

算法设计与分析(2024年春季学期)

Graph Algorithms

授课老师: 栗师

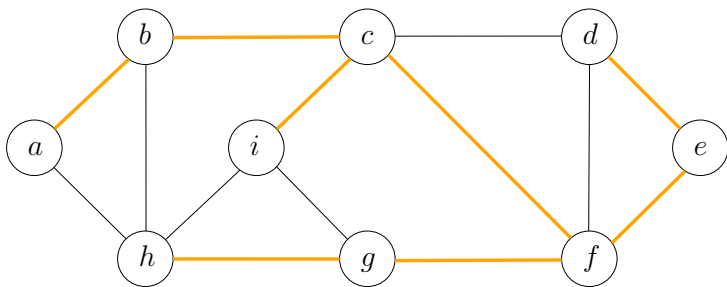
南京大学计算机科学与技术系

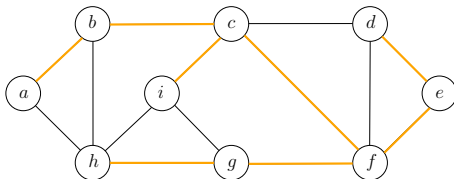
Outline

- 1 Minimum Spanning Tree
 - Kruskal's Algorithm
 - Reverse-Kruskal's Algorithm
 - Prim's Algorithm
- 2 Single Source Shortest Paths
 - Dijkstra's Algorithm
- 3 Shortest Paths in Graphs with Negative Weights
- 4 All-Pair Shortest Paths and Floyd-Warshall
- 5 Minimum Cost Arborescence

Spanning Tree

Def. Given a connected graph $G = (V, E)$, a **spanning tree** $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V .





Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G ;
- T is acyclic and connected;
- T is connected and has $n - 1$ edges;
- T is acyclic and has $n - 1$ edges;
- T is minimally connected: removal of any edge disconnects it;
- T is maximally acyclic: addition of any edge creates a cycle;
- T has a unique simple path between every pair of nodes.

Minimum Spanning Tree (MST) Problem

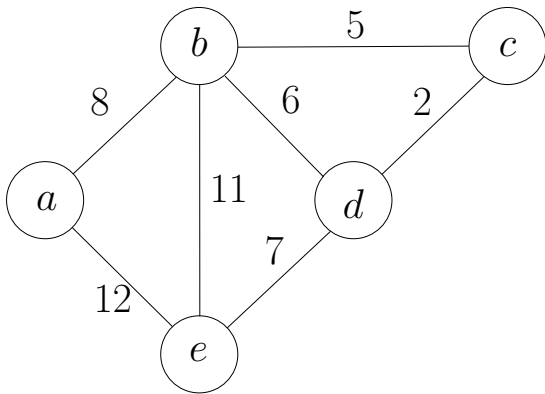
Input: Graph $G = (V, E)$ and edge weights $w : E \rightarrow \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight

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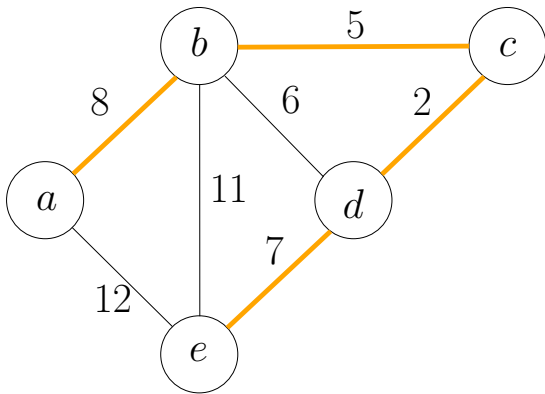
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Recall: Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice

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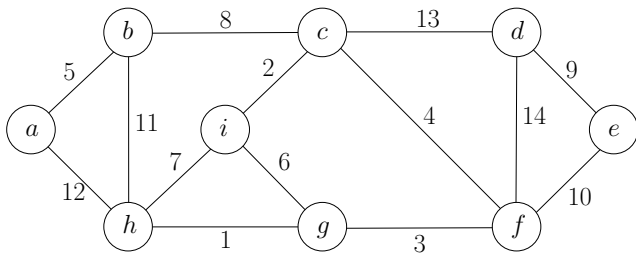
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Two Classic Greedy Algorithms for MST

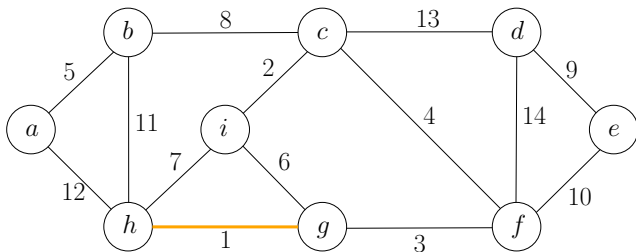
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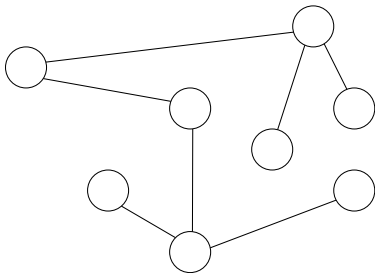
A: The edge with the smallest weight (lightest edge).

Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

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Proof.

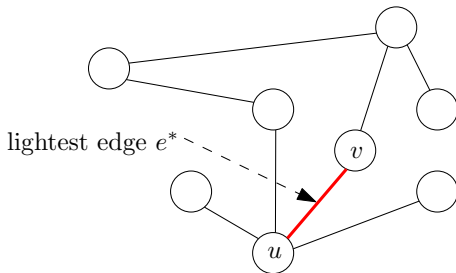
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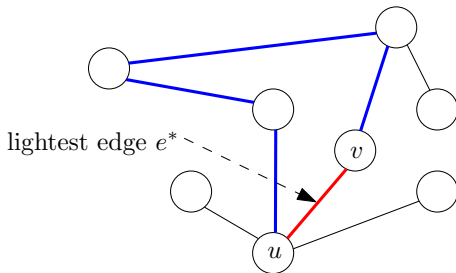
- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T



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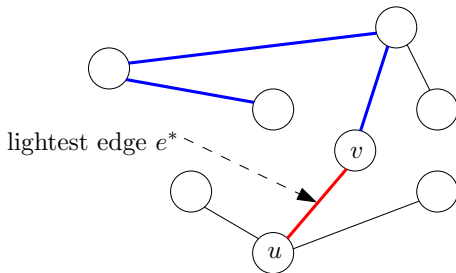
- Take a minimum spanning tree T
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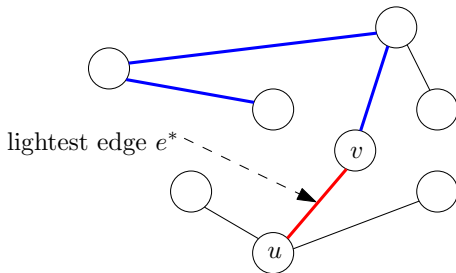
- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
- There is a unique path in T connecting u and v
- Remove any edge e in the path to obtain tree T'



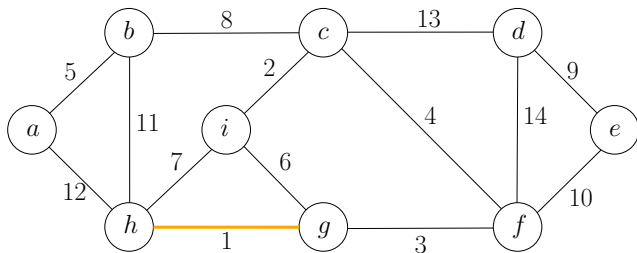
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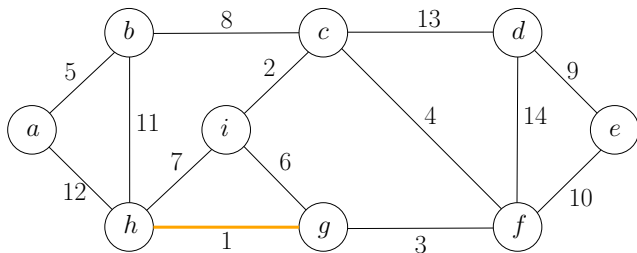
- Take a minimum spanning tree T
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- There is a unique path in T connecting u and v
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- $w(e^*) \leq w(e) \implies w(T') \leq w(T)$: T' is also a MST



Is the Residual Problem Still a MST Problem?

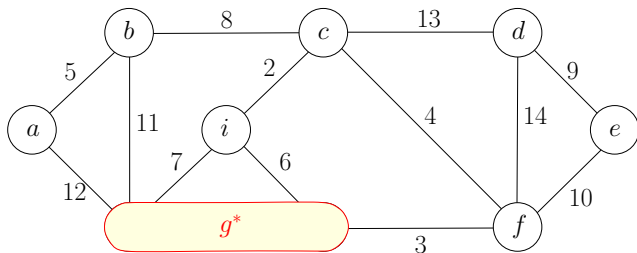


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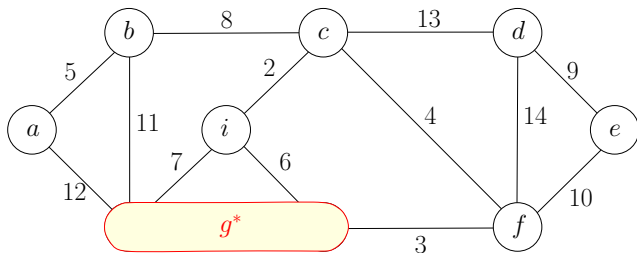
- Residual problem: find the minimum spanning tree that contains edge (g, h)

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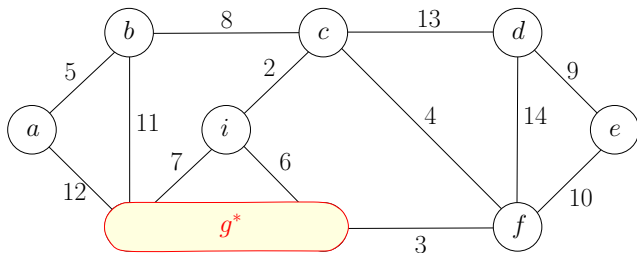
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- **Contract** the edge (g, h)

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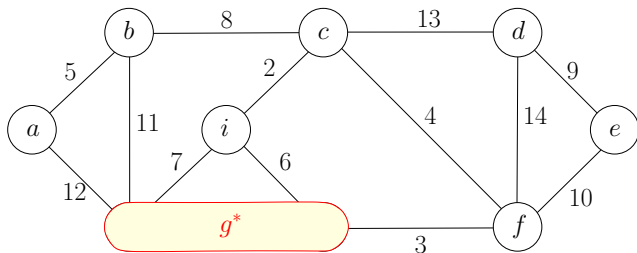


- Residual problem: find the minimum spanning tree that contains edge (g, h)
- **Contract** the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)

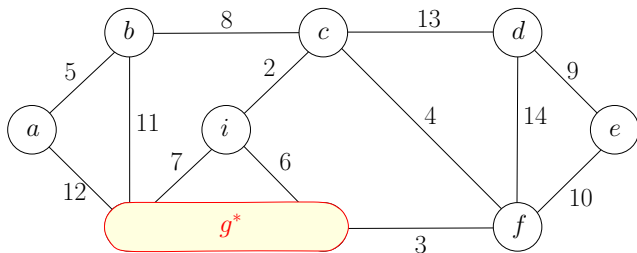


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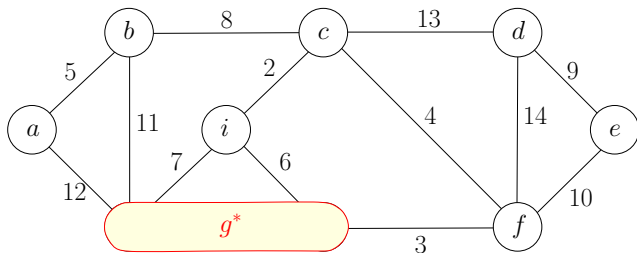
- Remove u and v from the graph, and add a new vertex u^*

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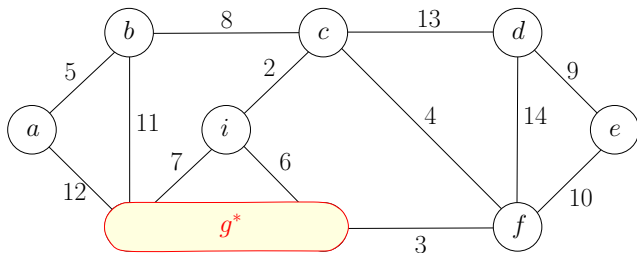
- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E

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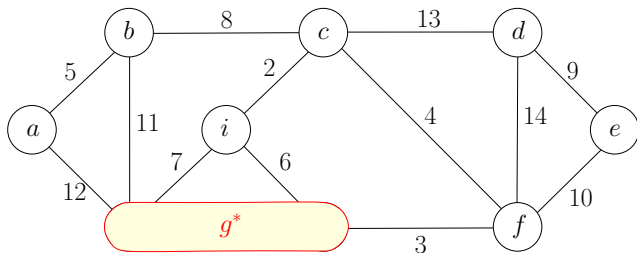
- Remove u and v from the graph, and add a new vertex u^*
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- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)

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- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- **May create parallel edges!** E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

- 1 Choose the lightest edge e^* , add e^* to the spanning tree
- 2 Contract e^* and update G be the contracted graph

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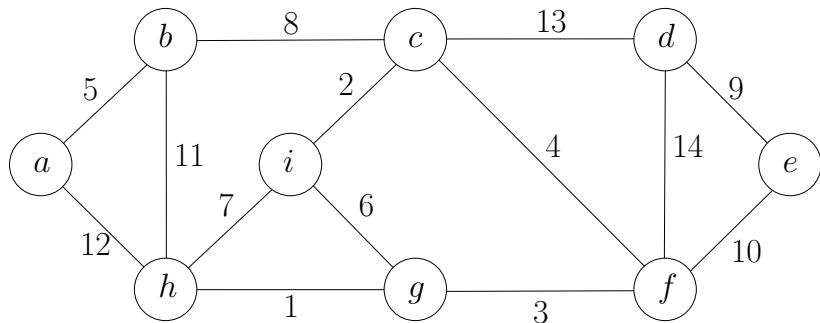
A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

MST-Greedy(G, w)

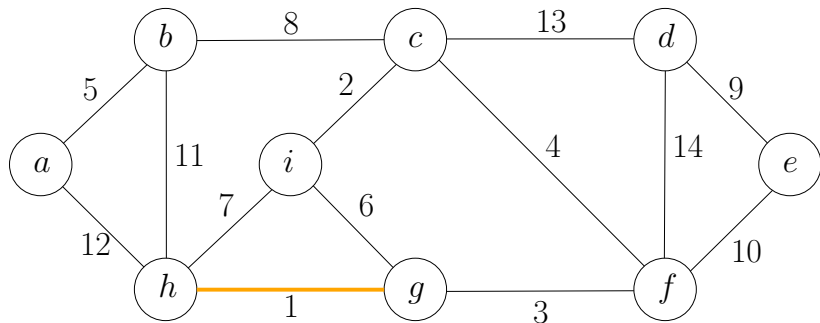
- 1: $F \leftarrow \emptyset$
- 2: sort edges in E in non-decreasing order of weights w
- 3: **for** each edge (u, v) in the order **do**
- 4: **if** u and v are not connected by a path of edges in F **then**
- 5: $F \leftarrow F \cup \{(u, v)\}$
- 6: **return** (V, F)

Kruskal's Algorithm: Example



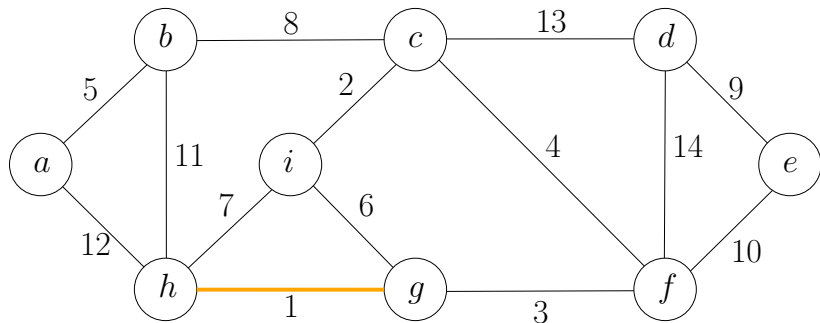
Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}$

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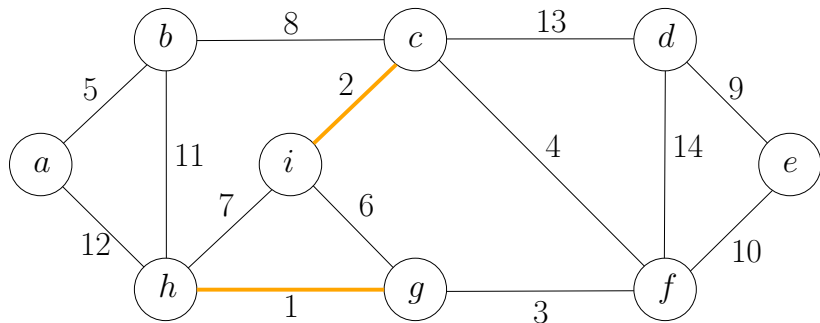
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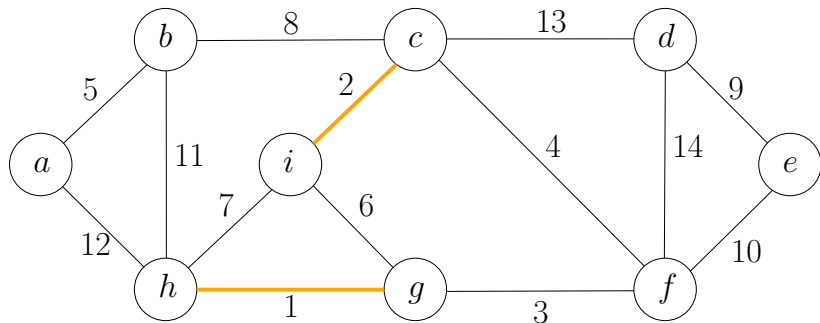
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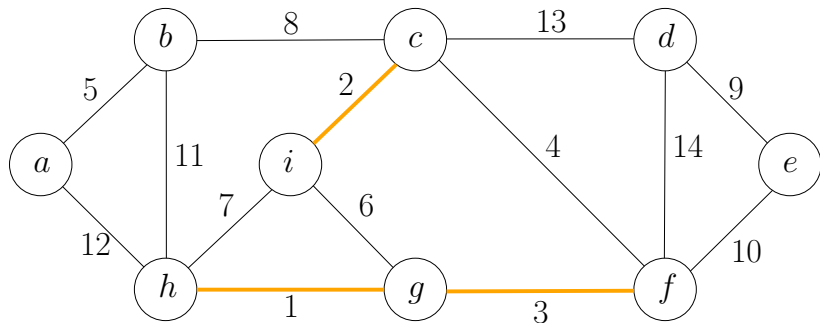
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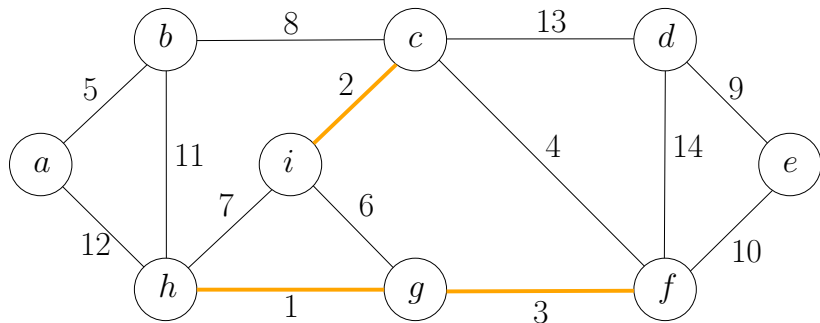
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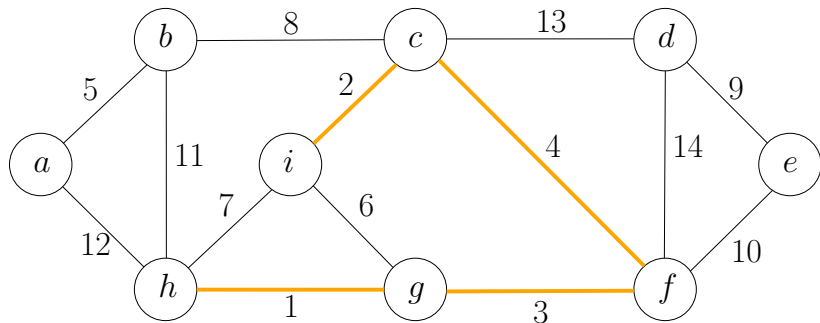
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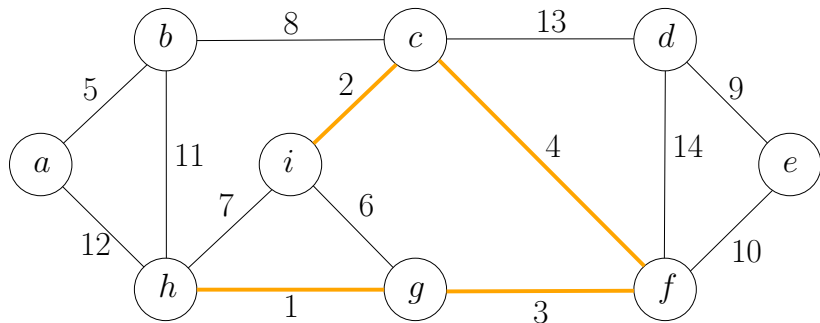
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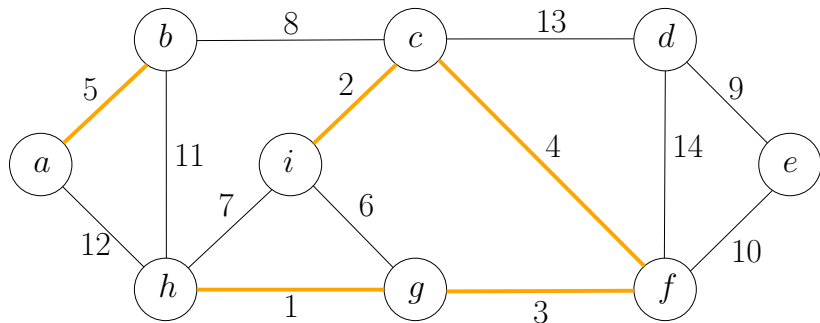
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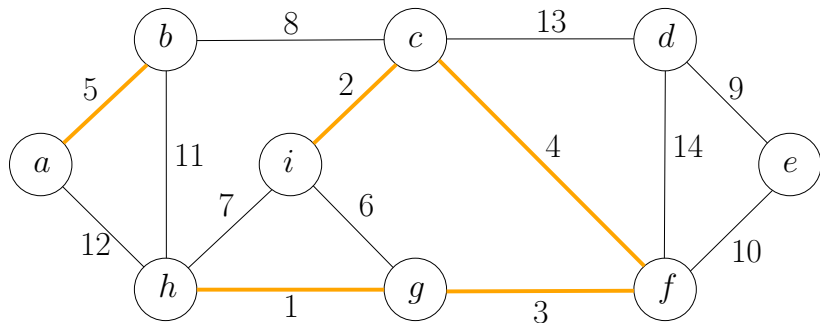
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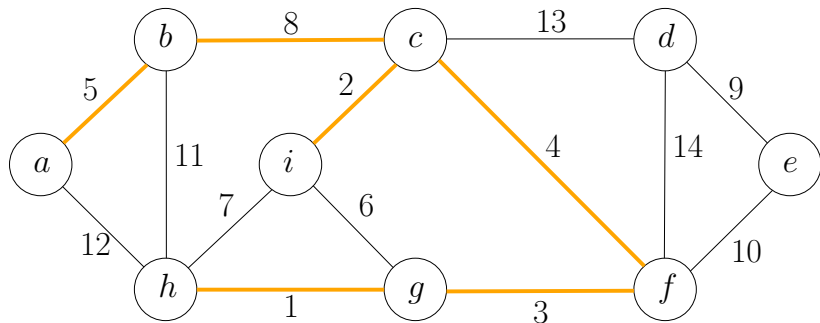
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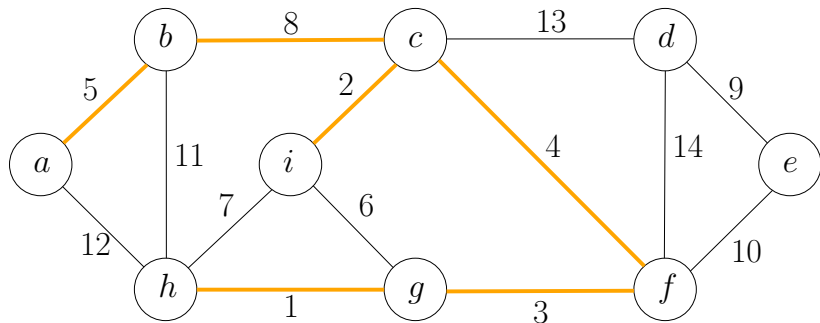
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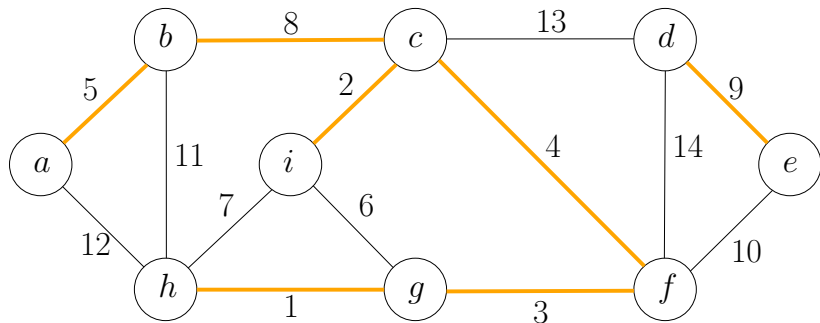
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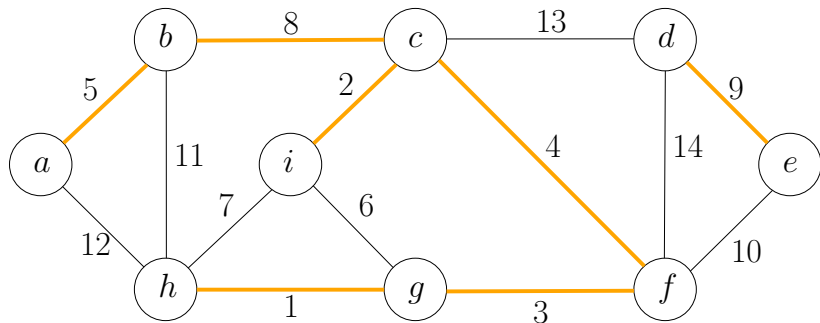
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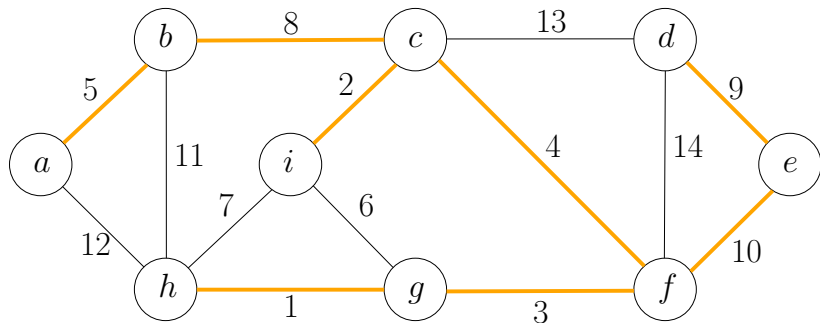
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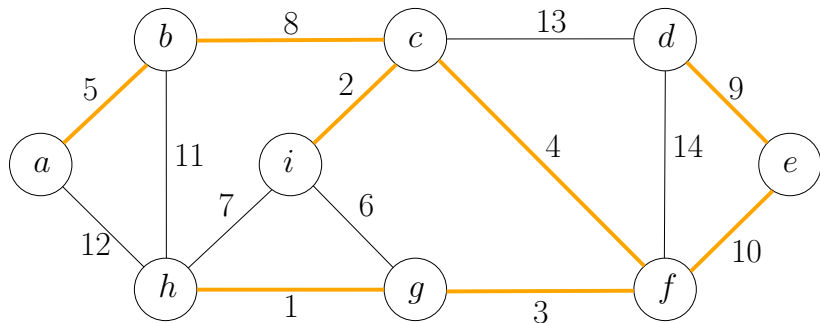
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Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

```
1:  $F \leftarrow \emptyset$ 
2:  $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$ 
3: sort the edges of  $E$  in non-decreasing order of weights  $w$ 
4: for each edge  $(u, v) \in E$  in the order do
5:    $S_u \leftarrow$  the set in  $\mathcal{S}$  containing  $u$ 
6:    $S_v \leftarrow$  the set in  $\mathcal{S}$  containing  $v$ 
7:   if  $S_u \neq S_v$  then
8:      $F \leftarrow F \cup \{(u, v)\}$ 
9:      $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$ 
10: return  $(V, F)$ 
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Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

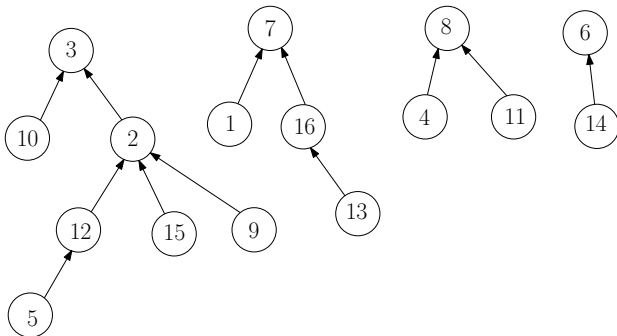
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Use **union-find** data structure to support ②, ⑤, ⑥, ⑦, ⑨.

Union-Find Data Structure

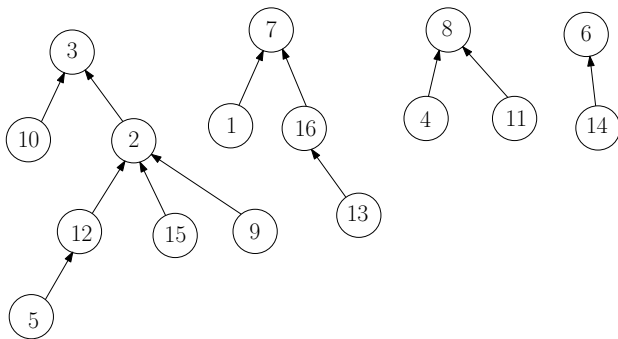
- V : ground set
- We need to maintain a partition of V and support following operations:
 - Check if u and v are in the same set of the partition
 - Merge two sets in partition

- $V = \{1, 2, 3, \dots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$

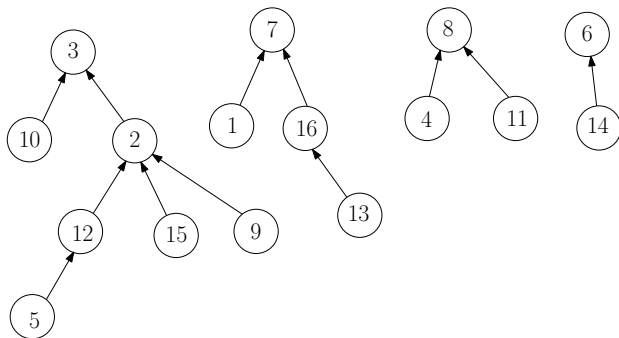


- $par[i]$: parent of i , ($par[i] = \perp$ if i is a root).

Union-Find Data Structure

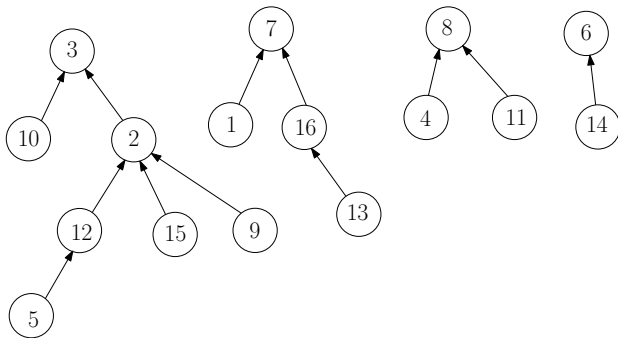


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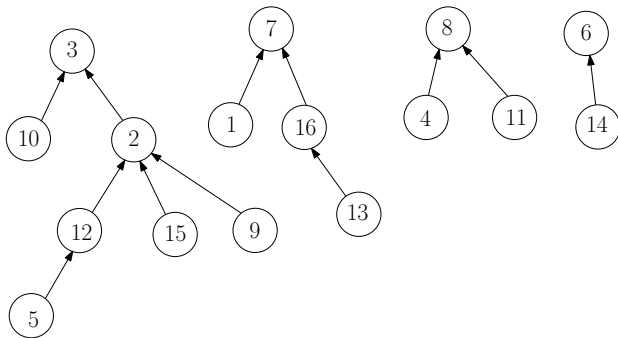
- Q: how can we check if u and v are in the same set?

Union-Find Data Structure



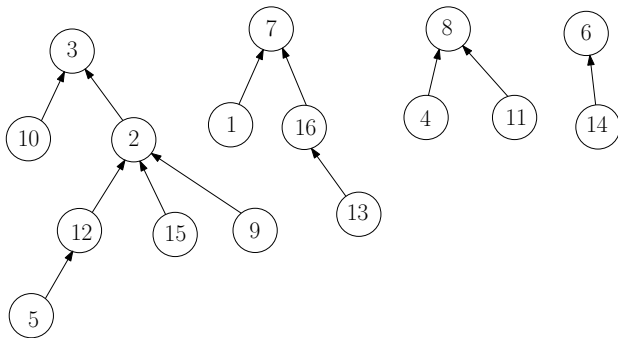
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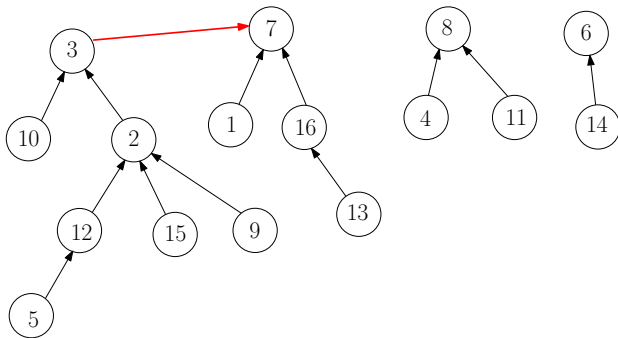
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Union-Find Data Structure



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Union-Find Data Structure

root(*v*)

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1: if  $par[v] = \perp$  then  
2:   return  $v$   
3: else  
4:   return  $root(par[v])$ 
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Union-Find Data Structure

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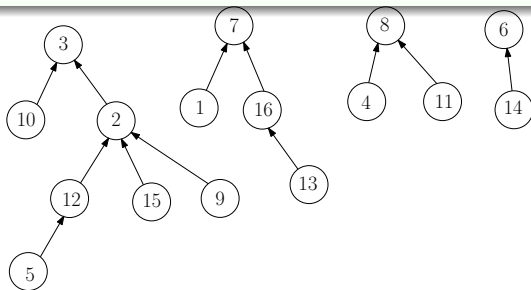
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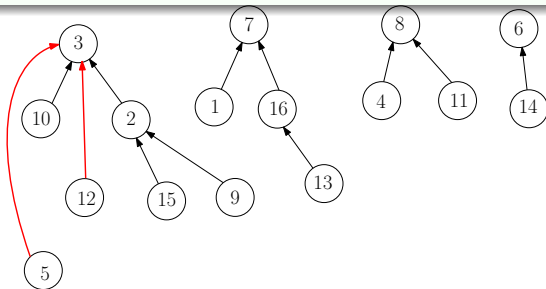
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5:   return  $par[v]$ 
```



