-Ford($G, w, s$)

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: for $\ell \leftarrow 1$ to $n$ do
3:     updated $\leftarrow$ false
4:     for each $(u, v) \in E$ do
5:         if $f[u] + w(u, v) < f[v]$ then
6:             $f[v] \leftarrow f[u] + w(u, v)$, $\pi[v] \leftarrow u$
7:         updated $\leftarrow$ true
8:     if not updated, then return $f$
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- $\pi[v]$: the parent of $v$ in the shortest path tree
- Running time = $O(nm)$
Outline

1. Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2. Single Source Shortest Paths
   - Dijkstra’s Algorithm

3. Shortest Paths in Graphs with Negative Weights

4. All-Pair Shortest Paths and Floyd-Warshall

5. Minimum Cost Arborescence
All-Pair Shortest Paths

**Input:** directed graph $G = (V, E)$,

$w : E \rightarrow \mathbb{R}$ (can be negative)

**Output:** shortest path from $u$ to $v$ for every $u, v \in V$

1: for every starting point $s \in V$
d2: run Bellman-Ford ($G, w, s$)

Running time $= O(n^2 m)$
All-Pair Shortest Paths

**Input:** directed graph $G = (V, E)$,

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**Output:** shortest path from $u$ to $v$ for every $u, v \in V$

1. **for** every starting point $s \in V$ **do**
2. run Bellman-Ford$(G, w, s)$
All-Pair Shortest Paths

**Input:** directed graph $G = (V, E)$,

$w : E \rightarrow \mathbb{R}$ (can be negative)

**Output:** shortest path from $u$ to $v$ for every $u, v \in V$

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2. run Bellman-Ford($G, w, s$)

Running time $= O(n^2m)$
## Summary of Shortest Path Algorithms we learned

<table>
<thead>
<tr>
<th>algorithm</th>
<th>graph</th>
<th>weights</th>
<th>SS?</th>
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<tr>
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- DAG = directed acyclic graph  
- U = undirected  
- D = directed  
- SS = single source  
- AP = all pairs
It is convenient to assume \( V = \{1, 2, 3, \ldots, n\} \)
Design a Dynamic Programming Algorithm

- It is convenient to assume $V = \{1, 2, 3, \cdots, n\}$
- For simplicity, extend the $w$ values to non-edges:

\[
w(i, j) = \begin{cases} 
0 & i = j \\
\text{weight of edge } (i, j) & i \neq j, (i, j) \in E \\
\infty & i \neq j, (i, j) \notin E
\end{cases}
\]
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**Cells for Floyd-Warshall Algorithm**

- First try: \( f[i, j] \) is length of shortest path from \( i \) to \( j \)
- Issue: do not know in which order we compute \( f[i, j] \)'s

\( f^k[i, j] \): length of shortest path from \( i \) to \( j \) that only uses vertices \( \{1, 2, 3, \ldots, k\} \) as intermediate vertices
Example for Definition of $f^k[i, j]$’s

\[
\begin{align*}
  f^0[1, 4] &= \infty \\
  f^1[1, 4] &= \infty \\
  f^2[1, 4] &= 140 \quad (1 \rightarrow 2 \rightarrow 4) \\
  f^3[1, 4] &= 90 \quad (1 \rightarrow 3 \rightarrow 2 \rightarrow 4) \\
  f^4[1, 4] &= 90 \quad (1 \rightarrow 3 \rightarrow 2 \rightarrow 4) \\
  f^5[1, 4] &= 60 \quad (1 \rightarrow 3 \rightarrow 5 \rightarrow 4)
\end{align*}
\]
\[ w(i, j) = \begin{cases} 
0 & \text{if } i = j \\
\text{weight of edge } (i, j) & \text{if } i \neq j, (i, j) \in E \\
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\[ f^k[i, j] = \begin{cases} 
k = 0 \\
k = 1, 2, \ldots, n
\end{cases} \]
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\[ f^k[i, j] = \begin{cases} 
w(i, j) & k = 0 \\
\min \left\{ f^{k-1}[i, j] \right\} & k = 1, 2, \ldots, n 
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- \( f^k[i, j] \): length of shortest path from \( i \) to \( j \) that only uses vertices \( \{1, 2, 3, \cdots, k\} \) as intermediate vertices

\[ f^k[i, j] = \begin{cases} 
w(i, j) & k = 0 \\
\min \left\{ f^{k-1}[i, j], f^{k-1}[i, k] + f^{k-1}[k, j] \right\} & k = 1, 2, \cdots, n 
\end{cases} \]
Floyd-Warshall($G, w$)