MST-Kruskal(G, w)

- 1: $F \leftarrow \emptyset$
- 2: $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
- 3: sort the edges of E in non-decreasing order of weights w
- 4: **for** each edge $(u, v) \in E$ in the order **do**
- 5: $S_u \leftarrow$ the set in \mathcal{S} containing u
- 6: $S_v \leftarrow$ the set in \mathcal{S} containing v
- 7: **if** $S_u \neq S_v$ **then**
- 8: $F \leftarrow F \cup \{(u, v)\}$
- 9: $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
- 10: **return** (V, F)

MST-Kruskal(G, w)

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1:  $F \leftarrow \emptyset$ 
2: for every  $v \in V$  do:  $par[v] \leftarrow \perp$ 
3: sort the edges of  $E$  in non-decreasing order of weights  $w$ 
4: for each edge  $(u, v) \in E$  in the order do
5:    $u' \leftarrow \text{root}(u)$ 
6:    $v' \leftarrow \text{root}(v)$ 
7:   if  $u' \neq v'$  then
8:      $F \leftarrow F \cup \{(u, v)\}$ 
9:      $par[u'] \leftarrow v'$ 
10: return  $(V, F)$ 
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- ②, ⑤, ⑥, ⑦, ⑨ takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.

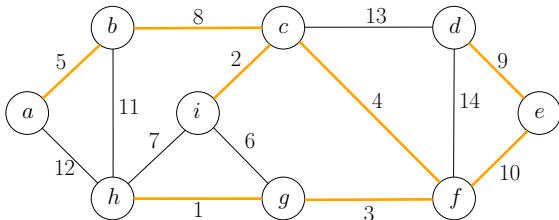
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- ②, ⑤, ⑥, ⑦, ⑨ takes time $O(m\alpha(n))$
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- Running time = time for ③ = $O(m \lg n)$.

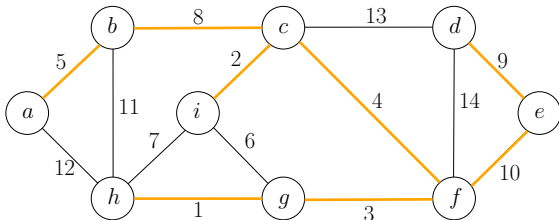
Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is **not** in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



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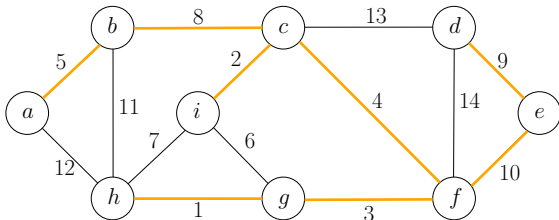
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- (i, g) is not in the MST because of cycle (i, c, f, g)

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is **not** in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

Outline

1 Minimum Spanning Tree

- Kruskal's Algorithm
- Reverse-Kruskal's Algorithm
- Prim's Algorithm

2 Single Source Shortest Paths

- Dijkstra's Algorithm

3 Shortest Paths in Graphs with Negative Weights

4 All-Pair Shortest Paths and Floyd-Warshall

5 Minimum Cost Arborescence

Two Methods to Build a MST

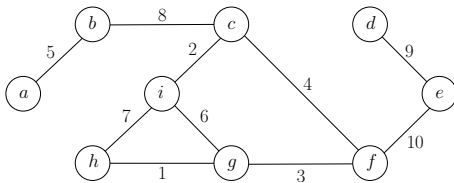
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Two Methods to Build a MST

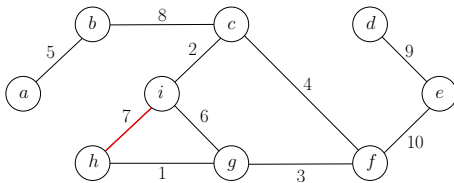
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Q: Which edge can be safely **excluded** from the MST?

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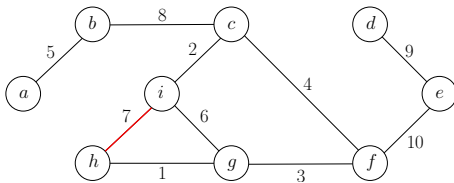


Q: Which edge can be safely **excluded** from the MST?

A: The heaviest non-**bridge** edge.

Two Methods to Build a MST

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A: The heaviest non-**bridge** edge.

Def. A **bridge** is an edge whose removal disconnects the graph.

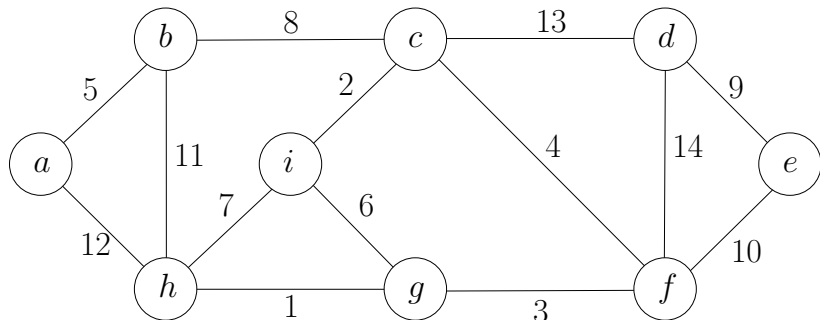
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

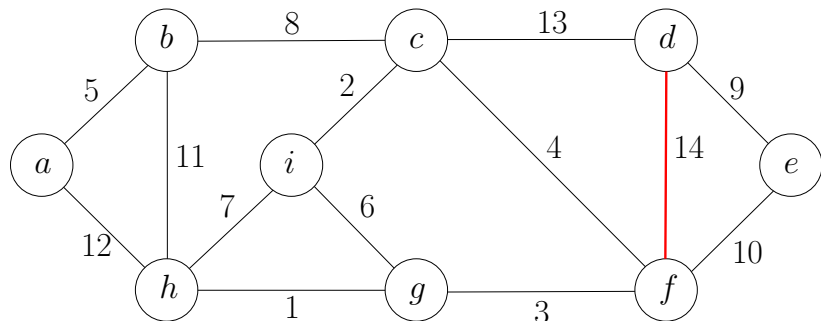
MST-Greedy(G, w)

- 1: $F \leftarrow E$
- 2: sort E in non-increasing order of weights
- 3: **for** every e in this order **do**
- 4: **if** $(V, F \setminus \{e\})$ is connected **then**
- 5: $F \leftarrow F \setminus \{e\}$
- 6: **return** (V, F)

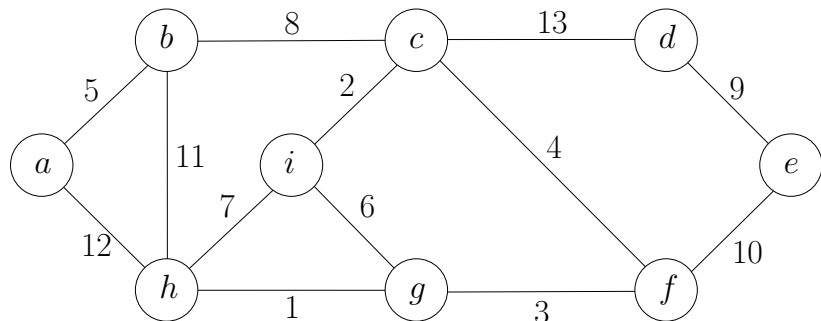
Reverse Kruskal's Algorithm: Example



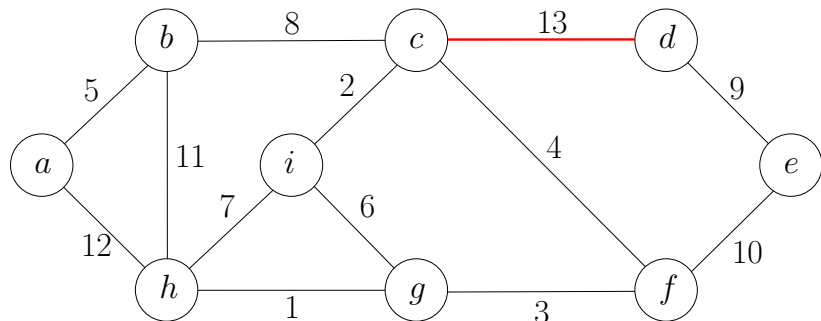
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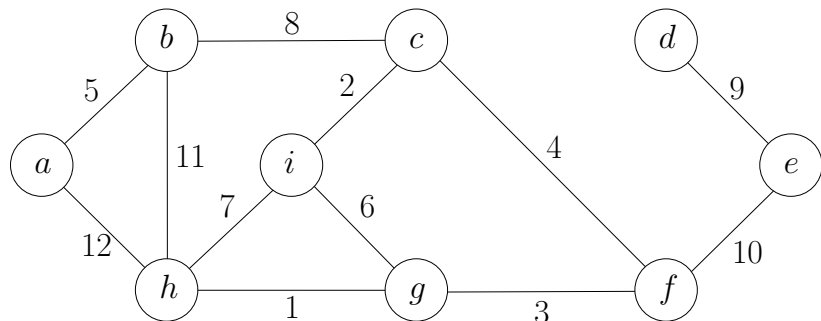
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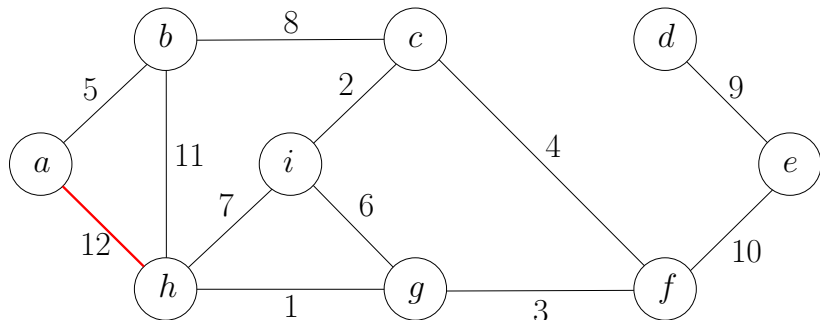
Reverse Kruskal's Algorithm: Example



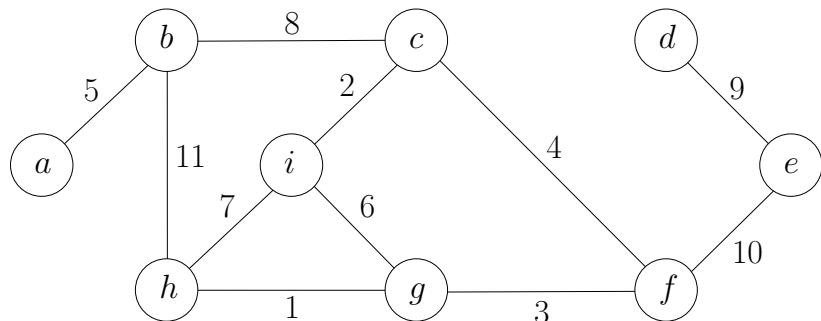
Reverse Kruskal's Algorithm: Example



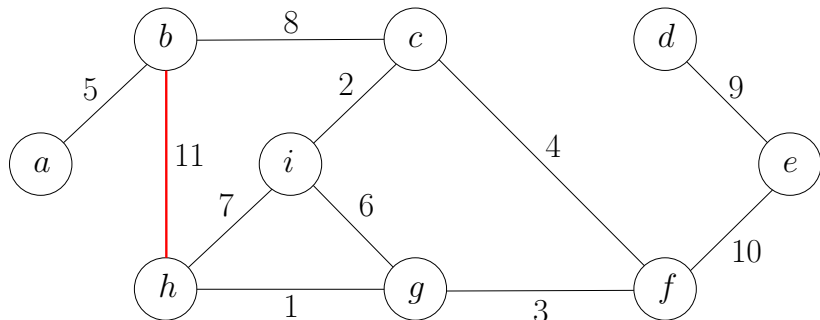
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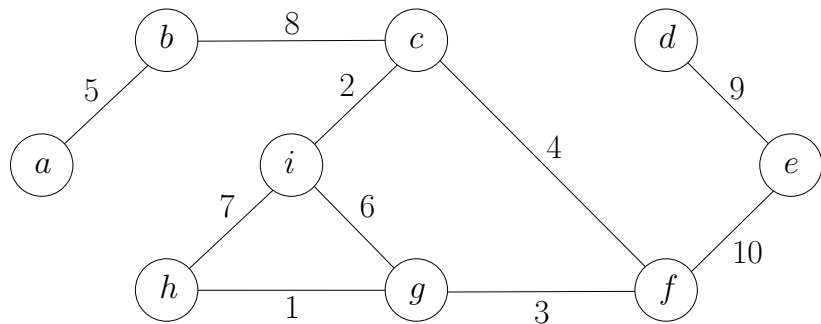
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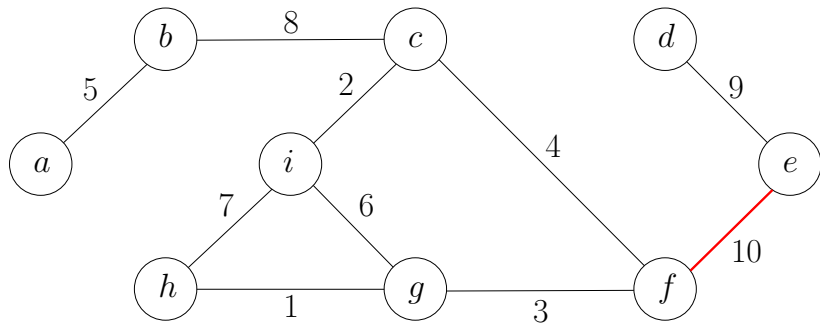
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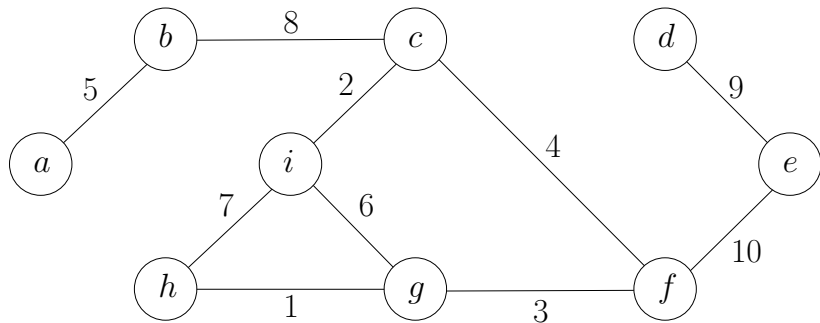
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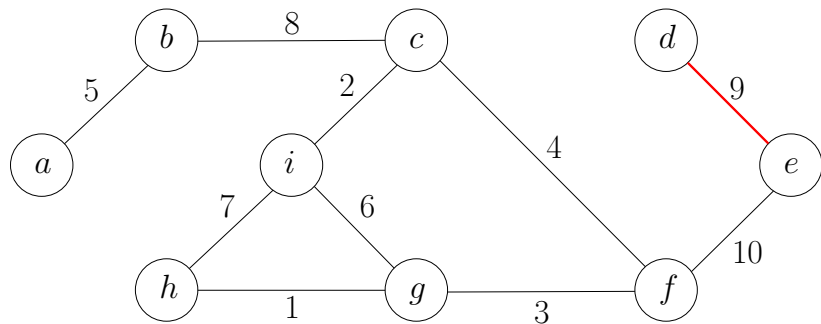
Reverse Kruskal's Algorithm: Example



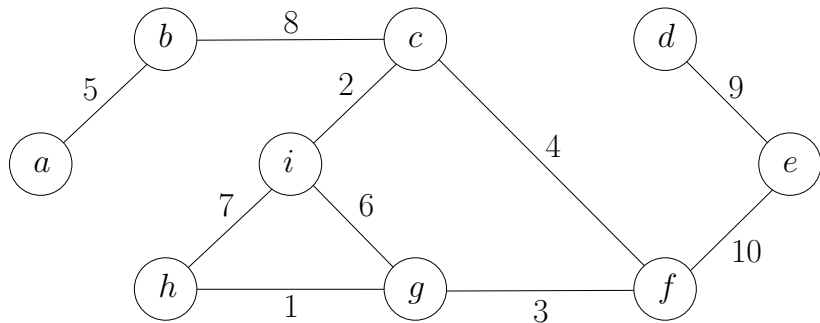
Reverse Kruskal's Algorithm: Example



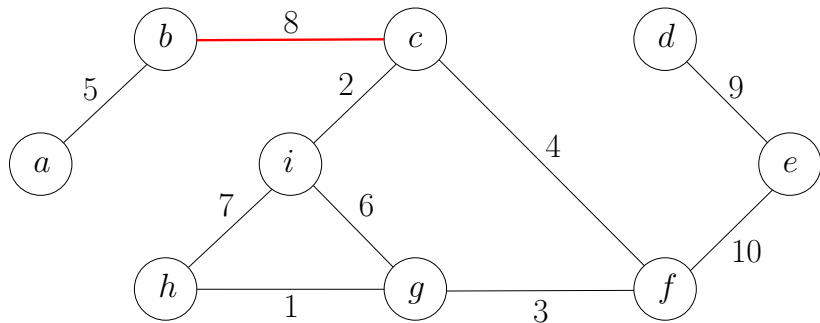
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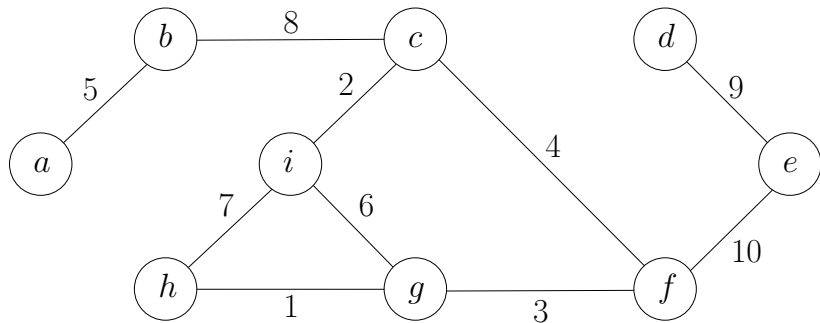
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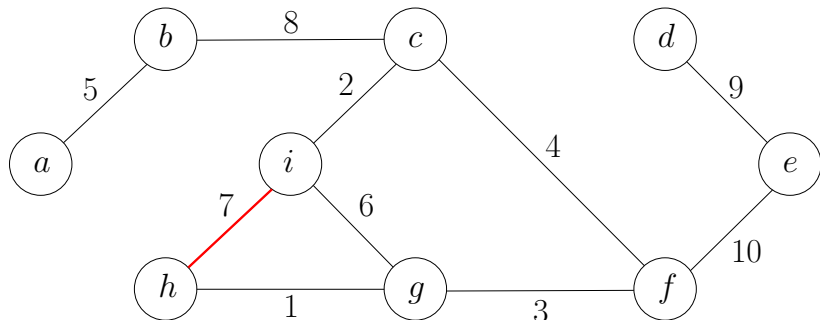
Reverse Kruskal's Algorithm: Example



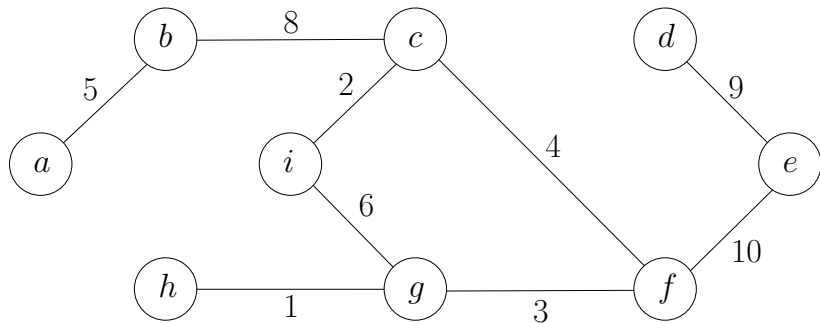
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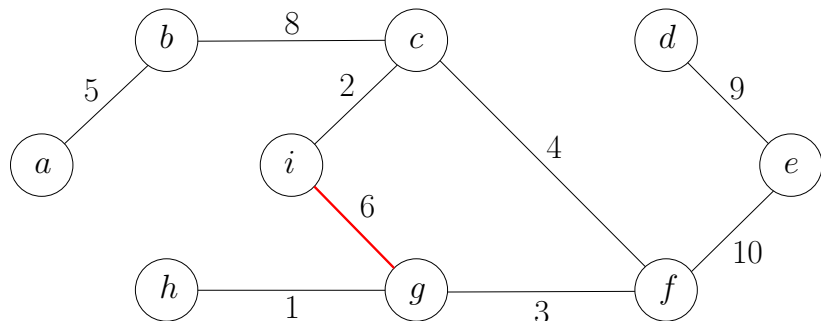
Reverse Kruskal's Algorithm: Example



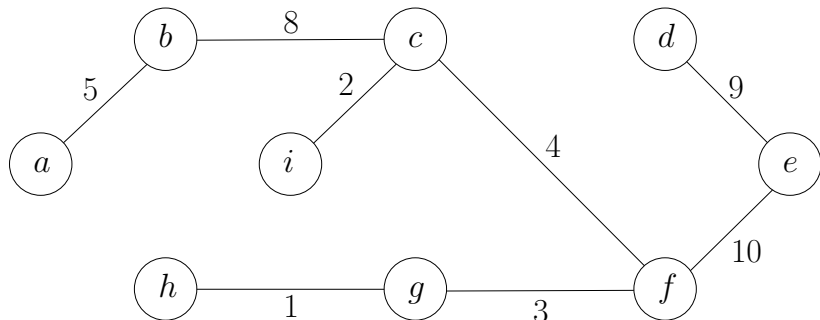
Reverse Kruskal's Algorithm: Example



Reverse Kruskal's Algorithm: Example



Reverse Kruskal's Algorithm: Example



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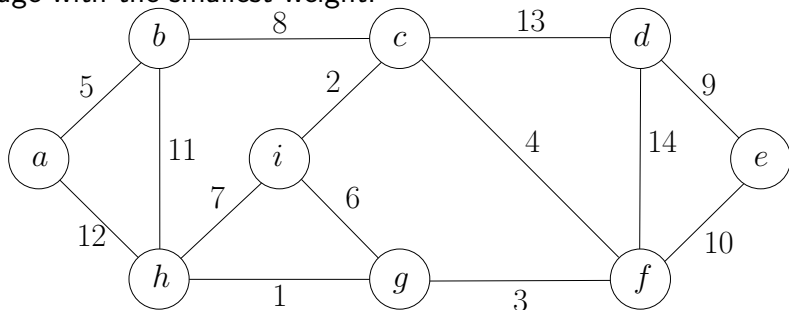
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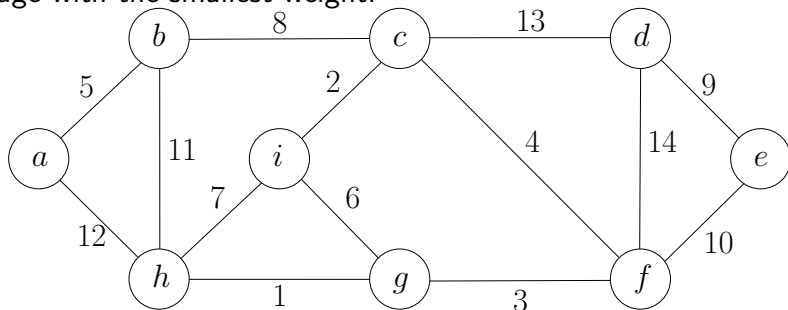
Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



Design Greedy Strategy for MST

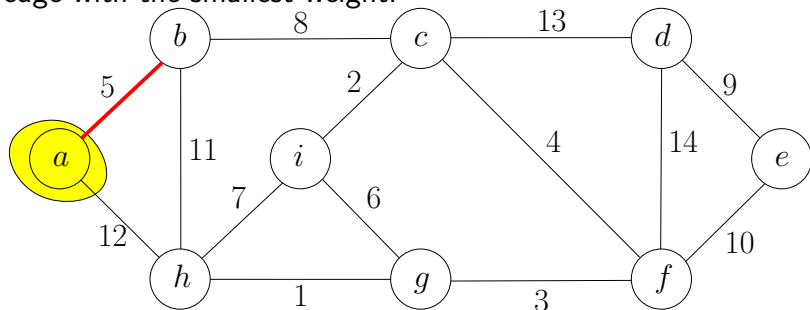
- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



- Greedy strategy for Prim's algorithm: choose the lightest edge incident to *a*.