1: $f^0 \leftarrow w$
2: for $k \leftarrow 1$ to $n$ do
3:   copy $f^{k-1} \rightarrow f^k$
4: for $i \leftarrow 1$ to $n$ do
5:   for $j \leftarrow 1$ to $n$ do
6:     if $f^{k-1}[i, k] + f^{k-1}[k, j] < f^k[i, j]$ then
7:         $f^k[i, j] \leftarrow f^{k-1}[i, k] + f^{k-1}[k, j]$
Floyd-Warshall($G, w$)

1: $f^{\text{old}} \leftarrow w$
2: \textbf{for } $k \leftarrow 1$ \textbf{to } $n$ \textbf{do}
3: \hspace{1em} copy $f^{\text{old}} \rightarrow f^{\text{new}}$
4: \textbf{for } $i \leftarrow 1$ \textbf{to } $n$ \textbf{do}
5: \hspace{1em} \textbf{for } $j \leftarrow 1$ \textbf{to } $n$ \textbf{do}
6: \hspace{2em} \textbf{if } $f^{\text{old}}[i, k] + f^{\text{old}}[k, j] < f^{\text{new}}[i, j]$ \textbf{then}
7: \hspace{3em} $f^{\text{new}}[i, j] \leftarrow f^{\text{old}}[i, k] + f^{\text{old}}[k, j]$
Floyd-Warshall\((G, w)\)

1: \(f^{\text{old}} \leftarrow w\)
2: \(\text{for } k \leftarrow 1 \text{ to } n \text{ do}\)
3: \(\text{copy } f^{\text{old}} \rightarrow f^{\text{new}}\)
4: \(\text{for } i \leftarrow 1 \text{ to } n \text{ do}\)
5: \(\text{for } j \leftarrow 1 \text{ to } n \text{ do}\)
6: \(\text{if } f^{\text{old}}[i, k] + f^{\text{old}}[k, j] < f^{\text{new}}[i, j] \text{ then}\)
7: \(f^{\text{new}}[i, j] \leftarrow f^{\text{old}}[i, k] + f^{\text{old}}[k, j]\)

Lemma
Assume there are no negative cycles in \(G\). After iteration \(k\), for \(i, j \in V\), \(f[i, j]\) is exactly the length of shortest path from \(i\) to \(j\) that only uses vertices in \(\{1, 2, 3, \cdots, k\}\) as intermediate vertices.

Running time \(= O(n^3)\).
**Floyd-Warshall**$(G, w)$

1: $f ← w$
2: for $k ← 1$ to $n$ do
3:     copy $f → f$
4:     for $i ← 1$ to $n$ do
5:         for $j ← 1$ to $n$ do
6:             if $f[i, k] + f[k, j] < f[i, j]$ then
7:                 $f[i, j] ← f[i, k] + f[k, j]$

**Lemma**
Assume there are no negative cycles in $G$. After iteration $k$, for $i, j ∈ V$, $f[i, j]$ is exactly the length of shortest path from $i$ to $j$ that only uses vertices in \{1, 2, 3, ···, $k$\} as intermediate vertices.

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1: $f \leftarrow w$
2: for $k \leftarrow 1$ to $n$ do
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Running time = $O(n^3)$. 
Floyd-Warshall \((G, w)\)

1: \(f \leftarrow w\)
2: \(\textbf{for} \ k \leftarrow 1 \ \text{to} \ n \ \textbf{do}\)
3: \(\quad \textbf{for} \ i \leftarrow 1 \ \text{to} \ n \ \textbf{do}\)
4: \(\quad \quad \textbf{for} \ j \leftarrow 1 \ \text{to} \ n \ \textbf{do}\)
5: \(\quad \quad \quad \textbf{if} \ f[i, k] + f[k, j] < f[i, j] \ \textbf{then}\)
6: \(\quad \quad \quad \quad f[i, j] \leftarrow f[i, k] + f[k, j]\)

**Lemma** Assume there are no negative cycles in \(G\). After iteration \(k\), for \(i, j \in V\), \(f[i, j]\) is exactly the length of shortest path from \(i\) to \(j\) that only uses vertices in \(\{1, 2, 3, \cdots, k\}\) as intermediate vertices.
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1: $f \leftarrow w$
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6:                 $f[i, j] \leftarrow f[i, k] + f[k, j]$

Lemma Assume there are no negative cycles in $G$. After iteration $k$, for $i, j \in V$, $f[i, j]$ is exactly the length of shortest path from $i$ to $j$ that only uses vertices in $\{1, 2, 3, \cdots, k\}$ as intermediate vertices.

- Running time $= O(n^3)$. 
The diagram shows a graph with five nodes labeled 1, 2, 3, 4, and 5. The edges between the nodes are labeled with the following weights:

- Edge from 1 to 2: 10
- Edge from 1 to 3: 60
- Edge from 1 to 4: 90
- Edge from 1 to 5: 30
- Edge from 2 to 3: 10
- Edge from 2 to 4: 50
- Edge from 2 to 5: ∞
- Edge from 3 to 2: 30
- Edge from 3 to 4: 70
- Edge from 3 to 5: 20
- Edge from 4 to 3: 10
- Edge from 4 to 5: 20
- Edge from 5 to 3: 70
- Edge from 5 to 4: 20
- Edge from 5 to 2: 10

The table below represents the adjacency matrix of the graph:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
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<td>0</td>
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<td>20</td>
</tr>
<tr>
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<td>∞</td>
<td>∞</td>
<td>∞</td>
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<td>20</td>
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<tr>
<td>5</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
<td>10</td>
<td>0</td>
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</table>
\[ i = 2, \, k = 1, \, j = 3 \]
\[ i = 2, \ k = 1, \ j = 3 \]
\( i = 1, \; k = 2, \; j = 4 \)
\( i = 1, \ k = 2, \ j = 4 \)
\[ i = 3, \quad k = 2, \quad j = 1, \]

\[
\begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
1 & 0 & 90 & 30 & 140 & \infty \\
2 & 10 & 0 & 40 & 50 & \infty \\
3 & 60 & 10 & 0 & 70 & 20 \\
4 & \infty & \infty & \infty & 0 & 20 \\
5 & \infty & \infty & \infty & 10 & 0 \\
\end{array}
\]
i = 3, k = 2, j = 1,
\[ i = 3, \ k = 2, \ j = 4 \]
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Floyd-Warshall($G, w$)

1: $f \leftarrow w$, $\pi[i, j] \leftarrow \bot$ for every $i, j \in V$
2: for $k \leftarrow 1$ to $n$ do
3:   for $i \leftarrow 1$ to $n$ do
4:     for $j \leftarrow 1$ to $n$ do
5:       if $f[i, k] + f[k, j] < f[i, j]$ then
6:         $f[i, j] \leftarrow f[i, k] + f[k, j]$, $\pi[i, j] \leftarrow k$

print-path($i, j$)
1: if $\pi[i, j] = \bot$ then
2:   if $i \neq j$ then
3:     print($i, \text{,} $)
4:   else
5:     print-path($i, \pi[i, j]$), print-path($\pi[i, j], j$)
Floyd-Warshall($G, w$)

1: $f ← w$, $\pi[i, j] ← \perp$ for every $i, j \in V$
2: for $k ← 1$ to $n$ do
3: for $i ← 1$ to $n$ do
4: for $j ← 1$ to $n$ do
5: if $f[i, k] + f[k, j] < f[i, j]$ then
6: $f[i, j] ← f[i, k] + f[k, j]$, $\pi[i, j] ← k$

print-path($i, j$)