Design Greedy Strategy for MST

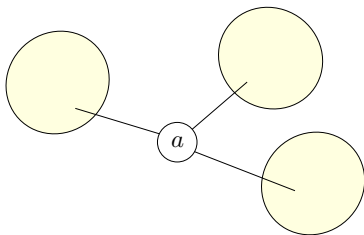
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Lemma It is safe to include the lightest edge incident to a .

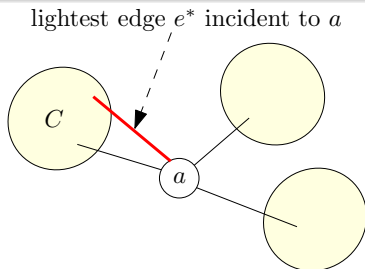
Lemma It is safe to include the lightest edge incident to a .



Proof.

- Let T be a MST
- Consider all components obtained by removing a from T

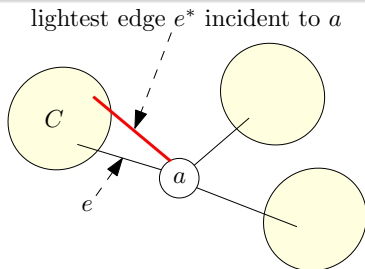
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Proof.

- Let T be a MST
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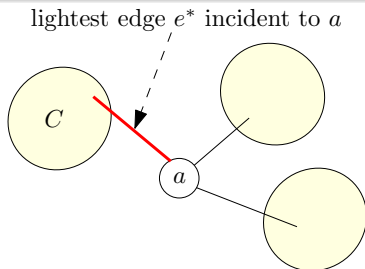
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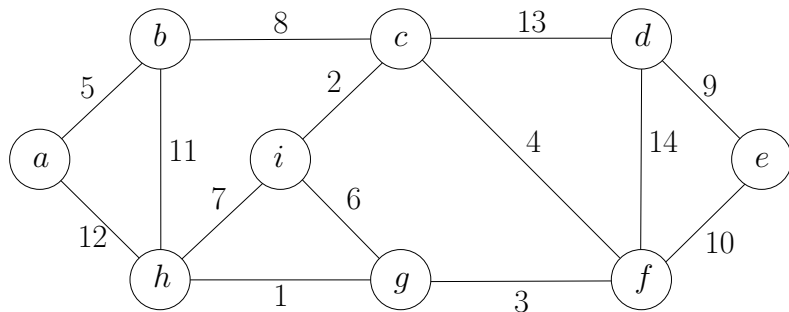


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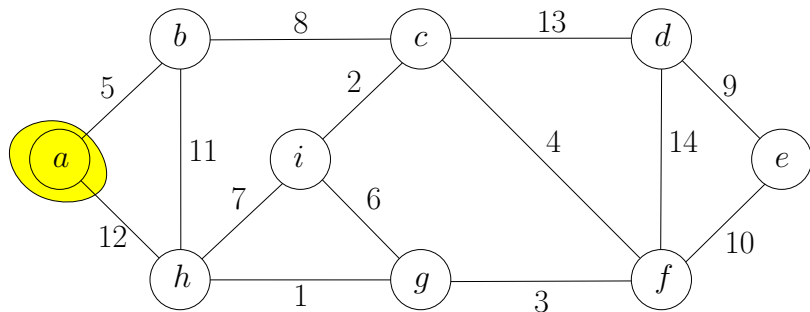
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- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$



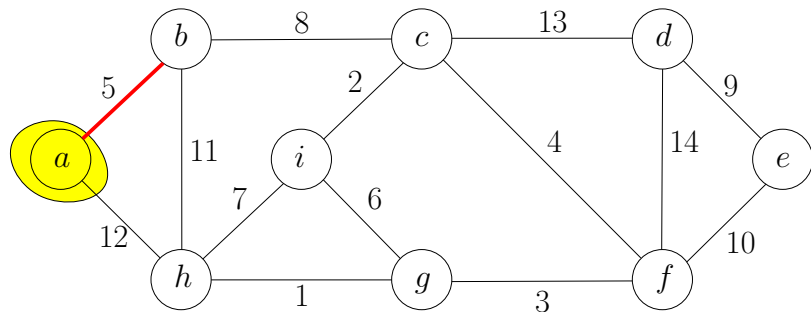
Prim's Algorithm: Example



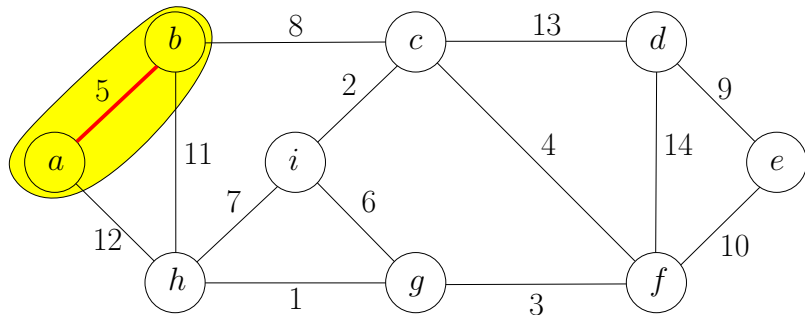
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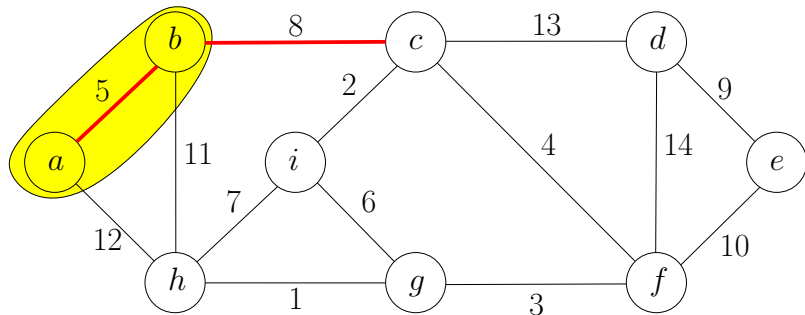
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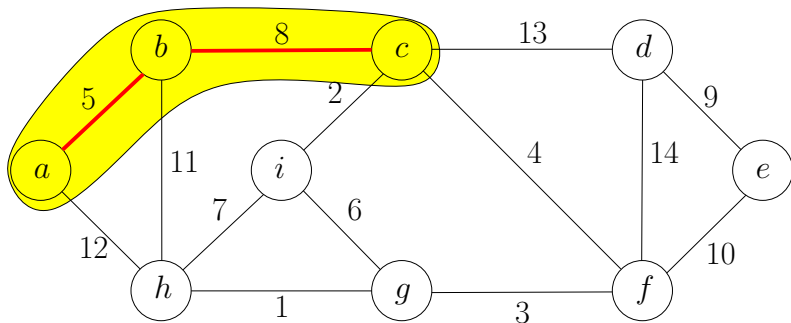
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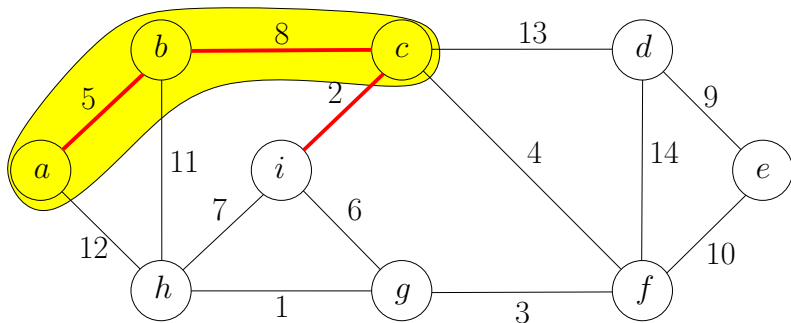
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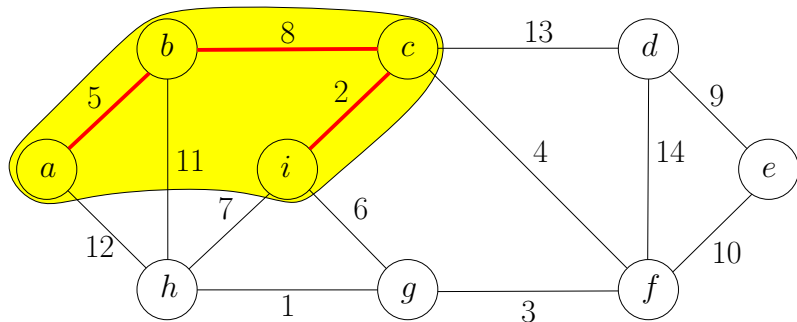
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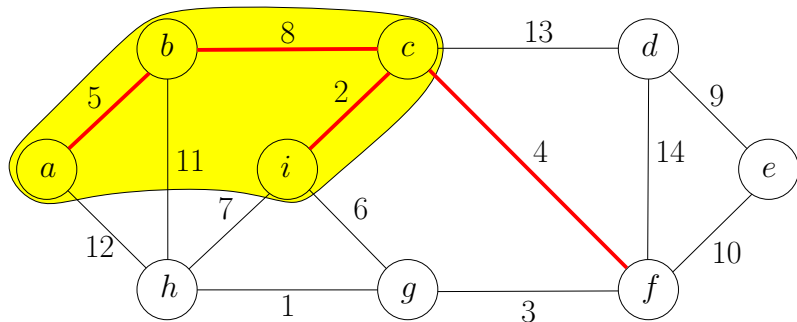
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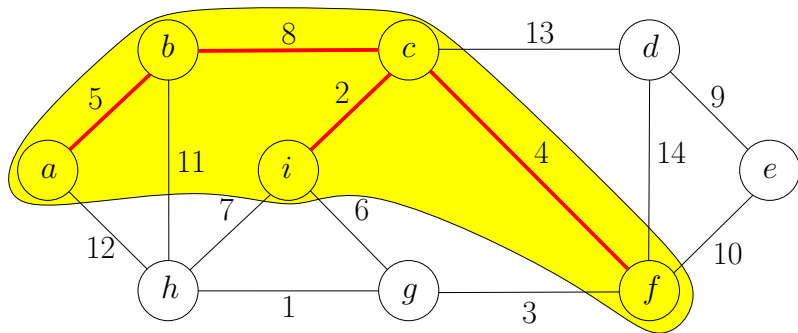
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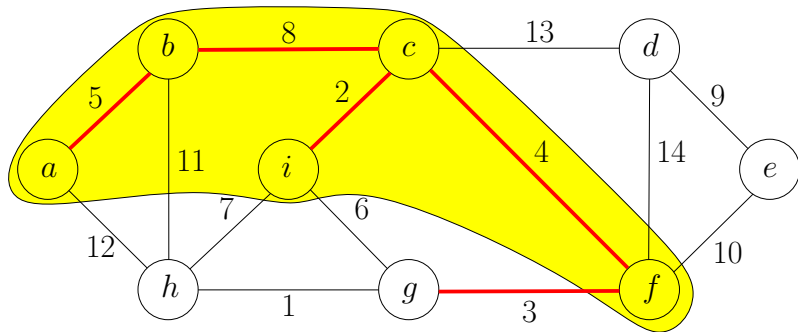
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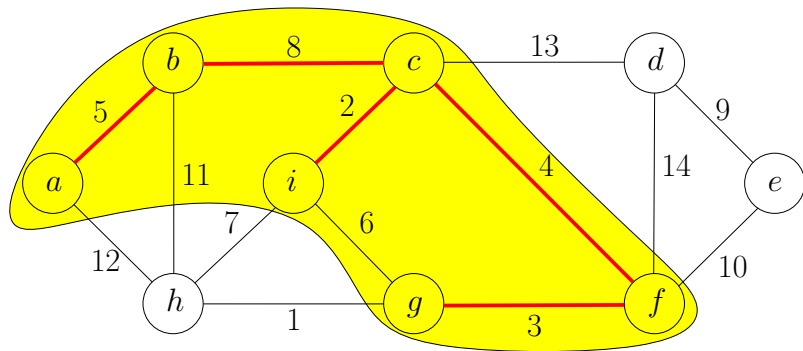
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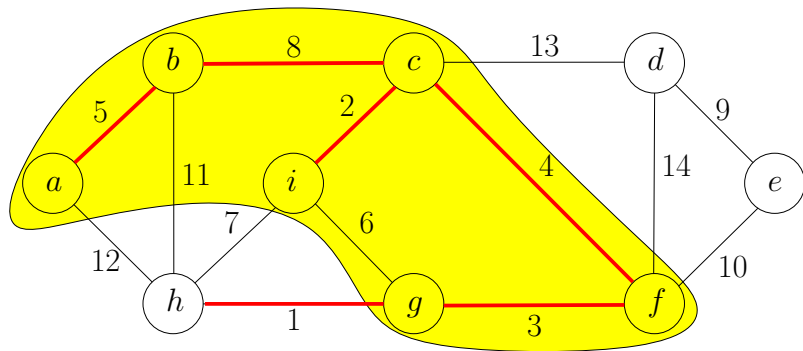
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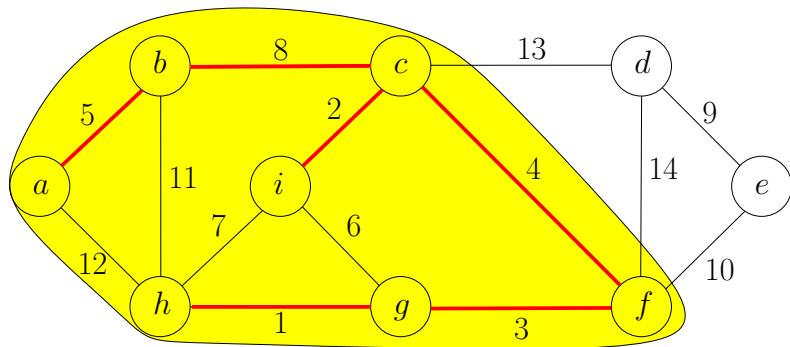
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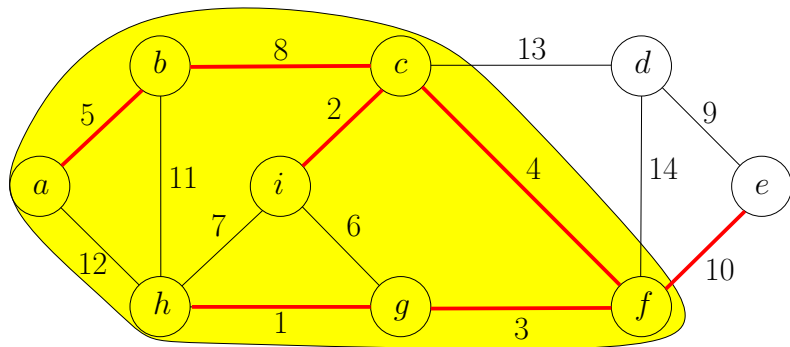
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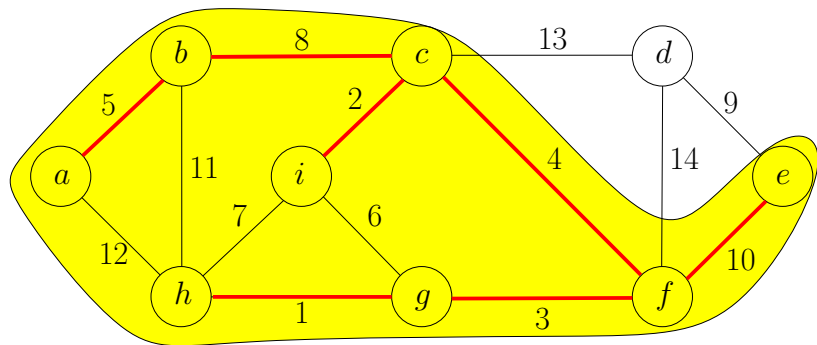
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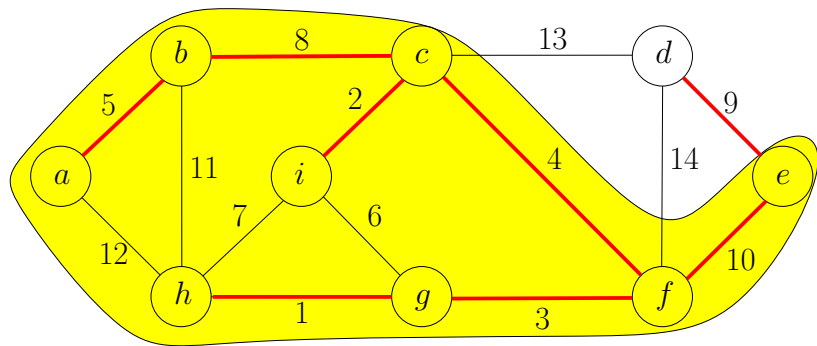
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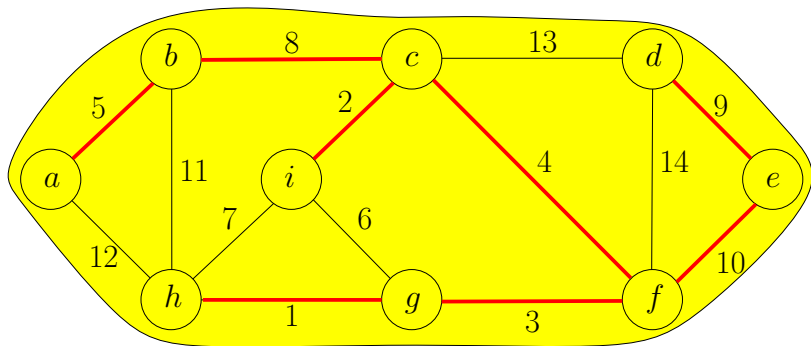
Prim's Algorithm: Example



Prim's Algorithm: Example



Prim's Algorithm: Example



Greedy Algorithm

MST-Greedy1(G, w)

1: $S \leftarrow \{s\}$, where s is arbitrary vertex in V

2: $F \leftarrow \emptyset$

3: **while** $S \neq V$ **do**

4: $(u, v) \leftarrow$ lightest edge between S and $V \setminus S$,
 where $u \in S$ and $v \in V \setminus S$

5: $S \leftarrow S \cup \{v\}$ 6: $F \leftarrow F \cup \{(u, v)\}$ 7: **return** (V, F)

Greedy Algorithm

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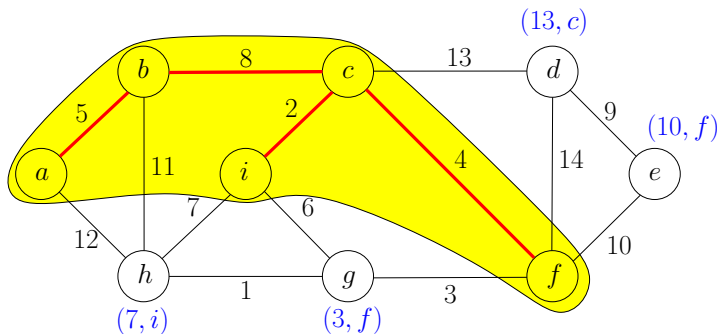
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- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S: (u,v) \in E} w(u, v)$:
the weight of the lightest edge between v and S
- $\pi[v] = \arg \min_{u \in S: (u,v) \in E} w(u, v)$:
 $(\pi[v], v)$ is the lightest edge between v and S



Prim's Algorithm: Efficient Implementation of Greedy Algorithm

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In every iteration

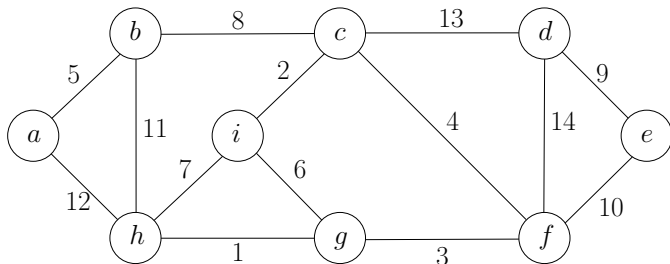
- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to F
- Add u to S , update d and π values.

Prim's Algorithm

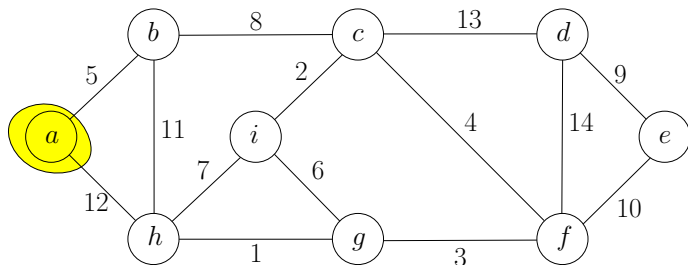
MST-Prim(G, w)

```
1:  $s \leftarrow$  arbitrary vertex in  $G$ 
2:  $S \leftarrow \emptyset, d(s) \leftarrow 0$  and  $d[v] \leftarrow \infty$  for every  $v \in V \setminus \{s\}$ 
3: while  $S \neq V$  do
4:    $u \leftarrow$  vertex in  $V \setminus S$  with the minimum  $d[u]$ 
5:    $S \leftarrow S \cup \{u\}$ 
6:   for each  $v \in V \setminus S$  such that  $(u, v) \in E$  do
7:     if  $w(u, v) < d[v]$  then
8:        $d[v] \leftarrow w(u, v)$ 
9:        $\pi[v] \leftarrow u$ 
10: return  $\{(u, \pi[u]) \mid u \in V \setminus \{s\}\}$ 
```

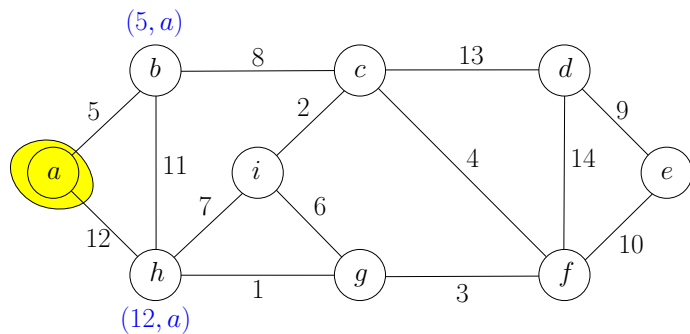
Example



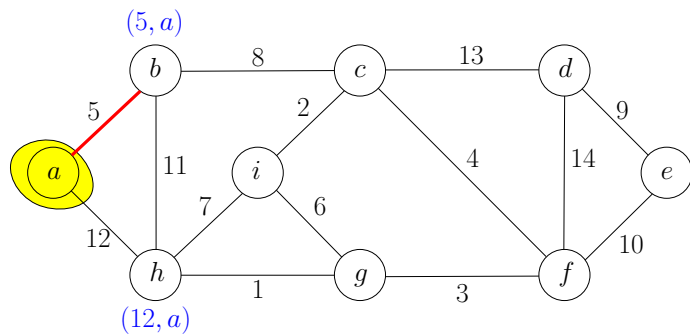
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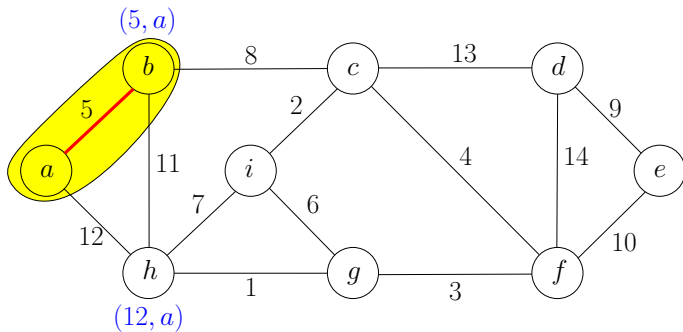
Example