1: if $\pi[i, j] = \perp$ then then
2: if $i \neq j$ then print($i, "\”, j$)
3: else
4: print-path($i, \pi[i, j]$), print-path($\pi[i, j], j$)
Detecting Negative Cycles

Floyd-Warshall($G, w$)

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7: for $k \leftarrow 1$ to $n$ do
8: for $i \leftarrow 1$ to $n$ do
9: for $j \leftarrow 1$ to $n$ do
10: if $f[i, k] + f[k, j] < f[i, j]$ then
11: report “negative cycle exists” and exit
## Summary of Shortest Path Algorithms

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  - D = directed  
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  - AP = all pairs
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Def. An arborescence is directed rooted tree, where all edges are directed away from the root.
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**Minimum Cost Arborescence Problem**

**Input:**
- A directed graph $G = (V, E)$,
- Edge weights $w : E \rightarrow \mathbb{R}_{\geq 0}$
- Root $r \in V$

**Output:**
- A minimum-cost sub-graph $T = (V, E')$ of $G$ that is an arborescence with root $r$
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Minimum Cost Arborescence Problem

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**Output:**
- A minimum-cost sub-graph \( T = (V, E') \) of \( G \) that is an arborescence with root \( r \).
Assumptions

- the root $r$ does not have incoming edges.
- every vertex is reachable from the root $r$. 

Lemma

The instances $(G, w, r)$ and $(G, w', r)$ have the same optimum solution.
Assumptions

- the root $r$ does not have incoming edges.
- every vertex is reachable from the root $r$.

For every $v \in V \setminus \{r\}$, define $l_v = \min_{e \in \delta_v^{\text{in}}} w(e)$.

For every $v \in V \setminus \{r\}$ and $e \in \delta_v^{\text{in}}$, define $w'(e) = w(e) - l_v$. 
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- $l_a = 10$
- $l_b = 1$
- $l_c = 5$
- $l_d = 3$
- $l_e = 6$
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\[
\begin{array}{c}
\text{Table 1: Graph Attributes} \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quarter\end{array}
\]

\[
\begin{array}{c}
\text{Table 2: Graph Attributes} \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quarter\end{array}
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**Lemma** The instances $(G, w, r)$ and $(G, w', r)$ have the same optimum solution.
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Proof.  
Given any tree solution \(T\), \(w(T) - w'(T)\) is always \(\sum_{v \in V \setminus \{r\}} l_v\). \(\square\)
**Lemma** The instances \((G, w, r)\) and \((G, w', r)\) have the same optimum solution.

**Proof.**

Given any tree solution \(T\), \(w(T) - w'(T)\) is always \(\sum_{v \in V \setminus \{r\}} l_v\).

**Lemma** Let \((v_0, v_1, v_2, \ldots, v_p = v_0)\) be a cycle \(C\) of 0-cost edges in \(G\). Then there is an optimum solution \(T\), that contains all but one edges in \(C\).
MCA($G, r, w$)

1: $F^* \leftarrow \emptyset$
2: for every $v \in V \setminus \{r\}$ do
3: \hspace{1em} $l_v \leftarrow \min_{e \in \delta^\text{in}_v} w(e)$
4: for every edge $e$ entering $v$ do: $w'(e) \leftarrow w(e) - l_v$
5: choose a 0-cost edge entering $v$, add it to $(V, F^*)$
6: if $F^*$ form an arborescence then return $F^*$
7: else
8: for every cycle $C$ in $F^*$ do: contract $C$ into a single node
9: let $G' = (V', E')$ be the obtained graph.
10: $T' \leftarrow \text{MCA}(G', r, w')$
11: extend $T'$ to an aborescence $T$ in $G$, by keeping all but one edges in every cycle $C$ in $F^*$, and return $T$
The running time of the algorithm is $O(mn)$
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[Tarjan (1971)]: $O(\min(m \log n, n^2))$

[Gabow, Galil, Spencer, Tarjan (1986)]: $O(n \log n + m)$

[Mendelson, Tarjan, Thorup, Zwick (2006)]: $O(m \log \log n)$