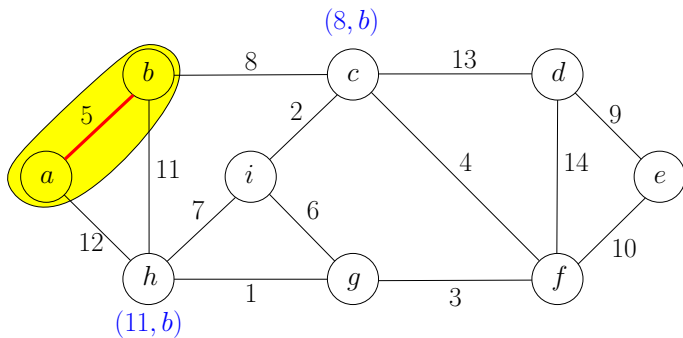
Example



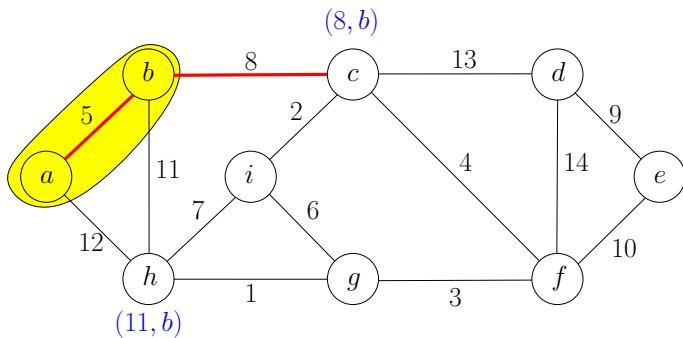
Example



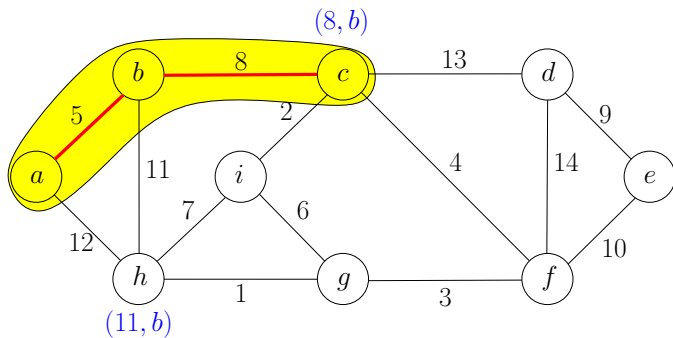
Example



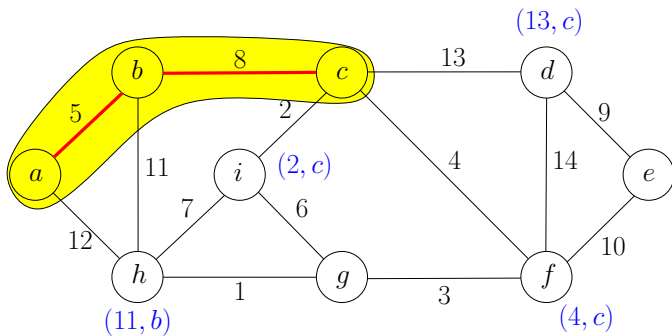
Example



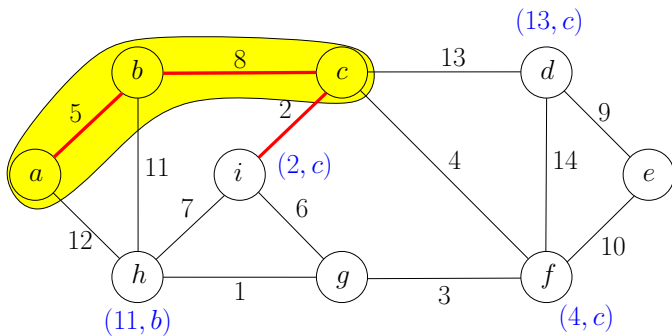
Example



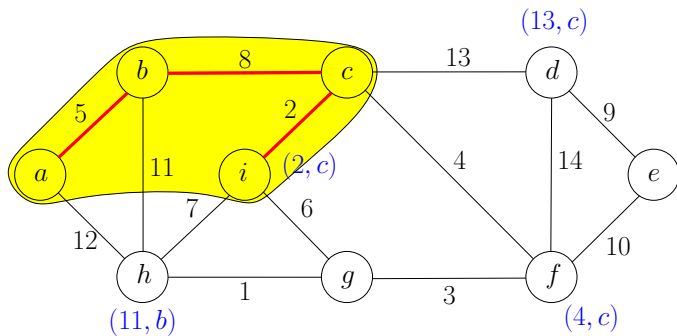
Example



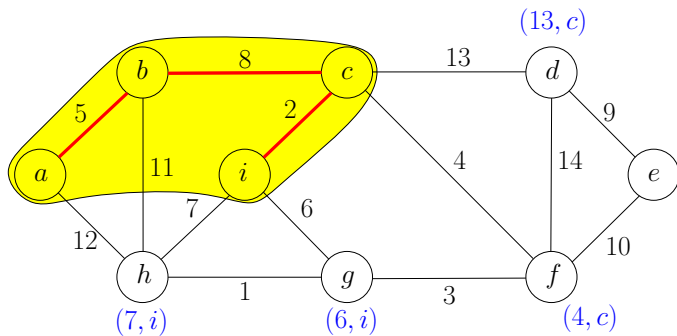
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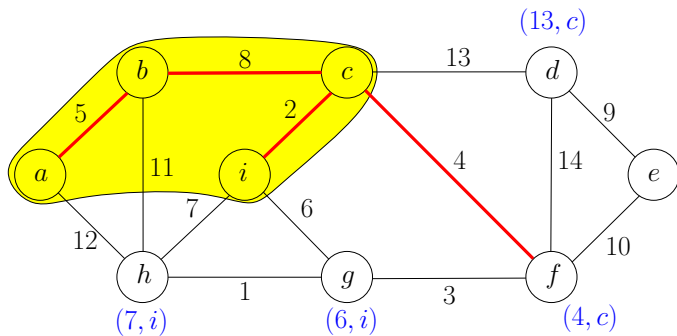
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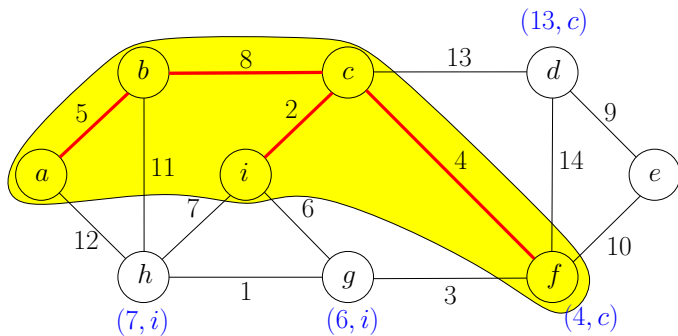
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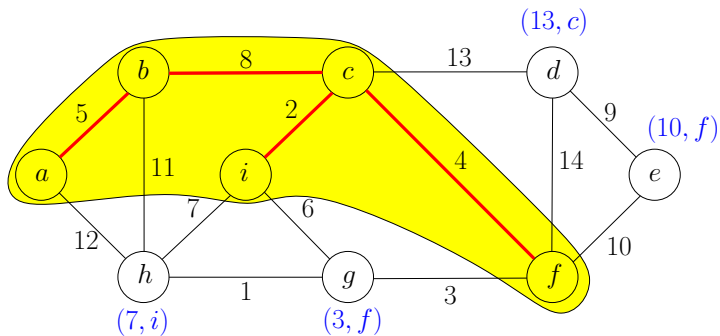
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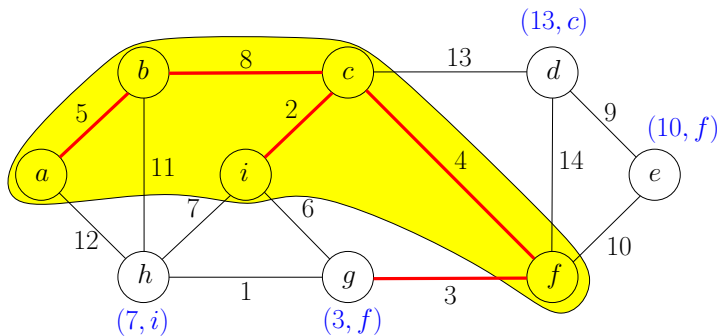
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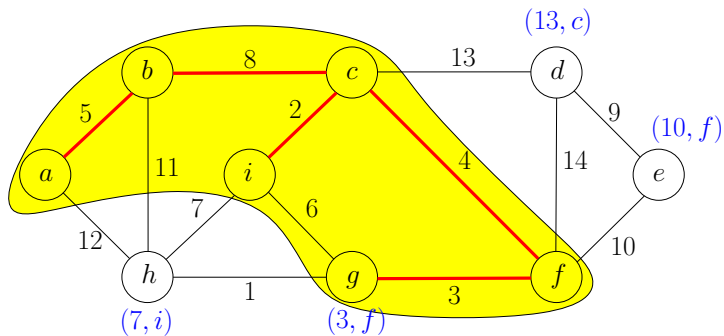
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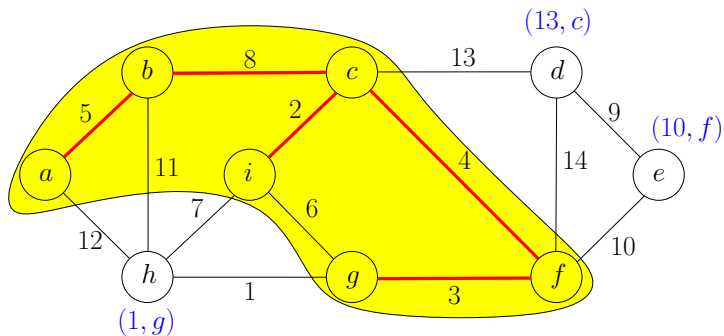
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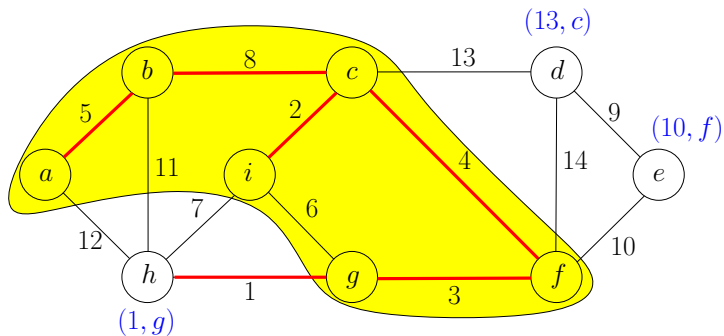
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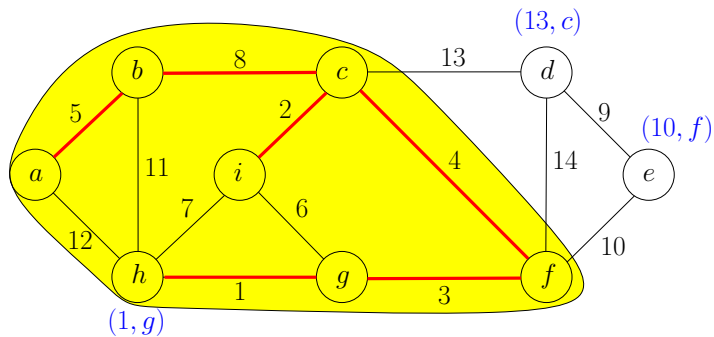
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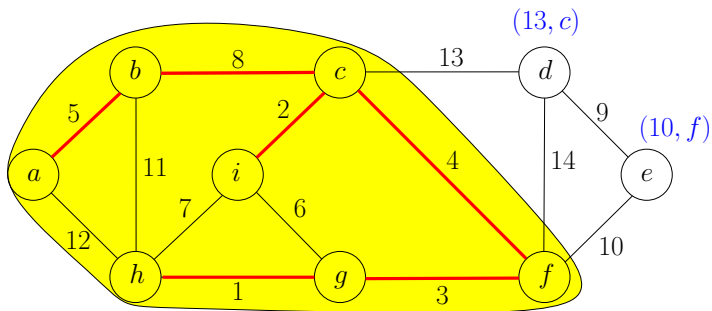
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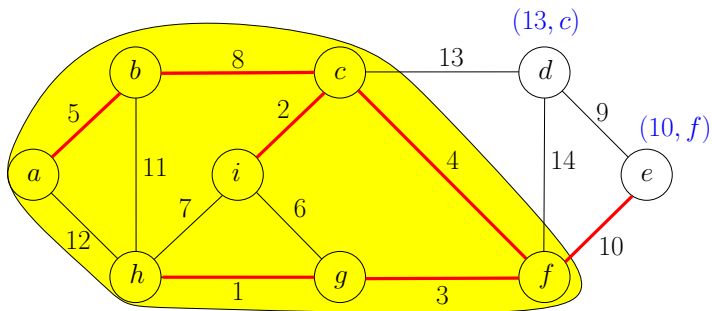
Example



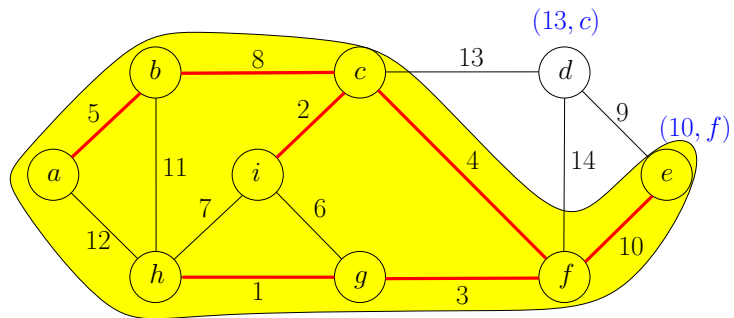
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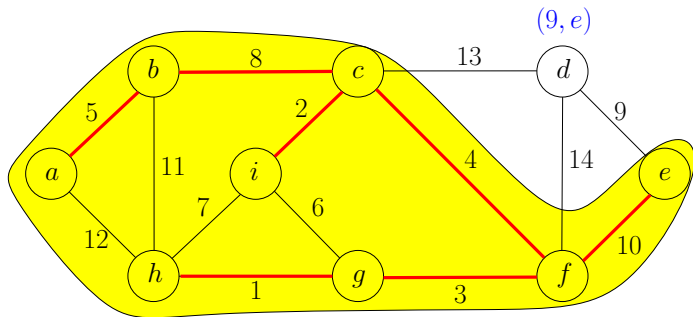
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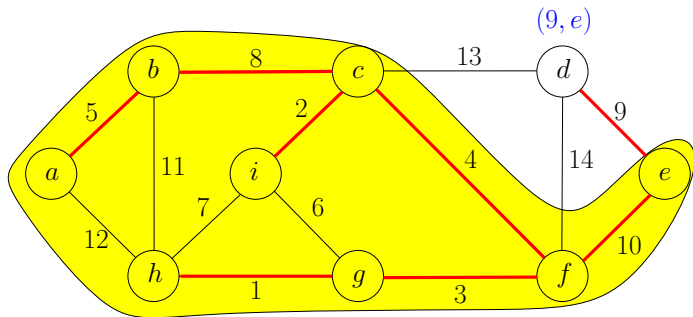
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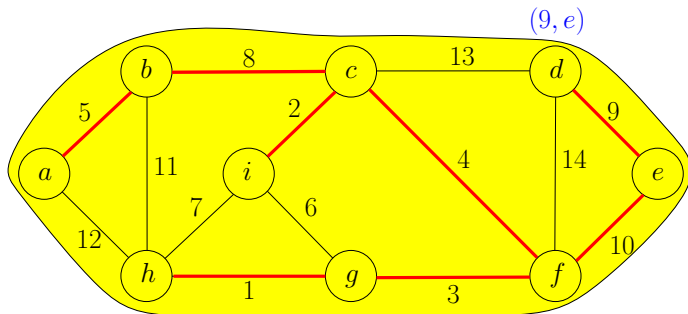
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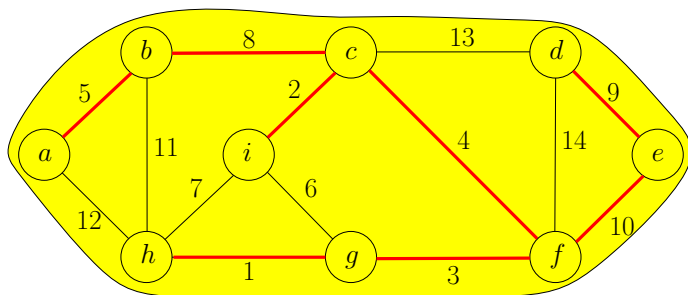
Example



Example



Example



Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S: (u,v) \in E} w(u, v)$:
the weight of the lightest edge between v and S
- $\pi[v] = \arg \min_{u \in S: (u,v) \in E} w(u, v)$:
 $(\pi[v], v)$ is the lightest edge between v and S

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to F
- Add u to S , update d and π values.

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In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value extract_min
- Add $(\pi[u], u)$ to F
- Add u to S , update d and π values. decrease_key

Use a priority queue to support the operations

Def. A **priority queue** is an **abstract** data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- $\text{insert}(v, \text{key_value})$: insert an element v , whose associated key value is key_value .
- $\text{decrease_key}(v, \text{new_key_value})$: decrease the key value of an element v in queue to new_key_value
- $\text{extract_min}()$: return and remove the element in queue with the smallest key value
- ...

Prim's Algorithm

MST-Prim(G, w)

```
1:  $s \leftarrow$  arbitrary vertex in  $G$ 
2:  $S \leftarrow \emptyset, d(s) \leftarrow 0$  and  $d[v] \leftarrow \infty$  for every  $v \in V \setminus \{s\}$ 
3:
4: while  $S \neq V$  do
5:    $u \leftarrow$  vertex in  $V \setminus S$  with the minimum  $d[u]$ 
6:    $S \leftarrow S \cup \{u\}$ 
7:   for each  $v \in V \setminus S$  such that  $(u, v) \in E$  do
8:     if  $w(u, v) < d[v]$  then
9:        $d[v] \leftarrow w(u, v)$ 
10:       $\pi[v] \leftarrow u$ 
11: return  $\{(u, \pi[u]) \mid u \in V \setminus \{s\}\}$ 
```

Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

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1:  $s \leftarrow$  arbitrary vertex in  $G$ 
2:  $S \leftarrow \emptyset, d(s) \leftarrow 0$  and  $d[v] \leftarrow \infty$  for every  $v \in V \setminus \{s\}$ 
3:  $Q \leftarrow$  empty queue, for each  $v \in V$ :  $Q.\text{insert}(v, d[v])$ 
4: while  $S \neq V$  do
5:    $u \leftarrow Q.\text{extract\_min}()$ 
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11: return  $\{(u, \pi[u]) \mid u \in V \setminus \{s\}\}$ 
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Running Time of Prim's Algorithm Using Priority Queue

$$O(n) \times (\text{time for extract_min}) + O(m) \times (\text{time for decrease_key})$$

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concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

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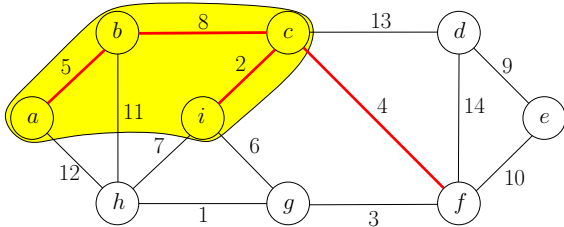
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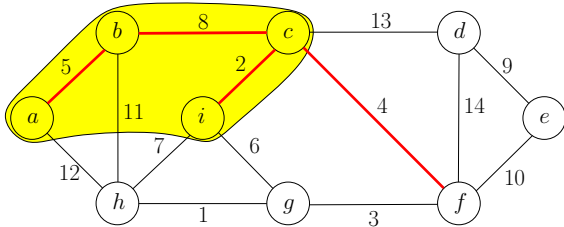
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- (c, f) is in MST because of cut $(\{a, b, c, i\}, V \setminus \{a, b, c, i\})$
- (i, g) is not in MST because no such cut exists

“Evidence” for $e \in \text{MST}$ or $e \notin \text{MST}$

Assumption Assume all edge weights are different.

- $e \in \text{MST} \leftrightarrow$ there is a cut in which e is the lightest edge
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Thus, the minimum spanning tree is unique with assumption.

Outline

- 1 Minimum Spanning Tree
 - Kruskal's Algorithm
 - Reverse-Kruskal's Algorithm
 - Prim's Algorithm
- 2 Single Source Shortest Paths
 - Dijkstra's Algorithm
- 3 Shortest Paths in Graphs with Negative Weights
- 4 All-Pair Shortest Paths and Floyd-Warshall
- 5 Minimum Cost Arborescence

algorithm	graph	weights	SS?	running time
Simple DP	DAG	\mathbb{R}	SS	$O(n + m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	\mathbb{R}	SS	$O(nm)$
Floyd-Warshall	U/D	\mathbb{R}	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

s - t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$$w : E \rightarrow \mathbb{R}_{\geq 0}$$

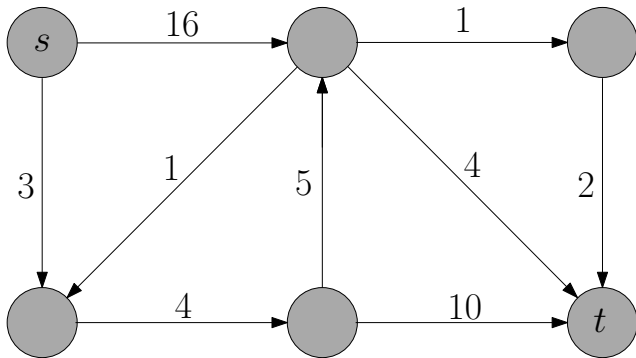
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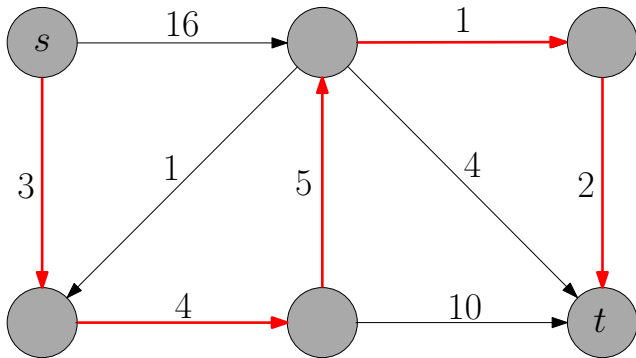


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Single Source Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s \in V$

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Output: shortest paths from s to all other vertices $v \in V$

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- We do not know how to solve s - t shortest path problem more efficiently than solving single source shortest path problem

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Reason for Considering Single Source Shortest Paths Problem

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Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$$w : E \rightarrow \mathbb{R}_{\geq 0}$$

Output: $\pi[v], v \in V \setminus s$: the parent of v in shortest path tree

$d[v], v \in V \setminus s$: the length of shortest path from s to v

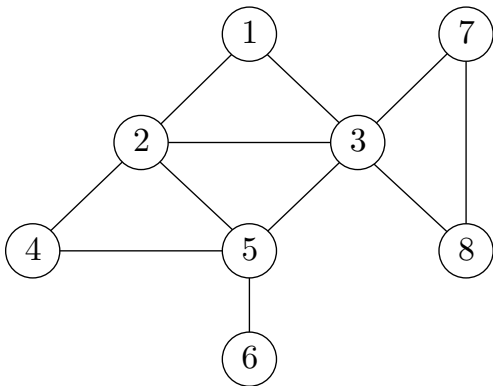
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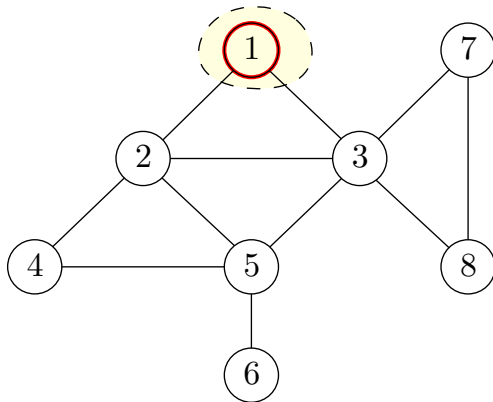
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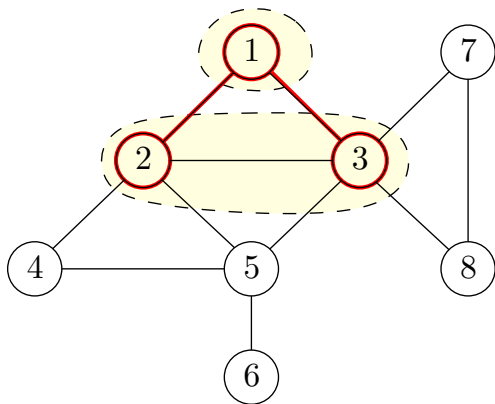
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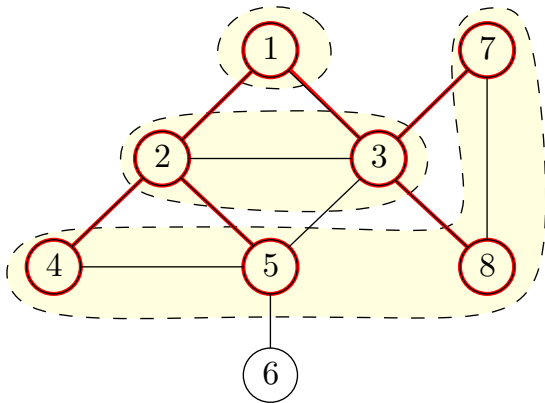
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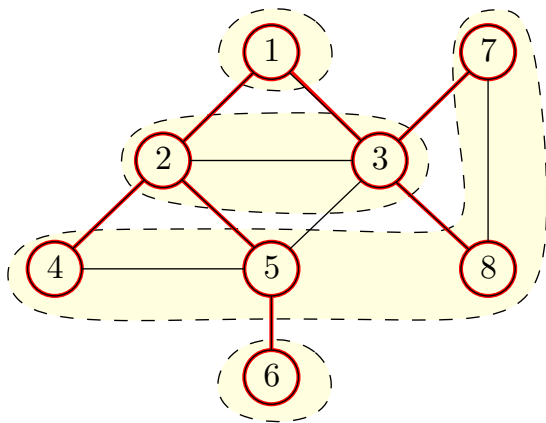
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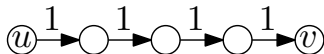
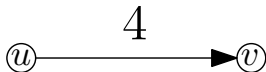
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Assumption Weights $w(u, v)$ are integers (w.l.o.g.).

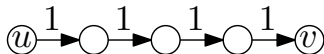
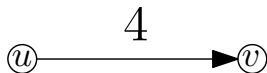
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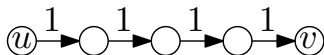
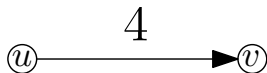


Shortest Path Algorithm by Running BFS

- 1: replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
- 2: run BFS
- 3: $\pi[v] \leftarrow$ vertex from which v is visited
- 4: $d[v] \leftarrow$ index of the level containing v

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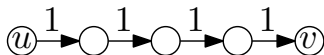
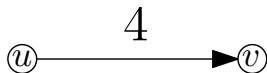
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Shortest Path Algorithm by Running BFS

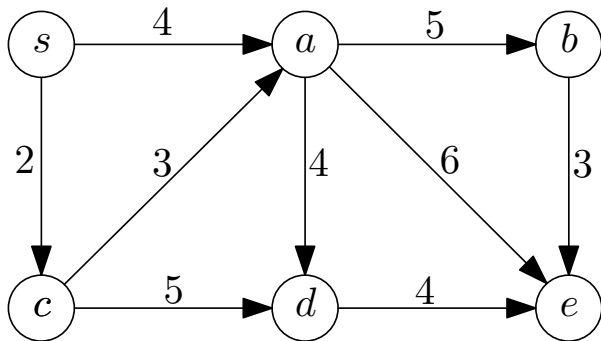
- 1: replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
- 2: run BFS **virtually**
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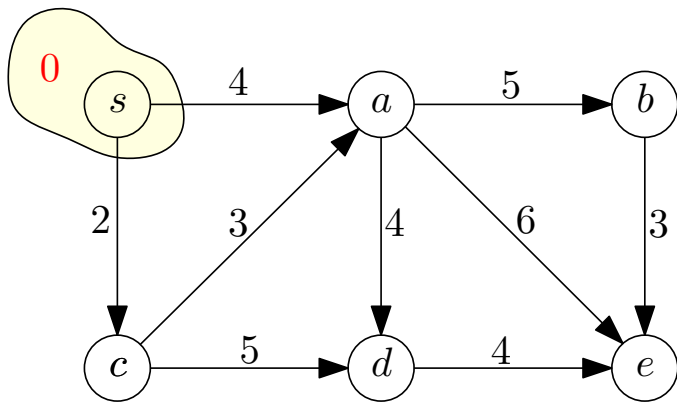
Shortest Path Algorithm by Running BFS Virtually

- 1: $S \leftarrow \{s\}, d(s) \leftarrow 0$
- 2: **while** $|S| \leq n$ **do**
- 3: find a $v \notin S$ that minimizes $\min_{u \in S: (u,v) \in E} \{d[u] + w(u, v)\}$
- 4: $S \leftarrow S \cup \{v\}$
- 5: $d[v] \leftarrow \min_{u \in S: (u,v) \in E} \{d[u] + w(u, v)\}$

Virtual BFS: Example

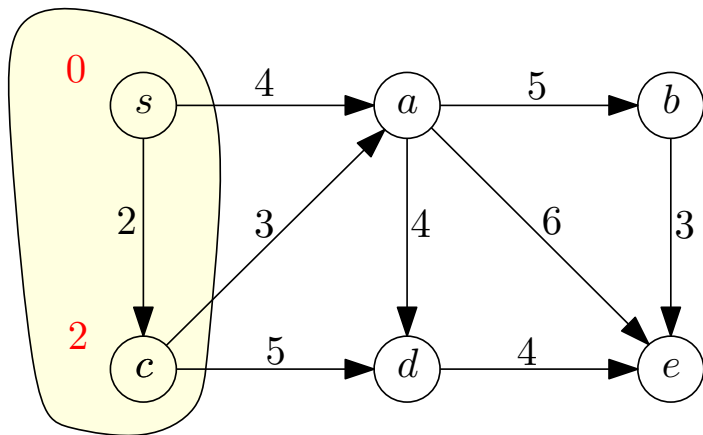


Virtual BFS: Example



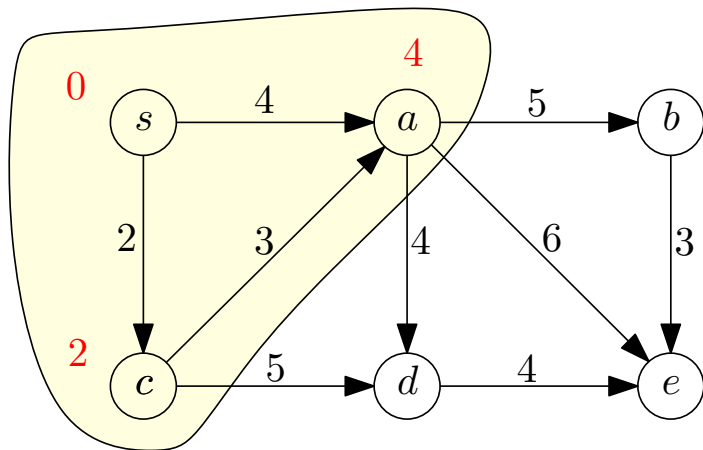
Time 0

Virtual BFS: Example



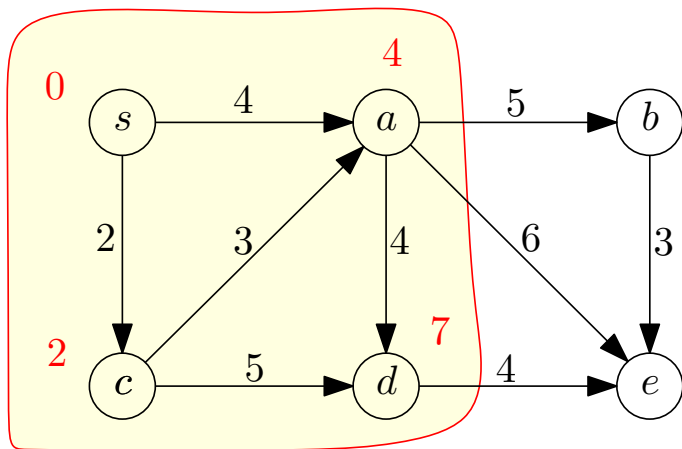
Time 2

Virtual BFS: Example



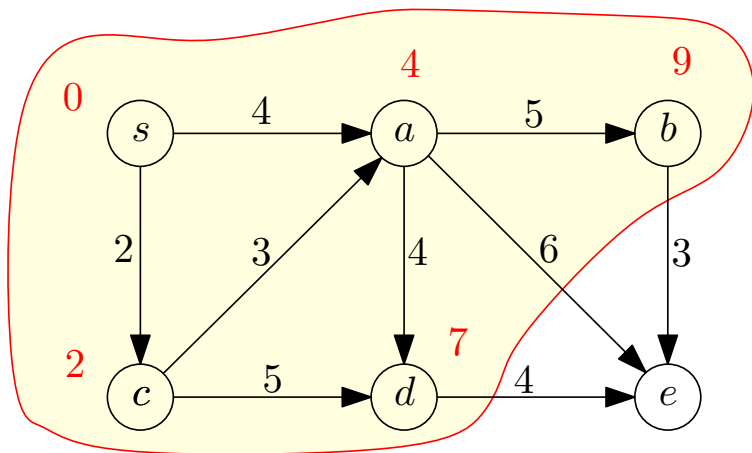
Time 4

Virtual BFS: Example



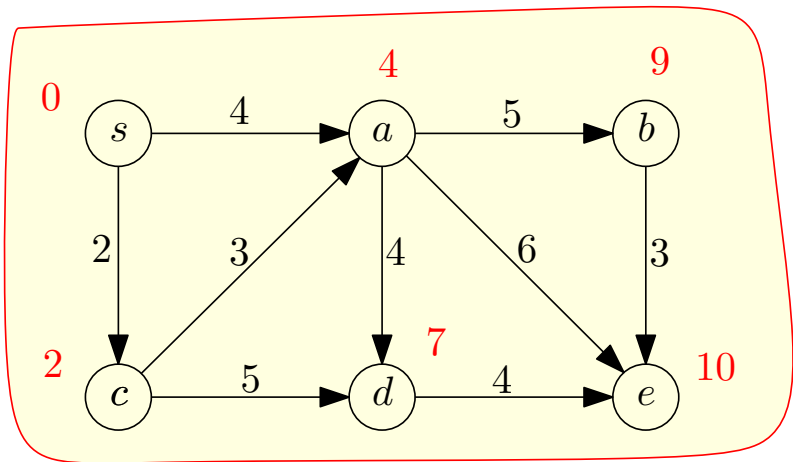
Time 7

Virtual BFS: Example



Time 9

Virtual BFS: Example



Time 10

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Dijkstra's Algorithm

Dijkstra(G, w, s)

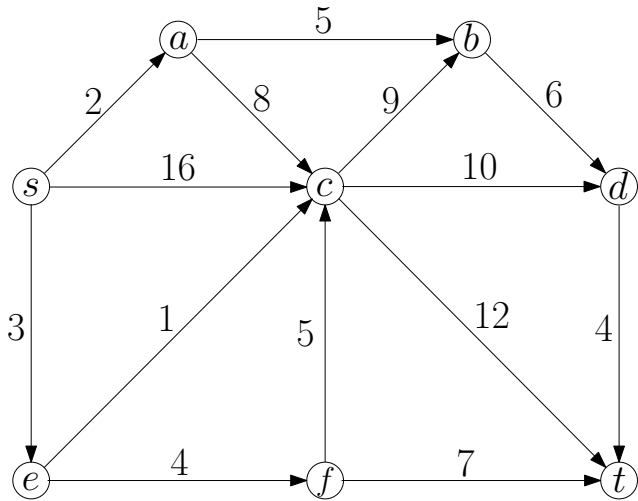
```
1:  $S \leftarrow \emptyset, d(s) \leftarrow 0$  and  $d[v] \leftarrow \infty$  for every  $v \in V \setminus \{s\}$ 
2: while  $S \neq V$  do
3:    $u \leftarrow$  vertex in  $V \setminus S$  with the minimum  $d[u]$ 
4:   add  $u$  to  $S$ 
5:   for each  $v \in V \setminus S$  such that  $(u, v) \in E$  do
6:     if  $d[u] + w(u, v) < d[v]$  then
7:        $d[v] \leftarrow d[u] + w(u, v)$ 
8:        $\pi[v] \leftarrow u$ 
9: return  $(d, \pi)$ 
```

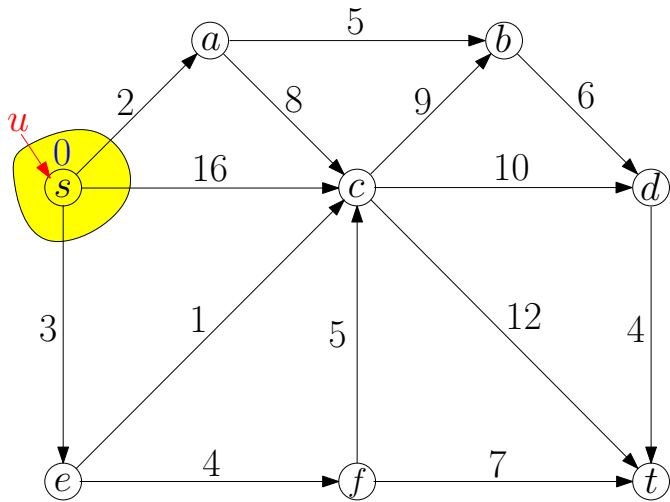
Dijkstra's Algorithm

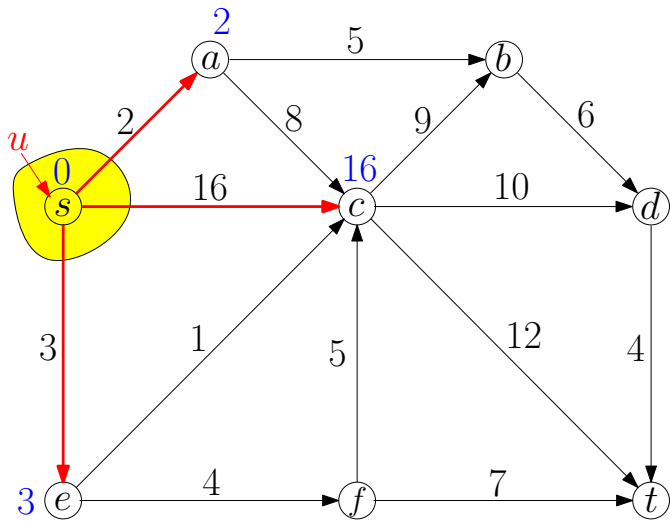
Dijkstra(G, w, s)

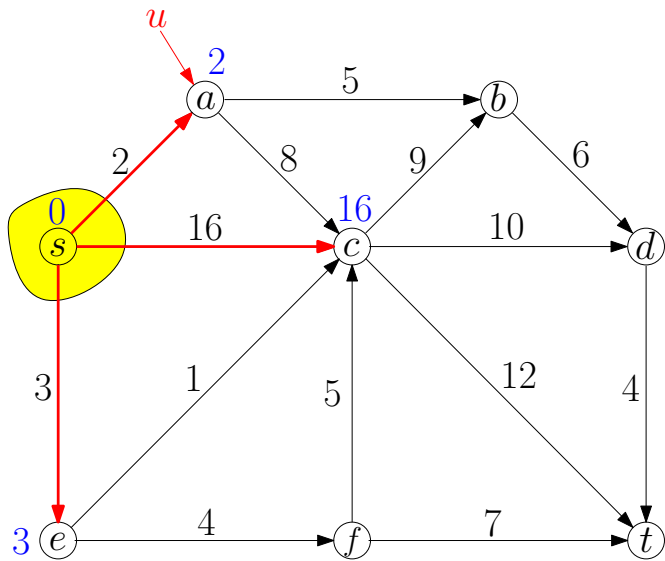
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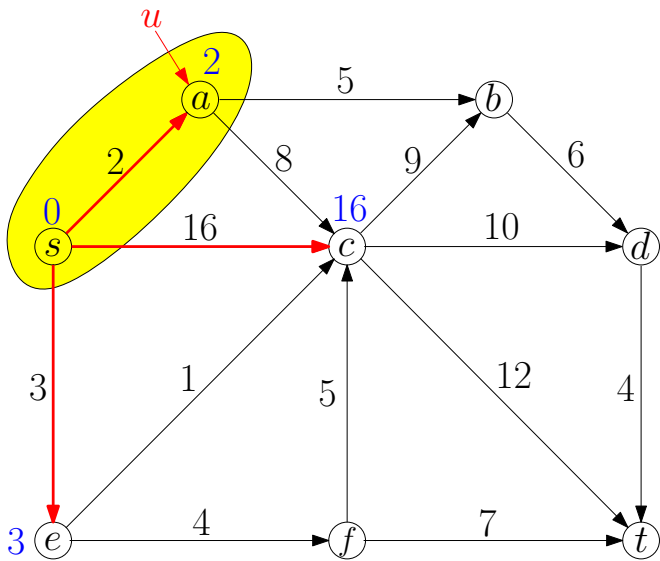
- Running time = $O(n^2)$

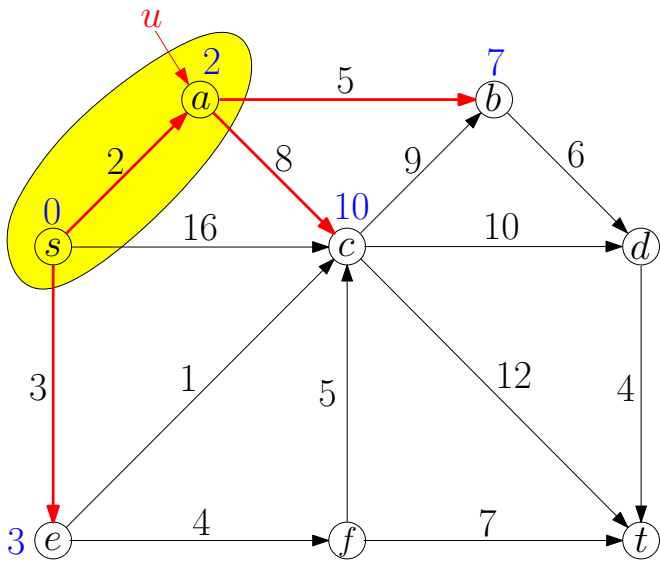


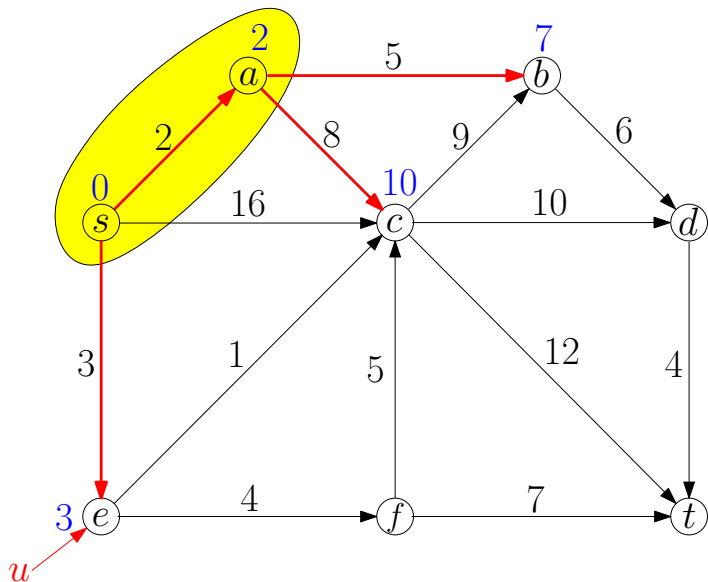


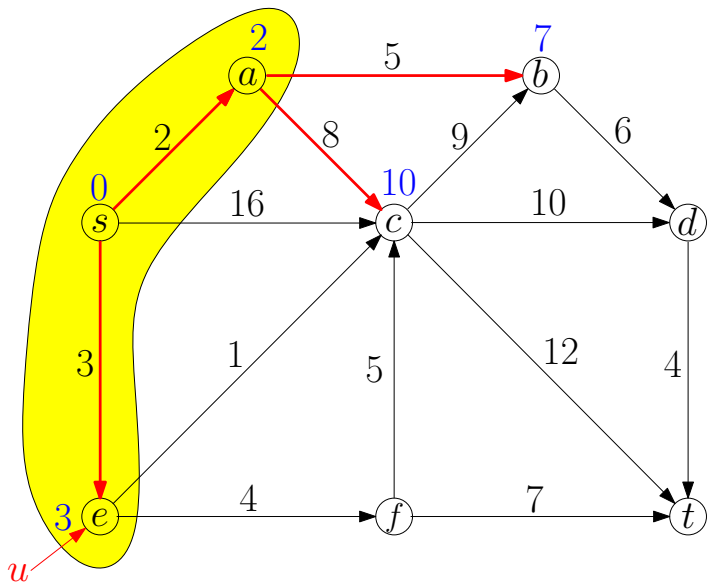


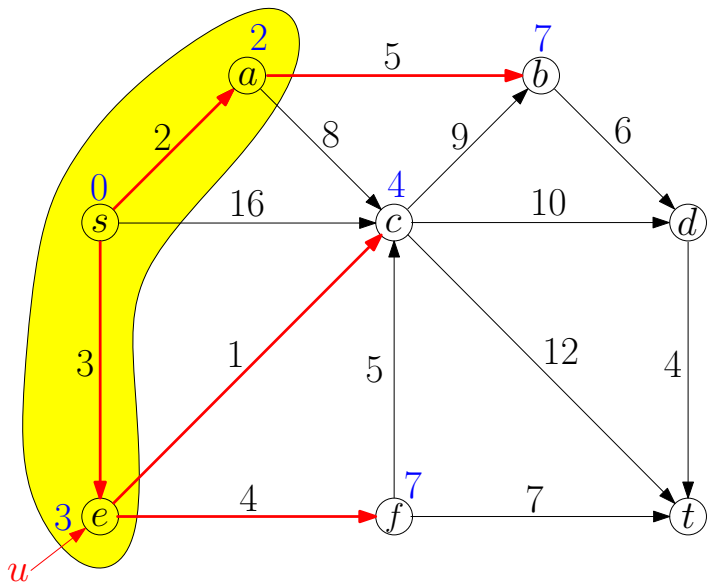


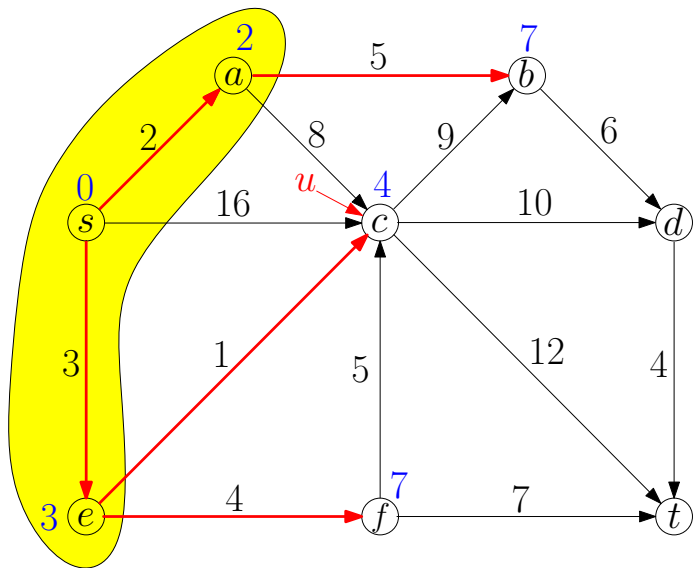


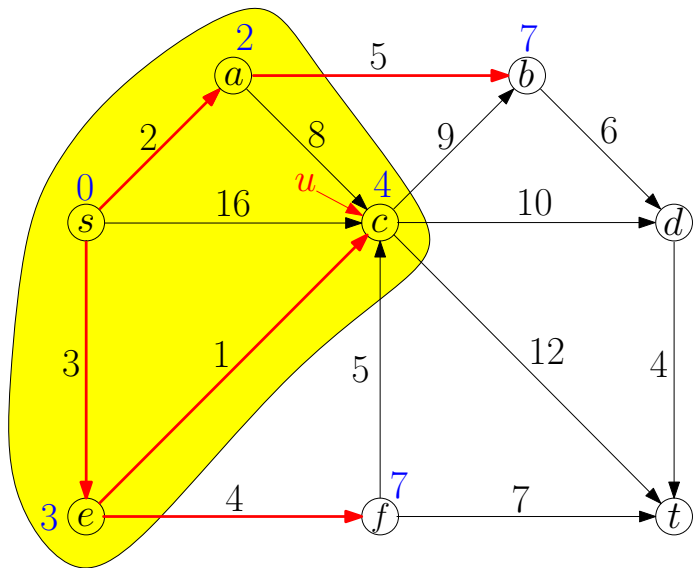


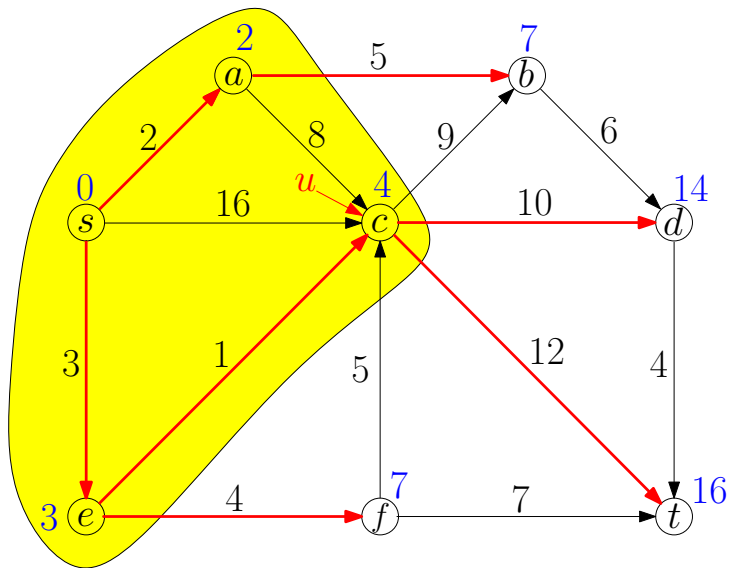


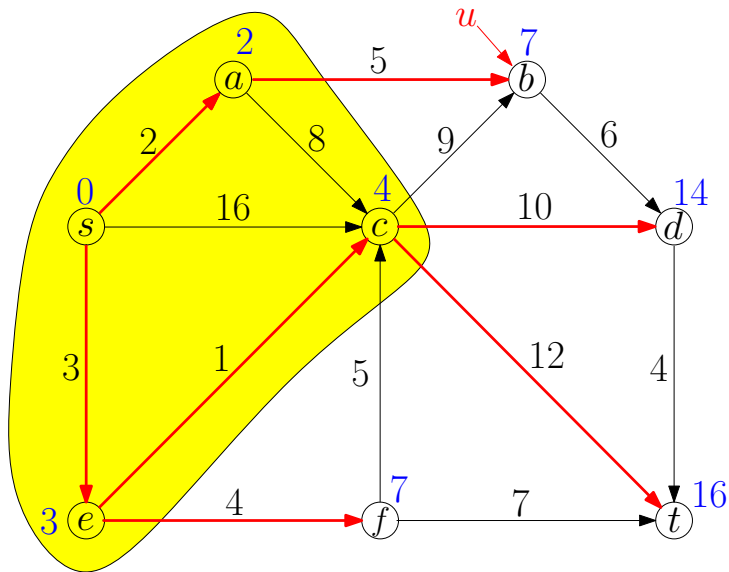


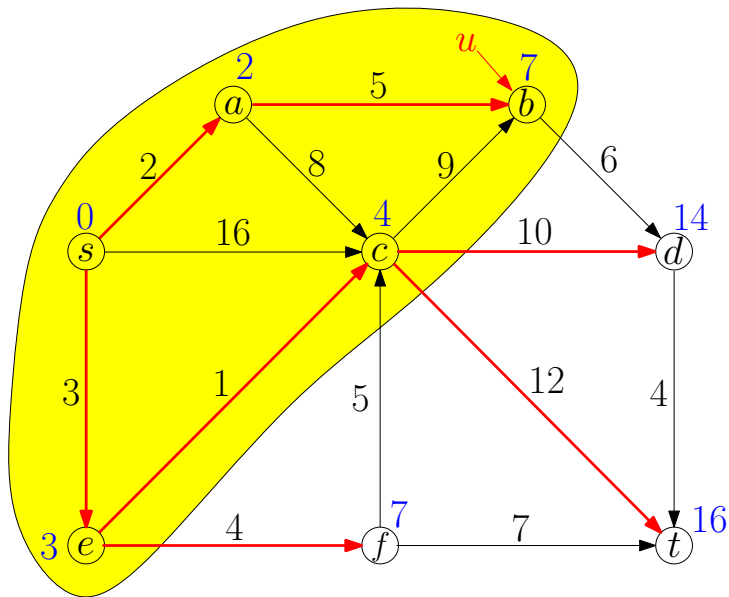


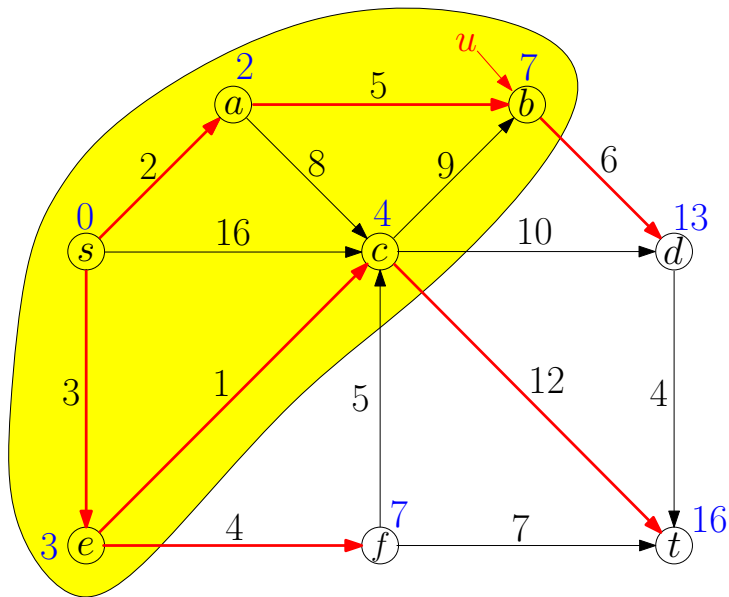


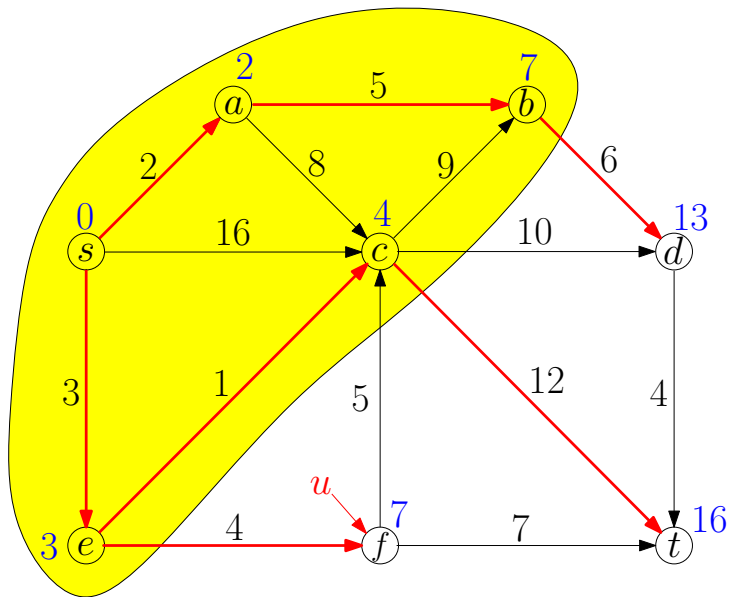


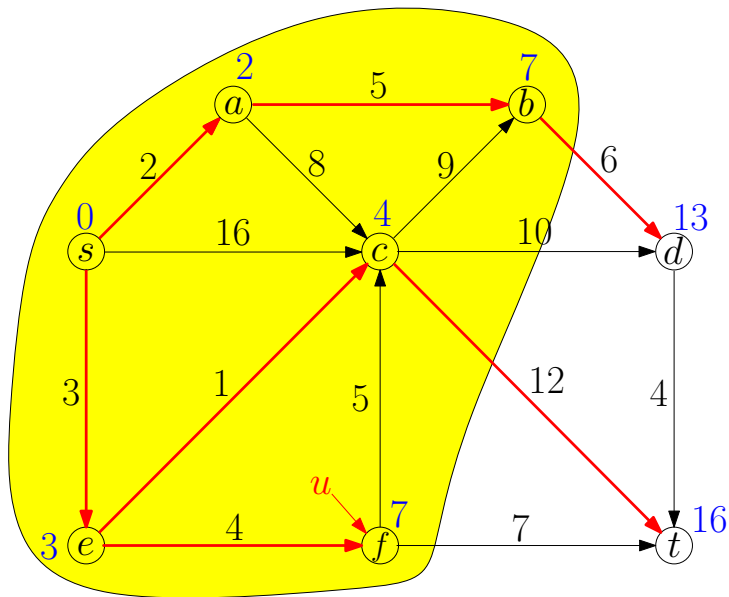


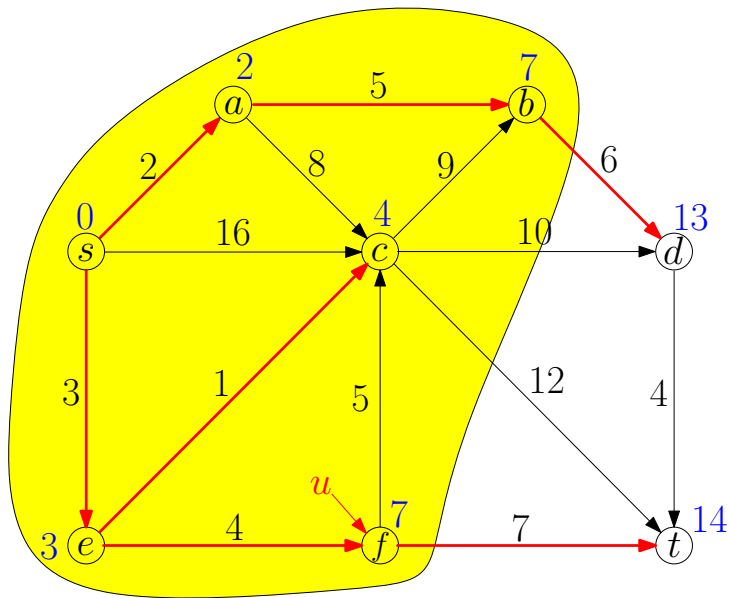


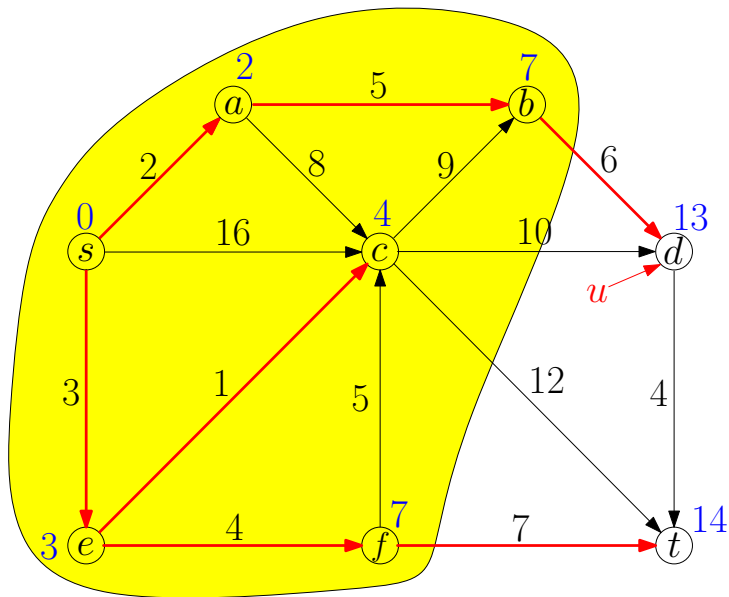


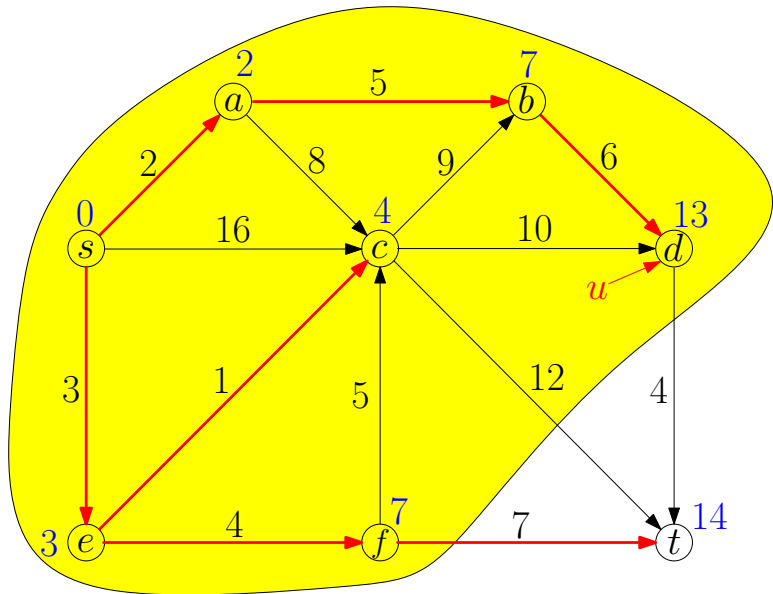


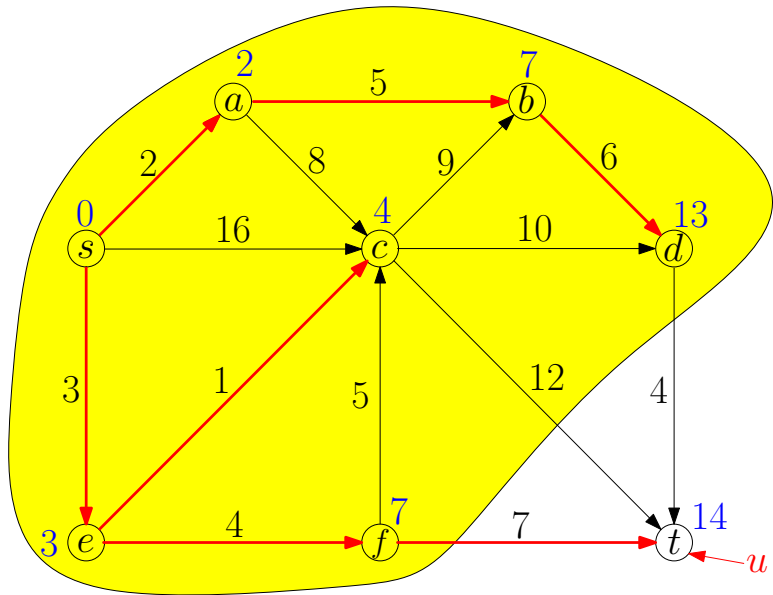


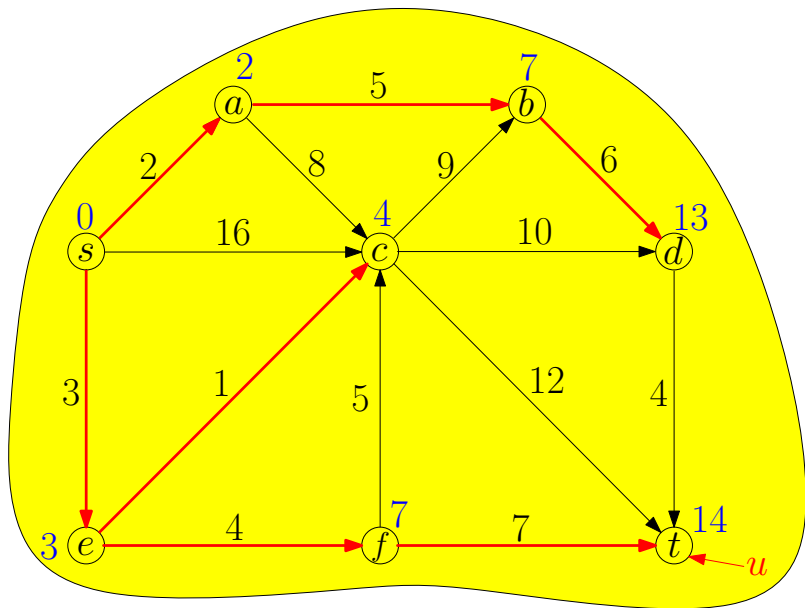












Improved Running Time using Priority Queue

Dijkstra(G, w, s)

```
1:  $s \leftarrow$  arbitrary vertex in  $G$ 
2:  $S \leftarrow \emptyset, d(s) \leftarrow 0$  and  $d[v] \leftarrow \infty$  for every  $v \in V \setminus \{s\}$ 
3:  $Q \leftarrow$  empty queue, for each  $v \in V$ :  $Q.\text{insert}(v, d[v])$ 
4: while  $S \neq V$  do
5:    $u \leftarrow Q.\text{extract\_min}()$ 
6:    $S \leftarrow S \cup \{u\}$ 
7:   for each  $v \in V \setminus S$  such that  $(u, v) \in E$  do
8:     if  $d[u] + w(u, v) < d[v]$  then
9:        $d[v] \leftarrow d[u] + w(u, v), Q.\text{decrease\_key}(v, d[v])$ 
10:       $\pi[v] \leftarrow u$ 
11: return  $(\pi, d)$ 
```

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

```
1:  $s \leftarrow$  arbitrary vertex in  $G$ 
2:  $S \leftarrow \emptyset, d(s) \leftarrow 0$  and  $d[v] \leftarrow \infty$  for every  $v \in V \setminus \{s\}$ 
3:  $Q \leftarrow$  empty queue, for each  $v \in V$ :  $Q.\text{insert}(v, d[v])$ 
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8:     if  $w(u, v) < d[v]$  then
9:        $d[v] \leftarrow w(u, v), Q.\text{decrease\_key}(v, d[v])$ 
10:       $\pi[v] \leftarrow u$ 
11: return  $\{(u, \pi[u]) \mid u \in V \setminus \{s\}\}$ 
```

Improved Running Time

Running time:

$$O(n) \times (\text{time for extract_min}) + O(m) \times (\text{time for decrease_key})$$

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Outline

- 1 Minimum Spanning Tree
 - Kruskal's Algorithm
 - Reverse-Kruskal's Algorithm
 - Prim's Algorithm
- 2 Single Source Shortest Paths
 - Dijkstra's Algorithm
- 3 Shortest Paths in Graphs with Negative Weights
- 4 All-Pair Shortest Paths and Floyd-Warshall
- 5 Minimum Cost Arborescence

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \rightarrow \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

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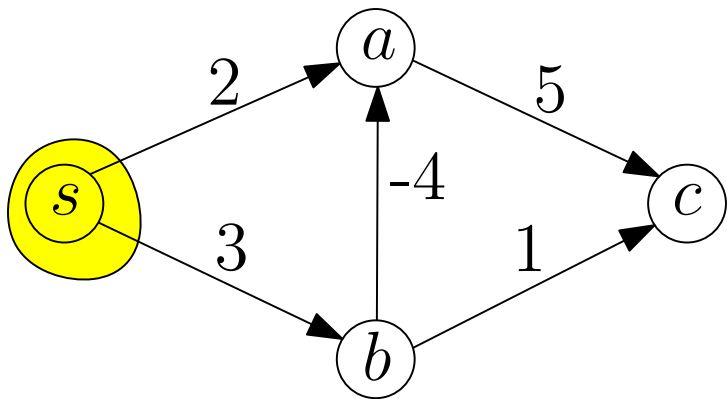
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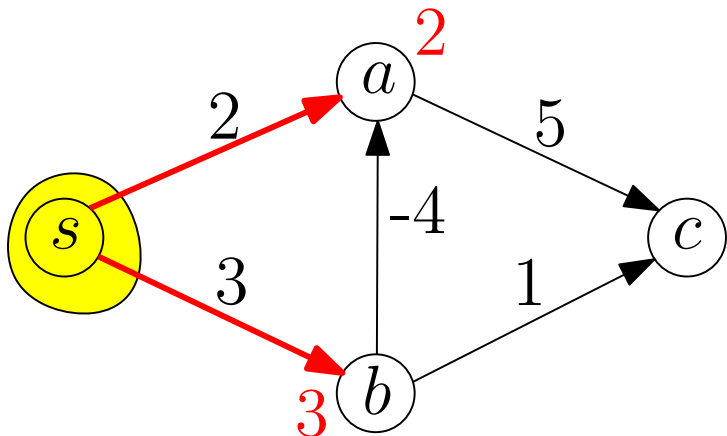
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- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' \rightarrow 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

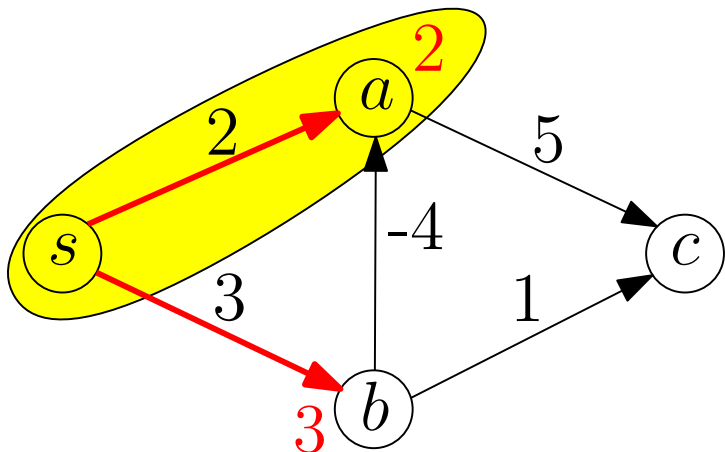
Dijkstra's Algorithm Fails if We Have Negative Weights



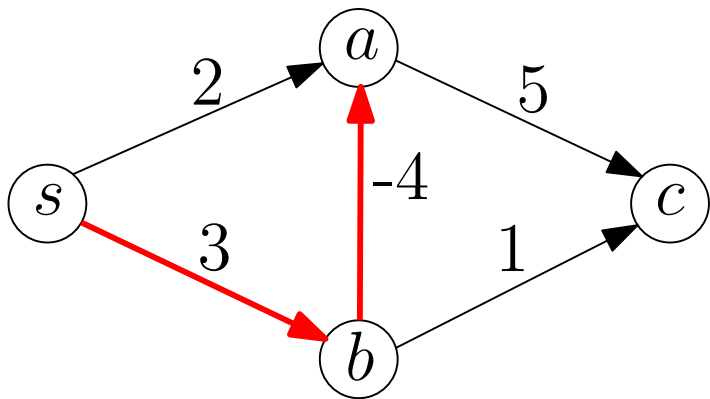
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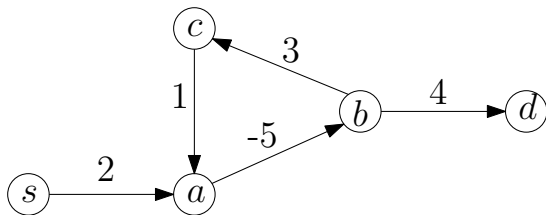


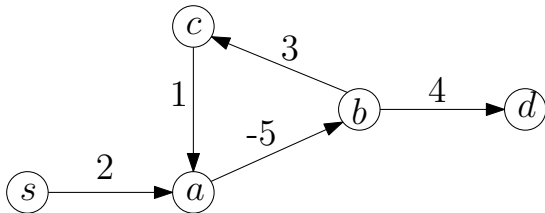
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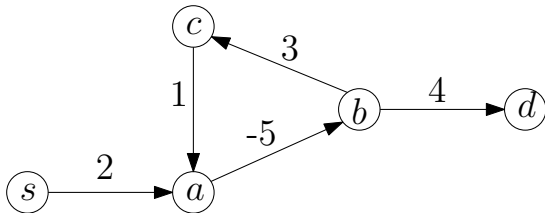
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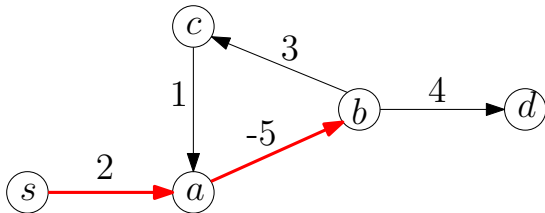


Q: What is the length of the shortest path from s to d ?



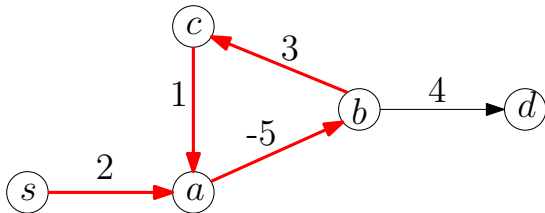
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A: $-\infty$



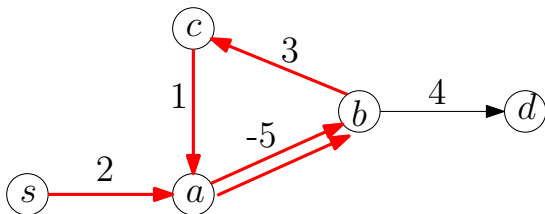
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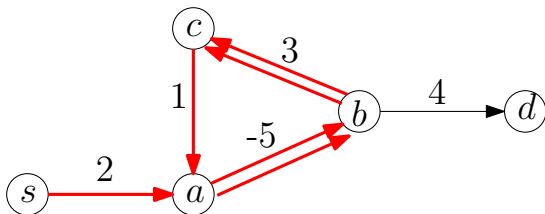
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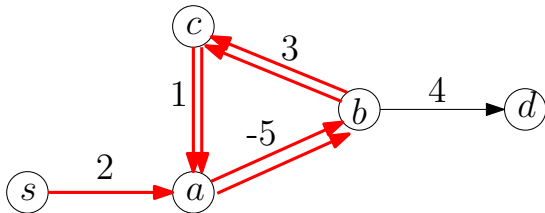
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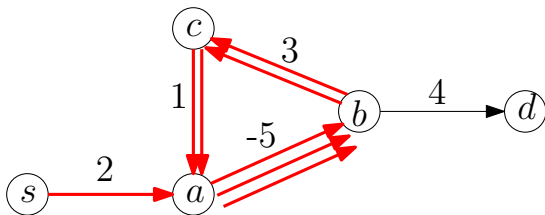
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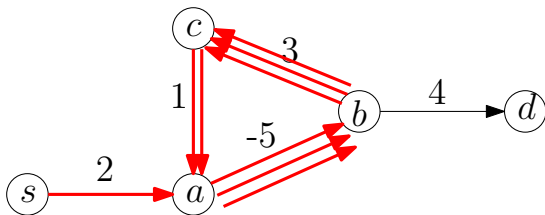
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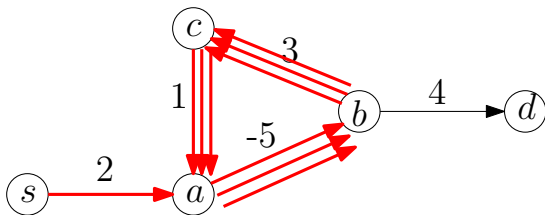
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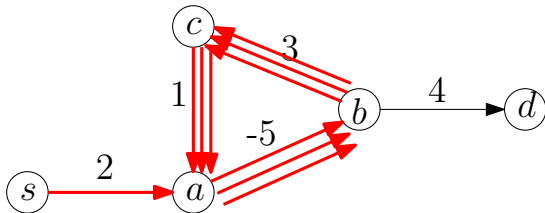
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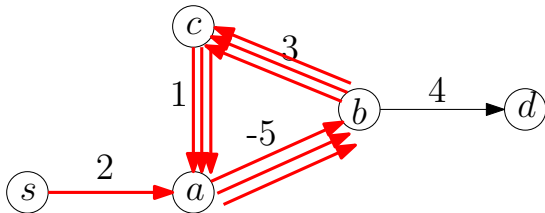
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Def. A negative cycle is a cycle in which the total weight of edges is negative.

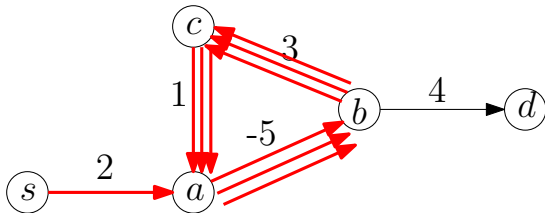


Q: What is the length of the shortest path from s to d ?

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Def. A negative cycle is a cycle in which the total weight of edges is negative.

Q: What is the length of the shortest **simple** path from s to d ?



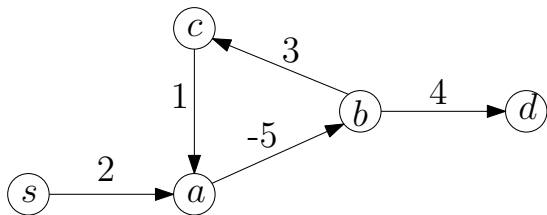
Q: What is the length of the shortest path from s to d ?

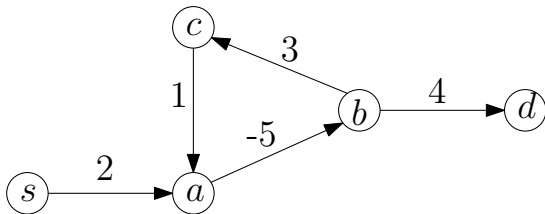
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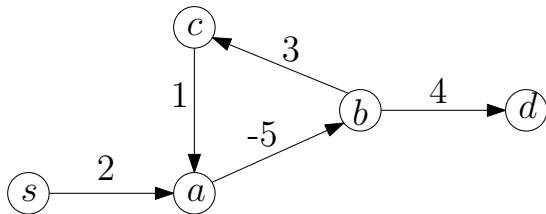
Q: What is the length of the shortest **simple** path from s to d ?

A: 1



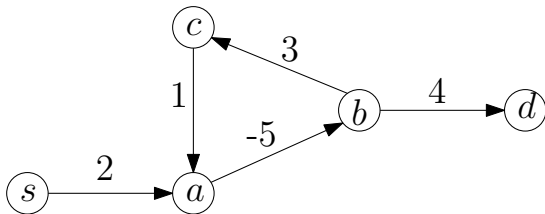


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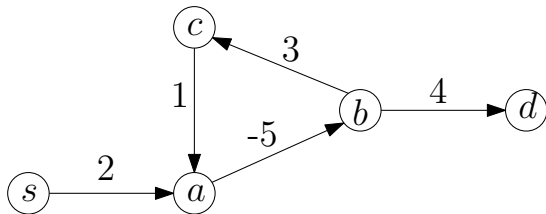
Dealing with Negative Cycles



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Dealing with Negative Cycles

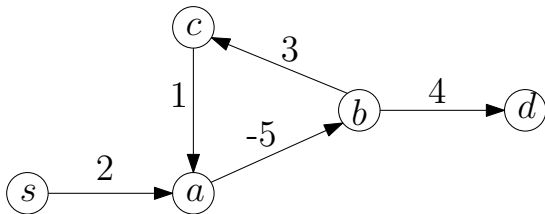
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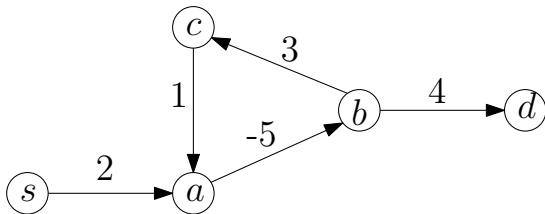
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- Easier: if negative cycle exists, allow algorithm to report “negative cycle exists” without computing distances



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Dealing with Negative Cycles

- We need to compute the shortest paths, among both simple and complex paths.
- Hardest: output $-\infty$ as a distance
- Easier: if negative cycle exists, allow algorithm to report “negative cycle exists” without computing distances
- Easiest: assume negative cycles do not exist; all shortest paths are automatically simple paths

algorithm	graph	weights	SS?	running time
Simple DP	DAG	\mathbb{R}	SS	$O(n + m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	\mathbb{R}	SS	$O(nm)$
Floyd-Warshall	U/D	\mathbb{R}	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

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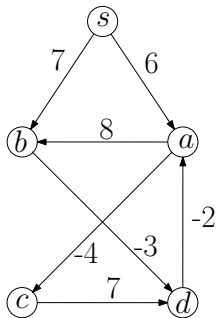
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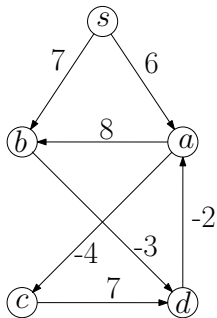
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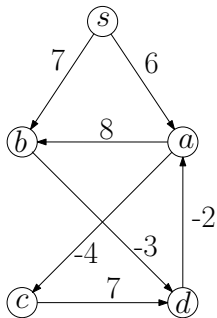
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- issue: do not know in which order we compute $f[v]$'s
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \dots, n-1\}$, $v \in V$: length of shortest path from s to v **that uses at most ℓ edges**



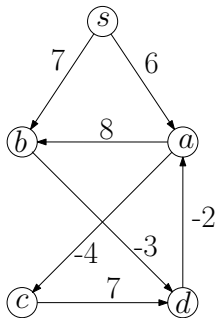
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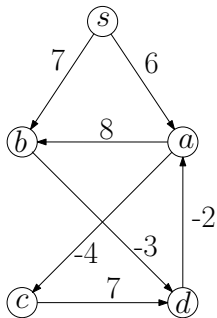
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- $f^2[a] =$



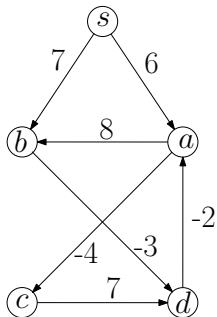
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- $f^2[a] = 6$



- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \dots, n-1\}$, $v \in V$:
length of shortest path from s to v that uses
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- $f^2[a] = 6$
- $f^3[a] =$



- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \dots, n-1\}$, $v \in V$:
length of shortest path from s to v that uses
at most ℓ edges
- $f^2[a] = 6$
- $f^3[a] = 2$



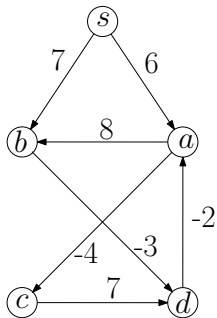
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- $f^2[a] = 6$
- $f^3[a] = 2$

$$f^\ell[v] = \left\{ \begin{array}{l} \end{array} \right.$$

$$\ell = 0, v = s$$

$$\ell = 0, v \neq s$$

$$\ell > 0$$



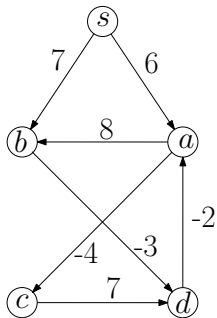
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- $f^2[a] = 6$
- $f^3[a] = 2$

$$f^\ell[v] = \begin{cases} 0 \\ \end{cases}$$

$$\ell = 0, v = s$$

$$\ell = 0, v \neq s$$

$$\ell > 0$$



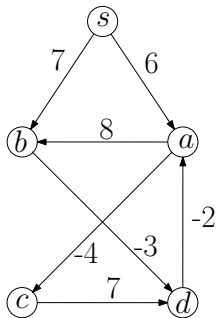
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \dots, n-1\}$, $v \in V$:
length of shortest path from s to v that uses
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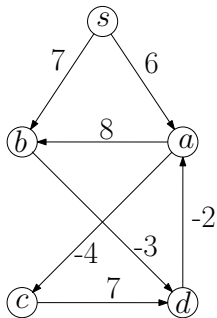
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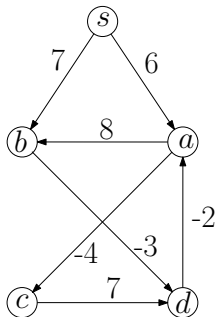
$$f^\ell[v] = \begin{cases} 0 \\ \infty \\ \min \left\{ \right. \end{cases}$$

$$f^{\ell-1}[v]$$

$$\ell = 0, v = s$$

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$$\ell > 0$$

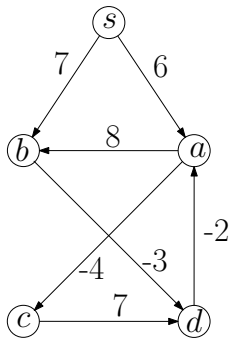


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$$f^\ell[v] = \begin{cases} 0 & \ell = 0, v = s \\ \infty & \ell = 0, v \neq s \\ \min \left\{ \begin{array}{l} f^{\ell-1}[v] \\ \min_{u:(u,v) \in E} (f^{\ell-1}[u] + w(u, v)) \end{array} \right. & \ell > 0 \end{cases}$$

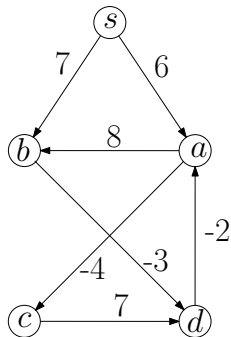
Dynamic Programming: Example

$$f^0 \quad \begin{array}{c} s \\ \textcircled{0} \end{array} \quad \begin{array}{c} a \\ \textcircled{\infty} \end{array} \quad \begin{array}{c} b \\ \textcircled{\infty} \end{array} \quad \begin{array}{c} c \\ \textcircled{\infty} \end{array} \quad \begin{array}{c} d \\ \textcircled{\infty} \end{array}$$

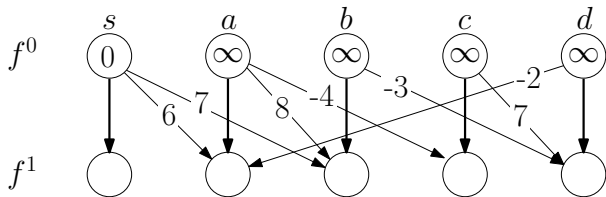


↓ length-0 edge

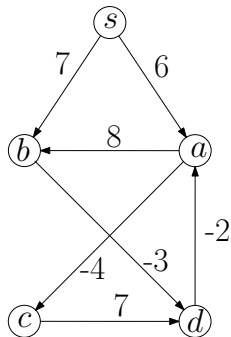
Dynamic Programming: Example



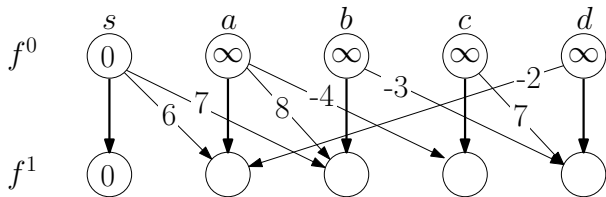
↓ length-0 edge



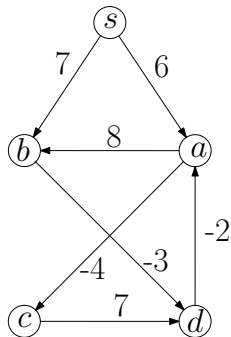
Dynamic Programming: Example



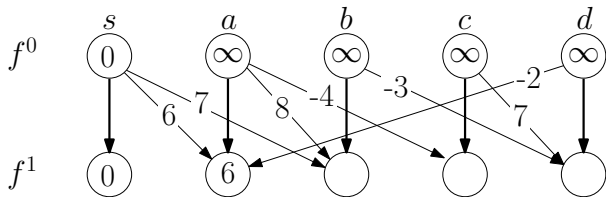
↓ length-0 edge



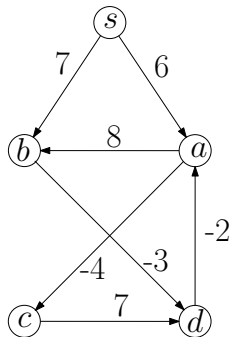
Dynamic Programming: Example



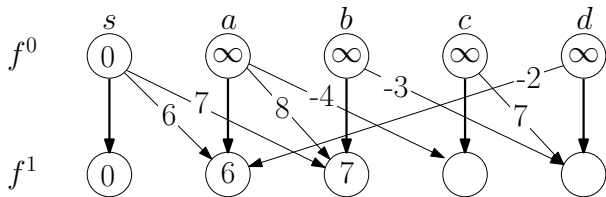
↓ length-0 edge



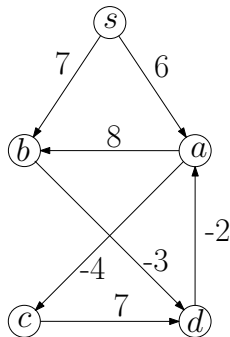
Dynamic Programming: Example



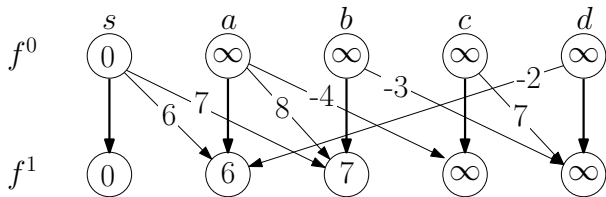
↓ length-0 edge



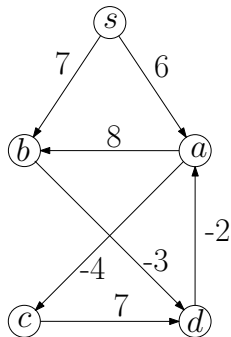
Dynamic Programming: Example



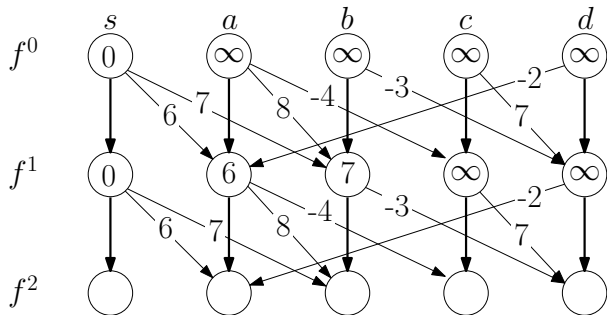
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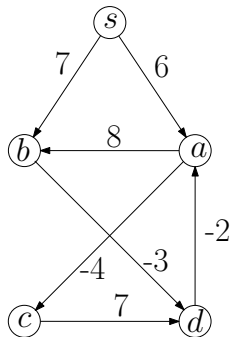
Dynamic Programming: Example



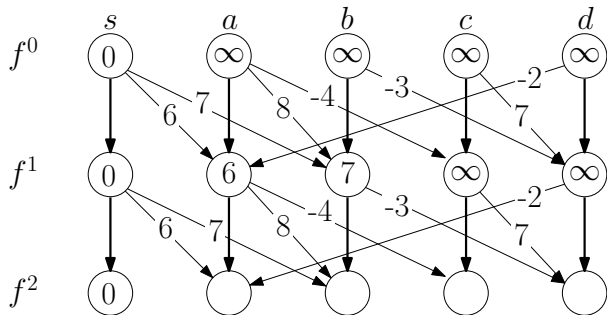
↓ length-0 edge



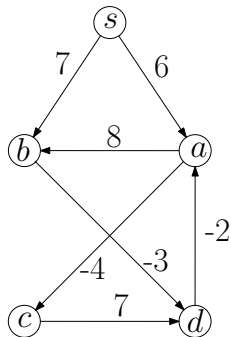
Dynamic Programming: Example



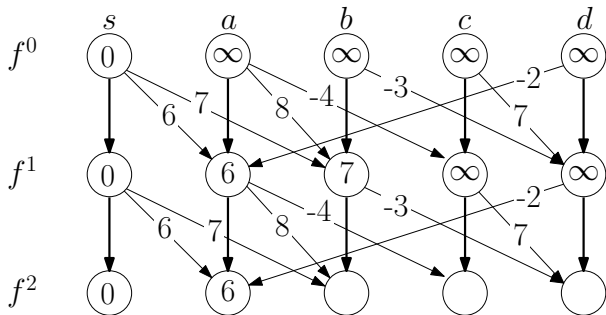
↓ length-0 edge



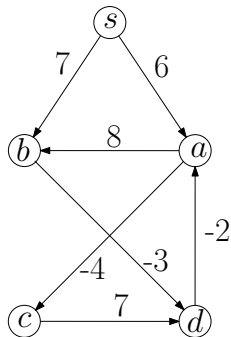
Dynamic Programming: Example



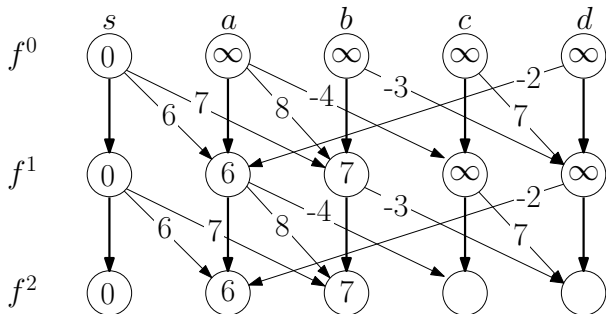
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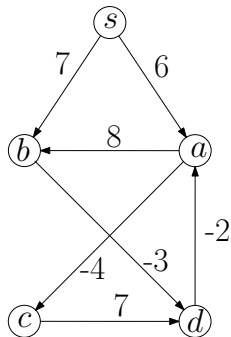
Dynamic Programming: Example



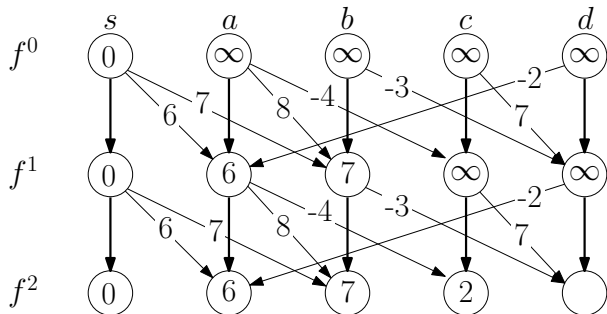
↓ length-0 edge



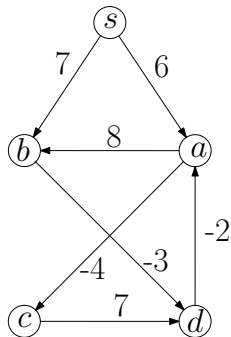
Dynamic Programming: Example



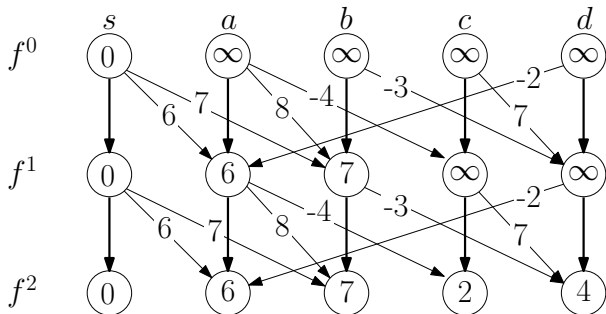
↓ length-0 edge



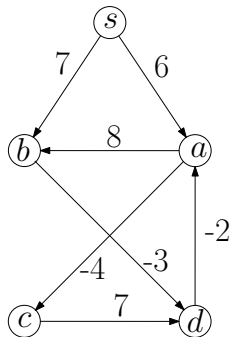
Dynamic Programming: Example



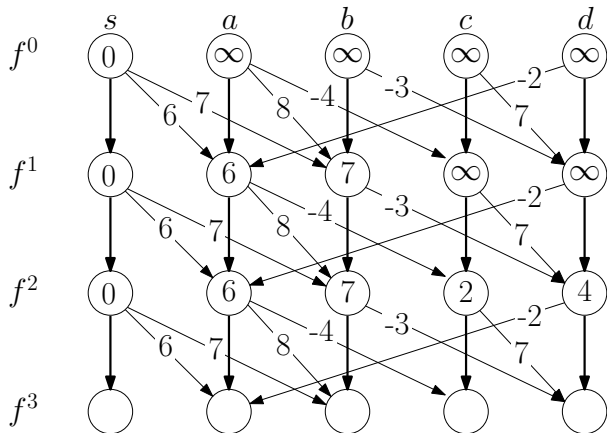
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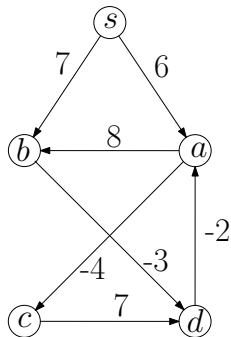
Dynamic Programming: Example



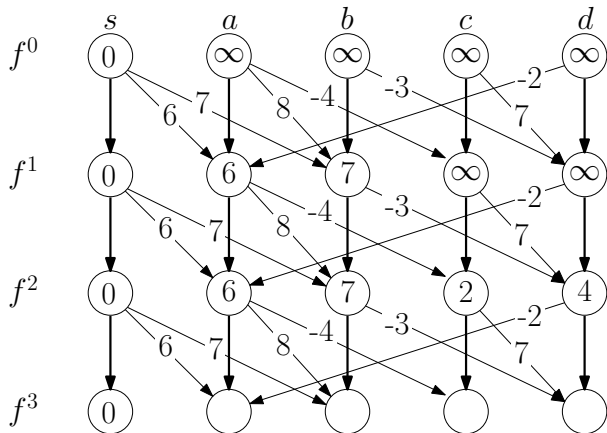
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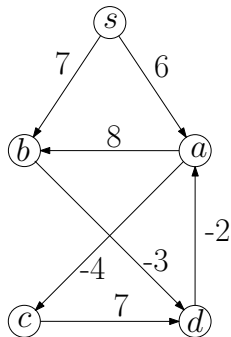
Dynamic Programming: Example



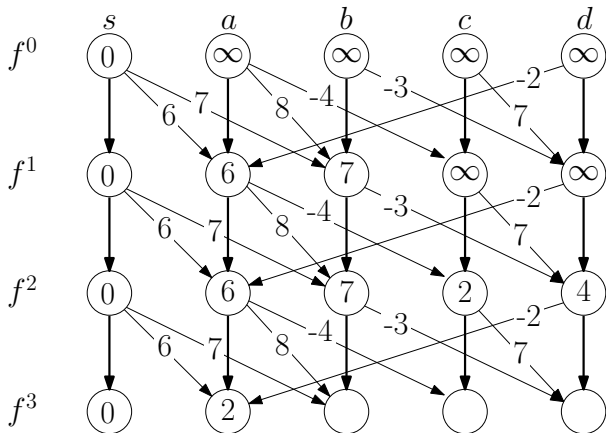
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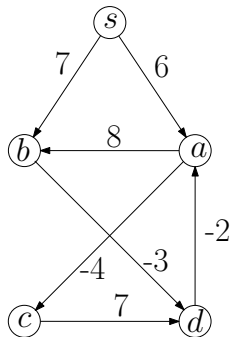
Dynamic Programming: Example



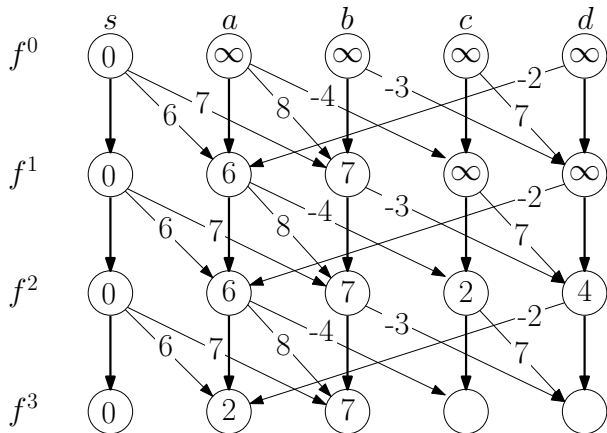
↓ length-0 edge



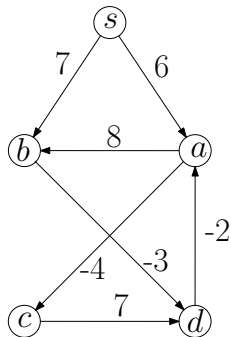
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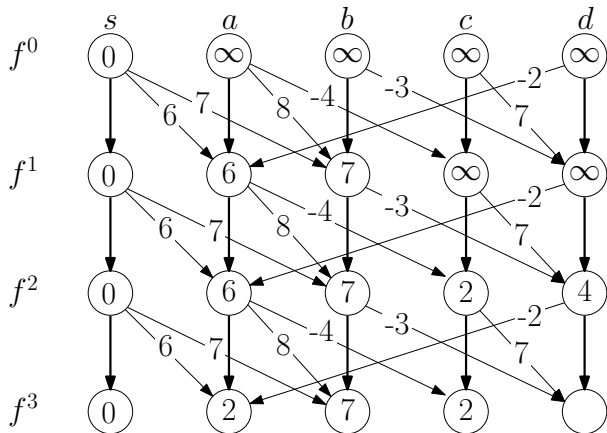
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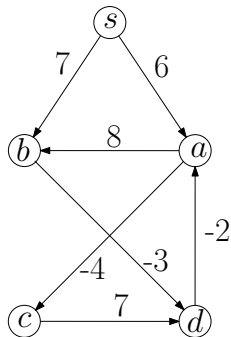
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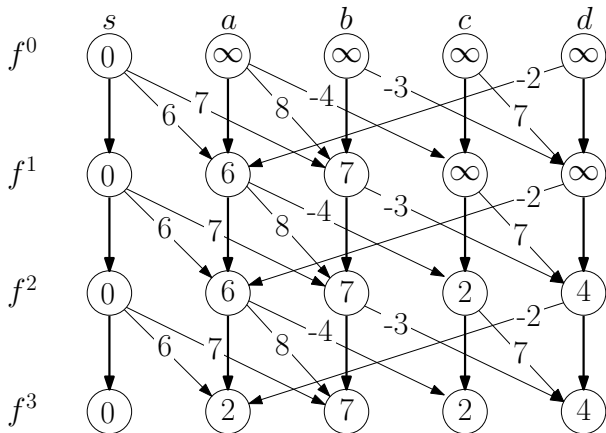
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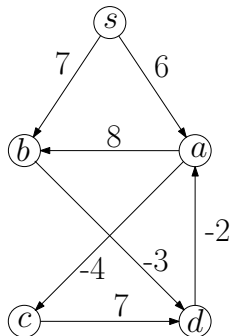
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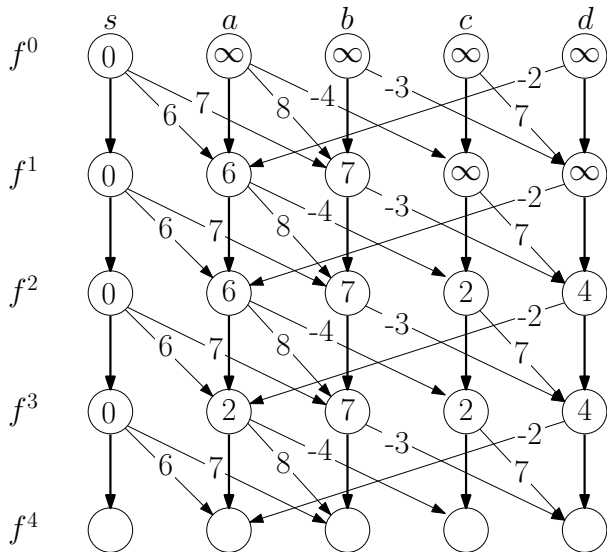
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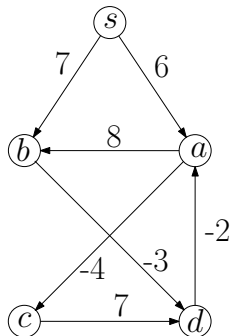
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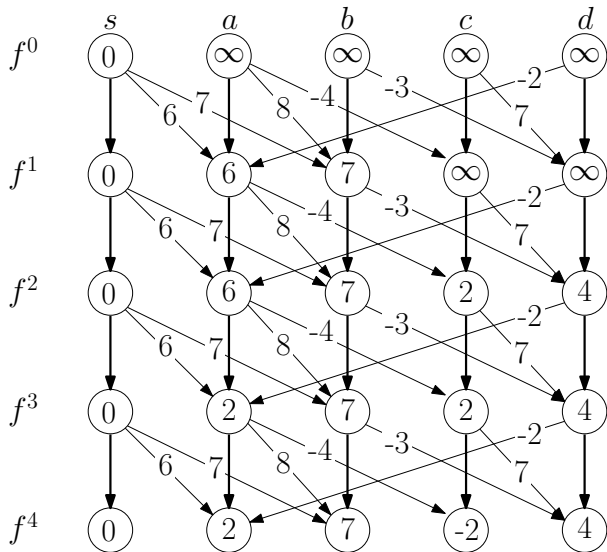
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Dynamic Programming: Example



↓ length-0 edge



dynamic-programming(G, w, s)

- 1: $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
- 2: **for** $\ell \leftarrow 1$ to $n - 1$ **do**
- 3: copy $f^{\ell-1} \rightarrow f^\ell$
- 4: **for** each $(u, v) \in E$ **do**
- 5: **if** $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$ **then**
- 6: $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
- 7: **return** $(f^{n-1}[v])_{v \in V}$

dynamic-programming(G, w, s)

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Obs. Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges

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Obs. Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges

Proof.

If there is a path containing at least n edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length. \square

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

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1:  $f^{\text{old}}[s] \leftarrow 0$  and  $f^{\text{old}}[v] \leftarrow \infty$  for any  $v \in V \setminus \{s\}$ 
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3:   copy  $f^{\text{old}} \rightarrow f^{\text{new}}$ 
4:   for each  $(u, v) \in E$  do
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6:        $f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v)$ 
7:   copy  $f^{\text{new}} \rightarrow f^{\text{old}}$ 
8: return  $f^{\text{old}}$ 
```

- f^ℓ only depends on $f^{\ell-1}$: only need 2 vectors

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Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

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- After iteration ℓ , $f[v]$ is **at most** the length of the shortest path from s to v that uses at most ℓ edges

Bellman-Ford Algorithm

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```

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only “accelerate” the process!
- After iteration ℓ , $f[v]$ is **at most** the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of **some path** from s to v

Bellman-Ford Algorithm

- After iteration ℓ :

$$\begin{aligned} & \text{length of shortest } s\text{-}v \text{ path} \\ & \leq f[v] \\ & \leq \text{length of shortest } s\text{-}v \text{ path using at most } \ell \text{ edges} \end{aligned}$$

Bellman-Ford Algorithm

- After iteration ℓ :

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- Assuming there are no negative cycles:

$$\begin{aligned} & \text{length of shortest } s\text{-}v \text{ path} \\ & = \text{length of shortest } s\text{-}v \text{ path using at most } n - 1 \text{ edges} \end{aligned}$$

Bellman-Ford Algorithm

- After iteration ℓ :

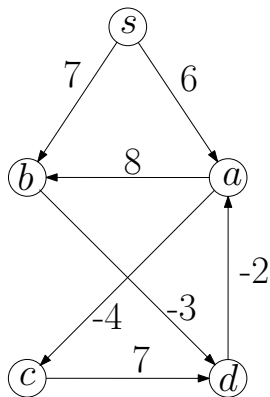
$$\begin{aligned} & \text{length of shortest } s\text{-}v \text{ path} \\ & \leq f[v] \\ & \leq \text{length of shortest } s\text{-}v \text{ path using at most } \ell \text{ edges} \end{aligned}$$

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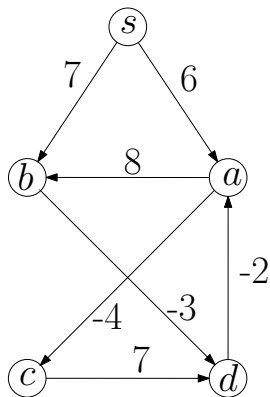
- So, assuming there are no negative cycles, after iteration $n - 1$:

$$f[v] = \text{length of shortest } s\text{-}v \text{ path}$$



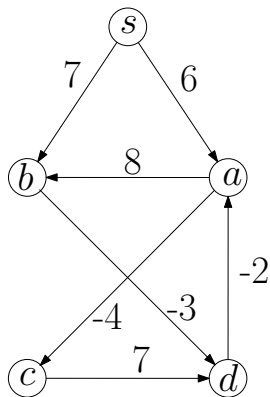
- order in which we consider edges:
 (s, a) , (s, b) , (a, b) , (a, c) , (b, d) ,
 (c, d) , (d, a)

vertices	s	a	b	c	d
f	0	∞	∞	∞	∞



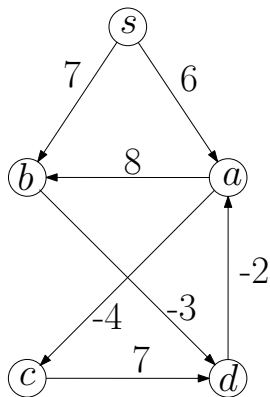
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vertices	s	a	b	c	d
f	0	∞	∞	∞	∞



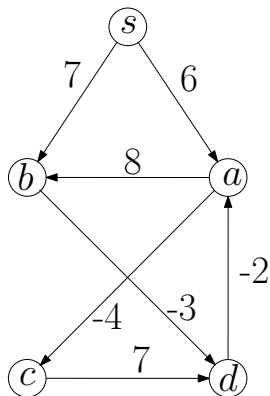
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vertices	<i>s</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>f</i>	0	6	∞	∞	∞



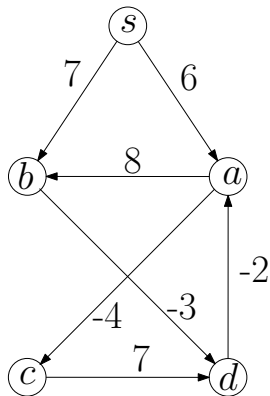
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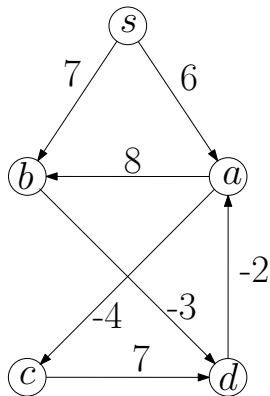
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vertices	s	a	b	c	d
f	0	6	7	∞	∞



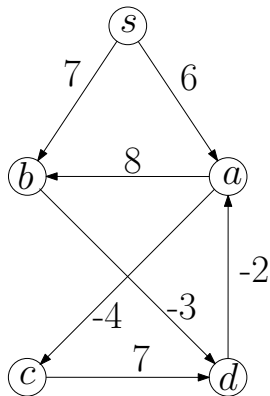
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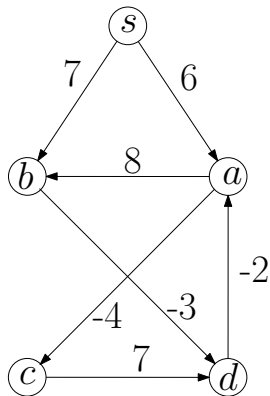
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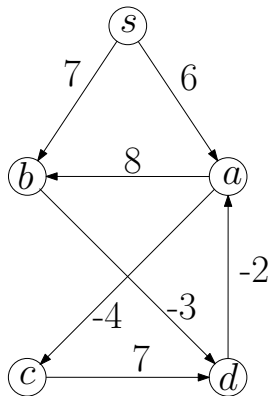
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vertices	s	a	b	c	d
f	0	6	7	2	∞



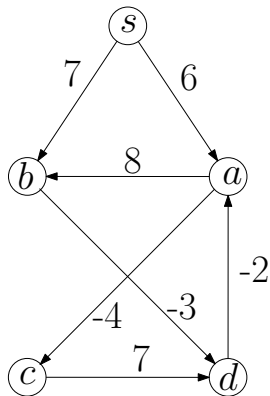
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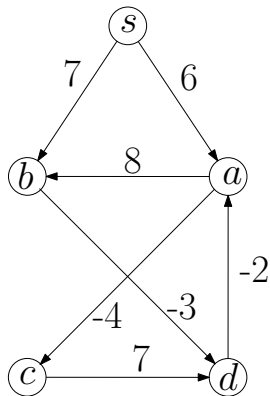
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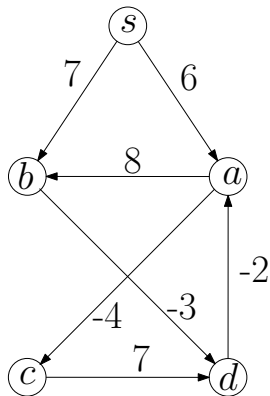
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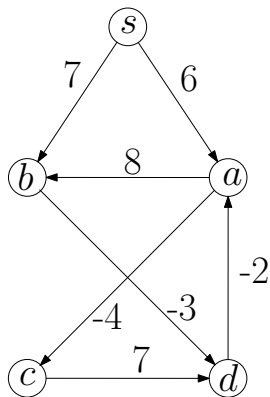
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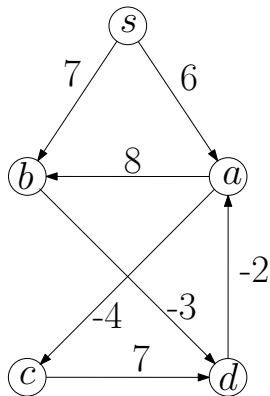
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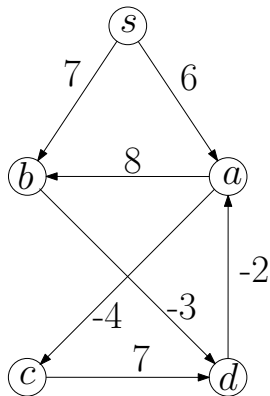
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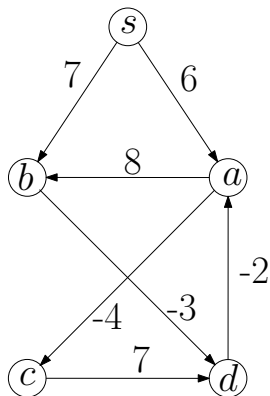
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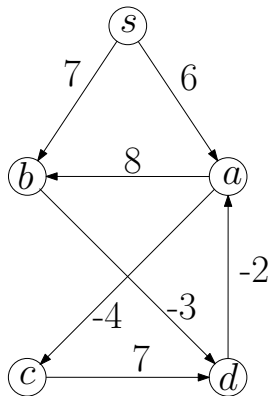
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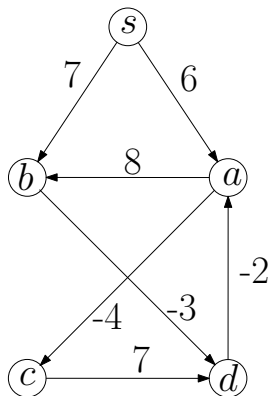
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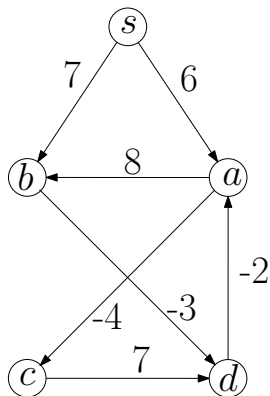
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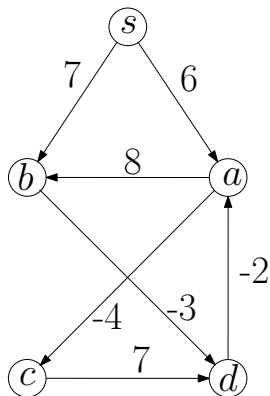
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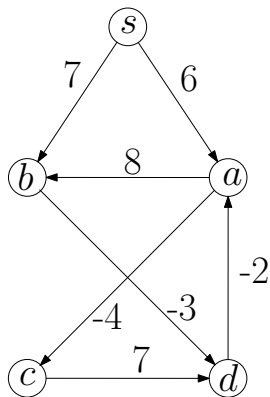
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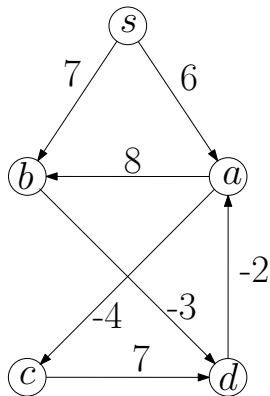
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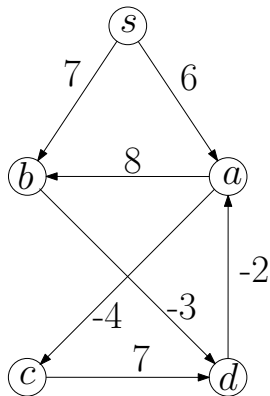
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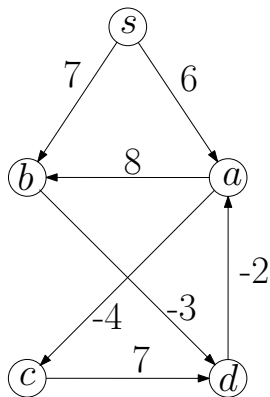
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- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4
- end of iteration 3: 0, 2, 7, -2, 4
- Algorithm terminates in 3 iterations, instead of 4.

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

- 1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
- 2: **for** $\ell \leftarrow 1$ to n **do**
- 3: $updated \leftarrow \text{false}$
- 4: **for each** $(u, v) \in E$ **do**
- 5: **if** $f[u] + w(u, v) < f[v]$ **then**
- 6: $f[v] \leftarrow f[u] + w(u, v)$
- 7: $updated \leftarrow \text{true}$
- 8: **if not** $updated$, then return f
- 9: output “negative cycle exists”

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- $\pi[v]$: the parent of v in the shortest path tree
- Running time = $O(nm)$

Outline

- 1 Minimum Spanning Tree
 - Kruskal's Algorithm
 - Reverse-Kruskal's Algorithm
 - Prim's Algorithm
- 2 Single Source Shortest Paths
 - Dijkstra's Algorithm
- 3 Shortest Paths in Graphs with Negative Weights
- 4 All-Pair Shortest Paths and Floyd-Warshall
- 5 Minimum Cost Arborescence

All-Pair Shortest Paths

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Input: directed graph $G = (V, E)$,
 $w : E \rightarrow \mathbb{R}$ (can be negative)

Output: shortest path from u to v for **every** $u, v \in V$

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- Running time = $O(n^2m)$

Summary of Shortest Path Algorithms we learned

algorithm	graph	weights	SS?	running time
Simple DP	DAG	\mathbb{R}	SS	$O(n + m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	\mathbb{R}	SS	$O(nm)$
Floyd-Warshall	U/D	\mathbb{R}	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

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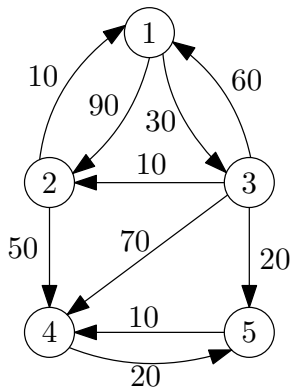
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- Issue: do not know in which order we compute $f[i, j]$'s
- $f^k[i, j]$: length of shortest path from i to j **that only uses vertices $\{1, 2, 3, \dots, k\}$ as intermediate vertices**

Example for Definition of $f^k[i, j]$'s



$$f^0[1, 4] = \infty$$

$$f^1[1, 4] = \infty$$

$$f^2[1, 4] = 140 \quad (1 \rightarrow 2 \rightarrow 4)$$

$$f^3[1, 4] = 90 \quad (1 \rightarrow 3 \rightarrow 2 \rightarrow 4)$$

$$f^4[1, 4] = 90 \quad (1 \rightarrow 3 \rightarrow 2 \rightarrow 4)$$

$$f^5[1, 4] = 60 \quad (1 \rightarrow 3 \rightarrow 5 \rightarrow 4)$$

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$$f^k[i, j] = \begin{cases} w(i, j) & k = 0 \\ \min \left\{ \begin{array}{l} f^k[i, v] + w(v, j) \\ f^k[v, i] + w(i, v) \end{array} \right\} & k = 1, 2, \dots, n \end{cases}$$

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$$f^k[i, j] = \begin{cases} w(i, j) & k = 0 \\ \min \left\{ \begin{array}{l} f^{k-1}[i, j] \\ \min_{1 \leq l < k} \{ f^{k-1}[i, l] + w(l, j) \} \end{array} \right. & k = 1, 2, \dots, n \end{cases}$$

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Floyd-Warshall(G, w)

```
1:  $f^0 \leftarrow w$ 
2: for  $k \leftarrow 1$  to  $n$  do
3:   copy  $f^{k-1} \rightarrow f^k$ 
4:   for  $i \leftarrow 1$  to  $n$  do
5:     for  $j \leftarrow 1$  to  $n$  do
6:       if  $f^{k-1}[i, k] + f^{k-1}[k, j] < f^k[i, j]$  then
7:          $f^k[i, j] \leftarrow f^{k-1}[i, k] + f^{k-1}[k, j]$ 
```


Floyd-Warshall(G, w)

```
1:  $f^{\text{old}} \leftarrow w$ 
2: for  $k \leftarrow 1$  to  $n$  do
3:   copy  $f^{\text{old}} \rightarrow f^{\text{new}}$ 
4:   for  $i \leftarrow 1$  to  $n$  do
5:     for  $j \leftarrow 1$  to  $n$  do
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Floyd-Warshall(G, w)

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2: for  $k \leftarrow 1$  to  $n$  do
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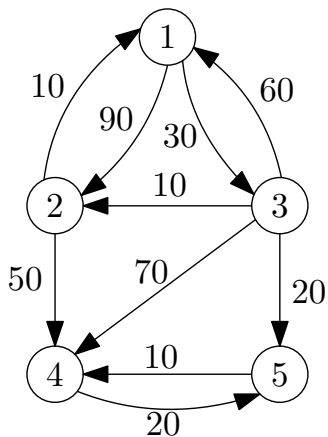
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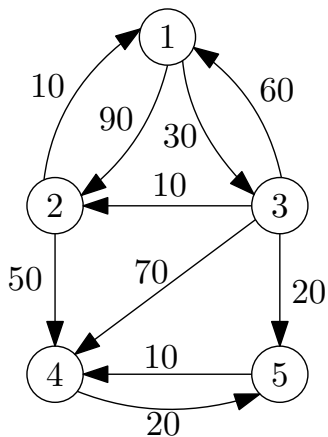
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- Running time = $O(n^3)$.



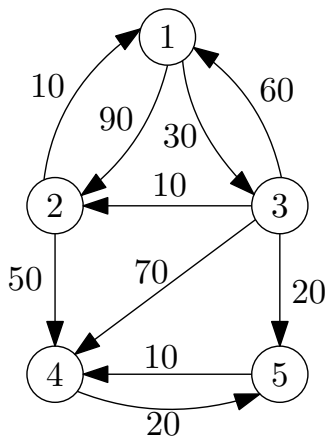
	1	2	3	4	5
1	0	90	30	∞	∞
2	10	0	∞	50	∞
3	60	10	0	70	20
4	∞	∞	∞	0	20
5	∞	∞	∞	10	0





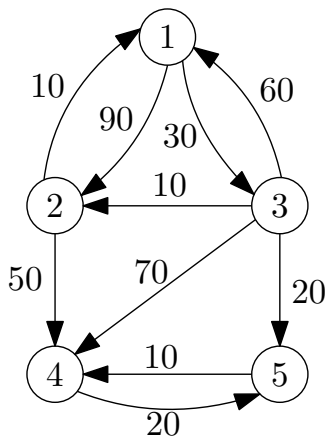
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1	0	90	30	∞	∞
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- $i = 2, k = 1, j = 3$



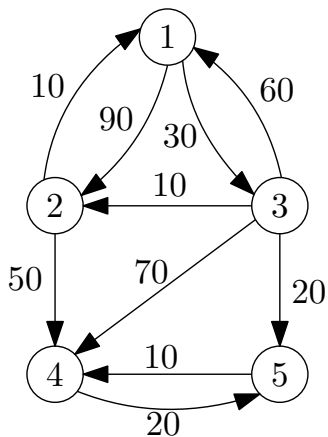
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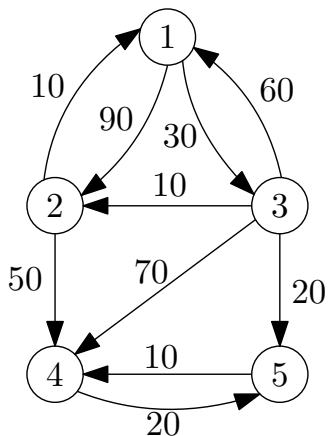
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4	∞	∞	∞	0	20
5	∞	∞	∞	10	0

- $i = 1, k = 2, j = 4$



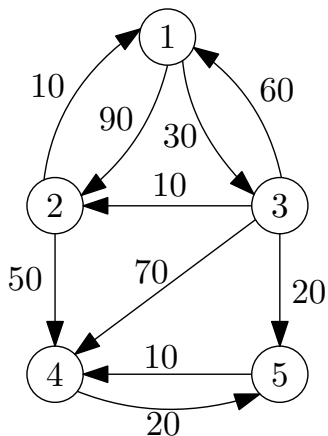
	1	2	3	4	5
1	0	90	30	140	∞
2	10	0	40	50	∞
3	60	10	0	70	20
4	∞	∞	∞	0	20
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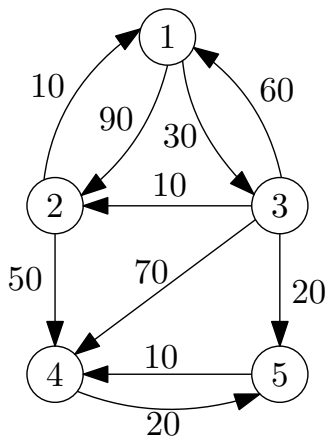
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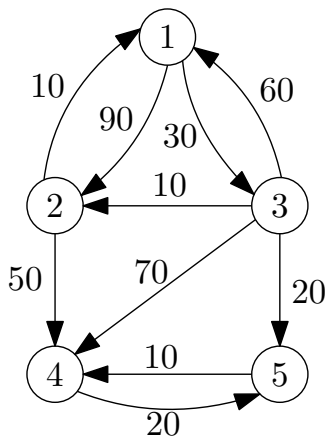
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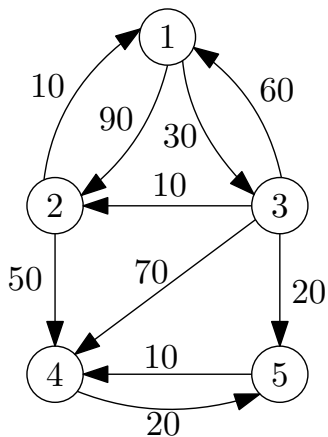
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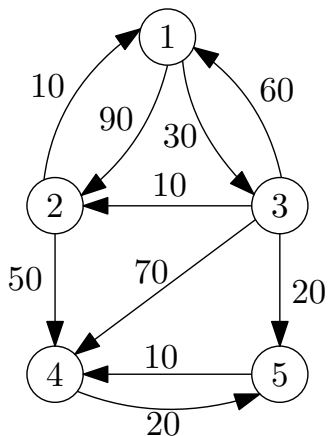
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Recovering Shortest Paths

Floyd-Warshall(G, w)

```
1:  $f \leftarrow w, \pi[i, j] \leftarrow \perp$  for every  $i, j \in V$ 
2: for  $k \leftarrow 1$  to  $n$  do
3:   for  $i \leftarrow 1$  to  $n$  do
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```

print-path(i, j)

```
1: if  $\pi[i, j] = \perp$  then then
2:   if  $i \neq j$  then print( $i, ","$ )
3: else
4:   print-path( $i, \pi[i, j]$ ), print-path( $\pi[i, j], j$ )
```

Detecting Negative Cycles

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7: for  $k \leftarrow 1$  to  $n$  do
8:   for  $i \leftarrow 1$  to  $n$  do
9:     for  $j \leftarrow 1$  to  $n$  do
10:      if  $f[i, k] + f[k, j] < f[i, j]$  then
11:        report "negative cycle exists" and exit
```

Summary of Shortest Path Algorithms

algorithm	graph	weights	SS?	running time
Simple DP	DAG	\mathbb{R}	SS	$O(n + m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	\mathbb{R}	SS	$O(nm)$
Floyd-Warshall	U/D	\mathbb{R}	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

Outline

- 1 Minimum Spanning Tree
 - Kruskal's Algorithm
 - Reverse-Kruskal's Algorithm
 - Prim's Algorithm
- 2 Single Source Shortest Paths
 - Dijkstra's Algorithm
- 3 Shortest Paths in Graphs with Negative Weights
- 4 All-Pair Shortest Paths and Floyd-Warshall
- 5 Minimum Cost Arborescence

Def. An arborescence is directed rooted tree, where all edges are directed away from the root.

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Minimum Cost Arborescence Problem

Input: a directed graph $G = (V, E)$,
edge weights $w : E \rightarrow \mathbb{R}_{\geq 0}$
root $r \in V$

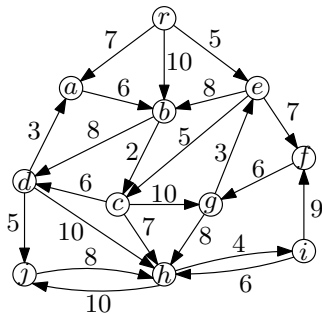
Output: a minimum-cost sub-graph
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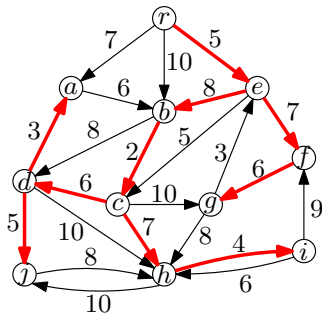


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Assumptions

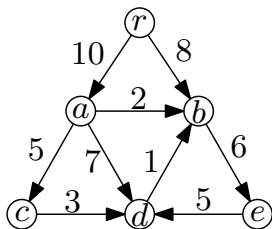
- the root r does not have incoming edges.
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- For every $v \in V \setminus \{r\}$, define $l_v = \min_{e \in \delta_v^{\text{in}}} w(e)$.
 - For every $v \in V \setminus \{r\}$ and $e \in \delta_v^{\text{in}}$, define $w'(e) = w(e) - l_v$.

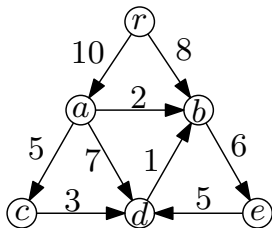
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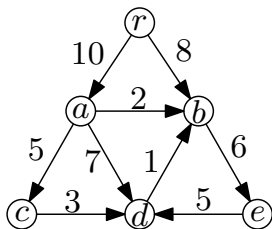
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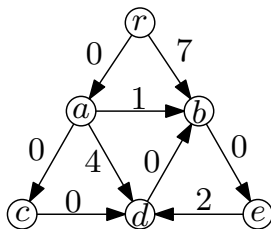
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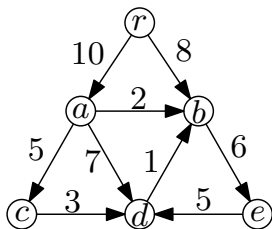
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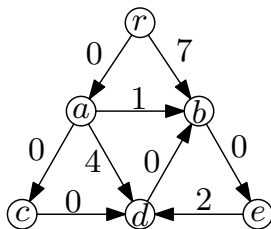
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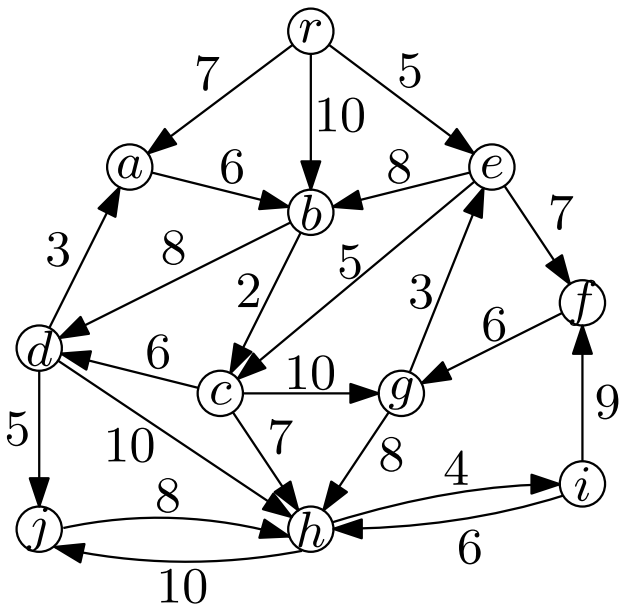
Given any tree solution T , $w(T) - w'(T)$ is always $\sum_{v \in V \setminus \{r\}} l_v$. \square

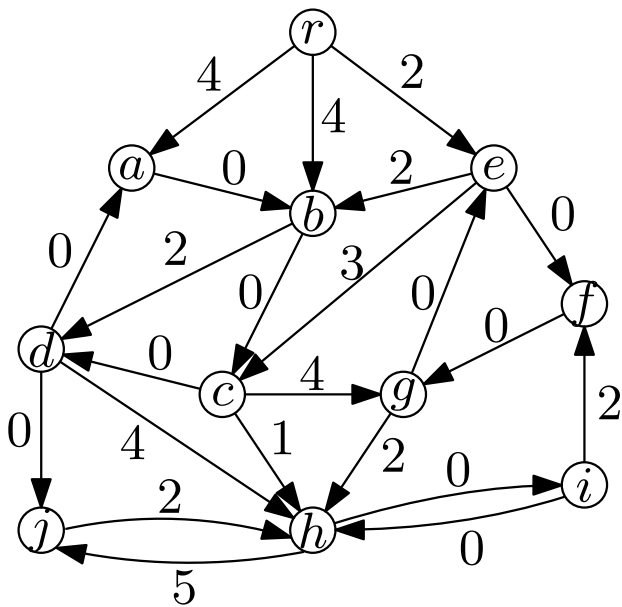
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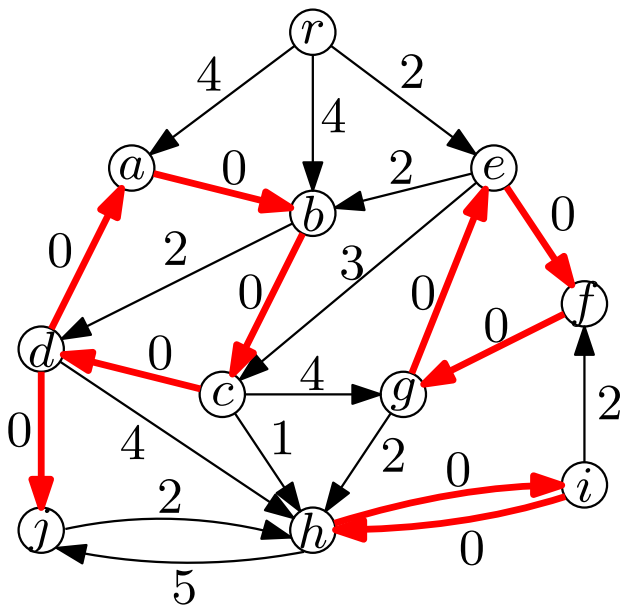
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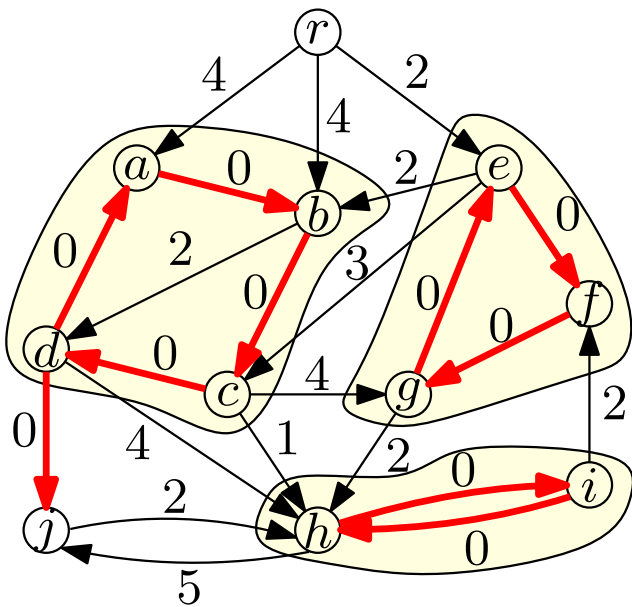
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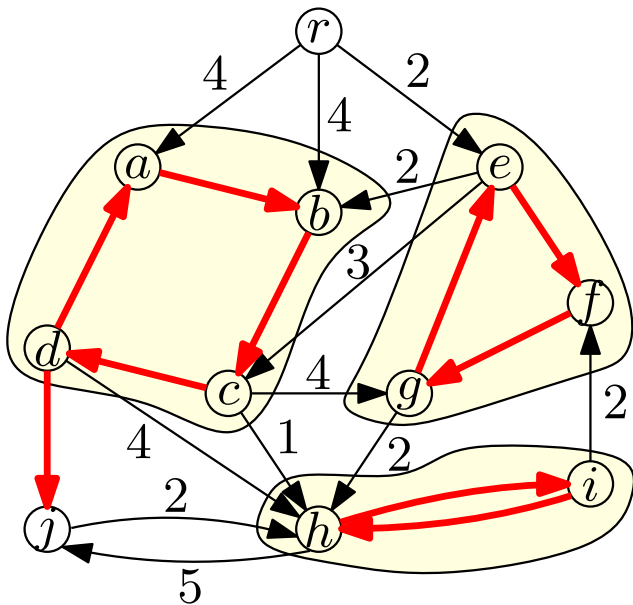
Lemma Let $(v_0, v_1, v_2, \dots, v_p = v_0)$ be a cycle C of 0-cost edges in G . Then there is an optimum solution T , that contains all but one edges in C .

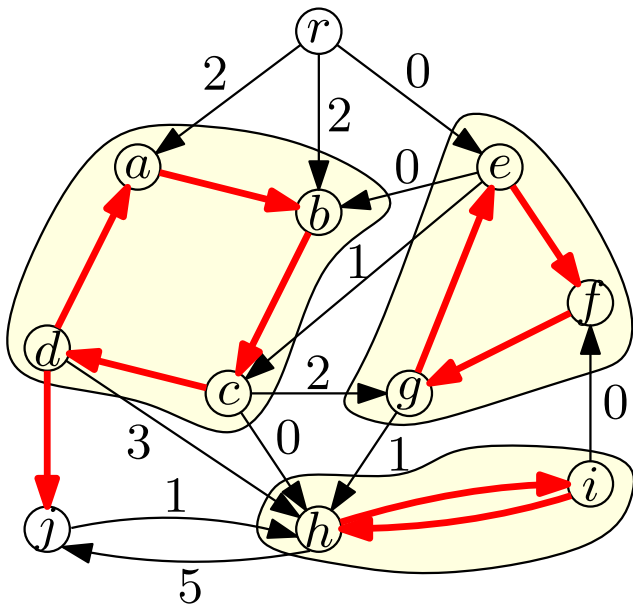


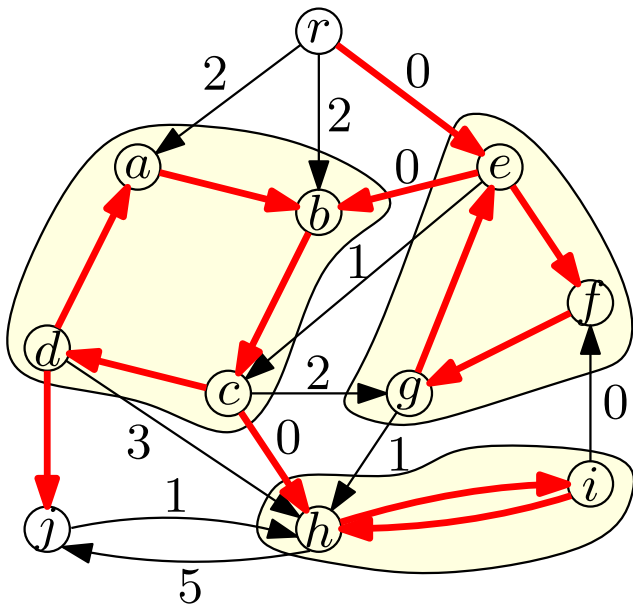


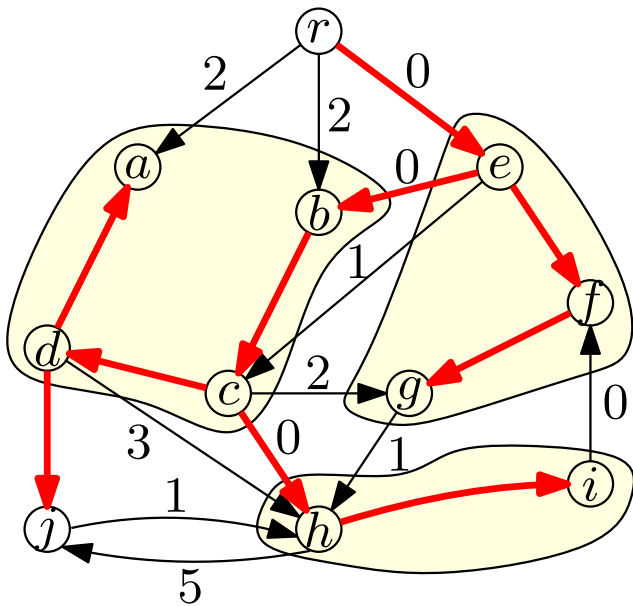


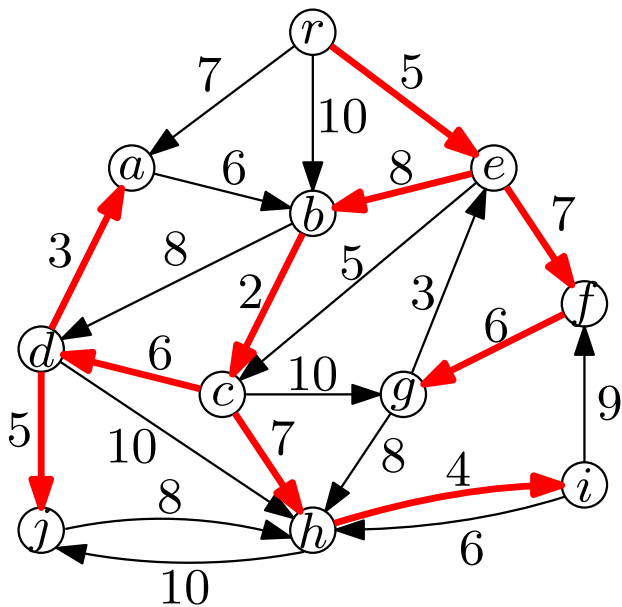












MCA(G, r, w)

- 1: $F^* \leftarrow \emptyset$
- 2: **for** every $v \in V \setminus \{r\}$ **do**
- 3: $l_v \leftarrow \min_{e \in \delta_v^{\text{in}}} w(e)$
- 4: **for** every edge e entering v **do**: $w'(e) \leftarrow w(e) - l_v$
- 5: choose a 0-cost edge entering v , add it to (V, F^*)
- 6: **if** F^* form an arborescence **then return** F^*
- 7: **else**
- 8: **for** every cycle C in F^* **do**: contract C into a single node
- 9: let $G' = (V', E')$ be the obtained graph.
- 10: $T' \leftarrow \text{MCA}(G', r, w')$
- 11: extend T' to an aborescence T in G , by keeping all but one edges in every cycle C in F^* , and **return** T

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- [Tarjan (1971)]: $O(\min(m \log n, n^2))$
- [Gabow, Galil, Spencer, Tarjan (1986)]: $O(n \log n + m)$
- [Mendelson, Tarjan, Thorup, Zwick (2006)]: $O(m \log \log n)$